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Local linear conditional cumulative distribution function with mixing data

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Abstract This paper investigates a conditional cumulative distribution of a scalar response given by a functional random variable with an α -mixing stationary sample using a local polynomial technique. The main purpose of this study is to establish asymptotic normality results under selected mixing conditions satisfied by many time-series analysis models in addition to the other appropriate conditions to confirm the planned prospects.

Mathematics Subject Classification 62G05 · 62G08 · 62G20 · 62G07 · 62G30 · 62H12

1 Introduction and motivations

With the evolution of the measuring instruments and the growth of research studies mainly since the publication of the pioneer paper of Ferraty and Vieu [19], functional data analysis (FDA) has attracted the attention of many works as in the recent monograph of Horváth and Kokoszka [23]. On the other hand, alternative conditional predictions of the classical regression have also gained a considerable interest in basically all the fields of statistics, especially for estimating conditional models using the kernel approach (or local constant) as investigated in the papers of Ferraty et al. [18], Dabo-Niang and Laksaci [8], or Ezzahrioui and Ould-Saïd [15].

In numerous nonparametric statistic problems, the estimation of a conditional distribution function (CDF) constitutes a key aspect of inference. Accordingly, the present study employs a specific CDF model for constructing prediction intervals that can be involved in many applications such as the survival analysis and reliability. Interestingly, it is well known that the CDF has the advantage to completely characterize the conditional law of the considered random variables. The determination of the CDF allows, in fact, to obtain the conditional density and conditional hazard functions. Moreover, several prediction tools can also be implemented for the nonparametric statistics modeling, taking the example of conditional mode, median, or quantile. In addition, an extensive literature including various nonparametric approaches has taken place in the conditional estimation of independent samples and dependent observations in finite- and infinite-dimensional spaces (see, for instance, Berline et al. [3], Honda [22], and Ferraty and Vieu [17]). In many situations, the

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kernel approach cannot adequately estimate the conditional models for the reason that this technique suffers from a large bias particularly at the boundary region.

However, the kernel approach can be improved using local polynomial smoothers, and especially local linear smoothers, because they correct the asymptotic bias that is adversely affected at the boundaries (see Fan and Gijbels [16] for more discussions on this subject in the real case). In recent years, there has been a strong interest in local linear smoothers in the infinite-dimensional space (see, for instance, Baïllo and Grané [1] and Barrientos-Marina et al. [2]). And it should be noted that the last precursor work has been extended in many directions, including asymptotic properties (see Demongeot et al. [11, 12] and Zhou and Lin [37]), nature of the variables (see Demongeot et al. [14]), or the dependence type (see Demongeot et al. [10] and Laksaci et al. [27]).

In this regard, our interest in this paper is to give a result concerning the limit in distribution of the estimate of the CDF, by the local linear fit. More precisely, we consider the case when the observations $(X_i, Y_i)_{i \geq 0}$ are strongly mixing. We prove the asymptotic normality of a local linear estimator of the CDF by utilizing the appropriate form of Bernstein's blocking arguments and a reduction analysis leading to the Lindeberg–Feller central limit Theorem. We point out that this contribution has a potential impact in practice as well as in the theory. Indeed, from a practical point of view, this asymptotic property is used to derive confidence intervals or to make statistical tests. On the other hand, from a theoretical point of view, the asymptotic normality is a basic ingredient to determine the mean quadratic error or to study the uniform integrability of the estimator.

Accordingly, this work is mainly structured as follows: Sect. 2 will present the model selected for study, describe the estimation method through giving the explicit solution to the minimization problem, and provide some basic assumptions and notations. Section 3 will state the main asymptotic normality results achieved by the conditional distribution function estimator, indicating that their accuracy will lead to interesting perspectives. Finally, Sect. 4 will discuss the applicability of the provided asymptotic results on some statistical problems such determination of confidence intervals. Detailed proofs of the main results will be consequently postponed to the appendix.

2 The model

The model is defined in the following way. Assuming that $(X_i, Y_i)_{1 \leq i \leq n}$ is stationary α -mixing process. The X_i are random variables with values in a functional space \mathfrak{F} where the random variables Y_i are real. In all the sequel \mathfrak{F} , a semi-metric space endowed with a semi-metric $d(\cdot, \cdot)$ is taken into consideration. For $x \in \mathfrak{F}$, the conditional probability distribution of Y_i which is given by $X_i = x$ is classically written as follows:

$$\forall y \in \mathbb{R}, \quad F^x(y) = \mathbb{P}(Y_i \leq y | X_i = x).$$

This distribution is absolutely continuous with respect to the Lebesgue measure on \mathbb{R} .

2.1 The estimate

The conditional cumulative distribution function F^x is estimated by \hat{a} where the couple (\hat{a}, \hat{b}) is obtained by the optimization rule:

$$\sum_{i=1}^n (H(h_H^{-1}(y - Y_i)) - a - b\beta(X_i, x))^2 K(h_K^{-1}\delta(x, X_i)), \quad (1)$$

where $\beta(\cdot, \cdot)$ and $\delta(\cdot, \cdot)$ are known functions from \mathfrak{F}^2 into \mathbb{R} , K is a kernel, H is a cumulative distribution function, and h_K and h_H are the bandwidths parameters. However, if the bi-functional operator β is such that, $\forall z \in \mathfrak{F}$, $\beta(z, z) = 0$, then the quantity $\hat{F}^x(y)$ is explicitly defined by the following:

$$\hat{F}^x(y) = \frac{\sum_{i,j=1}^n w_{ij} H(h_H^{-1}(y - Y_j))}{\sum_{i,j=1}^n w_{ij}} = \frac{\sum_{j=1}^n \Delta_j K_j H_j}{\sum_{j=1}^n \Delta_j K_j}, \quad (2)$$

with $w_{ij} = \beta_i(\beta_i - \beta_j) K_i K_j$, $\Delta_j = K_j^{-1}(\sum_{i=1}^n w_{ij}) = \sum_{i=1}^n \beta_i^2 K_i - (\sum_{i=1}^n \beta_i K_i) \beta_j$, where $\beta_i = \beta(X_i, x) K_i = K(h_K^{-1}(\delta(x, X_i)))$, and $H_j = H(h_H^{-1}(y - Y_j))$.



Several asymptotic properties of this estimation are recently obtained. It turns out that the existing literature addresses either the statement of almost-complete consistencies or a mean-square error (see Demongeot et al. [12]).

2.2 Assumptions and notations

In what follows, x (resp., y) will denote a fixed point in \mathfrak{F} (resp., in \mathbb{R}), \mathcal{N}_x (resp., \mathcal{N}_y) will denote a fixed neighborhood of x (resp., of y), and $\phi_x(r_1, r_2) = \mathbb{P}(r_2 \leq \delta(X, x) \leq r_1)$, and let G be the real valued function defined as for any $l \in \{0, 2\}$: $G_l(s) = \mathbb{E}[g_l(X, y) - g_l(x, y) | \beta(x, X) = s]$ with $g_l(x, y) = \frac{\partial^l F^x(y)}{\partial y^l}$.

We now state some conditions which ensure asymptotic normality of (2):

- (H1) (i) For any $r > 0$, $\phi_x(r) := \phi_x(-r, r) > 0$, and there exists a function $\Psi_x(\cdot)$, such that: for all $t \in [-1, 1]$, $\lim_{h_K \rightarrow 0} \frac{\phi_x(th_K, h_K)}{\phi_x(h_K)} = \Psi_x(t)$.
 (ii) For any $l \in \{0, 2\}$, the quantities $G_l^{(2)}(0)$ exist.
- (H3) $\sup_{i \neq j} \mathbb{P}((X_i, X_j) \in B(x, h_K) \times B(x, h_K)) \leq \psi_x(h_K)$, where $\psi_x(h_K) = O(\phi_x^2(h_K))$.
- (H4) The coefficients of α -mixing sequence $(X_i, Y_i)_{i \in \mathbb{N}}$ satisfy the following two conditions:
 - (i) $\exists s > 1, \exists c > 0 : \forall n \in \mathbb{N}; \alpha(n) \leq cn^{-s}$.
 - (ii) $\sum_{k=1}^{+\infty} k^\delta (\alpha(k))^{1/p} < \infty$ for some $p > 0$ and $\delta > \frac{1}{p}$.
- (H5) The bandwidth h_K satisfies:
 - (i) $\lim_{n \rightarrow +\infty} n \phi_x^m(h_K) = +\infty$ for $m = 1, 2$.
 - (ii) $\lim_{n \rightarrow +\infty} n^\zeta \phi_x(h_K) = 0$ for $\zeta = 2, 3$.
 - (iii) $\exists C_1 > 0, C_2 > 0, C_2 n^{1-p} \leq \phi_x(h_K) \leq C_1 n^{\frac{1}{1-p}}$ for $p \geq 2$.
- (H6) The locating operator β satisfies the following two conditions:
 - (i) $\forall z \in \mathfrak{F}, C_1 |\delta(x, z)| \leq |\beta(x, z)| \leq C_2 |\delta(x, z)|$, where $C_1 > 0, C_2 > 0$,
 - (ii) $h_K \int_{B(x, h_K)} \beta(u, x) dP(u) = o\left(\int_{B(x, h_K)} \beta^2(u, x) dP(u)\right)$, where $B(x, r) = \{z \in \mathfrak{F} | |\delta(z, x)| \leq r\}$ and $dP(x)$ is the cumulative distribution of X .
- (H7) (i) K is a positive, differentiable function with support $[-1, 1]$.
 (ii) H is a positive, bounded, and Lipschitzian continuous function, satisfying that $\int |t|^{b_2} H'(t) dt < \infty$, $\int H'(t) dt = 1$ and $\int H^2(t) dt < \infty$.
- (H8) Let (r_n) and (v_n) be sequences of positive integers tending to infinity, such that $(r_n + v_n) = O(n)$, and
 - (i) $v_n = o\left((n \phi_x(h_K))^{\frac{1}{2}}\right)$ and $\lim_{n \rightarrow +\infty} \left(\frac{n}{\phi_x(h_K)}\right)^{\frac{1}{2}} \alpha(v_n) = 0$,
 - (ii) $q_n v_n = o\left((n \phi_x(h_K))^{\frac{1}{2}}\right)$ and $\lim_{n \rightarrow +\infty} q_n \left(\frac{n}{\phi_x(h_K)}\right)^{\frac{1}{2}} \alpha(v_n) = 0$, where q_n is the largest integer, such that $q_n(r_n + v_n) = O(n)$.

2.2.1 Comments on the assumptions

It is observed that the assumptions listed above are standard in the FDA context. In particular, hypotheses (H1) and (H6) are not unduly restrictive and are common in the setting of the functional local linear fitting (see Barrientos-Marina et al. [2], and Demongeot et al. [12] among others). Concerning the first part of (H1), the reader will find, in Ferraty and Vieu [19], a deep discussion concerning the links between this assumption, the semi-metric d , and the small ball concentration properties. Moreover, this hypothesis intervenes to compute the exact constant terms involved in the asymptotic expansions. For example, the previous hypothesis is proposed to evaluate the constant $M_j = \mathbb{E}(K_1^j)$, where $j \in (1, 2)$. However, the second part of (H1) is needed to evaluate the bias of estimation in the asymptotic result. To avoid the expression of covariance in the rate of convergence, assumptions (H3) and (H4) are required; in addition, hypothesis (H3) can be differently seen based on the idea of the maximum concentration between the quantities $P(X_i \in B(x, h_K))$ and $P(X_j \in B(x, h_K))$ (see Ferraty et al. [18]). Concerning the hypothesis (H4), it is used to insure the absolute convergence of the series $\sum_{k \in \mathbb{Z}} \text{Cov}(X_0, X_k)$. Conditions on the smoothing parameters h_K and h_H are standard and will be stated along the theorem below. The boundedness of the Kernel K in (H7)(i) is standard; also assumptions (H7)(ii)

and (H8) constitute technical conditions for brevity proofs. Furthermore, the role of assumption (H8) is to use Bernstein’s big-block and small-block techniques to prove the asymptotic normality for the α -mixing sequence; nonetheless, the choice of the sequences (r_n) and (v_n) in hypothesis (H8) is not surprising. Another choice can be found in Masry [31].

3 Main results

Before announcing the main results, the quantities M_j and $N(a, b)$ are introduced to provide bias and variance dominant terms.

$$M_j = K^j(1) - \int_{-1}^1 (K^j(u))' \Psi_x(u) du, \text{ where } j = 1, 2,$$

$$N(a, b) = K^a(1) - \int_{-1}^1 (u^b K^a(u))' \Psi_x(u) du, \text{ For all } a > 0, \text{ and } b = 2, 4.$$

Moreover, let $\widehat{F}^x(y) = \frac{\widehat{F}_N^x(y)}{\widehat{F}_D^x}$, where $\widehat{F}_N^x(y) = \frac{1}{n \mathbb{E}(\Delta_1 K_1)} \sum_{j=1}^n \Delta_j K_j H_j$ and $\widehat{F}_D^x = \frac{1}{n \mathbb{E}(\Delta_1 K_1)} \sum_{j=1}^n \Delta_j K_j$.

Theorem 3.1 *Under assumptions (H1)–(H8), we have*

$$\sqrt{n \phi_x(h_K)} (\widehat{F}^x(y) - F^x(y) - B_n(x, y)) \xrightarrow{d} \mathcal{N}(0, V_{HK}(x, y)), \tag{3}$$

where

$$V_{HK}(x, y) = \frac{M_2}{M_1^2} F^x(y)(1 - F^x(y)), \tag{4}$$

and

$$B_n(x, y) = \mathbb{E}(\widehat{F}^x(y)) - F^x(y) = B_H(x, y)h_H^2 + B_K(x, y)h_K^2 + o(h_H^2) + o(h_K^2), \tag{5}$$

where

$$B_H(x, y) = \frac{1}{2} \frac{\partial^2 F^x(y)}{\partial y^2} \int t^2 H'(t) dt,$$

$$B_K(x, y) = \frac{1}{2} G_0^{(2)}(0) \frac{N(1, 2)}{M_1}.$$

Remark 3.2 1. If we impose the additional assumption:

$$(H9) \lim_{n \rightarrow \infty} \sqrt{n \phi_x(h_K)} B_n(x, y) = 0,$$

and, in addition, if we replace the function $\phi_x(h_K)$ by its empirical estimator defined by the following:

$$\widehat{\phi}_x(h_K) = \frac{\#\{i : |\delta(X_i, x)| \leq h_K\}}{n},$$

the bias term can be canceled to obtain the following Corollary:

Corollary 3.3 *When the assumptions (H1)–(H9) are held, the following asymptotic result is achieved:*

$$\sqrt{\frac{n \widehat{\phi}_x(h_K)}{V_{HK}(x, y)}} (\widehat{F}^x(y) - F^x(y)) \xrightarrow{d} \mathcal{N}(0, 1).$$



3.1 Proof of Theorem 3.1

Starting by

$$\widehat{F}^x(y) - F^x(y) - B_n(x, y) = \frac{\widehat{F}_N^x(y) - F^x(y)\widehat{F}_D^x - \widehat{F}_D^x B_n(x, y)}{\widehat{F}_D^x}.$$

Denote by

$$\begin{aligned} Q_n(x, y) &= \widehat{F}_N^x(y) - F^x(y)\widehat{F}_D^x - \mathbb{E}(\widehat{F}_N^x(y) - F^x(y)\widehat{F}_D^x) \\ &= \widehat{F}_N^x(y) - F^x(y)\widehat{F}_D^x - B_n(x, y), \end{aligned} \tag{6}$$

then

$$\widehat{F}^x(y) - F^x(y) - B_n(x, y) = \frac{Q_n(x, y) - B_n(x, y)(\widehat{F}_D^x - \mathbb{E}(\widehat{F}_D^x))}{\widehat{F}_D^x}. \tag{7}$$

The relationship (7) is important to establish the asymptotic normality of $\widehat{F}^x(y)$; moreover, the continuity of F^x insures the asymptotic negligibility of $B_n(x, y)$ and if \widehat{F}_D^x converges in probability to 1 as $n \rightarrow \infty$, then

$$\widehat{F}^x(y) - F^x(y) - B_n(x, y) = \frac{Q_n(x, y)}{\widehat{F}_D^x}(1 + o_p(1)),$$

is obtained. The proof of Theorem 3.1 will be completed from the above expression and the following results for which proofs are given in the appendix.

Lemma 3.4 *Under the assumptions of Theorem (3.1), we have*

$$\sqrt{n\phi_x(h_K)} Q_n(x, y) \xrightarrow{d} \mathcal{N}(0, V_{HK}(x, y)), \text{ as } n \rightarrow \infty,$$

where $V_{HK}(x, y)$ is defined in (4).

Lemma 3.5 *Under the assumptions (H3), (H5), we have*

$$\widehat{F}_D^x \xrightarrow{P} \mathbb{E}(\widehat{F}_D^x) = 1, \text{ as } n \rightarrow \infty.$$

Lemma 3.6 (see [12]) *Under the assumptions (H1), (H5), and (H7), we have the following:*

$$B_n(x, y) = B_H(x, y)h_H^2 + B_K(x, y)h_K^2 + o(h_K^2) + o(h_K^2), \text{ as } n \rightarrow \infty.$$

4 Confidence intervals

In parallel, the precise form of (3) is very useful to construct confidence intervals for $F^x(y)$ based on the normal-approximation method that requires to estimate the quantities \widehat{M}_1 , and \widehat{M}_2 by the following empirical estimators:

$$\widehat{M}_j = \frac{1}{n\widehat{\phi}_x(h_K)} \sum_{i=1}^n K^j \left(\frac{|\delta(X_i, x)|}{h_K} \right), \text{ where } j = 1, 2.$$

To show the asymptotic $(1 - \xi)$ confidence interval of $F^x(y)$, where $0 < \xi < 1$, it is necessary to consider the estimator of $\widehat{V}_{HK}(x, y)$ as follows:

$$\widehat{V}_{HK}(x, y) = \frac{\sum_{i=1}^n K_i^2 \widehat{F}^x(y)(1 - \widehat{F}^x(y))}{\widehat{\phi}_x(h_K)(\sum_{i=1}^n K_i)^2}.$$

In addition, a kernel K and a distribution function H are chosen to satisfy the condition (H7) by selecting the bandwidths h_K and h_H through adapting the cross-validation method. Choosing the locating functions $\beta(\cdot, \cdot)$ and $\delta(\cdot, \cdot)$ constitutes an important parameter for the practical utilization of the employed approach. There are several ways permitting to choose the operators $\beta(\cdot, \cdot)$ and $\delta(\cdot, \cdot)$ (see Barrientos–Marina–Marina et al. [2] for some examples), but the appropriate choice is determined with respect to the shape of the curves and depends on the purpose of the statistical study. For example, if the functional data are smooth curves, one can try to use the following family of locating functions:

$$\beta(x_1, x_2) = \int_0^1 \theta(t)(x_1^{(q)}(t) - x_2^{(q)}(t))dt \quad \text{and} \quad \delta(x_1, x_2) = \sqrt{\int_0^1 (x_1^{(q)}(t) - x_2^{(q)}(t))^2 dt},$$

where $x^{(q)}$ denotes the q th derivative of the curve x and $\theta(t)$ is the eigenfunction of the empirical covariance operator $\frac{1}{n} \sum_{i=1}^n (X_i^{(q)} - \overline{X^{(q)}})^t (X_i^{(q)} - \overline{X^{(q)}})$ associated with the q -greatest eigenvalue.

Finally, by Corollary (3.3), the asymptotic $(1 - \xi)$ confidence interval of $F^x(y)$ is given by the following:

$$\left[\widehat{F}^x(y) - \lambda_{\frac{\xi}{2}} \sqrt{\frac{\widehat{V}_{HK}(x, y)}{n \widehat{\phi}_x(h_K)}}, \widehat{F}^x(y) + \lambda_{\frac{\xi}{2}} \sqrt{\frac{\widehat{V}_{HK}(x, y)}{n \widehat{\phi}_x(h_K)}} \right],$$

where $\lambda_{\frac{\xi}{2}}$ is the $\frac{\xi}{2}$ quantile of the standard normal distribution.

5 Appendix

5.1 Proof of Lemma 3.4.

Let us first note that, in view of (6), we have

$$\sqrt{n \phi_x(h_K)} Q_n(x, y) = \frac{\sqrt{n \phi_x(h_K)}}{n \mathbb{E}(\Delta_1 K_1)} \left(\sum_{j=1}^n \Delta_j K_j (H_j - F^x(y)) - \mathbb{E} \left(\sum_{j=1}^n \Delta_j K_j (H_j - F^x(y)) \right) \right).$$

In the same way, using the definition of Δ_j , this equality can be rewritten as follows:

$$\begin{aligned} \sqrt{n \phi_x(h_K)} Q_n(x, y) &= \frac{1}{n \mathbb{E}(\beta_1^2 K_1)} \sum_{i=1}^n \beta_i^2 K_i \frac{\sqrt{n \phi_x(h_K)} \mathbb{E}(\beta_1^2 K_1)}{\mathbb{E}(\Delta_1 K_1)} \sum_{j=1}^n K_j (H_j - F^x(y)) \\ &\quad - \frac{1}{n \mathbb{E}(\beta_1 K_1)} \sum_{i=1}^n \beta_i K_i \frac{\sqrt{n \phi_x(h_K)} \mathbb{E}(\beta_1 K_1)}{\mathbb{E}(\Delta_1 K_1)} \sum_{j=1}^n \beta_j K_j (H_j - F^x(y)) \\ &\quad - \mathbb{E} \left(\frac{1}{n \mathbb{E}(\beta_1^2 K_1)} \sum_{i=1}^n \beta_i^2 K_i \frac{\sqrt{n \phi_x(h_K)} \mathbb{E}(\beta_1^2 K_1)}{\mathbb{E}(\Delta_1 K_1)} \sum_{j=1}^n K_j (H_j - F^x(y)) \right) \\ &\quad + \mathbb{E} \left(\frac{1}{n \mathbb{E}(\beta_1 K_1)} \sum_{i=1}^n \beta_i K_i \frac{\sqrt{n \phi_x(h_K)} \mathbb{E}(\beta_1 K_1)}{\mathbb{E}(\Delta_1 K_1)} \sum_{j=1}^n \beta_j K_j (H_j - F^x(y)) \right). \end{aligned}$$

Denote by

$$\begin{aligned} T_{1,i} &= \frac{1}{n \mathbb{E}(\beta_1^2 K_1)} \sum_{i=1}^n \beta_i^2 K_i, & T_{2,j} &= \frac{\sqrt{n \phi_x(h_K)} \mathbb{E}(\beta_1^2 K_1)}{\mathbb{E}(\Delta_1 K_1)} \sum_{j=1}^n K_j (H_j - F^x(y)), \\ T_{3,i} &= \frac{1}{n \mathbb{E}(\beta_1 K_1)} \sum_{i=1}^n \beta_i K_i \quad \text{and} \quad T_{4,j} &= \frac{\sqrt{n \phi_x(h_K)} \mathbb{E}(\beta_1 K_1)}{\mathbb{E}(\Delta_1 K_1)} \sum_{j=1}^n \beta_j K_j (H_j - F^x(y)), \end{aligned}$$

and thus

$$\begin{aligned} \sqrt{n \phi_x(h_K)} Q_n(x, y) &= T_{1,i} T_{2,j} - T_{3,i} T_{4,j} - \mathbb{E}(T_{1,i} T_{2,j} - T_{3,i} T_{4,j}) \\ &= T_{1,i} T_{2,j} - \mathbb{E}(T_{1,i} T_{2,j}) - (T_{3,i} T_{4,j} - \mathbb{E}(T_{3,i} T_{4,j})). \end{aligned}$$



Finally, the rest of the proof is based on the following statements:

$$T_{1,i}T_{2,j} - \mathbb{E}(T_{1,i}T_{2,j}) \xrightarrow{d} \mathcal{N}(0, V_{HK}(x, y)). \tag{8}$$

$$T_{3,i}T_{4,j} - \mathbb{E}(T_{3,i}T_{4,j}) \xrightarrow{P} 0. \tag{9}$$

Proof of (8).

Let us write the left-hand side of (8) as follows:

$$T_{1,i}T_{2,j} - \mathbb{E}(T_{1,i}T_{2,j}) = T_{2,j} - \mathbb{E}(T_{2,j}) + ((T_{1,i} - 1)T_{2,j} - \mathbb{E}((T_{1,i} - 1)T_{2,j})).$$

Hence, by Slutsky’s Theorem (see Theorem 11.1.5 in [25]), (8) is straightforward consequence of the following two claims:

$$T_{2,j} - \mathbb{E}(T_{2,j}) \xrightarrow{d} \mathcal{N}(0, V_{HK}(x, y)). \tag{10}$$

$$(T_{1,i} - 1)T_{2,j} - \mathbb{E}((T_{1,i} - 1)T_{2,j}) \xrightarrow{P} 0. \tag{11}$$

Proof of (10).

As a matter of fact, we need to evaluate the variance of $(T_{2,j})$. For this, we have the following:

$$\begin{aligned} \text{Var}(T_{2,j}) &= n \text{Var}(T_{2,1}) + 2 \frac{n\phi_x(h_K)\mathbb{E}^2(\beta_1^2 K_1)}{\mathbb{E}^2(\Delta_1 K_1)} \sum_{1 \leq i < j \leq n} \\ &\quad \text{cov}(K_i(H_i - F^x(y)), K_j(H_j - F^x(y))). \end{aligned}$$

Therefore, to prove that $\lim_{n \rightarrow +\infty} \text{Var}(T_{2,j}) = V_{HK}(x, y)$, it is necessary to establish the following results:

$$\lim_{n \rightarrow +\infty} n \text{Var}(T_{2,1}) = \frac{M_2}{M_1^2} F^x(y)(1 - F^x(y)). \tag{12}$$

$$\lim_{n \rightarrow +\infty} \left(\frac{n\phi_x(h_K)\mathbb{E}^2(\beta_1^2 K_1)}{\mathbb{E}^2(\Delta_1 K_1)} \sum_{1 \leq i < j \leq n} \text{Cov}(K_i(H_i - F^x(y)), K_j(H_j - F^x(y))) \right) = 0. \tag{13}$$

Proof of (12). One has

$$n \text{Var}(T_{2,1}) = \frac{n^2\phi_x(h_K)\mathbb{E}^2(\beta_1^2 K_1)}{\mathbb{E}^2(\Delta_1 K_1)} (\mathbb{E}(K_1^2(H_1 - F^x(y))^2) - \mathbb{E}^2(K_1(H_1 - F^x(y))))). \tag{14}$$

Concerning the second term on the right-hand side of (14), we have the following:

$$\mathbb{E}^2(K_1(H_1 - F^x(y))) = \mathbb{E}^2(K_1(\mathbb{E}(H_1/X_1) - F^x(y))),$$

and by the continuity of F^x we deduce that:

$$\mathbb{E}(H_1/X_1) - F^x(y) \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{15}$$

Now, we turn to the first term on the right-hand side of (14). Let us begin with writing:

$$\begin{aligned} \frac{n^2\phi_x(h_K)\mathbb{E}^2(\beta_1^2 K_1)}{\mathbb{E}^2(\Delta_1 K_1)} \mathbb{E}((H_1 - F^x(y))^2 K_1^2) &= \frac{n^2\phi_x(h)\mathbb{E}^2(\beta_1^2 K_1)}{E^2(\Delta_1 K_1)} \mathbb{E}(\mathbb{E}((H_1 - F^x(y))^2/X_1)K_1^2) \\ &= \frac{n^2\phi_x(h_K)\mathbb{E}^2(\beta_1^2 K_1)}{\mathbb{E}^2(\Delta_1 K_1)} \mathbb{E}(\text{Var}(H_1/X_1)K_1^2) \\ &\quad + \frac{n^2\phi_x(h_K)\mathbb{E}^2(\beta_1^2 K_1)}{E^2(\Delta_1 K_1)} \mathbb{E}((E(H_1/X_1) - F^x(y))^2 K_1^2). \end{aligned} \tag{16}$$

In view of (15), classical computations of the second term on the right-hand side of (16) give:

$$\frac{n^2 \phi_x(h_K) \mathbb{E}^2(\beta_1^2 K_1)}{\mathbb{E}^2(\Delta_1 K_1)} \mathbb{E}((E(H_1/X_1) - F^x(y))^2 K_1^2) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Concerning the first term on the right-hand side of (16), we use the following definition of the conditional variance:

$$\text{Var}(H_1/X_1) = \mathbb{E}(H_1^2/X_1) - \mathbb{E}^2(H_1/X_1). \tag{17}$$

Thus, using an integration by parts followed by a change of variable, we get:

$$\mathbb{E}\left(H^2\left(\frac{y - Y_1}{h_H}\right) / X_1\right) = \int_{\mathbb{R}} 2H'(t)H(t)(F^{X_1}(y - th_H) - F^x(y))dt + \int_{\mathbb{R}} 2H(t)'H(t)F^x(y)dt.$$

Remark that $\int_{\mathbb{R}} 2H'(t)H(t)F^x(y)dt = F^x(y)$, and by the continuity of F^x , we deduce that

$$\mathbb{E}\left(H^2\left(\frac{y - Y_1}{h_H}\right) / X_1\right) \rightarrow F^x(y) \text{ as } n \rightarrow \infty.$$

Therefore, the second term on the right-hand side of (17) tends to $(F^x(y))^2$ as n tends to infinity.

Finally, we have the following:

$$\mathbb{E}\left(K_1^2 \text{Var}\left(H\left(\frac{y - Y_1}{h_H}\right) / X_1\right)\right) \rightarrow \mathbb{E}(K_1^2)F^x(y)(1 - F^x(y)) \text{ as } n \rightarrow \infty.$$

Next, using Lemma A.1 of [37], we get:

$$\begin{aligned} n \text{Var}(T_{2,1}) &= \frac{n^2 \phi_x(h_K) \mathbb{E}^2(\beta_1^2 K_1)}{\mathbb{E}^2(\Delta_1 K_1)} \mathbb{E}((H_1 - F^x(y))^2 K_1^2) \\ &= \frac{n^2 M_2}{(n - 1)^2 M_1^2} F^x(y)(1 - F^x(y)) \rightarrow \frac{M_2}{M_1^2} F^x(y)(1 - F^x(y)) \\ &= V_{HK}(x, y) \text{ as } n \rightarrow \infty. \end{aligned}$$

Proof of (13).

First, the set E_1 and E_2 are defined by setting

$$E_1 = \{(i, j) \in \{1, 2, \dots, n\} \text{ such that } 1 \leq |i - j| \leq m_n\},$$

$$E_2 = \{(i, j) \in \{1, 2, \dots, n\} \text{ such that } m_n + 1 \leq |i - j| \leq n - 1\},$$

where m_n is a sequence of integers, such that $m_n \rightarrow +\infty$ as $n \rightarrow +\infty$.

Now, denote by

$$L_i = K_i(H_i - F^x(y)) \text{ and } L_j = K_j(H_j - F^x(y)),$$

then

$$\begin{aligned} \text{Cov}(T_{1,i}, T_{1,j}) &= \frac{n \phi_x(h_K) \mathbb{E}^2(\beta_1^2 K_1)}{\mathbb{E}^2(\Delta_1 K_1)} \sum_{E_1} \text{Cov}(L_i, L_j) \\ &\quad + \frac{n \phi_x(h_K) \mathbb{E}^2(\beta_1^2 K_1)}{\mathbb{E}^2(\Delta_1 K_1)} \sum_{E_2} \text{Cov}(L_i, L_j) \\ &=: A_{1,n} + A_{2,n}. \end{aligned}$$

Having the sum of covariance over the set E_1 , by stationarity:

$$\text{Cov}(L_i, L_j) = \mathbb{E}(K_i K_j \mathbb{E}((H_i - F^x(y))(H_j - F^x(y)) / (X_i, Y_j))) - \mathbb{E}^2(K_1(H_1 - F^x(y))). \tag{18}$$

Under (H7)(ii), Having $|H_i(y) - F^x(y)| \leq 1$, according to (H3), the following inequality is obtained as follows:

$$\begin{aligned} |\text{Cov}(L_i, L_j)| &\leq \mathbb{E}(K_i K_j) + \mathbb{E}^2(K_1), \\ &\leq C\mathbb{P}((X_i, X_j) \in B(x, h_K) \times B(x, h_K)) + M_1^2 \phi_x^2(h_K), \\ &\leq C\psi_x(h_K) + M_1^2 \phi_x^2(h_K). \end{aligned}$$

Now, by the application of Lemma A.1 [37], we have the following:

$$|A_{1,n}| \leq \frac{Cn^2 \phi_x(h_K) m_n}{(n-1)^2 M_1^2} \left(\left(\frac{\psi_x(h_K)}{\phi_x(h_K)^2} \right) + M_1^2 \right).$$

It follows that, by (H5)(iii) and taking $m_n = \left(\frac{1}{(\phi_x(h_K) \log n)^{\frac{1}{p}}} \right)^{\frac{1}{\delta}}$, we obtain $A_{1,n} = o(1)$ as $n \rightarrow +\infty$.

Concerning the sum over the set E_2 , the proposition A.10 (ii) of [19] is used to get:

$$\sum_{E_2} |\text{Cov}(L_i, L_j)| \leq \sum_{E_2} C(\alpha|j-i|)^{\frac{1}{p}} (\mathbb{E}|L_i|^q)^{\frac{1}{q}} (\mathbb{E}|L_j|^r)^{\frac{1}{r}}.$$

First, we evaluate the quantity $\mathbb{E}(|L_i|^q)$. Conditioning on X_i , and using the fact that $|H_i - F^x(y)| \leq 1$, we obtain:

$$\begin{aligned} \mathbb{E}(|L_i|^q) &= \mathbb{E}(|K_i|^q \mathbb{E}(|H_i - F^x(y)|^q / X_i)), \\ &\leq C(\phi_x(h_K))^{\frac{1}{q}}. \end{aligned}$$

Again, using Lemma A.1 of [37], we have:

$$|A_{2,n}| \leq \frac{Cn^2 (\phi_x(h_K))^{\frac{1}{q} + \frac{1}{r} - 1 - \frac{1}{p}}}{(n-1)^2 M_1^2 \phi_x(h_K) (m_n)^\delta} \sum_{|k| > m_n} k^\delta (\alpha(|k|))^{\frac{1}{p}}.$$

Finally, the obtained result is combined with assumptions (H4)(ii) and the sequence m_n previously chosen to get:

$$\lim_{x \rightarrow +\infty} A_{2,n} = o(1).$$

Now, the asymptotic normality of the conditional cumulative distribution estimation is established dealing with dependent random variables:

$$\begin{aligned} T_{2,j} - \mathbb{E}(T_{2,j}) &= \frac{\sqrt{n\phi_x(h_K)}\mathbb{E}(\beta_1^2 K_1)}{\mathbb{E}(\Delta_1 K_1)} \sum_{j=1}^n (K_j(H_j - F^x(y)) - \mathbb{E}(K_j(H_j - F^x(y)))) \\ &= \frac{\sum_{j=1}^n (Z_j - \mathbb{E}(Z_j))}{\sqrt{n}} =: S_n, \end{aligned}$$

where

$$Z_j = \frac{n\sqrt{\phi_x(h_K)}\mathbb{E}(\beta_1^2 K_1)}{\mathbb{E}(\Delta_1 K_1)} (K_j(H_j - F^x(y)) - \mathbb{E}(K_j(H_j - F^x(y)))).$$

Remark that (13) implies that:

$$\sum_{1 \leq i < j \leq n} \text{Cov}(Z_i, Z_j) = o(n).$$

Therefore, it suffices to show the following result:

$$S_n \xrightarrow{d} \mathcal{N}(0, V_{HK}(x, y)). \tag{19}$$

Bernstein’s big-block and small-block procedure is employed following similar arguments to those involved in Theorem 3.1 of Liang and Baek [29]. $(1, 2, \dots, n)$ is splitted into $2\kappa_n + 1$ subsets with large blocks of size (r_n) and small blocks of size (v_n) and by putting

$$\kappa = \left\lfloor \frac{n}{r_n + v_n} \right\rfloor.$$

Assumption (H8)(ii) permits to define the large block size as follows:

$$r_n = \left\lfloor \left(\frac{n\phi_x(h_\kappa)}{q_n} \right)^{\frac{1}{2}} \right\rfloor.$$

Moreover, some easy computations are obtained using the same hypothesis:

$$\lim_{n \rightarrow +\infty} \frac{v_n}{r_n} = 0, \quad \lim_{n \rightarrow +\infty} \frac{r_n}{n} = 0, \quad \lim_{n \rightarrow +\infty} \frac{r_n}{\sqrt{n\phi_x(h_\kappa)}} = 0, \tag{20}$$

and it can easily be deduced that, as $n \rightarrow +\infty$:

$$\frac{\kappa v_n}{n} \simeq \left(\frac{n}{r_n + v_n} \right) \frac{v_n}{n} \simeq \frac{v_n}{r_n + v_n} \simeq \frac{v_n}{r_n} = 0.$$

In addition, if v_n is replaced by r_n , we obtain

$$\lim_{n \rightarrow +\infty} \frac{\kappa r_n}{n} = 1.$$

The sum S_n is then splitted in the following way:

The set $I_j = \{j(r + v) + 1, \dots, j(r + v) + r\}$ contains r elements for each $j \in (0, 1, 2, \dots, k - 1)$, and the set

$I_j = \{j(r + v) + r + 1, \dots, (j + 1)(r + v)\}$ contains v elements for each $j \in (0, 1, 2, \dots, k - 1)$.

The following random variables are defined by the following:

$$\Upsilon_j = \sum_{i=j(r+v)+1}^{j(r+v)+r} Z_i, \quad \tilde{\Upsilon}_j = \sum_{i=j(r+v)+r+1}^{(j+1)(r+v)} Z_i, \quad R_n = \sum_{i=\kappa(r+v)+1}^n Z_i. \tag{21}$$

It is clear that:

$$\begin{aligned} S_n &= \sum_{j=0}^{\kappa-1} \frac{\Upsilon_j}{\sqrt{n}} + \sum_{j=0}^{\kappa-1} \frac{\tilde{\Upsilon}_j}{\sqrt{n}} + \frac{R_n}{\sqrt{n}} \\ &=: S_{1,n} + S_{2,n} + S_{3,n}. \end{aligned}$$

Then, (19) will be obtained as soon as the following assertions are checked:

$$[S_{2,n} + S_{3,n}] \xrightarrow{p} 0, \tag{22}$$

and

$$S_{1,n} \xrightarrow{d} \mathcal{N}(0, V_{HK}(x, y)). \tag{23}$$

Proof of (22).



By Markov’s Inequality, it remains to establish for all $\varepsilon > 0$:

$$\mathbf{P}(|S_{2,n}| > \varepsilon) \leq \frac{\mathbb{E}^2(S_{2,n})}{\varepsilon^2}, \tag{24}$$

$$\mathbf{P}(|S_{3,n}| > \varepsilon) \leq \frac{\mathbb{E}^2(S_{3,n})}{\varepsilon^2}. \tag{25}$$

To prove (24), it is clearly observed that:

$$\begin{aligned} \mathbb{E}^2(S_{2,n}) &= \frac{1}{n} \left(\sum_{j=0}^{k-1} \text{Var}(\tilde{Y}_j) + 2 \sum_{0 \leq i < j \leq k-1} \text{Cov}(\tilde{Y}_i, \tilde{Y}_j) \right) \\ &=: A_1 + A_2. \end{aligned}$$

Noting also that, by the second-order stationarity, it will be retained:

$$\begin{aligned} \text{Var}(\tilde{Y}_j) &= \text{Var} \left(\sum_{i=j(r+v)+r+1}^{(j+1)(r+v)} Z_i \right) \\ &= v_n \text{Var}(Z_1) + 2 \sum_{i \neq j}^{v_n} \text{Cov}(Z_i, Z_j). \end{aligned}$$

Consequently:

$$\begin{aligned} \frac{A_1}{n} &= \frac{\kappa v_n}{n} \text{Var}(Z_1) + \frac{2}{n} \sum_{j=0}^{k-1} \sum_{i \neq j}^{v_n} \text{Cov}(Z_i, Z_j) \\ &\leq \frac{\kappa v_n}{n} \underbrace{\text{Var}(Z_1)}_{V_{HK}(x,y)} + \underbrace{\frac{2}{n} \sum_{i \neq j}^n \text{Cov}(Z_i, Z_j)}_{o(n)}. \end{aligned}$$

By the assumption (H8), it is clear that:

$$\lim_{n \rightarrow +\infty} \frac{A_1}{n} \rightarrow 0.$$

Concerning A_2 , it has:

$$\begin{aligned} \frac{A_2}{n} &= \frac{2}{n} \sum_{|i-j|>0}^{k-1} \text{Cov}(\tilde{Y}_i, \tilde{Y}_j) \\ &= \frac{2}{n} \sum_{|i-j|>0}^{k-1} \sum_{l=1}^{v_n} \sum_{\acute{l}=1}^{v_n} \text{Cov}(Z_{\varsigma_i+l}, Z_{\varsigma_j+\acute{l}}), \end{aligned}$$

with $\varsigma_i = i(r_n + v_n) + r_n$. Since $i \neq j$, we have $|\varsigma_i - \varsigma_j + l - \acute{l}| \geq r_n$. It follows that

$$\frac{A_2}{n} \leq \frac{2}{n} \sum_{i=1}^n \sum_{\substack{j=1 \\ |i-j| \geq r_n}}^n |\text{Cov}(Z_i, Z_j)|,$$

which leads to

$$\lim_{n \rightarrow +\infty} \frac{A_2}{n} \rightarrow 0.$$

For (25), we have the following:

$$\begin{aligned}\mathbb{E}^2(S_{3,n}) &= \frac{1}{n} \text{Var}(R_n) \\ &= \frac{\mu_n}{n} \text{Var}(Z_1) + \underbrace{\frac{2}{n} \sum_{i \neq j}^{\mu_n} \text{Cov}(Z_i, Z_j)}_{o(n)},\end{aligned}$$

where $\mu_n = n - \kappa_n(r_n + v_n)$, and by the definition of κ_n , we have $\mu_n < r_n + v_n$. Hence

$$\mathbb{E}^2(S_{3,n}) \leq \frac{r_n + v_n}{n} \underbrace{\text{Var}(Z_1)}_{V_{HK}(x,y)} + o(1).$$

And, again hypothesis (H8), we get

$$\mathbb{E}^2(S_{3,n}) \longrightarrow 0 \quad \text{as } n \longrightarrow +\infty.$$

Proof of (23).

Making use of Volkonskii and Rozanov's Lemma [36] and the fact that the process (X_i, Y_i) is strong mixing; and that Υ_a is $\mathcal{A}_{i_a}^{j_a}$ measurable with $i_a = a(r_n + v_n) + 1$ and $j_a = a(r_n + v_n) + r_n$, hence, with $V_j = \exp(\frac{it\Upsilon_j}{\sqrt{n}})$, we have the following:

$$\left| \mathbb{E} \left(\exp \left(it \frac{S_{1,n}}{\sqrt{n}} \right) \right) - \prod_{j=0}^{\kappa-1} \mathbb{E} \left(\exp \left(it \frac{\Upsilon_j}{\sqrt{n}} \right) \right) \right| \longrightarrow 0 \quad \text{as } n \rightarrow +\infty. \quad (26)$$

Consequently, according to formula (26), Υ_j are asymptotically independent. Therefore, for the variance of $S_{1,n}$, we have the following:

$$\text{Var}(S_{1,n}) = \frac{\kappa r_n}{n} \underbrace{\text{Var}(Z_1)}_{V_{HK}(x,y)}.$$

Furthermore, from assumption (H8), $\frac{\kappa r_n}{n} \longrightarrow 1$ as $n \longrightarrow +\infty$.

Finally, we get

$$\frac{1}{n} \sum_{j=0}^{\kappa-1} \mathbb{E}[\Upsilon_j^2] \longrightarrow V_{HK}(x, y) \quad \text{as } n \rightarrow +\infty.$$

Now, to end the proof of (23), we focus on the central limit Theorem due to Linderberg. More precisely, by applying the Linderberg's version of central limit Theorem on Υ_j , it suffices to show that for any $\varepsilon > 0$:

$$\frac{1}{n} \sum_{j=0}^{\kappa-1} \mathbb{E} \left[\Upsilon_j^2 \mathbb{1}_{|\Upsilon_j| > \varepsilon \sqrt{n V_{HK}(x,y)}} \right] \longrightarrow 0 \quad \text{as } n \longrightarrow +\infty.$$

In view of the first summation of (21), classical computations give

$$\left| \frac{\Upsilon_j}{n} \right| \leq \frac{r_n}{n} |Z_1|.$$

Next, the application of Lemma A.1 of [37] together with (H5) leads to $|Z_1| \longrightarrow 0$ as $n \longrightarrow +\infty$, and noting that this last result combined with (20) ensures that $\left| \frac{\Upsilon_j}{n} \right| \longrightarrow 0$ as $n \longrightarrow +\infty$.



Therefore, for all ε and if n is great enough, the set $\{|Y_j| > \varepsilon\sqrt{nV_{HK}(x, y)}\}$ becomes empty; and the proof of (23) is, therefore, complete.

Proof of (11).

By Bienaymé–Tchebychev’s Inequality, it is sufficient to show that, for all $\varepsilon > 0$:

$$\mathbb{P}(|(T_{1,i} - 1)T_{2,j} - \mathbb{E}((T_{1,i} - 1)T_{2,j})| \geq \varepsilon) \leq \frac{\mathbb{E} |(T_{1,i} - 1)T_{2,j} - \mathbb{E}((T_{1,i} - 1)T_{2,j})|}{\varepsilon}.$$

In addition, Cauchy–Schwarz’s Inequality entails

$$\mathbb{E} |(T_{1,i} - 1)T_{2,j} - \mathbb{E}((T_{1,i} - 1)T_{2,j})| \leq 2\mathbb{E} |(T_{1,i} - 1)T_{2,j}| \leq \sqrt{E((T_{1,i} - 1)^2)} \sqrt{\mathbb{E}(T_{2,j}^2)}.$$

Then, (11) is a straightforward consequence of the following results:

$$\mathbb{E}((T_{1,i} - 1)^2) \longrightarrow 0, \quad \text{as } n \longrightarrow +\infty, \tag{27}$$

$$\mathbb{E}(T_{2,j}^2) \longrightarrow V_{HK}(x, y), \quad \text{as } n \longrightarrow +\infty. \tag{28}$$

Proof of (27).

First, we can write the following:

$$E((T_{1,i} - 1)^2) = n \text{Var}(T_{1,1}) + 2 \text{Cov}(T_{1,i}, T_{1,i}).$$

For the first term on the right-hand side of this equality, we have the following:

$$\begin{aligned} n \text{Var}(T_{1,1}) &= \frac{n \text{Var}(\beta_1^2 K_1)}{n^2 \mathbb{E}(\beta_1^2 K_1)} \\ &= O\left(\frac{1}{n\phi_x(h_K)}\right). \end{aligned}$$

Concerning the second term of the previous equality, we have the following:

$$\text{Cov}(T_{1,i}, T_{1,i}) = \frac{1}{n^2 \mathbb{E}^2(\beta_1^2 K_1)} \sum_{\substack{i=1 \\ i \neq l}}^n \sum_{l=1}^n \text{Cov}(\beta_i^2 K_i, \beta_l^2 K_l). \tag{29}$$

The proof of this result is very close to the proof of (13). Specifically, by keeping the same notations as those used in (13), and by splitting the sum into two separate summations over the sets E_1 and E_2 :

$$\begin{aligned} E_1 &= \{(i, l) \in \{1, 2, \dots, n\} \text{ such that } 1 \leq |i - l| \leq m_n\}, \\ E_2 &= \{(i, l) \in \{1, 2, \dots, n\} \text{ such that } m_n + 1 \leq |i - l| \leq n - 1\}, \end{aligned}$$

where the sequence m_n is chosen, such that $m_n \longrightarrow +\infty$ as $n \longrightarrow +\infty$. Denoting now by $A_{1,n}$ and $A_{2,n}$ the sum of covariances over E_1 and E_2 , respectively; so

$$\begin{aligned} A_{1,n} &= \frac{1}{n^2 \mathbb{E}^2(\beta_1^2 K_1)} \sum_{E_1} \text{Cov}(\beta_i^2 K_i, \beta_l^2 K_l), \\ A_{2,n} &= \frac{1}{n^2 \mathbb{E}^2(\beta_1^2 K_1)} \sum_{E_2} \text{Cov}(\beta_i^2 K_i, \beta_l^2 K_l). \end{aligned}$$

By stationarity, we have:

$$\text{Cov}(\beta_i^2 K_i, \beta_l^2 K_l) = \mathbb{E}(\beta_i^2 K_i \beta_l^2 K_l) - \mathbb{E}^2(\beta_1^2 K_1),$$

then

$$|\text{Cov}(\beta_i^2 K_i, \beta_l^2 K_l)| \leq |\mathbb{E}(\beta_i^2 K_i \beta_l^2 K_l)| + \mathbb{E}^2(\beta_1^2 K_1).$$

Moreover, assumptions (H1) and (H6) imply that:

$$K_i |\beta_i|^2 h_k^{-2} \leq C K_i |\delta(x, X_i)|^2 h_k^{-2} \leq C K_i.$$

On the other hand, we may apply Jensen’s Inequality and assumption (H3) to obtain:

$$\begin{aligned} |\mathbb{E}(\beta_i^2 K_i \beta_l^2 K_l)| &\leq \mathbb{E}|\beta_i^2 K_i \beta_l^2 K_l|, \\ &\leq C h_k^4 \mathbb{E}(K_i K_l), \\ &\leq C h_k^4 \psi_x(h_k). \end{aligned}$$

In the next step, we use the technical Lemma A.1 of [2] to get:

$$\begin{aligned} |A_{1,n}| &\leq n m_n (C h_k^4 \psi_x(h_k) + \mathbb{E}^2(\beta_1^2 K_1)) \frac{1}{n^2 \mathbb{E}^2(\beta_1^2 K_1)}, \\ &\leq \frac{C m_n}{n O(1)} \frac{\psi_x(h_k)}{\phi_x^2(h_k)} + \frac{m_n}{n}. \end{aligned}$$

Then, choosing $m_n = \sqrt{n}$, and since $\frac{\psi_x(h_k)}{\phi_x^2(h_k)}$ is bounded from assumption (H3), we arrive at:

$$A_{1,n} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Let us now treat the sum over E_2 . The application of the inequality for bounded mixing processes [see Proposition A.10(i) in [19]] for all $l \neq i$ leads to:

$$|\text{Cov}(\beta_i^2 K_i, \beta_l^2 K_l)| \leq C h_k^4 \alpha(|i - l|).$$

On the other hand, using the fact that $\sum_{j \geq x+1} j^{-s} \leq \int_{u \geq x} u^{-s} = ((1 - s)u^{s-1})^{-1}$, and under (H4)(i), it is easy to get:

$$\sum_{i=1}^n \sum_{m_n+1 \leq |i-l| \leq n-1} \alpha(|i - l|) \leq \frac{n (m_n)^{1-s}}{s - 1}.$$

Finally, we have:

$$|A_{2,n}| \leq \frac{C (m_n)^{1-s}}{s - 1} \frac{1}{O(n \phi_x^2(h_k))}.$$

We use the same choice of m_n as before, and using assumption (H5)(i), we obtain:

$$A_{2,n} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Proof of (28).

We start by writing

$$\mathbb{E}((T_{2,j})^2) = \text{Var}(T_{2,j}) + \mathbb{E}^2(T_{2,j}). \tag{30}$$

The first term on the right-hand side of (30) tends to $V_{HK}(x, y)$ as n tends to infinity, and the proof of this result was shown in (13). Concerning the second term on the right-hand side of (30), we have:

$$\mathbb{E}^2(T_{2,j}) = \frac{n^3 \phi_x(h_k) E^2(\beta_1^2 K_1)}{E^2(\Delta_1 K_1)} E^2(K_1(H_1 - F^x(y))),$$

where

$$\mathbb{E}[H_1 / X_1] = \int_R H(h_H^{-1}(y - z)) f^{X_1}(z) dz;$$

with an integration by parts followed by the change of variables $t = \frac{y-z}{h_H}$ allows to write:

$$\mathbb{E}(H_1/X_1) = \int_R H'(t) F^{X_1}(y - h_H t) dt.$$

Moreover, the latter integral can be rewritten as follows:

$$\mathbb{E}(H_1/X_1) = \int_R H'(t) (F^{X_1}(y - h_H t) - F^x(y)) + \int_R H'(t) F^x(y) dt.$$

Now, under assumption (H7)(ii), and by the continuity of F^x , we have the following:

$$\mathbb{E}(H_1/X_1) - F^x(y) \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{31}$$

In addition, by applying the technical Lemma A.1 in [37], we get $\lim_{n \rightarrow +\infty} \frac{E^2(\beta_1^2 K_1)}{E^2(\Delta_1 K_1)} = 0$. Finally, assumption (H5)(ii) allows us to deduce that:

$$\lim_{n \rightarrow +\infty} \mathbb{E}^2(T_{2,j}) = 0.$$

Proof of (9). By following the same ideas as those used in (11), we show that:

$$\mathbb{E} |T_{4,j} - \mathbb{E}(T_{4,j})| \xrightarrow{L^1} 0 \text{ as } n \rightarrow +\infty, \tag{32}$$

$$\mathbb{E} |(T_{3,i} - 1)T_{4,j} - \mathbb{E}((T_{3,i} - 1)T_{4,j})| \longrightarrow 0 \text{ as } n \rightarrow +\infty. \tag{33}$$

Proof (32).

To show the required result (32), it suffices to prove the L^2 consistency of $T_{4,j}$:

$$\begin{aligned} E \left((T_{4,j} - \mathbb{E}(T_{4,j}))^2 \right) &= n \text{Var}(T_{4,1}) \\ &+ 2 \frac{n\phi_x(h_K) \mathbb{E}^2(\beta_1^2 K_1)}{h_H \mathbb{E}^2(\Delta_1 K_1)} \sum_{1 \leq i < j \leq n} \text{Cov}(\beta_i K_i (H_i - F^x(y)), \beta_j K_j (H_j - F^x(y))). \end{aligned} \tag{34}$$

Concerning the first term on the right-hand side of (34), we have the following:

$$\begin{aligned} n \text{Var}(T_{4,1}) &= \frac{n\phi_x(h_K) \mathbb{E}^2(\beta_1 K_1)}{\mathbb{E}^2(\Delta_1 K_1)} n \text{Var}(\beta_1 K_1 (H_1 - F^x(y))), \\ &= \frac{n\phi_x(h_K) o(h_K^2 \phi_x^2(h_K))}{(n-1)^2 O(h_K^4 \phi_x^4(h_K))} n F^x(y) (1 - F^x(y)) O(h_K^2 \phi_x(h_K)), \\ &= F^x(y) (1 - F^x(y)) o(1). \end{aligned}$$

On the other hand, by exactly the same arguments at (13), the second term on the right-hand side of (34) tends to 0 as n tends to the infinity, and the desired result (32) is obtained.

Proof (33).

The *Cauchy–Schwarz’s* inequality implies that:

$$\mathbb{E} |(T_{3,i} - 1)T_{4,j} - \mathbb{E}((T_{3,i} - 1)T_{4,j})| \leq 2 \mathbb{E} | (T_{3,i} - 1)T_{4,j} | \leq 2 \sqrt{\mathbb{E}((T_{3,i} - 1)^2)} \sqrt{\mathbb{E}(T_{4,j}^2)}.$$

In a first attempt, we have:

$$\mathbb{E}((T_{3,i} - 1)^2) = n \text{Var}(T_{3,1}) + 2 \text{Cov}(T_{3,i}, T_{3,l}).$$

Hence:

$$\begin{aligned} n \operatorname{Var}(T_{2,1}) &= \frac{n \operatorname{Var}(\beta_1 K_1)}{n^2 \mathbb{E}^2(\beta_1 K_1)}, \\ &\leq \frac{\mathbb{E}(\beta_1^2 K_1^2)}{n \mathbb{E}^2(\beta_1 K_1)}, \\ &= O\left(\frac{1}{n\phi_x(h_K)}\right). \end{aligned}$$

Next, we use similar ideas as those used in (29) to deduce that:

$$\operatorname{Cov}(T_{3,i}, T_{3,l}) \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

5.2 Proof of Lemma 3.5

By the definition of \widehat{F}_D^x , we have the following:

$$\begin{aligned} \widehat{F}_D^x &= \frac{1}{n} \sum_{i=1}^n \beta_i^2 K_i \frac{1}{\mathbb{E}(\Delta_1 K_1)} \sum_{j=1}^n K_j - \frac{1}{n} \sum_{i=1}^n \beta_i K_i \frac{1}{\mathbb{E}(\Delta_1 K_1)} \sum_{j=1}^n \beta_j K_j \\ &= \frac{n\mathbb{E}(\beta_1^2 K_1)\mathbb{E}(K_1)}{\mathbb{E}(\Delta_1 K_1)} \frac{1}{n\mathbb{E}(\beta_1^2 K_1)} \sum_{i=1}^n \beta_i^2 K_i \frac{1}{n\mathbb{E}(K_1)} \sum_{j=1}^n K_j \\ &\quad - \frac{n\mathbb{E}^2(\beta_1 K_1)}{\mathbb{E}(\Delta_1 K_1)} \frac{1}{n\mathbb{E}(\beta_1 K_1)} \sum_{i=1}^n \beta_i K_i \frac{1}{n\mathbb{E}(\beta_1 K_1)} \sum_{j=1}^n \beta_j K_j. \end{aligned}$$

Let us write

$$\widehat{F}_D =: A_1 T_{1,i} T_{2,j} - A_2 T_{3,i} T_{4,j},$$

where

$$\begin{aligned} T_{1,i} &= \frac{1}{n\mathbb{E}(\beta_1^2 K_1)} \sum_{i=1}^n \beta_i^2 K_i, & T_{2,j} &= \frac{1}{n\mathbb{E}(K_1)} \sum_{j=1}^n K_j, \\ T_{3,i} &= \frac{1}{n\mathbb{E}(\beta_1 K_1)} \sum_{i=1}^n \beta_i K_i, & T_{4,j} &= \frac{1}{n\mathbb{E}(\beta_1 K_1)} \sum_{j=1}^n \beta_j K_j, \\ A_1 &= \frac{n\mathbb{E}(\beta_1^2 K_1) \mathbb{E}(K_1)}{\mathbb{E}(\Delta_1 K_1)}, & A_2 &= \frac{n\mathbb{E}^2(\beta_1 K_1)}{\mathbb{E}(\Delta_1 K_1)}. \end{aligned}$$

Finally, the claimed result will be obtained as soon as the two following claims have been checked:

Claim 5.1

$$T_{l,i} \xrightarrow{P} 1 \text{ as } n \rightarrow +\infty \text{ for } l \in \{1, 3, 4\}.$$

Claim 5.2

$$T_{2,j} \xrightarrow{P} 1 \text{ as } n \rightarrow +\infty, \quad A_1 \xrightarrow{P.S} 1 \text{ as } n \rightarrow +\infty,$$

$$\text{and } A_2 = o(1) \text{ as } n \rightarrow +\infty.$$

Proof of claim 5.1 By combining the sufficient convergence condition in probability with (27), we obtain the following:

$$\lim_{n \rightarrow +\infty} \mathbb{E}(T_{l,i}) = 1 \text{ and } \lim_{n \rightarrow +\infty} \operatorname{Var}(T_{l,i}) = 0 \text{ for } l \in \{1, 3, 4\}.$$

□

Proof of claim 5.2 By following the same ideas as those used in claim 1, we show that: $\lim_{n \rightarrow +\infty} \text{Var}(T_{2,j}) = 0$. □

First, we have the following:

$$\text{Var}(T_{2,j}) = \text{Var}(T_{2,1}) + 2 \text{Cov}(T_{2,i}, T_{2,j}),$$

where

$$\begin{aligned} \text{Var}(T_{2,1}) &= \frac{1}{n\mathbb{E}^2(K_1)} (\mathbb{E}(K_1^2) - \mathbb{E}^2(K_1)) \\ &= \frac{\mathbb{E}(K_1^2)}{n\mathbb{E}^2(K_1)} - \frac{1}{n}. \end{aligned}$$

Second, we use the technical Lemma A.1 of [2], to get:

$$\frac{\mathbb{E}(K_1^2)}{n\mathbb{E}^2(K_1)} = \frac{M_2}{M_1^2 n \phi_x(h_K)};$$

then, by the assumption (H5)(i), we have the following:

$$\lim_{n \rightarrow +\infty} \text{Var}(T_{2,1}) = 0.$$

Moreover:

$$\text{Cov}(T_{2,i}, T_{2,j}) = \frac{1}{n^2\mathbb{E}^2(K_1)} \sum_{\substack{i=1 \\ i \neq j}}^n \sum_{j=1}^n \text{Cov}(K_i, K_j).$$

Let us now define the sets E_1 and E_2 as follows:

$$\begin{aligned} E_1 &= \{(i, j) \in \{1, 2, \dots, n\} \text{ such that } 1 \leq |i - j| \leq m_n\}, \\ E_2 &= \{(i, j) \in \{1, 2, \dots, n\} \text{ such that } m_n + 1 \leq |i - j| \leq n - 1\}, \end{aligned}$$

where the sequence m_n is chosen, such that $m_n \rightarrow +\infty$ as $n \rightarrow +\infty$, and we denote by $A_{1,n}$ and $A_{2,n}$ the sum of covariances over E_1 and E_2 , respectively, then

$$A_{1,n} = \frac{1}{n^2\mathbb{E}^2(K_1)} \sum_{E_1} \text{Cov}(K_i, K_j).$$

By stationarity and assumption (H3), we have:

$$\begin{aligned} |\text{Cov}(K_i, K_j)| &\leq |\mathbb{E}(K_i K_j)| + \mathbb{E}^2(K_1), \\ &\leq C \psi_x(h_K) + \mathbb{E}^2(K_1). \end{aligned}$$

Now, we use the technical lemma A.1 of [37], to obtain:

$$\begin{aligned} |A_{1,n}| &\leq n m_n (\psi_x(h_K) + \mathbb{E}^2(K_1)) \frac{1}{n^2\mathbb{E}^2(K_1)} \\ &\leq \frac{C m_n \psi_x(h_K)}{M_1^2 n \phi_x^2(h_K)} + \frac{m_n}{n}. \end{aligned}$$

The fact that $\frac{\psi_x(h_K)}{\phi_x^2(h_K)}$ is bounded by assumption (H3), and by the choice $m_n = \sqrt{n}$ permits to get:

$$A_{1,n} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Concerning the sum over E_2 , by following the same ideas as those used in (29), we get:

$$|A_{2,n}| = \frac{1}{n^2 \mathbb{E}^2(K_1)} \sum_{E_2} |\text{Cov}(K_i, K_j)| \leq \frac{C(m_n)^{1-s}}{s-1} \frac{1}{M_1^2 n \phi_x^2(h_K)}.$$

We use the same choice of m_n as before, and by assumption (H5)(i), we obtain the following:

$$A_{2,n} \longrightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Finally, to show that $A_1 \xrightarrow{P.S.} 1$ and $A_2 = o(1)$ as $n \rightarrow +\infty$, it suffices to apply the technical Lemma A.1 of [37].

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