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O-Cross Gram matrices with respect to *g*-frames

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Abstract The matrix representation of operators in Hilbert spaces is a useful tool in applications. It is important to present the matrix representation by sequences other than orthonormal bases. In this paper, we extend the matrix representation of operators using *g*-frames and investigate their invertibility and stability.

Mathematics Subject Classification 41A58 · 43A35

1 Introduction and motivation

Frames, introduced by Duffin and Schaeffer [19] in 1952, are very important in signal processing, image processing, and many other fields [6, 17, 18, 28]. Several generalizations of frames as continuous frames [2], fusion frames [13], and *g*-frames [33] have been presented. The concept of *g*-frames, which was introduced by Sun [33], is a generalization of frames which was investigated by many authors [1, 4, 29, 36].

1.1 Motivation

Many models in applications are continuous, for example, in physics and sound vibration analysis [3, 8]. Some problems in differential or integral equations can be formulated as operator theory problems. Therefore, discretizing is useful for handling operators numerically [23, 24, 30]. One way to do this is to find (possibly infinite) matrices describing these operators using orthonormal bases. In some applications, it is not easy to find a ‘good’ orthonormal basis. For solving this problem, frames were introduced by Duffin and Schaeffer [19] and widely developed by many authors [17, 18, 21]. Frames have been developed rapidly in mathematics and have achieved successful applications in various areas of pure and applied sciences and engineering [5, 6, 9–12].

Many operator equations, such as typical diffusion problems, cannot be solved analytically [8, 26]. In this respect, numerical methods such as the finite difference method and the finite-element method have been developed for their solution [27]. More precisely, consider the operator equation $Tu = v$, where $T \in B(H)$. This equation can be rewritten of the form:

$$Ax = b, \quad (1.1)$$

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where $A_{i,j} = \langle Te_i, e_j \rangle$ and $\{e_i\}_{i \in I}$ is an orthonormal basis of H . Then, the standard methods to solve linear systems (1.1) such as variational method can be applied; for example, see Subsection 9.2 of [35]. Recently, bounded operators are analyzed as a matrix using Bessel sequences, frames, and Riesz bases [7, 15]. This representation of operators can lead to more accurate results [8]. In this paper, we present an operator T in $B(H)$ in the form of a g -matrix representation via the base of g -frames. This allows us to exploit the special structure of T to decouple the system into smaller subsystems which can be solved quite efficiently.

1.2 Notation and definitions

Throughout this paper, H is a Hilbert space and $\{H_i\}_{i \in I}$ is a sequence of closed subspaces of H , where I is a subset of $\mathbb{Z}$. In addition, $B(H, H_i)$ is the collection of all bounded linear operators from H into H_i . Moreover, ℓ^2 is the set of scalars $\{c_i\}_{i \in I}$, such that $\sum_{i \in I} |c_i|^2 < \infty$.

A sequence $\Lambda = \{\Lambda_i \in B(H, H_i) : i \in I\}$ is called a *generalized frame* or a *g -frame* for H with respect to $\{H_i\}_{i \in I}$ if there exist constants $A, B > 0$, such that

$$A\|f\|^2 \leq \sum_{i \in I} \|\Lambda_i f\|^2 \leq B\|f\|^2, \quad (f \in H). \quad (1.2)$$

The numbers A and B are called the *lower* and *upper g -frame bounds*. It is called a *g -Bessel sequence* if the right hand of (1.2) is satisfied. The family Λ is called [33]:

- (i) A *g -complete set* for H with respect to $\{H_i\}_{i \in I}$ if $H = \overline{\text{span}}\{\Lambda_i^*(H_i)\}_{i \in I}$.
- (ii) A *g -Riesz sequence* for H with respect to $\{H_i\}_{i \in I}$ if there are positive constants A and B , such that

$$A \sum_{i \in I_1} \|g_i\|^2 \leq \left\| \sum_{i \in I_1} \Lambda_i^* g_i \right\|^2 \leq B \sum_{i \in I_1} \|g_i\|^2,$$

for all finite subset I_1 of I and $\{g_i\}_{i \in I_1} \subseteq \{H_i\}_{i \in I_1}$. Moreover, if it is a g -complete set, then it is called *g -Riesz basis*.

- (iii) A *g -orthonormal system* for H with respect to $\{H_i\}_{i \in I}$, if

$$\langle \Lambda_i^* g, \Lambda_j^* g' \rangle = \delta_{ij} \langle g, g' \rangle, \quad (i, j \in I, g \in H_i, g' \in H_j).$$

- (iv) A *g -orthonormal basis* for H with respect to $\{H_i\}_{i \in I}$, if it is a g -orthonormal system for H with respect to $\{H_i\}_{i \in I}$ and $\{\Lambda_i^* e_{ij}\}_{i \in I, j \in J_i}$ is a basis for H , where $\{e_{ij}\}_{j \in J_i}$ is an orthonormal basis for H_i , for all $i \in I$.

It is easy to see that the space

$$\left(\sum_{i \in I} \oplus H_i \right)_{\ell^2} = \left\{ \{f_i\}_{i \in I} : f_i \in H_i, \sum_{i \in I} \|f_i\|^2 < \infty \right\}$$

with the inner product

$$\langle \{f_i\}_{i \in I}, \{g_i\}_{i \in I} \rangle = \sum_{i \in I} \langle f_i, g_i \rangle$$

is a Hilbert space. The *synthesis operator* of a g -Bessel sequence $\Lambda = \{\Lambda_i\}_{i \in I}$ is defined by the following:

$$T_\Lambda : \left(\sum_{i \in I} \oplus H_i \right)_{\ell^2} \rightarrow H, \quad T_\Lambda(\{f_i\}_{i \in I}) = \sum_{i \in I} \Lambda_i^*(f_i).$$

The associated adjoint operator given by

$$T_\Lambda^* : H \rightarrow \left(\sum_{i \in I} \oplus H_i \right)_{\ell^2}, \quad T_\Lambda^*(f) = \{\Lambda_i f\}_{i \in I}$$

is called the *analysis operator*. In addition, the *g-frame operator* S_Λ is defined by the following:

$$S_\Lambda : H \rightarrow H, \quad S_\Lambda f = \sum_{i \in I} \Lambda_i^* \Lambda_i f.$$

For a *g-frame* $\Lambda = \{\Lambda_i\}_{i \in I}$, S_Λ is a bounded, self adjoint, and invertible operator and so we have the following reconstruction formula:

$$f = \sum_{i \in I} \Lambda_i^* \tilde{\Lambda}_i f = \sum_{i \in I} \tilde{\Lambda}_i^* \Lambda_i f, \quad (f \in H), \quad (1.3)$$

where $\tilde{\Lambda}_i = \Lambda_i S_\Lambda^{-1}$ (see [33] for more details).

The following theorem frequently used is easy to obtain.

Theorem 1.1 *Let $\Lambda = \{\Lambda_i\}_{i \in I}$ be a g-Bessel sequences for H with respect to $\{H_i\}_{i \in I}$. Then:*

- (1) Λ is a *g-frame* if and only if T_Λ is onto.
- (2) Λ is a *g-Riesz basis* if and only if T_Λ is bijective.

Definition 1.2 Let $\{\Lambda_i\}_{i \in I}$ and $\{\Theta_i\}_{i \in I}$ be *g-Bessel sequences* for H with respect to $\{H_i\}_{i \in I}$. We call $\{\Theta_i\}_{i \in I}$ a *dual g-frame* of $\{\Lambda_i\}_{i \in I}$, if

$$f = \sum_{i \in I} \Lambda_i^* \Theta_i f, \quad (f \in H).$$

In this case, $\{\Lambda_i\}_{i \in I}$ and $\{\Theta_i\}_{i \in I}$ are also *g-frames* for H with respect to $\{H_i\}_{i \in I}$.

By (1.3), it is obvious that $\{\tilde{\Lambda}_i\}_{i \in I}$ is a dual of $\{\Lambda_i\}_{i \in I}$, which is called the *canonical dual g-frame* of $\{\Lambda_i\}_{i \in I}$.

Let $\Lambda_i \in B(H, H_i)$ and $\{e_{ij}\}_{j \in J_i}$ be orthonormal bases for H_i , where J_i is a subset of $\mathbb{Z}$. Then, there are always elements:

$$u_{ij} = \Lambda_i^* e_{ij}, \quad (i \in I, j \in J_i), \quad (1.4)$$

such that

$$\Lambda_i f = \sum_{j \in J_i} \langle f, u_{ij} \rangle e_{ij}, \quad (f \in H). \quad (1.5)$$

The sequence $\{u_{ij}\}_{i \in I, j \in J_i}$ which is called the *sequence induced* by $\{\Lambda_i\}_{i \in I}$ with respect to $\{e_{ij}\}_{i \in I, j \in J_i}$ plays a key role in this topic, see [33].

Theorem 1.3 [33] *Let $\{\Lambda_i\}_{i \in I} \subseteq B(H, H_i)$ and $\{u_{ij}\}_{i \in I, j \in J_i}$ be defined by (1.4). Then:*

- (1) $\{\Lambda_i\}_{i \in I}$ is a *g-frame* (resp. *g-Bessel sequence*, *g-Riesz basis*, and *g-orthonormal basis*) for H if and only if $\{u_{ij}\}_{i \in I, j \in J_i}$ is a *frame* (resp. *Bessel sequence*, *Riesz basis*, and *orthonormal basis*) for H .
- (2) The *g-frame operator* of $\{\Lambda_i\}_{i \in I}$ coincides with the *frame operator* of $\{u_{ij}\}_{i \in I, j \in J_i}$.
- (3) $\{\Lambda_i\}_{i \in I}$ and $\{\Theta_i\}_{i \in I}$ are a pair of (canonical and weakly) dual *g-frames* if and only if their induced sequences are a pair of (canonical and weakly) dual frames.

We will use the following criterion for the invertibility of operators.

Proposition 1.4 [24] *Let $F : H \rightarrow H$ be invertible on H . Suppose that $G : H \rightarrow H$ is a bounded operator and $\|Gh - Fh\| \leq v\|h\|$ for all $h \in H$, where $v \in [0, \frac{1}{\|F^{-1}\|})$. Then, G is invertible on H and $G^{-1} = \sum_{k=0}^{\infty} [F^{-1}(F - G)]^k F^{-1}$.*

2 O-Gram matrices

The matrix representation of operators on a Hilbert space H with Bessel sequences, frames, and Riesz bases was proposed by Balazs and developed in [7, 31]. In this section, we present an operator in $B(H)$ in the form of a g -matrix representation in terms of the base of g -frames.

Let $\Psi = \{\psi_i\}_{i \in I}$ and $\Phi = \{\phi_i\}_{i \in I}$ be Bessel sequences in Hilbert spaces H_1 and H_2 , respectively. For $U \in B(H_1, H_2)$, the matrix $\mathbf{G}_{U, \Phi, \Psi}$ given by

$$(\mathbf{G}_{U, \Phi, \Psi})_{i,j} = \langle U\psi_j, \phi_i \rangle, \quad (i, j \in I) \quad (2.1)$$

is called the U -cross Gram matrix, [7, 31]. If $H_1 = H_2$ and $U = I_{H_1}$, it is called cross Gram matrix and denoted by $\mathbf{G}_{\Phi, \Psi}$. We use $\mathbf{G}_{\Psi}$ for $\mathbf{G}_{\Psi, \Psi}$, so called the Gram matrix [14].

Definition 2.1 Let $\Lambda = \{\Lambda_i \in B(H, H_i) : i \in I\}$ and $\Theta = \{\Theta_i \in B(H, H_i) : i \in I\}$ be g -Bessel sequences, and $O \in B(H)$. The g -matrix representation $\mathbb{G}_{O, \Lambda, \Theta}$, for which the elements are in $B(\sum_{i \in I} \oplus H_i, H_i)$, given by

$$(\mathbb{G}_{O, \Lambda, \Theta})_{ij} = \Lambda_i O \Theta_j^*$$

is called the O -cross Gram matrix. If $O = I_H$, it is called cross Gram matrix and is denoted by $\mathbb{G}_{\Lambda, \Theta}$. We use $\mathbb{G}_{\Lambda}$ for $\mathbb{G}_{\Lambda, \Lambda}$, so called the Gram matrix.

Lemma 2.2 Let $\Lambda = \{\Lambda_i\}_{i \in I}$ and $\Theta = \{\Theta_i\}_{i \in I}$ be two g -Bessel sequences in H with respect to $\{H_i\}_{i \in I}$. In addition, let $O \in B(H)$. The following assertions hold.

- (1) $\mathbb{G}_{O, \Lambda, \Theta} = T_{\Lambda}^* O T_{\Theta}$. In particular, the O -cross Gram matrix $\mathbb{G}_{O, \Lambda, \Theta}$ defines a bounded operator on $(\sum_{i \in I} \oplus H_i)_{\ell^2}$ and $\|\mathbb{G}_{O, \Lambda, \Theta}\| \leq \sqrt{B_{\Lambda} B_{\Theta}} \|O\|$.
- (2) $(\mathbb{G}_{O, \Lambda, \Theta})^* = \mathbb{G}_{O^*, \Theta, \Lambda}$.
- (3) If $\mathbb{G}_{O, \Lambda, \Theta}$ is a compact operator, then $\lim_{i \rightarrow \infty} \|\Lambda_i O \Theta_i^*\| = 0$.
- (4) If $\sum_{i \in I} \sum_{j \in I} \|\Lambda_i O \Theta_j^*\|^2 < \infty$, then $\mathbb{G}_{O, \Lambda, \Theta}$ is compact.

Proof (1) For every $f = \{f_i\}_{i \in I} \in (\sum_{i \in I} \oplus H_i)_{\ell^2}$, we have the following:

$$\begin{aligned} (\mathbb{G}_{O, \Lambda, \Theta} f)_j &= \sum_{k \in I} (\mathbb{G}_{O, \Lambda, \Theta})_{jk} f_k \\ &= \sum_{k \in I} \Lambda_j O \Theta_k^* f_k \\ &= \Lambda_j O \sum_{k \in I} \Theta_k^* f_k = (T_{\Lambda}^* O T_{\Theta})_j. \end{aligned}$$

In particular:

$$\|\mathbb{G}_{O, \Lambda, \Theta}\| = \|T_{\Lambda}^* O T_{\Theta}\| \leq \sqrt{B_{\Lambda} B_{\Theta}} \|O\|.$$

(2) Using the last part, we obtain the following:

$$(\mathbb{G}_{O, \Lambda, \Theta})^* = (T_{\Lambda}^* O T_{\Theta})^* = T_{\Theta}^* O^* T_{\Lambda} = \mathbb{G}_{O^*, \Theta, \Lambda}.$$

As a consequence of Theorem 2.5 of [22], an operator O is compact if and only if $\lim_{k \rightarrow \infty} \|O e_k\| = 0$. This is true if and only if $\lim_{k \rightarrow \infty} \sum_l |\langle O e_k, f_l \rangle|^2 = \lim_{k \rightarrow \infty} \|O e_k\|^2 = 0$ for any orthonormal bases $\{e_n\}_{n \in I}$ and $\{f_n\}_{n \in I}$. From this and

$$(\mathbb{G}_{O, \Lambda, \Theta})_{ij} = \Lambda_i O \Theta_j^*,$$

(3) and (4) follow immediately. $\square$

Now, we state a key result.

Proposition 2.3 Let $\Lambda = \{\Lambda_i\}_{i \in I}$ be a g -Bessel sequence for H with respect to $\{H_i\}_{i \in I}$ and $u = \{u_{ij}\}_{i \in I, j \in J_i}$ its induced Bessel sequence. There exists a unitary operator $U \in B((\sum_{i \in I} \oplus H_i)_{\ell^2}, \ell^2)$, such that $T_\Lambda = T_u U$.

Proof Define $U : (\sum_{i \in I} \oplus H_i)_{\ell^2} \rightarrow \ell^2$ by the following:

$$U(\{f_i\}_{i \in I}) = \{\langle f_i, e_{ij} \rangle\}_{i \in I, j \in J_i},$$

where $\{e_{ij}\}_{j \in J_i}$ is an orthonormal basis for H_i , for all $i \in I$. Using (1.5), we obtain

$$\begin{aligned} T_\Lambda(\{f_i\}_{i \in I}) &= \sum_{i \in I} \Lambda_i^* f_i \\ &= \sum_{i \in I} \sum_{j \in J_i} \langle f_i, e_{ij} \rangle u_{ij} = T_u U(\{f_i\}_{i \in I}), \end{aligned}$$

for all $\{f_i\}_{i \in I} \in (\sum_{i \in I} \oplus H_i)_{\ell^2}$. Moreover:

$$\begin{aligned} \|U(\{f_i\}_{i \in I})\|^2 &= \sum_{i \in I, j \in J_i} |\langle f_i, e_{ij} \rangle|^2 \\ &= \sum_{i \in I} \|f_i\|^2 = \|\{f_i\}_{i \in I}\|^2. \end{aligned}$$

In addition, it is easy to check that

$$U^*(\{c_{ij}\}_{i \in I, j \in J_i}) = \left\{ \sum_{j \in J_i} c_{ij} e_{ij} \right\}_{i \in I}$$

for every $\{c_{ij}\}_{i \in I, j \in J_i} \in \ell^2$. In fact, U^* is one to one and $U^* = U^{-1}$, i.e., U is a unitary operator. $\square$

Theorem 2.4 Let $\Lambda = \{\Lambda_i\}_{i \in I}$ and $\Theta = \{\Theta_i\}_{i \in I}$ be g -Bessel sequences for H with respect to $\{H_i\}_{i \in I}$. In addition, let $u_\Lambda = \{u_{ij}^\Lambda\}_{i \in I, j \in J_i}$ and $u_\Theta = \{u_{ij}^\Theta\}_{i \in I, j \in J_i}$ be their induced Bessel sequences. Then, for $O \in B(H)$ the following assertions hold.

- (1) $\mathbb{G}_{O, \Lambda, \Theta}$ is invertible if and only if $\mathbf{G}_{O, u_\Lambda, u_\Theta}$ is invertible.
- (2) $\mathbb{G}_{O, \Lambda, \Theta}$ is the identity on $(\sum_{i \in I} \oplus H_i)_{\ell^2}$ if and only if $\mathbf{G}_{O, u_\Lambda, u_\Theta}$ is the identity on ℓ^2 .
- (3) If $\mathbb{G}_{O, \Lambda, \Theta}$ is invertible, then Λ and Θ are g -Riesz sequences. Moreover, if O is an invertible operator and Λ or Θ is g -frame, then

$$\mathbb{G}_{O, \Lambda, \Theta}^{-1} = \mathbb{G}_{O^{-1}, \tilde{\Theta}, \tilde{\Lambda}}. \quad (2.2)$$

Proof Using Proposition 2.3, it follows that

$$\begin{aligned} \mathbb{G}_{O, \Lambda, \Theta} &= T_\Lambda^* O T_\Theta \\ &= U^{-1} T_{u_\Lambda}^* O T_{u_\Theta} U \\ &= U^{-1} \mathbf{G}_{O, u_\Lambda, u_\Theta} U. \end{aligned} \quad (2.3)$$

This immediately gives (1) and (2). To show (3), note that T_Λ and T_Θ are injective by the invertibility of $\mathbb{G}_{O, \Lambda, \Theta}$. Hence, Λ and Θ are g -Riesz sequences. Suppose that Λ is a g -frame. Using Theorem 1.1, it follows that Λ is a g -Riesz basis, since O and $\mathbb{G}_{O, \Lambda, \Theta}$ are invertible. In addition, $T_\Theta = O^{-1} (T_\Lambda^*)^{-1} \mathbb{G}_{O, \Lambda, \Theta}$ is invertible and Θ is a g -Riesz basis by Theorem 1.1. Now, by Theorem 1.3, u_Λ and u_Θ are Riesz bases. On the other hand, using (1.4) and (1.5) $S_\Lambda = S_{u_\Lambda}$. Indeed:

$$\begin{aligned}
S_{\Lambda} f &= \sum_{i \in I} \Lambda_i^* \Lambda_i f \\
&= \sum_{i \in I} \Lambda_i^* \sum_{j \in J_i} \langle f, u_{ij} \rangle e_{ij} \\
&= \sum_{i \in I} \sum_{j \in J_i} \langle f, u_{ij} \rangle u_{ij} = S_{u_{\Lambda}} f,
\end{aligned}$$

for all $f \in H$. Similarly, $S_{\Theta} = S_{u_{\Theta}}$.

$$\begin{aligned}
(\mathbb{G}_{O, \Lambda, \Theta})^{-1} &= U^{-1} (\mathbf{G}_{O, u_{\Lambda}, u_{\Theta}})^{-1} U \\
&= U^{-1} \mathbf{G}_{O^{-1}, \tilde{u}_{\Theta}, \tilde{u}_{\Lambda}} U \\
&= U^{-1} (S_{u_{\Theta}}^{-1} T_{u_{\Theta}})^* O^{-1} S_{u_{\Lambda}}^{-1} T_{u_{\Lambda}} U \\
&= (T_{u_{\Theta}} U)^* S_{\Theta}^{-1} O^{-1} S_{\Lambda}^{-1} T_{\Lambda} \\
&= T_{\Theta}^* O^{-1} T_{\Lambda} = \mathbb{G}_{O^{-1}, \tilde{\Theta}, \tilde{\Lambda}}.
\end{aligned}$$

□

Corollary 2.5 (1) Let $\Lambda = \{\Lambda_i\}_{i \in I}$ and $\Theta = \{\Theta_i\}_{i \in I}$ be g -frames. Then, $\mathbb{G}_{O, \Lambda, \Theta} = I_{\ell^2}$ if and only if Λ and Θ are g -Riesz bases and $\Theta_i = \Lambda_i O S_{\Theta}$, for all $i \in I$.

(2) Let $\Lambda = \{\Lambda_i\}_{i \in I}$ and $\Theta = \{\Theta_i\}_{i \in I}$ be dual g -frames. If $\mathbb{G}_{O, \Lambda, \Theta} = I_{\ell^2}$, then $\Lambda = \tilde{\Theta}$.

Proof (1) Suppose that $\mathbb{G}_{O, \Lambda, \Theta} = I_{\ell^2}$. Then, (2.3) shows that $\mathbf{G}_{O, u_{\Lambda}, u_{\Theta}} = I_{\ell^2}$. Hence, $T_{u_{\Lambda}}^*$ is onto and $T_{u_{\Theta}}$ is injective. By using Theorem 5.2.2 of [14], u_{Λ} and u_{Θ} are Riesz bases, and so, Λ and Θ are g -Riesz bases:

$$\begin{aligned}
\Theta_i f &= \sum_{j \in J_i} \langle f, u_{ij}^{\Theta} \rangle e_{ij} \\
&= \sum_{j \in J_i} \langle f, S_{\Theta} O u_{ij}^{\Lambda} \rangle e_{ij} \\
&= \sum_{j \in J_i} \langle O^* S_{\Theta} f, u_{ij}^{\Lambda} \rangle e_{ij} = \Lambda_i O^* S_{\Theta} f,
\end{aligned}$$

for all $f \in H$. Conversely, by the above computations and the fact that $\{e_{ij}\}_{j \in J_i}$ is an orthonormal basis for H_i , we obtain $u_{ij}^{\Theta} = S_{\Theta} O u_{ij}^{\Lambda}$. Hence, $\mathbf{G}_{O, u_{\Lambda}, u_{\Theta}} = I_{\ell^2}$ by Proposition 2.5 of [31], and therefore, (2.3) implies that $\mathbb{G}_{O, \Lambda, \Theta} = I_{\ell^2}$.

(2) If $\mathbb{G}_{O, \Lambda, \Theta} = I_{\ell^2}$, then $\mathbf{G}_{O, u_{\Lambda}, u_{\Theta}} = I_{\ell^2}$ by (2.3). Using Corollary 2.6 of [31], we have $u_{\Lambda} = \tilde{u}_{\Theta}$. Therefore, by the fact that $S_{\Theta} = S_{u_{\Theta}}$, we obtain

$$\begin{aligned}
\Lambda_i f &= \sum_{j \in J_i} \langle f, u_{ij}^{\Lambda} \rangle e_{ij} \\
&= \sum_{j \in J_i} \langle f, S_{\Theta}^{-1} u_{ij}^{\Theta} \rangle e_{ij} \\
&= \Theta_i S_{\Theta}^{-1} f = \tilde{\Theta}_i f.
\end{aligned}$$

□

3 Stability of O -cross Gram matrices with respect to g -Bessel sequences

In this section, we discuss sufficient conditions for the invertibility of O -cross Gram matrices with respect to g -frames.

Proposition 3.1 Let $\Lambda = \{\Lambda_i\}_{i \in I}$ be a g -Bessel sequence in H with Bessel bound B_{Λ} . If $O, O_1, O_2 \in B(H)$ are such that $\mathbb{G}_{O, \Lambda, \Lambda}$ is an invertible operator, then the following assertions hold.

(1) If

$$\|O_1 - I_H\| < \frac{1}{\|\mathbb{G}_{O,\Lambda,\Lambda}^{-1}\| B_\Lambda \|O\|}, \quad (3.1)$$

then $\mathbb{G}_{O,\Lambda,O_1\Lambda}$ and $\mathbb{G}_{O,O_1\Lambda,\Lambda}$ are also invertible.

(2) If

$$\|O_1^* O O_2 - O\| < \frac{1}{\|\mathbb{G}_{O,\Lambda,\Lambda}^{-1}\| B_\Lambda}, \quad (3.2)$$

then $\mathbb{G}_{O,O_1\Lambda,O_2\Lambda}$ is also invertible. Moreover, if Λ is a g -frame, then

$$\mathbb{G}_{O,O_1\Lambda,O_2\Lambda}^{-1} = T_\Lambda^{-1} \sum_{k=0}^{\infty} (O^{-1} (O_1^* O O_2 - O))^k O^{-1} T_\Lambda.$$

Proof (1) Using the assumption (3.1), we obtain

$$\begin{aligned} \|\mathbb{G}_{O,\Lambda,O_1\Lambda} - \mathbb{G}_{O,\Lambda,\Lambda}\| &= \|\mathbb{G}_{O O_1,\Lambda,\Lambda} - \mathbb{G}_{O,\Lambda,\Lambda}\| \\ &= \|T_\Lambda^* O (O_1 - I_H) T_\Lambda\| \\ &\leq \|T_\Lambda\|^2 \|O\| \|O_1 - I_H\| \\ &\leq B_\Lambda \|O\| \|O_1 - I_H\| < \|\mathbb{G}_{O,\Lambda,\Lambda}^{-1}\|^{-1}. \end{aligned}$$

Using Proposition 1.4 implies that $\mathbb{G}_{O,\Lambda,O_1\Lambda}$ is invertible. The same concerns the invertibility of $\mathbb{G}_{O,O_1\Lambda,\Lambda}$.

(2) The assumption (3.2) yields

$$\begin{aligned} \|\mathbb{G}_{O,O_1\Lambda,O_2\Lambda} - \mathbb{G}_{O,\Lambda,\Lambda}\| &= \|\mathbb{G}_{O_1^* O O_2,\Lambda,\Lambda} - \mathbb{G}_{O,\Lambda,\Lambda}\| \\ &= \|T_\Lambda^* (O_1^* O O_2 - O) T_\Lambda\| \\ &\leq \|T_\Lambda\|^2 \|O_1^* O O_2 - O\| < \|\mathbb{G}_{O,\Lambda,\Lambda}^{-1}\|^{-1}. \end{aligned}$$

Therefore, $\mathbb{G}_{O,O_1\Lambda,O_2\Lambda}$ is invertible by Proposition 1.4. Moreover, if Λ is a g -frame, then $\mathbb{G}_{O,\Lambda,\Lambda}^{-1} = T_\Lambda^{-1} O^{-1} (T_\Lambda^*)^{-1}$, and Theorem 2.4 implies that Λ is a g -Riesz basis and, therefore, O is invertible. By Proposition 1.4, we obtain

$$\begin{aligned} \mathbb{G}_{O,O_1\Lambda,O_2\Lambda}^{-1} &= \sum_{k=0}^{\infty} \left(\mathbb{G}_{O,\Lambda,\Lambda}^{-1} (\mathbb{G}_{O,O_1\Lambda,O_2\Lambda} - \mathbb{G}_{O,\Lambda,\Lambda}) \right)^k \mathbb{G}_{O,\Lambda,\Lambda}^{-1} \\ &= \sum_{k=0}^{\infty} \left(T_\Lambda^{-1} O^{-1} (T_\Lambda^*)^{-1} (T_\Lambda^* (O_1^* O O_2 - O) T_\Lambda) \right)^k \mathbb{G}_{O,\Lambda,\Lambda}^{-1} \\ &= \sum_{k=0}^{\infty} \left(T_\Lambda^{-1} O^{-1} (O_1^* O O_2 - O) T_\Lambda \right)^k \mathbb{G}_{O,\Lambda,\Lambda}^{-1} \\ &= \sum_{k=0}^{\infty} T_\Lambda^{-1} (O^{-1} (O_1^* O O_2 - O))^k T_\Lambda T_\Lambda^{-1} O^{-1} (T_\Lambda^*)^{-1} \\ &= T_\Lambda^{-1} \sum_{k=0}^{\infty} (O^{-1} (O_1^* O O_2 - O))^k O^{-1} T_\Lambda. \end{aligned}$$

□

Theorem 3.2 Let $\Lambda = \{\Lambda_i\}_{i \in I}$ and $\Theta = \{\Theta_i\}_{i \in I}$ be g -Bessel sequences for H with respect to $\{H_i\}_{i \in I}$. In addition, let $O \in B(H)$, such that $\mathbb{G}_{O,\Lambda,\Theta}$ is invertible.

(1) Let V be a bounded operator on H , such that

$$\|O - V\| < \frac{1}{\|\mathbb{G}_{O,\Lambda,\Theta}^{-1}\| \sqrt{B_\Lambda B_\Theta}}, \quad (3.3)$$

then $\mathbb{G}_{V,\Lambda,\Theta}$ is also invertible.

(2) If $\Gamma = \{\Gamma_i\}_{i \in I}$ is a g -Bessel sequence for H with respect to $\{H_i\}_{i \in I}$, such that

$$\left(\sum_{i \in I, j \in J_i} \|u_{ij}^\Theta - u_{ij}^\Gamma\|^2 \right)^{1/2} < \frac{1}{\|\mathbb{G}_{O,\Lambda,\Theta}^{-1}\| \sqrt{B_\Lambda} \|O\|}, \quad (3.4)$$

then $\mathbb{G}_{O,\Lambda,\Gamma}$ is invertible.

(3) If $\Xi = \{\Xi_i\}_{i \in I}$ is a g -Bessel sequence for H with respect to $\{H_i\}_{i \in I}$, such that

$$\left(\sum_{i \in I, j \in J_i} \|u_{ij}^\Lambda - u_{ij}^\Xi\|^2 \right)^{1/2} < \frac{1}{\|\mathbb{G}_{O,\Lambda,\Theta}^{-1}\| \sqrt{B_\Theta} \|O\|}, \quad (3.5)$$

then $\mathbb{G}_{O,\Xi,\Theta}$ is invertible.

Proof The assumption (3.3) yields

$$\begin{aligned} \|\mathbb{G}_{O,\Lambda,\Theta} - \mathbb{G}_{V,\Lambda,\Theta}\| &= \|U^{-1} \mathbb{G}_{O,u_\Lambda,u_\Theta} U - U^{-1} \mathbb{G}_{V,u_\Lambda,u_\Theta} U\| \\ &= \|U^{-1} T_{u_\Lambda}^* (O - V) T_{u_\Theta} U\| \\ &\leq \sqrt{B_\Lambda B_\Theta} \|O - V\| < \frac{1}{\|\mathbb{G}_{O,\Lambda,\Theta}^{-1}\|}. \end{aligned}$$

An application of Proposition 1.4 shows that $\mathbb{G}_{V,\Lambda,\Theta}$ is also invertible. To obtain (2), note that

$$\|T_{u_\Theta} - T_{u_\Gamma}\|^2 \leq \sum_{i \in I, j \in J_i} \|u_{ij}^\Theta - u_{ij}^\Gamma\|^2.$$

Using (3.4), we get that

$$\begin{aligned} \|\mathbb{G}_{O,\Lambda,\Theta} - \mathbb{G}_{O,\Lambda,\Gamma}\| &= \|U^{-1} T_{u_\Lambda}^* O T_{u_\Theta} U - U^{-1} T_{u_\Lambda}^* O T_{u_\Gamma} U\| \\ &= \|U^{-1} T_{u_\Lambda}^* O (T_{u_\Theta} - T_{u_\Gamma}) U\| \\ &\leq \sqrt{B_\Lambda} \|O\| \left(\sum_{i \in I, j \in J_i} \|u_{ij}^\Theta - u_{ij}^\Gamma\|^2 \right)^{1/2} < \frac{1}{\|\mathbb{G}_{O,\Lambda,\Theta}^{-1}\|}. \end{aligned}$$

Hence, $\mathbb{G}_{O,\Lambda,\Gamma}$ is invertible by Proposition 1.4. The proof of (3) is similar. $\square$

Corollary 3.3 Let $\Lambda = \{\Lambda_i\}_{i \in I}$ and $\Theta = \{\Theta_i\}_{i \in I}$ be a g -Riesz basis and a g -Bessel sequence, respectively, for H with respect to $\{H_i\}_{i \in I}$. In addition, let $O \in B(H)$, such that

$$\left(\sum_{i \in I} \|O \Theta_i - \Lambda_i\|^2 \right)^{1/2} < \frac{A}{\sqrt{B}},$$

where A and B are g -Riesz bounds of Λ . Then, $\mathbb{G}_{O,\Lambda,\Theta}$ is invertible and

$$\mathbb{G}_{O,\Lambda,\Theta}^{-1} = \sum_{k=0}^{\infty} \left(T_\Lambda^{-1} O T_\Theta - I_{(\sum_{i \in I} \oplus H_i)_{\ell^2}} \right)^k \mathbb{G}_\Lambda^{-1}.$$

Proof Since Λ is a g -Riesz basis, the Gram matrix $\mathbb{G}_\Lambda$ is invertible by Theorem 2.4. Moreover:

$$\|\mathbb{G}_\Lambda^{-1}\| = \|T_\Lambda^{-1}(T_\Lambda^*)^{-1}\| \leq A^{-1}.$$

Hence:

$$\begin{aligned} \|I_{(\sum_{i \in I} \oplus H_i)_{\ell^2}} - \mathbb{G}_\Lambda^{-1} \mathbb{G}_{O, \Lambda, \Theta}\| &= \|\mathbb{G}_\Lambda^{-1} (\mathbb{G}_\Lambda - \mathbb{G}_{O, \Lambda, \Theta})\| \\ &\leq \|\mathbb{G}_\Lambda^{-1}\| \|T_\Lambda^* O T_\Theta - T_\Lambda^* T_\Lambda\| \\ &\leq \frac{1}{A} \|T_\Lambda^*\| \|O T_\Theta - T_\Lambda\| \\ &\leq \frac{\sqrt{B}}{A} \left(\sum_{i \in I} \|O \Theta_i - \Lambda_i\|^2 \right)^{1/2} < 1. \end{aligned}$$

Therefore, $\mathbb{G}_{O, \Lambda, \Theta}$ is invertible. Moreover, by Proposition 1.4, we have the following:

$$\begin{aligned} \mathbb{G}_{O, \Lambda, \Theta}^{-1} &= \sum_{k=0}^{\infty} \left(\mathbb{G}_\Lambda^{-1} (\mathbb{G}_{O, \Lambda, \Theta} - \mathbb{G}_\Lambda) \right)^k \mathbb{G}_\Lambda^{-1} \\ &= \sum_{k=0}^{\infty} \left(\mathbb{G}_\Lambda^{-1} T_\Lambda^* (O T_\Theta - T_\Lambda) \right)^k \mathbb{G}_\Lambda^{-1} \\ &= \sum_{k=0}^{\infty} \left(T_\Lambda^{-1} O T_\Theta - I_{(\sum_{i \in I} \oplus H_i)_{\ell^2}} \right)^k \mathbb{G}_\Lambda^{-1}. \end{aligned}$$

□

Theorem 3.4 Let $\Lambda = \{\Lambda_i\}_{i \in I}$ and $\Theta = \{\Theta_i\}_{i \in I}$ be g -frames for H with respect to $\{H_i\}_{i \in I}$. If $\Xi = \{\Xi_i\}_{i \in I}$ and $\Gamma = \{\Gamma_i\}_{i \in I}$ are g -Bessel sequences for H with respect to $\{H_i\}_{i \in I}$, such that

$$\begin{aligned} \left\| \sum_{i \in I} (\Theta_i^* - \Gamma_i^*) f_i \right\| + \left\| \sum_{i \in I} (\Lambda_i^* - \Xi_i^*) f_i \right\| &\leq \lambda_1 \left\| \sum_{i \in I} \Theta_i^*(f_i) \right\| \\ + \lambda_2 \left\| \sum_{i \in I} \Lambda_i^*(f_i) \right\| + \lambda_3 \left\| \sum_{i \in I} \Xi_i^*(f_i) \right\| + \lambda_4 \left\| \sum_{i \in I} \Gamma_i^*(f_i) \right\|. \end{aligned} \quad (3.6)$$

for all $\{f_i\}_{i \in I} \in (\sum_{i \in I} \oplus H_i)_{\ell^2}$ and

$$\|O - V\| < \mu, \quad \mu + 2\|O\|\lambda < \frac{\sqrt{A_\Theta A_\Lambda}}{\|O^{-1}\|B}, \quad (3.7)$$

where $B = \max\{B_\Lambda, B_\Theta, B_\Xi, B_\Gamma\}$, $\lambda = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$ and A_Λ, A_Θ are lower bounds of Λ and Θ , respectively. Then, $\mathbb{G}_{V, \Xi, \Gamma}$ is invertible if $\mathbb{G}_{O, \Lambda, \Theta}$ is invertible. Moreover, Ξ and Γ are g -Riesz sequences.

Proof Using (3.6), we obtain

$$\|T_\Theta - T_\Gamma\| + \|T_\Lambda - T_\Xi\| \leq \lambda_1 \sqrt{B_\Theta} + \lambda_2 \sqrt{B_\Lambda} + \lambda_3 \sqrt{B_\Xi} + \lambda_4 \sqrt{B_\Gamma}. \quad (3.8)$$

On the other hand, using Theorem 2.4, Λ and Θ are g -Riesz bases and O is invertible. Therefore

$$\mathbb{G}_{O, \Lambda, \Theta}^{-1} = \mathbb{G}_{O^{-1}, \tilde{\Theta}, \tilde{\Lambda}}.$$

Moreover

$$\frac{\sqrt{A_\Theta A_\Lambda}}{\|O^{-1}\|} \leq \frac{1}{\|\mathbb{G}_{O, \Lambda, \Theta}^{-1}\|}. \quad (3.9)$$

Combining (3.7), (3.8), and (3.9), we have

$$\begin{aligned}
 \|\mathbb{G}_{V,\Xi,\Gamma} - \mathbb{G}_{O,\Lambda,\Theta}\| &= \|\mathbb{G}_{V,\Xi,\Gamma} - \mathbb{G}_{O,\Xi,\Gamma} + \mathbb{G}_{O,\Xi,\Gamma} - \mathbb{G}_{O,\Xi,\Theta} + \mathbb{G}_{O,\Xi,\Theta} - \mathbb{G}_{O,\Lambda,\Theta}\| \\
 &\leq \|T_{\Xi}^*(V - O)T_{\Gamma}\| + \|T_{\Xi}^*O(T_{\Gamma} - T_{\Theta})\| + \|(T_{\Xi}^* - T_{\Lambda}^*)OT_{\Theta}\| \\
 &\leq \mu\sqrt{B_{\Xi}B_{\Gamma}} + \sqrt{B_{\Xi}}\|O\|(T_{\Gamma} - T_{\Theta}) + \|O\|\sqrt{B_{\Theta}}\|T_{\Xi} - T_{\Lambda}\| \\
 &\leq \mu\sqrt{B_{\Xi}B_{\Gamma}} + (\sqrt{B_{\Xi}} + \sqrt{B_{\Theta}})\|O\|[\lambda_1\sqrt{B_{\Theta}} + \lambda_2\sqrt{B_{\Lambda}} \\
 &\quad + \lambda_3\sqrt{B_{\Xi}} + \lambda_4\sqrt{B_{\Gamma}}] \\
 &< B(\mu + 2\|O\|\lambda) \\
 &\leq \frac{\sqrt{A_{\Theta}A_{\Lambda}}}{\|O^{-1}\|} \leq \frac{1}{\|\mathbb{G}_{O,\Lambda,\Theta}^{-1}\|}.
 \end{aligned}$$

Hence, $\mathbb{G}_{V,\Xi,\Gamma}$ is invertible by Proposition 1.4. $\square$

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