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Chebyshev collocation treatment of Volterra–Fredholm integral equation with error analysis

Abstract This work reports a collocation algorithm for the numerical solution of a Volterra–Fredholm integral equation (V-FIE), using shifted Chebyshev collocation (SCC) method. Some properties of the shifted Chebyshev polynomials are presented. These properties together with the shifted Gauss–Chebyshev nodes were then used to reduce the Volterra–Fredholm integral equation to the solution of a matrix equation. Nextly, the error analysis of the proposed method is presented. We compared the results of this algorithm with others and showed the accuracy and potential applicability of the given method.

Mathematics Subject Classification 65R20 · 65M70 · 42C10

1 Introduction

The Volterra–Fredholm integral equations are derived from parabolic boundary value problems and can also be obtained from spatiotemporal epidemic modeling [15,27].

Consider the following Volterra–Fredholm integral equation [17]:

$$M(x)u(x) + W(x)u(Q(x)) = g(x) + \gamma_1 \int_0^{Q(x)} \theta_1(x,t)u(t)\mathrm{d}t + \gamma_2 \int_0^L \theta_2(x,t)u(Q(t))\mathrm{d}t \qquad (1)$$

where the functions $\theta_1(x,t)$ and $\theta_2(x,t)$ are known kernel functions on the interval $[0,L] \times [0,L]$ and the functions $M(x)$, $W(x)$, $Q(x)$ and $g(x)$ are known functions defined on the interval $[0,L]$ and $0 \le Q(x) < \infty$, $u(x)$ is the unknown function and γ_1, γ_2 are real constants such that $\gamma_1{}^2 + \gamma_2{}^2 \neq 0$. When $Q(x)$ is a first-order polynomial, Eq. (1) is called functional integral equation with proportional delay.

Numerical methods such as Jacobi collocation method [12,13], fifth kind Chebyshev method [2], Laguerre spectral method [8], and general orthogonal spectral method [9] are powerful techniques that can be used in applied mathematics and scientific computation to solve different types of differential problems. This work presents an approximation method for a class of Volterra–Fredholm integral equations on the interval $[0,L]$ via the shifted Chebyshev polynomials, and finite difference methods [18–22].

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Recently, various works have focused on the development of highly advanced and efficient methods for integral equations such as rationalized Haar functions method [23], He's variational iteration method [28], an adaptive method [5], hybrid function method [14], collocation Method [24], Legendre collocation method for solving Volterra–Fredholm integral equations [17], and second kind Chebyshev quadrature collocation algorithm [1].

The structure of this paper is as follows. In Sect. 2, we give an overview of shifted Chebyshev polynomials and their relevant properties needed hereafter. In Sect. 3, the way of constructing the collocation technique for Volterra–Fredholm integral equations is described using the shifted Chebyshev polynomials. In Sect. 4, we discuss in depth the convergence and error analysis of the suggested expansion. In Sect. 5, we present some numerical results exhibiting the accuracy and efficiency of our numerical algorithms. Also, a brief conclusion is given in Sect. 6.

2 Preliminaries

The conventional Chebyshev polynomials are defined on $[-1, 1]$ and can be generated with the aid of the following recursive formulae:

$$T_0(x) = 1, \ T_1(x) = x,$$
$$T_{i+1}(x) = 2x T_i(x) - T_{i-1}(x), \ i = 1, 2, \ldots, \ -1 \leq x \leq 1.$$

2.1 Shifted Chebyshev polynomials

To use these polynomials over the interval $[0, L]$, we define the so-called shifted Chebyshev polynomials of degree i as follows:

$$\chi_i(x) = T_i\left(\frac{2}{L}x - 1\right), \ i = 0, 1, 2, \ldots.$$

We consider the space $L^2_{\omega_L}[0, L]$ equipped with the following inner product and norm:

$$\langle f, g \rangle = \int_0^L f(x)g(x)\mathrm{d}x, \ \|u\|_2 = \langle u, u \rangle^{1/2}.$$

The set of shifted Chebyshev polynomials forms a complete $L^2_{\omega_L}[0, L]$-orthogonal system with the orthogonality condition

$$\int_0^L \chi_k(x)\chi_j(x)\omega_L(x)\mathrm{d}x = \delta_{kj}h_k,$$

where

$$h_k = \begin{cases} \frac{\epsilon_k}{2}\pi, & k = j, \\ 0, & k \neq j, \end{cases} \quad \epsilon_0 = 2, \quad \epsilon_k = 1, \quad k \geq 1 \text{ and } \omega_L(x) = \frac{1}{\sqrt{Lx - x^2}}.$$

The analytic form of the shifted Chebyshev polynomials $\chi_i(x)$ of degree i is given by

$$\chi_i(x) = \sum_{k=0}^i \mu_k^{(i)} x^k, \tag{2}$$

where

$$\mu_k^{(i)} = i \frac{(-1)^{i-k}(i+k-1)! 2^{2k}}{(i-k)!(2k)! L^k}.$$

The polynomials $\chi_i(x)$ may be generated with the aid of the following recurrence relation:

$$\chi_{i+1}(x) = 2\left(\frac{2}{L}x - 1\right)\chi_i(x) - \chi_{i-1}(x), \ i = 1, 2, \ldots, \ 0 \leq x \leq L$$

where $\chi_0(x) = 1$ and $\chi_1(x) = \frac{2}{L}x - 1$.

2.2 Function approximation

A function $u(x)$ in $L^2_{\omega_L}[0, L]$ may be expanded in terms of the shifted Chebyshev polynomials as:

$$u(x) = \sum_{i=0}^{\infty} c_i \chi_i(x), \tag{3}$$

where the coefficients c_i are given by

$$c_i = \frac{1}{h_i} \int_0^L u(x)\chi_i(x)\omega_L(x)\,\mathrm{d}x, \, i = 0, 1, 2, \ldots. \tag{4}$$

Considering only the first $(N + 1)$-terms of shifted Chebyshev polynomials, we have the following approximation:

$$u(x) \simeq u_N(x) = \sum_{i=0}^{N} c_i \chi_i(x) = C^T \phi(x),$$

where the shifted Chebyshev coefficient vector C and the shifted Chebyshev vector $\phi(x)$ are given by

$$C^T = [c_0, c_1, \ldots, c_N], \tag{5}$$

$$\phi(x) = [\chi_0, \chi_1, \ldots, \chi_N]^T, \tag{6}$$

respectively.

3 Shifted Chebyshev collocation treatment of V-FIE

In this section, we approximate the solution of Eq. (1) using shifted Chebyshev polynomials. We assume that the known functions in Eq. (1) are such that this equation has a unique solution [4,6,7,10,11] and also we assume that $0 \leq Q(x) \leq L$. We approximate the function $u(x)$ using the way mentioned in the previous section as follows:

$$u(x) \simeq \sum_{i=0}^{N} c_i \chi_i(x) = C^T \phi(x), \tag{7}$$

where the coefficients c_i, $i = 0, 1, \ldots, N$ are the unknowns to be determined and C and $\phi(x)$ are defined as in (5) and (6), respectively. Using (7), we can consider that

$$u(Q(x)) \simeq \sum_{i=0}^{N} c_i \chi_i(Q(x)). \tag{8}$$

Substituting Eqs. (7) and (8) into Eq. (1) yields:

$$
\begin{aligned}
M(x) \sum_{i=0}^{N} c_i \chi_i(x) &+ W(x) \sum_{i=0}^{N} c_i \chi_i(Q(x)) = g(x) \\
&+ \gamma_1 \int_0^{Q(x)} \theta_1(x,t) \sum_{i=0}^{N} c_i \chi_i(t)\mathrm{d}t + \gamma_2 \int_0^L \theta_2(x,t) \sum_{i=0}^{N} c_i \chi_i(Q(t))\mathrm{d}t.
\end{aligned}
\tag{9}
$$

Suppose that:

$$f_i(x) = M(x)\chi_i(x) + W(x)\chi_i(Q(x)) - \gamma_1 \int_0^{Q(x)} \theta_1(x,t)\chi_i(t)\mathrm{d}t - \gamma_2 \int_0^L \theta_2(x,t)\chi_i(Q(t))\mathrm{d}t,$$

then, Eq. (9) can be rewritten as:

$$\sum_{i=0}^{N} c_i f_i(x) = g(x). \tag{10}$$

Now, we collocate Eq. (10) at the distinct $N + 1$ roots of the shifted Chebyshev polynomial $T_{L,N+1}(x)$ and, consequently, the following system of algebraic equations

$$\sum_{i=0}^{N} c_i f_i(x_j) = g(x_j), \text{ for } j = 0, 1, \ldots, N, \tag{11}$$

was obtained, which can be written in the following matrix form:

$$\mathbf{F}^T \mathbf{C} = \mathbf{G},$$

where

$$\mathbf{G} = [g(x_0), g(x_1), \ldots, g(x_N)]^T,$$

and

$$\mathbf{F} = (f_{ij}), \ i, j = 0, 1, \ldots, N.$$

The elements of the matrix $\mathbf{F}$ are determined as follows:

$$f_{ij} = f_i(x_j), \ i, j = 0, 1, \ldots, N.$$

Finally, the unknown vector $\mathbf{C}$ can be computed by

$$\mathbf{C} = (\mathbf{F}^T)^{-1} \mathbf{G}.$$

Therefore, the approximate solution of Eq. (1) is given by $u(x) = C^T \phi(x)$.

4 Error analysis

Here, we discuss in depth the convergence and error analysis of the suggested truncated series expansion. For this target, we define the following errors:

- Absolute error function $e_N(x) = |u(x) - u_N(x)|$,
- Q-Absolute error function $e_N^Q(x) = e_N(Q(x))$,
- Maximum absolute error $E_N = \max_{0 \le x \le L} e_N(x)$,
- Maximum Q-absolute error $E_N^Q = \max_{0 \le x \le L} e_N^Q(x)$.

The following lemmas are needed:

Lemma 4.1 *For $p > 1$, the following sum is valid*

$$\sum_{s=n+1}^{\infty} \frac{1}{(s - p + 1)_p} = \frac{1}{(p-1)(n-p+2)_{p-1}}, \tag{12}$$

where $(a)_p = \Gamma(a + p)/\Gamma(a), a = s - p + 1$ is the pochhammer symbol.

Proof The L.H.S. of (12) is a telescopic series. By splitting $\dfrac{1}{(s - p + 1)_p}$ into partial fractions and taking the limit of the partial sums, we get the R.H.S. $\qquad\square$

Theorem 4.1 *If $u^{(p)}(x)$ is bounded for some $p > 1$, then the expansion coefficients in (3) satisfy the following estimate:*

$$|c_i| \le C/(i - p + 1)_p; \quad \forall i > p,$$

where C is a generic constant independent of i and $C < i - p + 1$.

Proof Starting from Eq. (4), using the substitution $x = \frac{L}{2}(1 + \cos\theta)$, following the procedures in [3] and by the assumptions of the theorem, we get the desired result. □

Theorem 4.2 *If* u, u_N *are the exact and approximate solutions of Eq.* (1)*, respectively, and* u *satisfies the hypothesis of Theorem* 4.1*, then we have the following error estimate:*

$$\max\{E_N, E_N^Q\} \le \frac{C}{(p-1)\,(N-p+2)_{p-1}}.$$

Proof It suffices to prove that $E_N^Q \le \dfrac{C}{(p-1)\,(N-p+2)_{p-1}}$. We have

$$e_N(Q(x)) = |u(Q(x)) - u_N(Q(x))| = |\sum_{i=N+1}^{\infty} c_i \,\chi_i(Q(x))|$$

$$\le \sum_{i=N+1}^{\infty} |c_i|\,|\chi_i(Q(x))|.$$

Now, since $|\chi_i(Q(x))| \le 1$ an application of Theorem 4.1 gives

$$E_N^Q \le \sum_{i=N+1}^{\infty} \frac{C}{(i-p+1)_p}.$$

Finally, a direct application of Lemma 4.1 yields the result. □

Theorem 4.3 *Let*

$$R_N(x) = |M(x)u_N(x) + W(x)u_N(Q(x)) - \gamma_1 \int_0^{Q(x)} \theta_1(x,t)u_N(t)dt$$

$$-\gamma_2 \int_0^L \theta_2(x,t)u_N(Q(t))dt - g(x)|,$$

$$R_N = \max_{0 \le x \le L} R_N(x).$$

If $|M(x)| \le M_1, |W(x)| \le W_1, |\theta_1(x,t)| \le \Theta_1, |\theta_2(x,t)| \le \Theta_2$ *and* $|Q(x)| \le q$, *where,* $M_1, W_1, \Theta_1, \Theta_2$ *and* q *are positive constants, then we have the following residual estimate:*

$$R_N \le \frac{\rho\,C}{(p-1)\,(N-p+2)_{p-1}},$$

where $\rho = \max\{M_1, W_1, |\gamma_1|\Theta_1\,q, |\gamma_2|\Theta_2\,L\}$.

Proof From Eq. 1, we have

$$g(x) = M(x)u(x) + W(x)u(Q(x)) - \gamma_1 \int_0^{Q(x)} \theta_1(x,t)u(t)dt - \gamma_2 \int_0^L \theta_2(x,t)u(Q(t))dt$$

therefore

$$R_N(x) \le |M(x)e_N(x)| + |W(x)e_N^Q(x)| + |\gamma_1 \int_0^{Q(x)} \theta_1(x,t)e_N(t)dt| + |\gamma_2 \int_0^L \theta_2(x,t)e_N^Q(t)dt|.$$

Noticing that, $|\int_A^B f| \le \int_A^B |f|$, by the result of Theorem 4.2, and by the hypotheses of the theorem, we get the desired result. □

5 Test problems

To justify the validity and accuracy of the presented algorithm, we apply it to solve some examples of the Volterra–Fredholm integral equations. Also, we compare our numerical results with those obtained using other methods and with the exact solutions of such problems. The following tables and figures contain the values of the exact solution $u(x)$, the approximate solution $u_N(x)$, and the absolute error functions $e_N(x)$ at the selected points.

Example 5.1 Consider the following V-FIE [17]

$$(\sin x)u(x) + (\cos x)u(e^x) = f(x) + \int_0^{e^x} e^{x+t}u(t)dt - \int_0^1 e^{x+t}u(e^t)dt, \qquad (13)$$

where $f(x) = \frac{1}{3}e^x(-1+e^3) + e^x\{2 - e^{e^x}[2 + e^x(-2+e^x)]\} + e^{2x}\cos x + x^2 \sin x$. The exact solution of this equation is $u(x) = x^2$. In Table 1, we compare the absolute errors of the present method with the Taylor collocation (TC) method of [26], the Taylor polynomial (TP) method of [16] and the Lagrange collocation (LC) [25]. The numerical results for this example are displayed in Fig. 1. The graphs of the analytical solution and the approximate solution at $N = 10$ and $L = 100$ are displayed in Fig 2 to make the comparison easier.

Example 5.2 Consider the following V-FIE

$$x^2 u(x) + e^x u(2x) = f(x) + \int_0^{2x} e^{x+t}u(t)dt - \int_0^1 e^{x-2t}u(2t)dt, \qquad (14)$$

where $f(x) = -\frac{e^x}{4} - \frac{1}{4}e^{-2+x}\cos 2 + \frac{1}{2}e^{3x}\cos 2x - \frac{1}{4}e^{-2+x}\sin 2 + x^2 \sin x + e^x \sin 2x - \frac{1}{2}e^{3x}\sin 2x$. Its exact solution is $u(x) = \sin x$. Table 2 shows that the absolute error obtained by the SCC method is significantly better than that obtained by the Taylor collocation (TC) method of [26], the Taylor polynomial (TP) method of [16] and the Lagrange collocation (LC) [25]. The absolute error obtained by the SCC method at $N = 14$ is plotted in Fig 3. The graphs of the analytical solution and the approximate solution at $N = 10$ and $L = 8$ are displayed in Fig 4 to make the comparison easier.

Example 5.3 Consider the V-FIE

$$u(x) = f(x) + \int_0^{h(x)} k_1(x,t)u(t)dt + \int_0^1 k_2(x,t)u(h(t))dt, \qquad (15)$$

Table 1 Comparison of the absolute errors with various choices of N, for Example 5.1

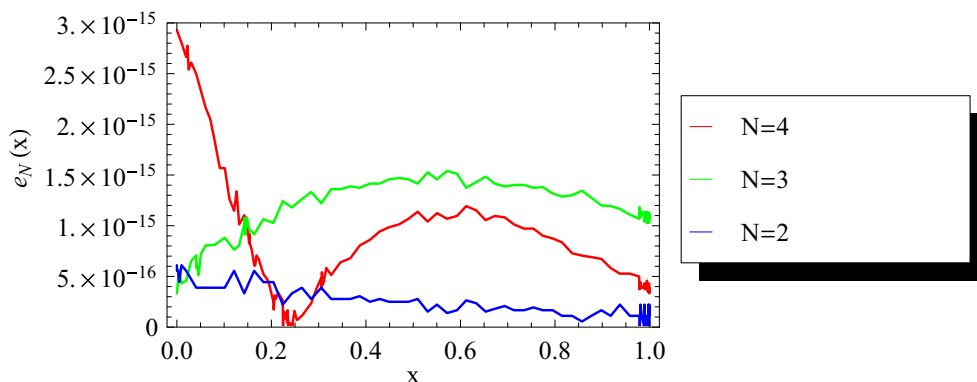| N	SCC method	TC method	TP method	LC method
2	4.99×10^{-16}	7.64×10^{-15}	1.96×10^{-15}	2.82×10^{-15}
3	1.44×10^{-15}	1.22×10^{-14}	2.75×10^{-15}	1.36×10^{-14}
4	3.05×10^{-15}	3.41×10^{-14}	4.42×10^{-15}	1.91×10^{-13}

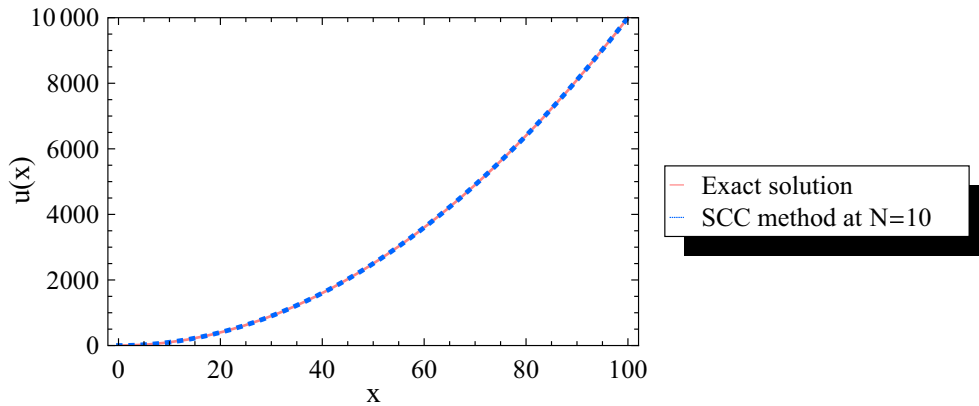**Fig. 1** Graph of the $e_N(x)$, with $N = 2, 3, 4$ for Example 5.1

Fig. 2 Graph of exact solution and approximate solution at $N = 10$ for Example 5.1

Table 2 Comparison of the absolute errors with various choices of N, for Example 5.2

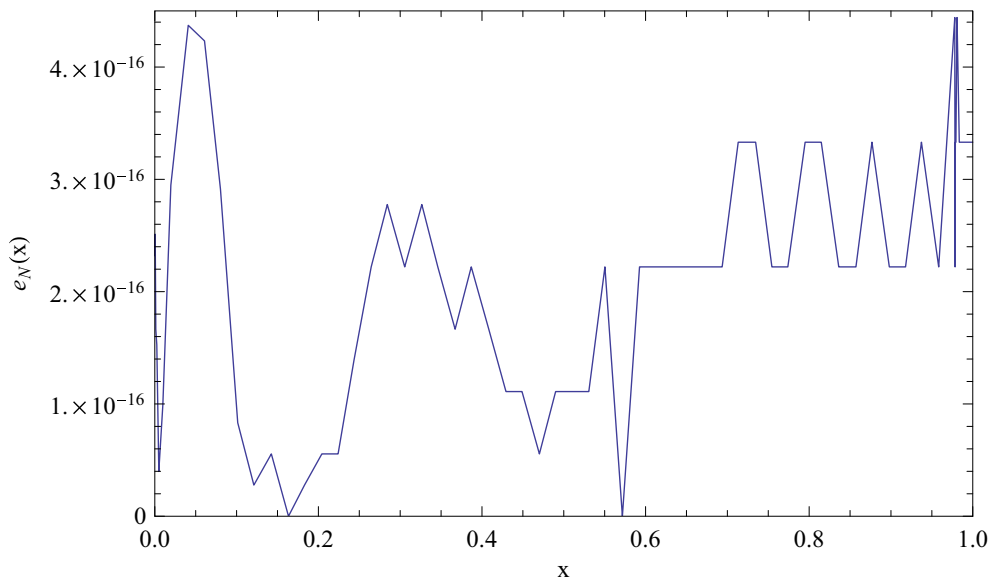| N	SCC method	TC method	TP method	LC method
2	3.35×10^{-2}	7.87×10^{-2}	3.41×10^{-2}	7.87×10^{-2}
5	5.54×10^{-5}	6.23×10^{-5}	3.68×10^{-4}	6.23×10^{-5}
8	7.95×10^{-9}	1.89×10^{-8}	1.24×10^{-5}	1.77×10^{-7}
9	5.47×10^{-10}	2.35×10^{-8}	3.46×10^{-7}	7.21×10^{-6}

Fig. 3 Graph of absolute error at $N = 14$ for Example 5.2

where $h(x) = x$, $k_1(x, t) = xt$, $k_2(x, t) = (x - t)$, $f(x) = -\frac{2x^{7/2}}{5} - \frac{2x}{3} + \sqrt{x} + \frac{2}{5}$. The exact solution of this problem is $u(x) = x^{\frac{1}{2}}$. Table 3 shows the numerical results of the proposed method of this example, with $N = 8, 12, 16$.

Example 5.4 As a final test problem, consider the following V-FIE

$$u(x) = f(x) + \int_0^{h(x)} k_1(x, t)u(t)\mathrm{d}t - \int_0^1 k_2(x, t)u(h(t))\mathrm{d}t, \tag{16}$$

where $h(x) = \ln(x+1)$, $k_1(x, t) = e^{x+t}$, $k_2(x, t) = e^{x+h(t)}$, $f(x) = e^{-x} - e^x(h(x) - 1)$. The exact solution of this problem is $u(x) = e^{-x}$. The numerical results obtained by the present method for $N = 2, 5, 8, 9$ are
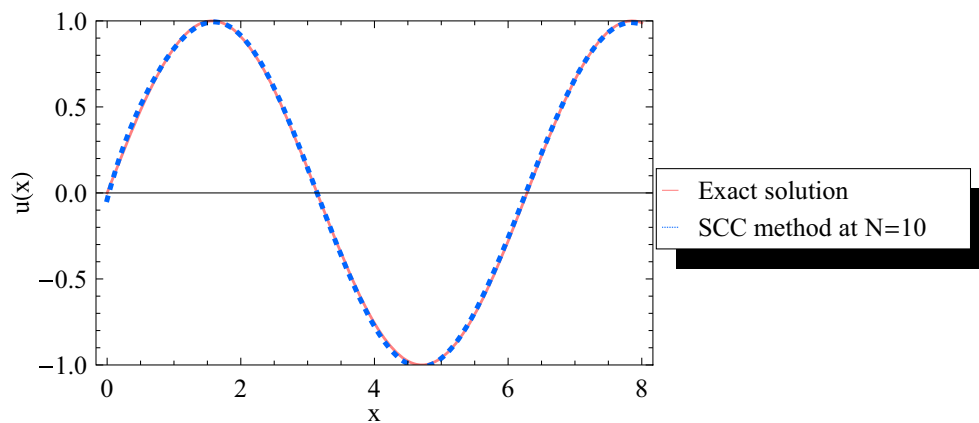
Fig. 4 Graph of exact solution and approximate solution at $N = 10$ for Example 5.2

Table 3 The absolute errors with various choices of x and N, for Example 5.3

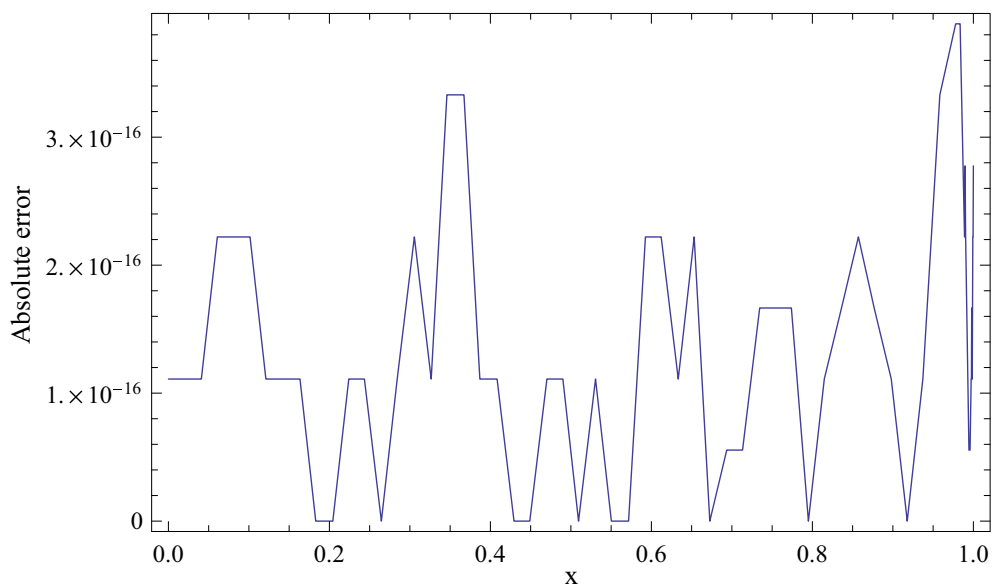| x	$N = 8$	$N = 12$	$N = 16$
0.1	2.96×10^{-3}	5.97×10^{-4}	3.02×10^{-5}
0.2	8.79×10^{-4}	5.75×10^{-4}	3.00×10^{-4}
0.3	7.10×10^{-4}	3.63×10^{-4}	1.33×10^{-4}
0.4	9.59×10^{-4}	1.64×10^{-4}	4.62×10^{-5}
0.5	1.75×10^{-5}	6.72×10^{-6}	3.24×10^{-6}
0.6	7.12×10^{-4}	1.31×10^{-4}	2.19×10^{-5}
0.7	2.48×10^{-4}	1.43×10^{-4}	6.51×10^{-5}
0.8	1.87×10^{-4}	1.75×10^{-4}	6.74×10^{-5}
0.9	4.93×10^{-4}	4.87×10^{-5}	9.79×10^{-6}
1	5.12×10^{-4}	1.71×10^{-4}	7.71×10^{-5}

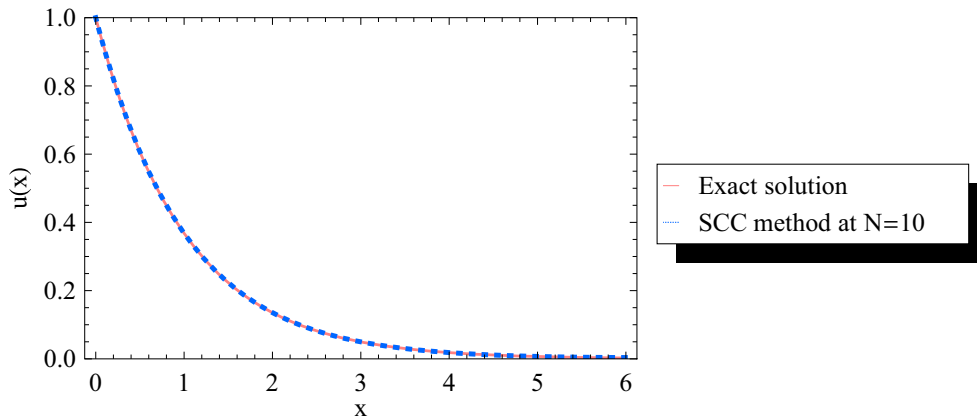**Fig. 5** Graph of absolute error at $N = 11$ for Example 5.4

Fig. 6 Graph of exact solution and approximate solution at $N = 10$ for Example 5.4

Table 4 Comparison of the absolute errors with various choices of N, for Example 5.4

N	SCC method	TC method	TP method	LC method
2	4.78×10^{-3}	3.27×10^{-3}	3.59×10^{-2}	3.27×10^{-3}
5	4.83×10^{-7}	4.30×10^{-7}	3.05×10^{-4}	4.30×10^{-7}
8	1.43×10^{-11}	5.96×10^{-8}	5.61×10^{-7}	5.78×10^{-7}
9	4.24×10^{-13}	8.84×10^{-8}	1.41×10^{-7}	1.86×10^{-5}

compared with the Taylor collocation (TC) method of [26], the Taylor polynomial (TP) method of [16] and the Lagrange collocation (LC) [25]. The absolute error obtained by the SCC method at $N = 11$ is plotted in Fig 5 and in Fig. 6, we compare the analytic solution with the approximate solution at $N = 10$ and $L = 6$ (Table 4).

6 Concluding remarks

A Chebyshev collocation method was applied to solve a special class of Volterra–Fredholm integral equation. This method uses the shifted Gauss–Chebyshev nodes to reduce the considered Volterra–Fredholm integral equations to the solution of a matrix equation. In the given examples, through the selection of a relatively few shifted Gauss–Chebyshev nodes, we were able to obtain very accurate approximations, demonstrating the utility of our approach over other analytical or numerical methods. The proposed method is a powerful tool for obtaining novel numerical solutions of such equations.

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