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Weighted averages of n-convex functions via extension of Montgomery's identity

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Abstract Using an extension of Montgomery's identity and the Green's function, we obtain new identities and related inequalities for weighted averages of n-convex functions, i.e. the sum $\sum_{i=1}^{m} \rho_i h(\lambda_i)$ and the integral $\int_a^b \rho(\lambda)h(\gamma(\lambda))d\lambda$ where h is an n-convex function.

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1 Introduction

In this paper we will give some identities and related inequalities involving weighted discrete or integral averages of functions, i. e., containing sums $\sum_{i=1}^{m} \rho_i h(\lambda_i)$ or integrals $\int_a^b \rho(\lambda)h(\gamma(\lambda))\,d\lambda$. As a consequence we will give conditions on numbers $\lambda_1, \ldots, \lambda_m, \ \rho_1, \ldots, \rho_m$ under which the inequality $\sum_{i=1}^{m} \rho_i h(\lambda_i) \geq 0$ holds for every function h from a particular class of functions. For example, for the class of convex functions such results were studied in [5], while Popoviciu [7–9] gave results for the class of n-convex functions (see [6, Chapter 9] also). We will extend the results of Popoviciu and we would start first with some basic definitions and properties of n-convex functions.

Definition 1.1 The *nth order divided difference* of a function $h : [a, b] \to \mathbb{R}$ at distinct points $\lambda_i, \lambda_{i+1}, \ldots, \lambda_{i+n} \in [a, b] \subset \mathbb{R}$ for some $i \in \mathbb{N}$ is defined recursively by:

$$\left[\lambda_j; h\right] = h\left(\lambda_j\right), \quad j \in \{i, \ldots, i+n\}$$
$$\left[\lambda_i, \ldots, \lambda_{i+n}; h\right] = \frac{\left[\lambda_{i+1}, \ldots, \lambda_{i+n}; h\right] - \left[\lambda_i, \ldots, \lambda_{i+n-1}; h\right]}{\lambda_{i+n} - \lambda_i}.$$

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A function h is said to be *convex of order n* or *n-convex* if for all choices of distinct points $\lambda_i, \ldots, \lambda_{i+n}$, we have $\left[\lambda_i, \ldots, \lambda_{i+n}; h\right] \geq 0$.

From the definition, it is easy to see that 1-convex functions are nondecreasing functions, while 2-convex functions are the classical convex functions, so n-convex functions are a generalization of the notion of convexity. If the nth order derivative $h^{(n)}$ exists, then h is n-convex iff $h^{(n)} \geq 0$. For $1 \leq k \leq n-2$, a function h is n-convex iff $h^{(k)}$ exists and is $(n-k)$-convex.

The following result is due to Popoviciu [7,8] (see [6] also).

Proposition 1.2 *The inequality*

$$\sum_{i=1}^{m} \rho_i h(\lambda_i) \geq 0, \tag{1.1}$$

holds for all n-convex functions $h : [a, b] \to \mathbb{R}$, $n \in \mathbb{N}$, *iff the m-tuples* $\boldsymbol{\lambda} \in [a, b]^m$, $\boldsymbol{\rho} \in \mathbb{R}^m$ *satisfy*

$$\sum_{i=1}^{m} \rho_i \lambda_i^k = 0, \quad \textit{for all } k = 0, 1, \ldots, n-1, \tag{1.2}$$

$$\sum_{i=1}^{m} \rho_i (\lambda_i - t)_+^{n-1} \geq 0, \quad \textit{for every } t \in [a, b], \tag{1.3}$$

where $y_+ = \max(y, 0)$.

In fact, Popoviciu proved a stronger result that it is enough to assume that the inequality in (1.3) holds for every $t \in [\lambda_{(1)}, \lambda_{(m-n+1)}]$, where $\lambda_{(1)} \leq \cdots \leq \lambda_{(m)}$ is the ordered m-tuple $\boldsymbol{\lambda}$, since this, together with (1.2), implies that it holds for every $t \in [a, b]$ (see [9]). In the case of convex functions, i.e., $n = 2$, Pečarić [5] proved the result with the conditions (1.2) and (1.3) replaced with

$$\sum_{i=1}^{m} \rho_i = 0 \quad \text{and} \quad \sum_{i=1}^{m} \rho_i |\lambda_i - \lambda_k| \geq 0 \text{ for } k \in \{1, \ldots, m\}. \tag{1.4}$$

The integral analogue of Proposition 1.2 is given in the next proposition.

Proposition 1.3 *Let* $n \geq 2$, $\rho : [\alpha, \beta] \to \mathbb{R}$ *and* $\gamma : [\alpha, \beta] \to [a, b]$. *The inequality*

$$\int_{\alpha}^{\beta} \rho(\lambda) h(\gamma(\lambda))\, \mathrm{d}\lambda \geq 0, \tag{1.5}$$

holds for all n-convex functions $h : [a, b] \to \mathbb{R}$ *iff*

$$\begin{aligned} &\int_{\alpha}^{\beta} \rho(\lambda)\gamma(\lambda)^k\, \mathrm{d}\lambda = 0, \quad \textit{for all } k = 0, 1, \ldots, n-1, \\ &\int_{\alpha}^{\beta} \rho(\lambda)\, (\gamma(\lambda) - t)_+^{n-1}\, \mathrm{d}\lambda \geq 0, \quad \textit{for every } t \in [a, b]. \end{aligned} \tag{1.6}$$

We will also need the following extension of Montgomery's identity given in [1] which was derived using Taylor's formula.

Proposition 1.4 *Let* $n \in \mathbb{N}$, $h : I \to \mathbb{R}$ *be such that* $h^{(n-1)}$ *is absolutely continuous,* $I \subset \mathbb{R}$ *an open interval,* $a, b \in I$, $a < b$. *Then the following identity holds*

$$\begin{aligned} h(\lambda) = {} & \frac{1}{b-a} \int_a^b h(s)\, \mathrm{d}s + \sum_{k=0}^{n-2} \frac{h^{(k+1)}(a)}{k!\,(k+2)} \frac{(\lambda - a)^{k+2}}{b-a} \\ & - \sum_{k=0}^{n-2} \frac{h^{(k+1)}(b)}{k!\,(k+2)} \frac{(\lambda - b)^{k+2}}{b-a} + \frac{1}{(n-1)!} \int_a^b T_n(\lambda, t)\, h^{(n)}(t)\, \mathrm{d}t, \end{aligned} \tag{1.7}$$

where

$$T_n(\lambda, t) = \begin{cases} -\frac{(\lambda-t)^n}{n(b-a)} + \frac{\lambda-a}{b-a}(\lambda-t)^{n-1}, & a \le t \le \lambda, \\ -\frac{(\lambda-t)^n}{n(b-a)} + \frac{\lambda-b}{b-a}(\lambda-t)^{n-1}, & \lambda < t \le b. \end{cases} \tag{1.8}$$

In case $n = 1$ the sum $\sum_{k=0}^{n-2} \cdots$ is empty, so identity (1.7) reduces to the well-known Montgomery identity (see for instance [4])

$$h(\lambda) = \frac{1}{b-a}\int_a^b h(t)\,dt + \int_a^b P(\lambda, s)\,h'(s)\,ds,$$

where $P(\lambda, s)$ is the Peano kernel defined by

$$P(\lambda, s) = \begin{cases} \frac{s-a}{b-a}, & a \le s \le \lambda, \\ \frac{s-b}{b-a}, & \lambda < s \le b. \end{cases}$$

Let us denote by $G : [a, b] \times [a, b] \to \mathbb{R}$ the Green's function of the boundary value problem

$$z''(\lambda) = 0, \ z(a) = z(b) = 0.$$

The function G is given by

$$G(t, s) = \begin{cases} \frac{(t-b)(s-a)}{b-a} & \text{for } a \le s \le t, \\ \frac{(s-b)(t-a)}{b-a} & \text{for } t \le s \le b \end{cases} \tag{1.9}$$

and integration by parts easily yields that for any function $h \in C^2[a, b]$ the following identity holds

$$h(\lambda) = \frac{b-\lambda}{b-a}h(a) + \frac{\lambda-a}{b-a}h(b) + \int_a^b G(\lambda, s)h''(s)ds. \tag{1.10}$$

The function G is continuous, symmetric and convex with respect to both variables t and s.

The paper is organized as follows: in Sect. 2, we obtain new identities involving discrete and integral weighted averages of n-convex functions by appropriate use of the extended Montgomery's identity and the Green's function. Related Popoviciu type inequalities are also derived. In Sect. 3, we obtain new Grüss- and Ostrowski-type inequalities by obtaining bounds for the remainders of the identities from Sect. 2.

2 Identities and related Popoviciu-type inequalities for *n*-convex functions

We will first prove couple of identities which will have a key role in the rest of the paper.

Theorem 2.1 *Let $n \in \mathbb{N}$, $n \ge 3$, $h : I \to \mathbb{R}$ be a function such that $h^{(n-1)}$ is absolutely continuous, $I \subset \mathbb{R}$ an open interval, $a, b \in I$, $a < b$. Furthermore, let $\boldsymbol{\lambda} \in [a, b]^m$ and $\boldsymbol{\rho} \in \mathbb{R}^m$ satisfy $\sum_{i=1}^m \rho_i = 0$ and $\sum_{i=1}^m \rho_i\lambda_i = 0$ and let G be given by (1.9). Then*

$$\begin{aligned} \sum_{i=1}^m \rho_i h(\lambda_i) &= \frac{h'(a) - h'(b)}{b-a}\int_a^b \sum_{i=1}^m \rho_i G(\lambda_i, s)ds \\ &+ \sum_{k=2}^{n-1} \frac{k}{(k-1)!}\int_a^b \sum_{i=1}^m \rho_i G(\lambda_i, s)\frac{h^{(k)}(a)(s-a)^{k-1} - h^{(k)}(b)(s-b)^{k-1}}{b-a}ds \\ &+ \frac{1}{(n-3)!}\int_a^b h^{(n)}(t)\left(\int_a^b \sum_{i=1}^m \rho_i G(\lambda_i, s)\tilde{T}_{n-2}(s, t)ds\right)dt, \end{aligned} \tag{2.1}$$

where

$$\tilde{T}_{n-2}(s,t)=\begin{cases}\frac{1}{b-a}\left[\frac{(s-t)^{n-2}}{(n-2)}+(s-a)\,(s-t)^{n-3}\right],\ a\le t\le s,\\ \frac{1}{b-a}\left[\frac{(s-t)^{n-2}}{(n-2)}+(s-b)\,(s-t)^{n-3}\right],\ s<t\le b.\end{cases}\tag{2.2}$$

Moreover, the following identity holds

$$\begin{aligned}\sum_{i=1}^{m}\rho_i h(\lambda_i)&=\frac{h'(b)-h'(a)}{b-a}\int_a^b\sum_{i=1}^{m}\rho_i G(\lambda_i,s)\mathrm{d}s\\&+\sum_{k=3}^{n-1}\frac{k-2}{(k-1)!}\int_a^b\sum_{i=1}^{m}\rho_i G(\lambda_i,s)\frac{h^{(k)}(a)(s-a)^{k-1}-h^{(k)}(b)(s-b)^{k-1}}{b-a}\mathrm{d}s\\&+\frac{1}{(n-3)!}\int_a^b h^{(n)}(t)\left(\int_a^b\sum_{i=1}^{m}\rho_i G(\lambda_i,s)T_{n-2}(s,t)\mathrm{d}s\right)\mathrm{d}t,\end{aligned}\tag{2.3}$$

where T_n is as defined in (1.8).

Proof Using (1.10) in $\sum_{i=1}^{m}\rho_i h(\lambda_i)$ and the fact that $\sum_{i=1}^{m}\rho_i=0$ and $\sum_{i=1}^{m}\rho_i\lambda_i=0$, we get

$$\sum_{i=1}^{m}\rho_i h(\lambda_i)=\int_a^b\sum_{i=1}^{m}\rho_i G(\lambda_i,s)h''(s)\mathrm{d}s.\tag{2.4}$$

Differentiating the function h in (1.7) twice gives

$$\begin{aligned}h''(s)&=\frac{h'(a)-h'(b)}{b-a}+\sum_{k=2}^{n-1}\frac{k}{(k-1)!}\frac{h^{(k)}(a)(s-a)^{k-1}-h^{(k)}(b)(s-b)^{k-1}}{b-a}\\&+\frac{1}{(n-3)!}\int_a^b\tilde{T}_{n-2}(s,t)h^{(n)}(t)\mathrm{d}t.\end{aligned}\tag{2.5}$$

Inserting (2.5) in (2.4) yields

$$\begin{aligned}\sum_{i=1}^{m}\rho_i h(\lambda_i)&=\frac{h'(a)-h'(b)}{b-a}\int_a^b\sum_{i=1}^{m}\rho_i G(\lambda_i,s)\mathrm{d}s\\&+\sum_{k=2}^{n-1}\frac{k}{(k-1)!}\int_a^b\sum_{i=1}^{m}\rho_i G(\lambda_i,s)\frac{h^{(k)}(a)(s-a)^{k-1}-h^{(k)}(b)(s-b)^{k-1}}{b-a}\mathrm{d}s\\&+\frac{1}{(n-3)!}\int_a^b\sum_{i=1}^{m}\rho_i G(\lambda_i,s)\left(\int_a^b\tilde{T}_{n-2}(s,t)h^{(n)}(t)\mathrm{d}t\right)\mathrm{d}s,\end{aligned}$$

and then using Fubini's theorem in the last term we get (2.1).

Moreover, by applying formula (1.7) with h and n replaced by h'' and $n-2$, respectively, and rearranging the indices, we get

$$\begin{aligned}h''(s)&=\frac{h'(b)-h'(a)}{b-a}+\sum_{k=3}^{n-1}\frac{k-2}{(k-1)!}\frac{h^{(k)}(a)(s-a)^{k-1}-h^{(k)}(b)(s-b)^{k-1}}{b-a}\\&+\frac{1}{(n-3)!}\int_a^b T_{n-2}(s,t)h^{(n)}(t)\mathrm{d}t.\end{aligned}\tag{2.6}$$

Similarly, using (2.6) in (2.4) and applying Fubini's Theorem, we get (2.3). □

Next we will state some inequalities that can be derived from the obtained identities.

Theorem 2.2 *Let all the assumptions of Theorem 2.1 hold with the additional condition*

$$\int_a^b \sum_{i=1}^m \rho_i G(\lambda_i, s)\tilde{T}_{n-2}(s,t)\mathrm{d}s \geq 0, \quad \forall\, t \in [a,b], \tag{2.7}$$

where G and $\tilde{T}_{n-2}$ are defined in (1.9) and (2.2). If h is n-convex, then the following inequality holds

$$\begin{aligned} &\sum_{i=1}^m \rho_i h(\lambda_i) - \frac{h'(a)-h'(b)}{b-a}\int_a^b \sum_{i=1}^m \rho_i G(\lambda_i, s)\mathrm{d}s \\ &-\sum_{k=2}^{n-1}\frac{k}{(k-1)!}\int_a^b \sum_{i=1}^m \rho_i G(\lambda_i, s)\frac{h^{(k)}(a)(s-a)^{k-1}-h^{(k)}(b)(s-b)^{k-1}}{b-a}\mathrm{d}s \geq 0. \end{aligned} \tag{2.8}$$

Proof Since the function h is n-convex, we have $h^{(n)} \geq 0$. Using this fact and (2.7) in (2.1) we easily arrive at our required result. □

Theorem 2.3 *Let all the assumptions of Theorem 2.1 hold with the additional condition*

$$\int_a^b \sum_{i=1}^m \rho_i G(\lambda_i, s)T_{n-2}(s,t)\mathrm{d}s \geq 0, \quad \forall\, t \in [a,b], \tag{2.9}$$

where G and T_n are defined in (1.9) and (1.8). If h is n-convex, then the following inequality holds

$$\begin{aligned} &\sum_{i=1}^m \rho_i h(\lambda_i) - \frac{h'(b)-h'(a)}{b-a}\int_a^b \sum_{i=1}^m \rho_i G(\lambda_i, s)\mathrm{d}s \\ &-\sum_{k=3}^{n-1}\frac{k-2}{(k-1)!}\int_a^b \sum_{i=1}^m \rho_i G(\lambda_i, s)\frac{h^{(k)}(a)(s-a)^{k-1}-h^{(k)}(b)(s-b)^{k-1}}{b-a}\mathrm{d}s \geq 0 \end{aligned} \tag{2.10}$$

Proof Since the function h is n-convex we have $h^{(n)} \geq 0$. Using this fact and (2.9) in (2.3) we easily arrive at our required result. □

Now we state an important consequence.

Theorem 2.4 *Let all the assumptions from Theorem 2.1 hold with the additional assumptions $\sum_{i=1}^m \rho_i = 0$ and $\sum_{i=1}^m \rho_i |\lambda_i - \lambda_k| \geq 0$ for $k \in \{1,\ldots,m\}$. If h is n-convex and n is even, then inequalities (2.8) and (2.10) hold.*

Proof The Green's function $G(s,t)$ is convex with respect to t for every $s \in [a,b]$. Therefore, from Proposition 1.2, with conditions (1.2) and (1.3) replaced by (1.4) as in [5], we have

$$\sum_{i=1}^m \rho_i G(\lambda_i, s) \geq 0 \quad \text{for} \quad s \in [a,b]. \tag{2.11}$$

Also note that for even n both $\tilde{T}_{n-2}(s,t) \geq 0$ and $T_{n-2}(s,t) \geq 0$. Therefore, combining this fact with (2.11) we get inequalities (2.7) and (2.9). As h is n-convex, the results follow from Theorems 2.2 and 2.3. □

We will next state the integral versions of our main results. Since the proofs are of similar nature we will omit the details.

Theorem 2.5 *Let $n \in \mathbb{N}$, $n \geq 3$, $h : I \to \mathbb{R}$ be a function such that $h^{(n-1)}$ is absolutely continuous, $I \subset \mathbb{R}$ an open interval, $a, b \in I$, $a < b$. Furthermore, let $\gamma : [\alpha,\beta] \to [a,b]$ and $\rho : [\alpha,\beta] \to \mathbb{R}$ satisfy*

$\int_{\alpha}^{\beta} \rho(\lambda)\mathrm{d}\lambda = 0$ *and* $\int_{\alpha}^{\beta} \rho(\lambda)\gamma(\lambda)\mathrm{d}\lambda = 0$, *and let* G, $\tilde{T}_n$ *and* T_n *be given by* (1.9), (2.2) *and* (1.8). *Then the following two identities hold:*

$$\int_{\alpha}^{\beta} p(\lambda)\, h(\gamma(\lambda))\, \mathrm{d}\lambda = \frac{h'(a)-h'(b)}{b-a}\int_a^b\int_{\alpha}^{\beta} p(\lambda)\, G(\gamma(\lambda),s)\,\mathrm{d}\lambda\,\mathrm{d}s + \sum_{k=2}^{n-1}\frac{k}{(k-1)!}\int_a^b\left(\int_{\alpha}^{\beta} p(\lambda)\, G(\gamma(\lambda),s)\,\mathrm{d}\lambda\right) \frac{h^{(k)}(a)(s-a)^{k-1}-h^{(k)}(b)(s-b)^{k-1}}{b-a}\mathrm{d}s + \frac{1}{(n-3)!}\int_a^b h^{(n)}(t)\left(\int_a^b\left(\int_{\alpha}^{\beta} p(\lambda)\, G(\gamma(\lambda),s)\,\mathrm{d}\lambda\right)\tilde{T}_{n-2}(s,t)\mathrm{d}s\right)\mathrm{d}t, \tag{2.12}$$

and

$$\int_{\alpha}^{\beta} p(\lambda)\, h(\gamma(\lambda))\, \mathrm{d}\lambda = \frac{h'(b)-h'(a)}{b-a}\int_a^b\int_{\alpha}^{\beta} p(\lambda)\, G(\gamma(\lambda),s)\,\mathrm{d}\lambda\,\mathrm{d}s + \sum_{k=3}^{n-1}\frac{k-2}{(k-1)!}\int_a^b\left(\int_{\alpha}^{\beta} p(\lambda)\, G(\gamma(\lambda),s)\,\mathrm{d}\lambda\right) \frac{h^{(k)}(a)(s-a)^{k-1}-h^{(k)}(b)(s-b)^{k-1}}{b-a}\mathrm{d}s + \frac{1}{(n-3)!}\int_a^b h^{(n)}(t)\left(\int_a^b\left(\int_{\alpha}^{\beta} p(\lambda)\, G(\gamma(\lambda),s)\,\mathrm{d}\lambda\right)T_{n-2}(s,t)\mathrm{d}s\right)\mathrm{d}t. \tag{2.13}$$

Theorem 2.6 *Let all the assumptions of Theorem 2.5 hold with the additional condition*

$$\int_a^b\int_{\alpha}^{\beta} p(\lambda)\, G(\gamma(\lambda),s)\,\tilde{T}_{n-2}(s,t)\,\mathrm{d}\lambda\,\mathrm{d}s \geq 0, \quad \forall\, t\in[a,b], \tag{2.14}$$

where G *is defined in* (1.9) *and* $\tilde{T}_n$ *is defined in* (2.2). *If* h *is n-convex, then the following inequality holds:*

$$\int_{\alpha}^{\beta} p(\lambda)\, h(\gamma(\lambda))\, \mathrm{d}\lambda - \frac{h'(a)-h'(b)}{b-a}\int_a^b\int_{\alpha}^{\beta} p(\lambda)\, G(\gamma(\lambda),s)\,\mathrm{d}\lambda\,\mathrm{d}s \tag{2.15}$$

$$-\sum_{k=2}^{n-1}\frac{k}{(k-1)!}\int_a^b\left(\int_{\alpha}^{\beta} p(\lambda)\, G(\gamma(\lambda),s)\,\mathrm{d}\lambda\right)\frac{h^{(k)}(a)(s-a)^{k-1}-h^{(k)}(b)(s-b)^{k-1}}{b-a}\mathrm{d}s \geq 0. \tag{2.16}$$

Theorem 2.7 *Let all the assumptions of Theorem 2.5 hold with the additional condition*

$$\int_a^b\int_{\alpha}^{\beta} p(\lambda)\, G(\gamma(\lambda),s)\,T_{n-2}(s,t)\mathrm{d}\lambda\,\mathrm{d}s \geq 0, \quad \forall\, t\in[a,b], \tag{2.17}$$

where G *is defined in* (1.9) *and* T_n *is defined in* (1.8). *If* h *is n-convex, then the following inequality holds*

$$\int_{\alpha}^{\beta} p(\lambda)\, h(\gamma(\lambda))\, \mathrm{d}\lambda - \frac{h'(b)-h'(a)}{b-a}\int_a^b\int_{\alpha}^{\beta} p(\lambda)\, G(\gamma(\lambda),s)\,\mathrm{d}\lambda\,\mathrm{d}s - \sum_{k=3}^{n-1}\frac{k-2}{(k-1)!}\int_a^b\left(\int_{\alpha}^{\beta} p(\lambda)\, G(\gamma(\lambda),s)\,\mathrm{d}\lambda\right)\frac{h^{(k)}(a)(s-a)^{k-1}-h^{(k)}(b)(s-b)^{k-1}}{b-a}\mathrm{d}s \geq 0. \tag{2.18}$$

Theorem 2.8 *Let all the assumptions from Theorem 2.5 hold with the additional assumption that* $\gamma : [\alpha,\beta] \to [a,b]$ *and* $\rho : [\alpha,\beta] \to \mathbb{R}$ *satisfy* (1.6). *If* h *is* $n-$*convex and* n *is even, then inequalities* (2.15) *and* (2.18) *hold.*

3 Bounds for the remainders

In these sections, we will give bounds for the remainders which occur in certain representations of the sum $\sum_{i=1}^{m} \rho_i h(\lambda_i)$ and the integral $\int_\alpha^\beta \rho(\lambda) h(\gamma(\lambda))\, d\lambda$. Namely, we will give some Grüss- and Ostrowski-type inequalities.

Let $h, \gamma : [a, b] \to \mathbb{R}$ be two Lebesgue integrable functions. We consider the Čebyšev functional

$$T(h, g) = \frac{1}{b-a}\int_a^b h(\lambda)g(\lambda)d\lambda - \left(\frac{1}{b-a}\int_a^b h(\lambda)d\lambda\right)\left(\frac{1}{b-a}\int_a^b g(\lambda)d\lambda\right). \tag{3.1}$$

$L_\infty[a, b]$ denotes the space of essentially bounded functions on $[a, b]$ with the norm

$$\|h\|_\infty = \text{ess} \sup_{t\in[a,b]} |h(t)|.$$

The following results can be found in [3]:

Proposition 3.1 *Let $h : [a, b] \to \mathbb{R}$ be a Lebesgue integrable function and let $g : [a, b] \to \mathbb{R}$ be an absolutely continuous function with $(\cdot - a)(b - \cdot)[g']^2 \in L[a, b]$. Then we have the inequality*

$$|T(h, g)| \le \frac{1}{\sqrt{2}}\left(\frac{1}{b-a}|T(h, h)|\int_a^b (\lambda - a)(b-\lambda)[g'(\lambda)]^2 d\lambda\right)^{1/2}. \tag{3.2}$$

The constant $\frac{1}{\sqrt{2}}$ in (3.2) is the best possible.

Proposition 3.2 *Let $g : [a, b] \to \mathbb{R}$ be a monotonic nondecreasing function and let $h : [a, b] \to \mathbb{R}$ be an absolutely continuous function such that $h' \in L_\infty[a, b]$. Then we have the inequality*

$$|T(h, g)| \le \frac{1}{2(b-a)}\|h'\|_\infty \int_a^b (\lambda - a)(b-\lambda)dg(\lambda). \tag{3.3}$$

The constant $\frac{1}{2}$ in (3.3) is the best possible.

For the ease of notation, throughout this section Ω_j, $j \in \{1, 2, 3, 4\}$, will denote the following functions: under the assumption of Theorems 2.1 and 2.5 we define

$$\Omega_1(t) = \int_a^b \sum_{i=1}^{m} \rho_i G(\lambda_i, s)\tilde{T}_{n-2}(s, t)ds, \quad t \in [a, b],$$

$$\Omega_2(t) = \int_a^b \sum_{i=1}^{m} \rho_i G(\lambda_i, s)T_{n-2}(s, t)ds \ge 0, \quad t \in [a, b],$$

$$\Omega_3(t) = \int_a^b \int_\alpha^\beta p(\lambda)\, G(\gamma(\lambda), s)\, \tilde{T}_{n-2}(s, t)\, d\lambda\, ds, \quad t \in [a, b],$$

$$\Omega_4(t) = \int_a^b \int_\alpha^\beta p(\lambda)\, G(\gamma(\lambda), s)\, T_{n-2}(s, t) d\lambda\, ds, \quad t \in [a, b].$$

Theorem 3.3 *Let $n \in \mathbb{N}$, $n \ge 3$, $h : [a, b] \to \mathbb{R}$ be such that $h^{(n)}$ is an absolutely continuous function with $(\cdot - a)(b - \cdot)[h^{(n+1)}]^2 \in L[a, b]$ and let $\boldsymbol{\lambda} \in [a, b]^m$ and $\boldsymbol{\rho} \in \mathbb{R}^m$ satisfy $\sum_{i=1}^{m} \rho_i = 0$ and $\sum_{i=1}^{m} \rho_i\lambda_i = 0$. Then*

$$\begin{aligned}\sum_{i=1}^{m} \rho_i h(\lambda_i) &= \frac{h'(a) - h'(b)}{b-a}\int_a^b \sum_{i=1}^{m} \rho_i G(\lambda_i, s)ds \\ &+ \sum_{k=2}^{n-1} \frac{k}{(k-1)!}\int_a^b \sum_{i=1}^{m} \rho_i G(\lambda_i, s)\frac{h^{(k)}(a)(s-a)^{k-1} - h^{(k)}(b)(s-b)^{k-1}}{b-a}ds \\ &+ \frac{h^{(n-1)}(b) - h^{(n-1)}(a)}{(n-3)!(b-a)}\int_a^b \Omega(s)ds + R_n^1(h; a, b),\end{aligned} \tag{3.4}$$

and

$$\sum_{i=1}^{m}\rho_i h(\lambda_i)=\frac{h'(b)-h'(a)}{b-a}\int_a^b\sum_{i=1}^{m}\rho_i G(\lambda_i,s)\mathrm{d}s$$
$$+\sum_{k=3}^{n-1}\frac{k-2}{(k-1)!}\int_a^b\sum_{i=1}^{m}\rho_i G(\lambda_i,s)\frac{h^{(k)}(a)(s-a)^{k-1}-h^{(k)}(b)(s-b)^{k-1}}{b-a}\mathrm{d}s$$
$$+\frac{h^{(n-1)}(b)-h^{(n-1)}(a)}{(n-3)!(b-a)}\int_a^b\Omega_2(s)\mathrm{d}s+R_n^2(h;a,b), \tag{3.5}$$

where the remainders $R_n^j(h;a,b)$, $j=1,2$, satisfy the bounds

$$|R_n^j(h;a,b)|\le\frac{1}{(n-3)!}\left(\frac{(b-a)}{2}\left|T(\Omega_j,\Omega_j)\int_a^b(s-a)(b-s)[h^{(n+1)}(s)]^2\mathrm{d}s\right|\right)^{1/2}. \tag{3.6}$$

Proof We will prove the claim for $j=1$, while the proof for $j=2$ is analogous. Proposition 3.1 with $h\to\Omega_1$ and $g\to h^{(n)}$ yields

$$\left|\frac{1}{b-a}\int_a^b\Omega_1(t)h^{(n)}(t)\mathrm{d}t-\left(\frac{1}{b-a}\int_a^b\Omega_1(t)\mathrm{d}t\right)\left(\frac{1}{b-a}\int_a^b h^{(n)}(t)\mathrm{d}t\right)\right|$$
$$\le\frac{1}{\sqrt{2}}\left(\frac{1}{b-a}|T(\Omega_1,\Omega_1)|\int_a^b(t-a)(b-t)[h^{(n+1)}(t)]^2\mathrm{d}t\right)^{1/2}. \tag{3.7}$$

By identity (2.1) from Theorem 2.1

$$\sum_{i=1}^{m}\rho_i h(\lambda_i)-\frac{h'(a)-h'(b)}{b-a}\int_a^b\sum_{i=1}^{m}\rho_i G(\lambda_i,s)\mathrm{d}s$$
$$-\sum_{k=2}^{n-1}\frac{k}{(k-1)!}\int_a^b\sum_{i=1}^{m}\rho_i G(\lambda_i,s)\frac{h^{(k)}(a)(s-a)^{k-1}-h^{(k)}(b)(s-b)^{k-1}}{b-a}\mathrm{d}s$$
$$=\frac{1}{(n-3)!}\int_a^b\Omega_1(t)h^{(n)}(t)\mathrm{d}t$$

and since

$$\frac{1}{(n-3)!}\int_a^b\Omega_1(t)h^{(n)}(t)\mathrm{d}t=\frac{h^{(n-1)}(b)-h^{(n-1)}(a)}{(n-3)!(b-a)}\int_a^b\Omega_1(t)\mathrm{d}t+R_n^1(h;a,b),$$

the bound (3.6) for the remainder $R_n^1(h;a,b)$ follows from (3.7). □

Using Proposition 3.2, we obtain the following Grüss-type inequality.

Theorem 3.4 *Let $n\in\mathbb{N}$, $n\ge 3$, $h:[a,b]\to\mathbb{R}$ be such that $h^{(n)}$ is an absolutely continuous function with $h^{(n+1)}\ge 0$ and let $\boldsymbol{\lambda}\in[a,b]^m$ and $\boldsymbol{\rho}\in\mathbb{R}^m$ satisfy $\sum_{i=1}^m\rho_i=0$ and $\sum_{i=1}^m\rho_i\lambda_i=0$. Then representations (3.4) and (3.5) hold and the remainders $R_n^j(h;a,b)$, $j=1,2$, satisfy the bounds*

$$|R_n^j(h;a,b)|\le\frac{1}{(n-3)!}\|\Omega_j'\|_\infty\left\{\frac{b-a}{2}\left[h^{(n-1)}(b)+h^{(n-1)}(a)\right]\right.$$
$$\left.-\left[h^{(n-2)}(b)-h^{(n-2)}(a)\right]\right\}. \tag{3.8}$$

Proof Proposition 3.2 with $h \to \Omega_j$ and $g \to h^{(n)}$ yields

$$\left|\frac{1}{b-a}\int_a^b \Omega_j(t)h^{(n)}(t)\mathrm{d}t - \left(\frac{1}{b-a}\int_a^b \Omega_j(t)\mathrm{d}t\right)\left(\frac{1}{b-a}\int_a^b h^{(n)}(t)\mathrm{d}t\right)\right| \le \frac{1}{2(b-a)}\|\Omega_j'\|_\infty \int_a^b (t-a)(b-t)h^{(n+1)}(t)\mathrm{d}t.$$

Since

$$\int_a^b (t-a)(b-t)h^{(n+1)}(t)\mathrm{d}t = \int_a^b (2t-a-b)h^{(n)}(t)\mathrm{d}t = (b-a)\left[h^{(n-1)}(b)+h^{(n-1)}(a)\right] - 2\left[h^{(n-2)}(b)-h^{(n-2)}(a)\right], \tag{3.9}$$

using (3.9) and the identities from Theorem 2.1, we deduce (3.8). □

Here, the symbol $L_p[a,b]$ $(1 \le p < \infty)$ denotes the space of p-power integrable functions on the interval $[a,b]$ equipped with the norm

$$\|h\|_p = \left(\int_a^b |h(t)|^p \,\mathrm{d}t\right)^{\frac{1}{p}}$$

Now we state some Ostrowski-type inequalities related to the generalized linear inequalities.

Theorem 3.5 *Let* $n \in \mathbb{N}$, $n \ge 3$, $1 \le q, r \le \infty$, $\frac{1}{q} + \frac{1}{r} = 1$, $h^{(n)} \in L_q[a,b]$ *and let* $\boldsymbol{\lambda} \in [a,b]^m$ *and* $\boldsymbol{\rho} \in \mathbb{R}^m$ *satisfy* $\sum_{i=1}^m \rho_i = 0$ *and* $\sum_{i=1}^m \rho_i\lambda_i = 0$. *Then*

$$\left|\sum_{i=1}^m \rho_i h(\lambda_i) - \frac{h'(a)-h'(b)}{b-a}\int_a^b \sum_{i=1}^m \rho_i G(\lambda_i,s)\mathrm{d}s - \sum_{k=2}^{n-1}\frac{k}{(k-1)!}\int_a^b \sum_{i=1}^m \rho_i G(\lambda_i,s)\frac{h^{(k)}(a)(s-a)^{k-1}-h^{(k)}(b)(s-b)^{k-1}}{b-a}\mathrm{d}s\right| \le \frac{1}{(n-3)!}\|h^{(n)}\|_q \|\Omega_1\|_r. \tag{3.10}$$

and

$$\left|\sum_{i=1}^m \rho_i h(\lambda_i) - \frac{h'(b)-h'(a)}{b-a}\int_a^b \sum_{i=1}^m \rho_i G(\lambda_i,s)\mathrm{d}s - \sum_{k=3}^{n-1}\frac{k-2}{(k-1)!}\int_a^b \sum_{i=1}^m \rho_i G(\lambda_i,s)\frac{h^{(k)}(a)(s-a)^{k-1}-h^{(k)}(b)(s-b)^{k-1}}{b-a}\mathrm{d}s\right| \le \frac{1}{(n-3)!}\|h^{(n)}\|_q \|\Omega_2\|_r. \tag{3.11}$$

The constant on the right-hand sides of (3.10) and (3.11) is sharp for $1 < q \le \infty$ *and the best possible for* $q = 1$.

Proof Let us denote

$$\mu_j(t) = \frac{1}{(n-3)!}\Omega_j(t), \quad j = 1, 2.$$

Using identities (2.1) and (2.3) from Theorem 2.1 and Hölder's inequality, we obtain inequalities (3.10) and (3.11), i.e. that the left-hand sides of these inequalities are less than or equal to

$$\text{L.H.S.} \le \|h^{(n)}\|_q \left\|\mu_j\right\|_r. \tag{3.12}$$

For the proof of the sharpness of the constant $\left(\int_a^b \left|\mu_j(t)\right|^r \mathrm{d}t\right)^{1/r}$, let us find a function h for which the equality in (3.12) is obtained.

For $1 < q < \infty$ take h to be such that

$$h^{(n)}(t) = sgn\mu_j(t) \cdot |\mu_j(t)|^{1/(q-1)}.$$

For $q = \infty$, take h such that

$$h^{(n)}(t) = sgn\mu_j(t).$$

Finally, for $q = 1$, we prove that

$$\left|\int_a^b \mu_j(t)h^{(n)}(t)\mathrm{d}t\right| \leq \max_{t\in[a,b]} \left|\mu_j(t)\right| \int_a^b h^{(n)}(t)\mathrm{d}t \tag{3.13}$$

is the best possible inequality.

Suppose that $\left|\mu_j(t)\right|$ attains its maximum at $t_0 \in [a, b]$. First we consider the case $\mu_j(t_0) > 0$. For δ small enough we define $h_\delta(t)$ by

$$h_\delta(t) = \begin{cases} 0\,, & a \leq t \leq t_0, \\ \frac{1}{\delta n!}(t-t_0)^n\,, & t_0 \leq t \leq t_0+\delta, \\ \frac{1}{(n-1)!}(t-t_0)^{n-1}\,, & t_0+\delta \leq t \leq b. \end{cases}$$

Therefore, we have

$$\left|\int_a^b \mu_j(t)h_\delta^n(t)\mathrm{d}t\right| = \left|\int_{t_0}^{t_0+\delta} \mu_j(t)\frac{1}{\delta}\mathrm{d}t\right| = \frac{1}{\delta}\int_{t_0}^{t_0+\delta} \mu_j(t)\mathrm{d}t$$

Now from inequality (3.13), we have

$$\frac{1}{\delta}\int_{t_0}^{t_0+\delta} \mu_j(t)\mathrm{d}t \leq \mu_j(t_0)\frac{1}{\delta}\int_{t_0}^{t_0+\delta} \mathrm{d}t = \mu_j(t_0)$$

Since

$$\lim_{\delta\to 0}\frac{1}{\delta}\int_{t_0}^{t_0+\delta} \mu_j(t)\mathrm{d}t = \mu_j(t_0)$$

the statement follows.

In the case $\mu_j(t_0) < 0$, we define $h_\delta(t)$ by

$$h_\delta(t) = \begin{cases} \frac{1}{(n-1)!}(t-t_0-\delta)^{n-1}\,, & a \leq t \leq t_0, \\ -\frac{1}{\delta n!}(t-t_0-\delta)^n, & t_0 \leq t \leq t_0+\delta, \\ 0, & t_0+\delta \leq t \leq b, \end{cases}$$

and the rest of the proof is the same as above. □

We will end this section with the integral versions of the results, the proofs of which are analogues to the discrete case and are omitted.

Theorem 3.6 *Let $n \in \mathbb{N}$, $n \geq 3$, $h : [a, b] \to \mathbb{R}$ be such that $h^{(n)}$ is an absolutely continuous function with $(\cdot - a)(b - \cdot)[h^{(n+1)}]^2 \in L[a, b]$ and let $\gamma : [\alpha, \beta] \to [a, b]$ and $\rho : [\alpha, \beta] \to \mathbb{R}$ satisfy $\int_\alpha^\beta \rho(\lambda)\mathrm{d}\lambda = 0$ and $\int_\alpha^\beta \rho(\lambda)\gamma(\lambda)\mathrm{d}\lambda = 0$. Then*

$$\begin{aligned}\int_\alpha^\beta p(\lambda)\, h(\gamma(\lambda))\,\mathrm{d}\lambda &= \frac{h'(a)-h'(b)}{b-a}\int_a^b\int_\alpha^\beta p(\lambda)\, G(\gamma(\lambda), s)\,\mathrm{d}\lambda\,\mathrm{d}s \\ &+\sum_{k=2}^{n-1}\frac{k}{(k-1)!}\int_a^b\left(\int_\alpha^\beta p(\lambda)\, G(\gamma(\lambda), s)\,\mathrm{d}\lambda\right)\frac{h^{(k)}(a)(s-a)^{k-1}-h^{(k)}(b)(s-b)^{k-1}}{b-a}\mathrm{d}s \\ &+\frac{h^{(n-1)}(b)-h^{(n-1)}(a)}{(n-3)!(b-a)}\int_a^b \Omega_3(s)\mathrm{d}s + R_n^3(h; a, b),\end{aligned} \tag{3.14}$$

and

$$\int_{\alpha}^{\beta} p(\lambda)\, h(\gamma(\lambda))\, \mathrm{d}\lambda = \frac{h'(b)-h'(a)}{b-a} \int_a^b \int_{\alpha}^{\beta} p(\lambda)\, G(\gamma(\lambda), s)\, \mathrm{d}\lambda\, \mathrm{d}s$$
$$+ \sum_{k=3}^{n-1} \frac{k-2}{(k-1)!} \int_a^b \left(\int_{\alpha}^{\beta} p(\lambda)\, G(\gamma(\lambda), s)\, \mathrm{d}\lambda \right) \frac{h^{(k)}(a)(s-a)^{k-1} - h^{(k)}(b)(s-b)^{k-1}}{b-a} \mathrm{d}s$$
$$+ \frac{h^{(n-1)}(b) - h^{(n-1)}(a)}{(n-3)!(b-a)} \int_a^b \Omega_4(s)\mathrm{d}s + R_n^4(h; a, b), \tag{3.15}$$

where the remainders $R_n^j(h; a, b)$, $j = 3, 4$, *satisfy the bounds*

$$|R_n^j(h; a, b)| \le \frac{1}{(n-3)!} \left(\frac{(b-a)}{2} \left| T(\Omega_j, \Omega_j) \int_a^b (s-a)(b-s)[h^{(n+1)}(s)]^2 \mathrm{d}s \right| \right)^{1/2}.$$

Theorem 3.7 *Let* $n \in \mathbb{N}$, $n \ge 3$, $h : [a, b] \to \mathbb{R}$ *be such that* $h^{(n)}$ *is an absolutely continuous function with* $h^{(n+1)} \ge 0$ *and let* $\gamma : [\alpha, \beta] \to [a, b]$ *and* $\rho : [\alpha, \beta] \to \mathbb{R}$ *satisfy* $\int_{\alpha}^{\beta} \rho(\lambda)\mathrm{d}\lambda = 0$ *and* $\int_{\alpha}^{\beta} \rho(\lambda)\gamma(\lambda)\mathrm{d}\lambda = 0$. *Then representations (3.14) and (3.15) hold and the remainders* $R_n^j(h; a, b)$, $j = 3, 4$, *satisfy the bounds*

$$|R_n^j(h; a, b)| \le \frac{1}{(n-3)!} \|\Omega_j'\|_{\infty} \left\{ \frac{b-a}{2} \left[h^{(n-1)}(b) + h^{(n-1)}(a) \right] - \left[h^{(n-2)}(b) - h^{(n-2)}(a) \right] \right\}.$$

Theorem 3.8 *Let* $n \in \mathbb{N}$, $n \ge 3$, $1 \le q, r \le \infty$, $\frac{1}{q} + \frac{1}{r} = 1$, $h^{(n)} \in L_q[a, b]$ *and let* $\gamma : [\alpha, \beta] \to [a, b]$ *and* $\rho : [\alpha, \beta] \to \mathbb{R}$ *satisfy* $\int_{\alpha}^{\beta} \rho(\lambda)\mathrm{d}\lambda = 0$ *and* $\int_{\alpha}^{\beta} \rho(\lambda)\gamma(\lambda)\mathrm{d}\lambda = 0$. *Then*

$$\left| \int_{\alpha}^{\beta} p(\lambda)\, h(\gamma(\lambda))\, \mathrm{d}\lambda - \frac{h'(a)-h'(b)}{b-a} \int_a^b \int_{\alpha}^{\beta} p(\lambda)\, G(\gamma(\lambda), s)\, \mathrm{d}\lambda\, \mathrm{d}s \right.$$
$$\left. - \sum_{k=2}^{n-1} \frac{k}{(k-1)!} \int_a^b \left(\int_{\alpha}^{\beta} p(\lambda)\, G(\gamma(\lambda), s)\, \mathrm{d}\lambda \right) \frac{h^{(k)}(a)(s-a)^{k-1} - h^{(k)}(b)(s-b)^{k-1}}{b-a} \mathrm{d}s \right|$$
$$\le \frac{1}{(n-3)!} \|h^{(n)}\|_q \, \|\Omega_3\|_r \tag{3.16}$$

and

$$\left| \int_{\alpha}^{\beta} p(\lambda)\, h(\gamma(\lambda))\, \mathrm{d}\lambda - \frac{h'(b)-h'(a)}{b-a} \int_a^b \int_{\alpha}^{\beta} p(\lambda)\, G(\gamma(\lambda), s)\, \mathrm{d}\lambda\, \mathrm{d}s \right.$$
$$\left. - \sum_{k=3}^{n-1} \frac{k-2}{(k-1)!} \int_a^b \left(\int_{\alpha}^{\beta} p(\lambda)\, G(\gamma(\lambda), s)\, \mathrm{d}\lambda \right) \frac{h^{(k)}(a)(s-a)^{k-1} - h^{(k)}(b)(s-b)^{k-1}}{b-a} \mathrm{d}s \right|$$
$$\le \frac{1}{(n-3)!} \|h^{(n)}\|_q \, \|\Omega_4\|_r . \tag{3.17}$$

The constant on the right hand sides of (3.16) and (3.17) is sharp for $1 < q \le \infty$ *and the best possible for* $q = 1$.

Remark 3.9 Left-hand sides of the inequalities (2.8), (2.10), (2.15) and (2.18) can be defined as linear functionals in h. Using similar methods as in [2] we can prove mean value results for these functionals, as well as construct new families of exponentially convex functions and Cauchy-type means. Then, using some known properties of exponentially convex functions, we can derive new inequalities and prove monotonicity of the obtained Cauchy-type means analogously as in [2].

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References

1. Aljinović, A.Aglić; Pečarić, J.; Vukelić, A.: On some Ostrowski type inequalities via Montgomery identity and Taylor's formula II. Tamkang Jour. Math **36**(4), 279–301 (2005)
2. Butt, S.I.; Pečarić, J.; Praljak, M.: Reversed Hardy inequality for C-monotone functions. J. Math. Inequal. **10**(3), 603–622 (2016)
3. Cerone, P.; Dragomir, S.S.: Some new Ostrowski-type bounds for the Čebyšev functional and applications. J. Math. Inequal. **8**(1), 159–170 (2014)
4. Mitrinović, D.S.; Pečarić, J.E.; Fink, A.M.: Inequalities for Functions and Their Integrals and Derivatives. Kluwer Academic Publishers, Dordrecht (1994)
5. Pečarić, J.: On Jessens inequality for convex functions. III. J. Math. Anal. Appl. **156**, 231–239 (1991)
6. Pečarić, J.E.; Proschan, F.; Tong, Y.L.: Convex Functions, Partial Orderings and Statistical Applications. Academic Press, New York (1992)
7. Popoviciu, T.: Notes sur les fonctions convexes d'orde superieur III. Mathematica (Cluj) **16**, 74–86 (1940)
8. Popoviciu, T.: Notes sur les fonctions convexes d'orde superieur IV. Disqusitiones Math. **1**, 163–171 (1940)
9. Popoviciu, T.: Les Fonctions Convexes. Herman and Cie, Editeurs (1944)