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Approximation by (p,q) Szász-beta–Stancu operators

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Abstract Motivated by recent investigations, in this paper we introduce (p, q)-Szász-beta–Stancu operators and investigate their local approximation properties in terms of modulus of continuity. We also obtain a weighted approximation and Voronovskaya-type asymptotic formula.

Mathematics Subject Classification 41A30 · 41A35

1 Introduction

During the period of 1997–2017, the application of quantum-calculus came out as a new area of research in the field of approximation of functions by positive linear operators. Lupas [11] presented the first q-analogue of the Bernstein polynomials using q-integers, after a decade Phillips [21] gave another q-analogue of the Bernstein polynomials. Since then diverse operators have been generalized to their quantum variants and their approximation properties were discussed in [5,10,12,13,22].

Post-quantum calculus is an advanced extention of quantum-calculus and symbolized by (p, q)-calculus. Mursaleen et al. [19] introduced the Bernstein polynomials using (p, q)-calculus, which was further improved in [18].

Gupta and Aral introduced the Durrmeyer-type generalization of (p, q)-Bernstein operators in [9]. The quantum variant and post-quantum variant of Szász–Mirakyan operators were introduced and studied in [14] and [1]. Some approximation properties on (p, q)-analogue of Stancu-type generalization of linear positive operators were studied in [6,15,20].

We also consider some more results on approximation of functions by positive linear operators using (p, q)-calculus given in [2,16,17].

We mention some notations, definitions of (p, q)-calculus as follows (see for details [24,25]).

The (p, q)-bracket is defined as

$$[d]_{p,q} = \frac{p^d - q^d}{p - q}, \quad d = 0, 1, 2 \ldots, [0]_{p,q} = 0.$$

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The (p,q)-factorial is defined as

$$[d]_{p,q}! = \prod_{k=1}^{d}[k]_{p,q}, \quad d \geq 1, \quad [0]_{p,q}! = 1.$$

The (p,q)-binomial coefficient is given by

$$\begin{bmatrix} d \\ k \end{bmatrix}_{p,q} = \frac{[d]_{p,q}!}{[d-k]_{p,q}![k]_{p,q}!}, \quad 0 \leq k \leq d.$$

Definition 1.1 The (p,q)-power basis is defined below:

$$(x \oplus y)_{p,q}^{\mathrm{d}} = (x+y)(px+qy)(p^2x+q^2y)\cdots(p^{d-1}x+q^{d-1}y)$$
$$(x \ominus y)_{p,q}^{\mathrm{d}} = (x-y)(px-qy)(p^2x-q^2y)\cdots(p^{d-1}x-q^{d-1}y).$$

Definition 1.2 [23] For $d \geq 0$, the (p,q)-gamma function is given as

$$\Gamma_{p,q}(d+1) = \frac{(p \ominus q)_{p,q}^{\mathrm{d}}}{(p-q)^d} = [d]_{p,q}!, \quad 0 < q < p.$$

Definition 1.3 The (p,q) derivative of the function f is defined as

$$D_{p,q}f(y) = \frac{f(py)-f(qy)}{(p-q)y}, \quad y \neq 0$$

and $D_{p,q}f(0) = f'(0)$, provided that f is differentiable at zero.

Proposition 1.4 [23] *The (p,q)-integration by parts is given by*

$$\int_a^b g(px)D_{p,q}h(x)d_{p,q}x = g(b)h(b) - g(a)h(a) - \int_a^b h(qx)D_{p,q}g(x)d_{p,q}x.$$

The (p,q)-beta function of second kind [4] *is given by*

$$B_{p,q}(m,n) = \int_0^\infty \frac{x^{m-1}}{(1 \oplus px)_{p,q}^{m+n}} d_{p,q}x,$$

where $m, n \in N$.

The relation between (p,q)-beta and (p,q)-gamma functions is given as

$$B_{p,q}(m,n) = q^{\frac{2-m(m-1)}{2}} p^{\frac{-m(m+1)}{2}} \frac{\Gamma_{p,q}m\Gamma_{p,q}n}{\Gamma_{p,q}(m+n)}.$$

For $0 \leq x < \infty$, $0 < q < p \leq 1$, Aral and Gupta [3] *defined the (p,q)-analogue of Szász-beta operators as follows:*

$$D_n^{p,q}(f;x) = \sum_{k=1}^{\infty} s_{n,k}^{p,q}(x) \frac{1}{B_{p,q}(k,n+1)} \int_0^\infty \frac{t^{k-1}}{(1 \oplus pt)_{p,q}^{k+n+1}} f(p^{k+1}qt)d_{p,q}t + \frac{f(0)}{E_{p,q}([n]_{p,q}x)}, \quad (1)$$

where

$$s_{n,k}^{p,q}(x) = \frac{1}{E_{p,q}([n]_{p,q}x)} \frac{q^{\frac{k(k-1)}{2}}}{[k]_{p,q}!}([n]_{p,q}x)^k.$$

The aim of this paper is to generalize the operator in (1) using Stancu-type parameters, (i.e. assuming $0 \le \alpha \le \beta$*), we define:*

$$D^{p,q}_{n,\alpha,\beta}(f;x) = \sum_{k=1}^{\infty} s^{p,q}_{n,k}(x) \frac{1}{B_{p,q}(k,n+1)} \int_0^{\infty} \frac{t^{k-1}}{(1 \oplus pt)^{k+n+1}_{p,q}} f\left(\frac{p^{k+1}qt[n]_{p,q}+\alpha}{[n]_{p,q}+\beta}\right) d_{p,q}t + \frac{f\left(\frac{\alpha}{[n]_{p,q}+\beta}\right)}{E_{p,q}\left([n]_{p,q}x\right)}, \tag{2}$$

where $s^{p,q}_{n,k}(x)$ *is given in (1).*

In particular case, if $\alpha = \beta = 0$*, then the operators* $D^{p,q}_{n,\alpha,\beta}(f;x)$ *turn out to be the one defined by (1).*

2 Moments

Lemma 2.1 [3] *For the operator defined in (1),* $x \ge 0$ *and for* $\alpha = \beta = 0$*, the following equalities hold for* $0 < q < p \le 1$

(i) $D^{p,q}_n(1;x) = 1$,
(ii) $D^{p,q}_n(t;x) = x$,
(iii) $D^{p,q}_n(t^2;x) = \frac{[2]_{p,q}qx}{p[n-1]_{p,q}} + \frac{p[n]_{p,q}x^2}{[n-1]_{p,q}}$.

Lemma 2.2 *For* $x \in [0,\infty)$, $0 < q < p \le 1$*, then for the operator in (2), we have the following moments.*

(i) $D^{p,q}_{n,\alpha,\beta}(1;x) = 1$,
(ii) $D^{p,q}_{n,\alpha,\beta}(t;x) = \frac{[n]_{p,q}x+\alpha}{[n]_{p,q}+\beta}$,
(iii) $D^{p,q}_{n,\alpha,\beta}(t^2;x) = \frac{1}{([n]_{p,q}+\beta)^2}\left(\frac{p[n]^3_{p,q}}{[n-1]_{p,q}}x^2 + \left(\frac{q[n]^2_{p,q}[2]_{p,q}}{p[n-1]_{p,q}} + 2\alpha[n]_{p,q}\right)x + \alpha^2\right)$.

Proof By the definition of operator (2) and Lemma 2.1, we have

$$\begin{aligned}
(i) D^{p,q}_{n,\alpha,\beta}(1;x) &= \sum_{k=1}^{\infty} s^{p,q}_{n,k}(x) \frac{1}{B_{p,q}(k,n+1)} \int_0^{\infty} \frac{t^{k-1}}{(1 \oplus pt)^{k+n+1}_{p,q}} d_{p,q}t + \frac{1}{E_{p,q}([n]_{p,q}x)} \\
&= D^{p,q}_n(1;x) \\
&= 1. \\
(ii) D^{p,q}_{n,\alpha,\beta}(t;x) &= \sum_{k=1}^{\infty} s^{p,q}_{n,k}(x) \frac{1}{B_{p,q}(k,n+1)} \int_0^{\infty} \frac{t^{k-1}}{(1 \oplus pt)^{k+n+1}_{p,q}} \left(\frac{p^{k+1}qt[n]_{p,q}+\alpha}{[n]_{p,q}+\beta}\right) d_{p,q}t \\
&\quad + \frac{\frac{\alpha}{[n]_{p,q}+\beta}}{E_{p,q}([n]_{p,q}x)} \\
&= \frac{[n]_{p,q}}{[n]_{p,q}+\beta} D^{p,q}_n(t;x) + \frac{\alpha}{[n]_{p,q}+\beta} D^{p,q}_n(1;x) \\
&= \frac{[n]_{p,q}x+\alpha}{[n]_{p,q}+\beta}. \\
(iii) D^{p,q}_{n,\alpha,\beta}(t^2;x) &= \sum_{k=1}^{\infty} s^{p,q}_{n,k}(x) \frac{1}{B_{p,q}(k,n+1)} \int_0^{\infty} \frac{t^{k-1}}{(1 \oplus pt)^{k+n+1}_{p,q}} \left(\frac{p^{k+1}qt[n]_{p,q}+\alpha}{[n]_{p,q}+\beta}\right)^2 d_{p,q}t \\
&\quad + \frac{\left(\frac{\alpha}{[n]_{p,q}+\beta}\right)^2}{E_{p,q}([n]_{p,q}x)}
\end{aligned}$$

$$= \frac{[n]_{p,q}^2}{([n]_{p,q}+\beta)^2} D_n^{p,q}(t^2;x) + \frac{2\alpha[n]_{p,q}}{([n]_{p,q}+\beta)^2} D_n^{p,q}(t;x) + \frac{\alpha^2}{([n]_{p,q}+\beta)^2} D_n^{p,q}(1;x)$$

$$= \frac{1}{([n]_{p,q}+\beta)^2}\left(\frac{p[n]_{p,q}^3}{[n-1]_{p,q}}x^2 + \left(\frac{q[n]_{p,q}^2[2]_{p,q}}{p[n-1]_{p,q}} + 2\alpha[n]_{p,q}\right)x + \alpha^2\right).$$

This completes the proof. □

Using Lemma 2.2, we can obtain

$$D_{n,\alpha,\beta}^{p,q}((t-x);x) = \frac{\alpha-\beta x}{[n]_{p,q}+\beta},$$

and

$$D_{n,\alpha,\beta}^{p,q}((t-x)^2;x) = \beta_1(n)x^2 + \beta_2(n)x + \beta_3(n),$$

where

$$\beta_1(n) = \left(\frac{p[n]_{p,q}^3}{([n]_{p,q}+\beta)^2[n-1]_{p,q}} - \frac{2[n]_{p,q}}{[n]_{p,q}+\beta} + 1\right),$$

$$\beta_2(n) = \left(\frac{q[2]_{p,q}[n]_{p,q}^2}{([n]_{p,q}+\beta)^2 p[n-1]_{p,q}} + \frac{2\alpha[n]_{p,q}}{([n]_{p,q}+\beta)^2} - \frac{2\alpha}{[n]_{p,q}+\beta}\right),$$

$$\beta_3(n) = \frac{(\alpha)^2}{([n]_{p,q}+\beta)^2}.$$

Assuming $\beta^*(n) = \max[\beta_1(n), \frac{\beta_2(n)}{2}, \beta_3(n)]$, we can write

$$D_{n,\alpha,\beta}^{p,q}((t-x)^2;x) \leq \beta^*(n)(1+x)^2. \tag{3}$$

3 Local approximation

Let us consider the space of all real valued continuous and bounded functions on $\mathbb{R}_+$ and denote this space by $C_B(\mathbb{R}_+)$ under the norm:

$$\| f \| = \sup_{x\in\mathbb{R}_+} | f(x) |,$$

where $\mathbb{R}_+ = [0,\infty)$.

Let $W^2 = \{s \in C_B(\mathbb{R}_+) : s', s'' \in C_B(\mathbb{R}_+)\}$. Then, Peetre's K-functional is defined as

$$K_2(f,\delta) = \inf\{\| f - s \| + \delta \| g'' \| : s \in W^2\}.$$

Then as in [7], there exists a positive constant C such that

$$K_2(f,\delta) \leq C\omega_2(f,\sqrt{\delta}), \quad \delta > 0. \tag{4}$$

The second-order modulus of smoothness of $f \in C_B(\mathbb{R}_+)$ is

$$\omega_2(f,\sqrt{\delta}) = \sup_{0\leq h\leq\delta}\ \sup_{0\leq x<\infty} | f(x+2h) - 2f(x+h) + f(x) |,$$

and the usual modulus of continuity is given by

$$\omega(f,\delta) = \sup_{0\leq h\leq\delta}\ \sup_{0\leq x<\infty} |f(x+h) - f(x)|.$$

Lemma 3.1 *For $f \in C_B(\mathbb{R}_+)$ and for $s \in W^2$, we have*

$$|D^{*p,q}_{n,\alpha,\beta}(s;x) - s(x)| \leq \| s'' \| \, (\beta^*(n)(1+x)^2 + \mu_n^{*2}(p,q,x)),$$

where the auxiliary operators are given by

$$D^{*p,q}_{n,\alpha,\beta}(s;x) = D^{p,q}_{n,\alpha,\beta}(s;x) + s(x) - s(D^{p,q}_{n,\alpha,\beta}(t;x)), \tag{5}$$

and

$$\mu_n^*(p,q,x) = \frac{\alpha - \beta x}{[n]_{p,q} + \beta}. \tag{6}$$

Proof By the definition of auxiliary operators, it can be shown that

$$D^{*p,q}_{n,\alpha,\beta}((t-x);x) = 0. \tag{7}$$

Let $s \in W^2$. Then from the Taylor's expansion, we have

$$s(t) = s(x) + s'(x)(t-x) + \int_x^t (t-u)s''(u)\mathrm{d}u. \tag{8}$$

Operating (8) with (5) and using (7), we get

$$D^{*p,q}_{n,\alpha,\beta}(s;x) = s(x) + D^{*p,q}_{n,\alpha,\beta}\left(\int_x^t (t-u)s''(u)\mathrm{d}u; x\right)$$

$$(D^{*p,q}_{n,\alpha,\beta}(s;x) - s(x)) = D^{p,q}_{n,\alpha,\beta}\left(\int_x^t (t-u)s''(u)\mathrm{d}u; x\right) - \int_x^{D^{p,q}_{n,\alpha,\beta}(t;x)} (D^{p,q}_{n,\alpha,\beta}(t;x) - u)s''(u)\mathrm{d}u$$

$$= D^{p,q}_{n,\alpha,\beta}\left(\int_x^t (t-u)s''(u)\mathrm{d}u; x\right) - \int_x^{D^{p,q}_{n,\alpha,\beta}(t;x)} \left(\frac{[n]_{p,q}+\alpha}{[n]_{p,q}+\beta} - u\right)s''(u)\mathrm{d}u$$

$$|D^{*p,q}_{n,\alpha,\beta}(s;x) - s(x)| \leq D^{p,q}_{n,\alpha,\beta}\left(\int_x^t |(t-u)||s''(u)|\mathrm{d}u; x\right)$$

$$+ \int_x^{D^{p,q}_{n,\alpha,\beta}(t;x)} \left|\frac{[n]_{p,q}+\alpha}{[n]_{p,q}+\beta} - u)\right| |s''(u)|\mathrm{d}u$$

$$\leq \| s''(u) \| \, (\beta^*(n)(1+x)^2 + \mu_n^{*2}(p,q,x)).$$

Therefore,

$$|D^{*p,q}_{n,\alpha,\beta}(s;x) - s(x)| \leq \| s''(u) \| \, (\beta^*(n)(1+x)^2 + \mu_n^{*2}(p,q,x)). \tag{9}$$

Hence the proof is completed. □

Theorem 3.2 *For $f \in C_B(\mathbb{R}_+)$ and $x \in [0,\infty)$, there exists a constant $C > 0$ such that*

$$|D^{p,q}_{n,\alpha,\beta}(f;x) - f(x)| \leq C\omega_2(f; \sqrt{\beta^*(n)(1+x)^2 + \mu_n^{*2}(p,q,x)}) + \omega(f; \frac{\alpha - \beta x}{[n]_{p,q} + \beta}).$$

Proof Using (2), (5) and Lemma 2.2, we have

$$|D^{*p,q}_{n,\alpha,\beta}(f;x) - f(x)| \leq 4 \parallel f \parallel, \tag{10}$$

for any $s \in W^2$ and using (5), (9) and (10), we get

$$\begin{aligned}
|D^{p,q}_{n,\alpha,\beta}(f;x) - f(x)| &\leq |D^{*p,q}_{n,\alpha,\beta}((f-s);x) - (f-s)(x)| + |D^{*p,q}_{n,\alpha,\beta}(s;x) - s(x)| \\
&\quad + \left| f\left(x + \frac{\alpha - \beta x}{[n]_{p,q} + \beta}\right) - f(x) \right| \\
&\leq 4 \parallel f - s \parallel + \left| f\left(x + \frac{\alpha - \beta x}{[n]_{p,q} + \beta}\right) - f(x) \right| \\
&\quad + \parallel s'' \parallel (\beta^*(n)(1+x)^2 + \mu_n^{*2}(p,q,x)).
\end{aligned}$$

Now taking infimum on right-hand side over all $s \in W^2$ and using (4), we get

$$\begin{aligned}
|D^{p,q}_{n,\alpha,\beta}(f;x) - f(x)| &\leq C K_2(f; \beta^*(n)(1+x)^2 + \mu_n^{*2}(p,q,x)) + \omega\left(f; \frac{\alpha - \beta x}{[n]_{p,q} + \beta}\right) \\
&\leq C\omega_2\left(f; \sqrt{\beta^*(n)(1+x)^2 + \mu_n^{*2}(p,q,x)}\right) + \omega\left(f; \frac{\alpha - \beta x}{[n]_{p,q} + \beta}\right).
\end{aligned}$$

Hence the proof of the theorem. □

Remark If $f \in C[0,\infty)$, $0 \leq x < \infty$ and $\omega(f;\delta)$ is the modulus of continuity, then

$$|D^{p,q}_{n,\alpha,\beta}(f;x) - f(x)| \leq \left[1 + \sqrt{\varphi_n(x)}\right] \omega(f;\delta_{n,\beta}),$$

where

$$\delta_{n,\beta} = \frac{1}{[n]_{p,q} + \beta}$$

and

$$\begin{aligned}
\varphi_n(x) = \frac{1}{[n]_{p,q} + \beta} \Bigg[&\frac{p[n]^3_{p,q}}{[n-1]_{p,q}} x^2 + \left(\frac{[n]^2_{p,q}[2]_{p,q} q}{p[n-1]_{p,q}} + 2\alpha[n]_{p,q} \right) x \\
&+ (\alpha)^2 - 2x([n]_{p,q} x + \alpha)([n]_{p,q} + \beta) + ([n]_{p,q} + \beta)^2 x^2 \Bigg].
\end{aligned}$$

Proof Let $f \in C[0,\infty)$ and $0 \leq x < \infty$. Then using monotonicity of the operator defined in (2), we can easily obtain for every $\delta > 0$ that

$$\begin{aligned}
|D^{p,q}_{n,\alpha,\beta}(f;x) - f(x)| &\leq \left[1 + \frac{1}{\delta_{n,\beta}} \sqrt{D^{p,q}_{n,\alpha,\beta}((t-x)^2;x)}\right] \omega(f;\delta_{n,\beta}) \\
&\leq \left[1 + \sqrt{\varphi_n(x)}\right] \omega(f;\delta_{n,\beta}),
\end{aligned}$$

which is obtained using Lemma 2.2 and choosing $\delta_{n,\beta} = \sqrt{\frac{1}{[n]_{p,q}+\beta}}$. Hence we arrive at the result. □

If we put $\alpha = \beta = 0$, we can find the similar results for the operators defined by (1):

$$|D^{p,q}_n(f;x) - f(x)| \leq \left[1 + \sqrt{\frac{p[n]^3_{p,q}}{[n-1]_{p,q}} x^2 + \frac{[n]_{p,q}[2]_{p,q} q}{p[n-1]_{p,q}} x - [n]_{p,q} x^2}\right] \omega(f;\delta_n),$$

where $\delta_n = \sqrt{\frac{1}{[n]_{p,q}}}$ and it is observed that $\delta_{n,\beta} \leq \delta_n$.

Therefore, rate of convergence of $D^{p,q}_{n,\alpha,\beta}$ is better than $D^{p,q}_n$.

4 Weighted approximation

Let us consider the functions in weighted space defined as

(1) $H_2(\mathbb{R}_+)$ denotes the set of all functions f defined on $[0,\infty)$, such that $|f(x)| \leq M_f(1+x^2)$ where $M_f > 0$ depending only on f.

(2) $C(\mathbb{R}_+)$ be the set of all continuous functions f defined on $[0,\infty)$.

(3) $C_2(\mathbb{R}_+)$ denotes the subspace of all continuous functions in $H_2(\mathbb{R}_+)$.

(4) $C_2^*(\mathbb{R}_+)$ means the subspace of all functions $f \in C_2(\mathbb{R}_+)$ for which $\lim_{x\to\infty} \frac{f(x)}{1+x^2}$ is finite.

$H_2(\mathbb{R}_+)$ is a normed vector space under the norm:

$$\|f\|_2 = \sup_{x\geq 0} \frac{|f(x)|}{1+x^2}.$$

Theorem 4.1 *Let $p = p_n$ and $q = q_n$ such that $0 < q_n < p_n \leq 1$ and $p_n \to 1, q_n \to 1$, $p_n^n \to 1, q_n^n \to 1$ as $n \to \infty$. Then for each $f \in C_2^*(\mathbb{R}_+)$, we have*

$$\lim_{n\to\infty} \|D_{n,\alpha,\beta}^{p_n,q_n}(f;x) - f\|_2 = 0.$$

Proof By ([8], Theorem 4.1.4), it is sufficient to verify the following three conditions:

$$\lim_{n\to\infty} \|D_{n,\alpha,\beta}^{p_n,q_n}(t^\lambda;x) - x^\lambda\|_2 = 0, \quad \lambda = 0,1,2. \tag{11}$$

Applying Lemma 2.2, (11) is true for $\lambda = 0$.

Next by Lemma 2.2, we have

$$\begin{aligned}\|D_{n,\alpha,\beta}^{p_n,q_n}(t;x) - x\|_2 &= \sup_{x\geq 0} \frac{|\mu_n^*(p_n,q_n,x)|}{1+x^2}\\ &\leq \frac{\alpha}{[n]_{p_n,q_n}+\beta} \sup_{x\geq 0}\frac{1}{1+x^2} + \frac{\beta}{[n]_{p_n,q_n}+\beta}\sup_{x\geq 0}\frac{x}{1+x^2};\end{aligned}$$

this means (11) is true for $\lambda = 1$.

Finally, considering the same, we have

$$\begin{aligned}\|D_{n,\alpha,\beta}^{p_n,q_n}(t^2;x) - x^2\|_2 \leq& \left(\frac{p_n[n]_{p_n,q_n}^3}{([n]_{p_n,q_n}+\beta)^2[n-1]_{p_n,q_n}} - 1\right)\sup_{x\geq 0}\frac{x^2}{1+x^2}\\ &+\left(\frac{q_n[2]_{p_n,q_n}[n]_{p_n,q_n}^2}{p_n([n]_{p_n,q_n}+\beta)^2[n-1]_{p_n,q_n}} + \frac{2\alpha[n]_{p_n,q_n}}{([n]_{p_n,q_n}+\beta)^2}\right)\sup_{x\geq 0}\frac{x}{1+x^2}\\ &+\frac{\alpha^2}{([n]_{p_n,q_n}+\beta)^2}\sup_{x\geq 0}\frac{1}{1+x^2}.\end{aligned}$$

Hence (11) holds for $\lambda = 2$

Hence the theorem . □

5 Voronovskaya type asymptotic formula

Theorem 5.1 *Let $p = p_n$ and $q = q_n$ such that $0 < q_n < p_n \leq 1$ and $p_n \to 1, q_n \to 1$, $p_n^n \to 1, q_n^n \to 1$ as $n \to \infty$. Then for any $f \in C_2^*(\mathbb{R}_+)$, such that $f', f'' \in C_2^*(\mathbb{R}_+)$, we have*

$$\lim_{n\to\infty}[n]_{p_n,q_n}\left(D^{p_n,q_n}_{n,\alpha,\beta} - f(x)\right) = (\alpha - \beta x)f'(x) + \frac{f''(x)}{2}[2x + x^2(1 + A)],$$

where

$$A = \lim_{n\to\infty}[n]_{p_n,q_n}\left(\frac{[n]^2_{p_n,q_n}p_n^2}{([n]_{p_n,q_n}+\beta)^2} - \frac{2[n]_{p_n,q_n}}{([n]_{p_n,q_n}+\beta)} + 1\right),$$

uniformly on any $[0, L]$, $L > 0$.

Proof Let $f, f', f'' \in C_2^*(\mathbb{R}_+)$ and $x \geq 0$. Then by Taylor's formula

$$f(t) = f(x) + f'(x)(t - x) + \frac{1}{2}f''(x)(t - x)^2 + v(t, x)(t - x)^2, \tag{12}$$

where $v(t, x)$ is the Peano form of the remainder.

Since $v(\cdot, x) \in C_2^*(\mathbb{R}_+)$, for sufficiently large n

$$\lim_{t\to x} v(t, x) = 0.$$

Applying the operator (2) on both sides of (12), we get

$$D^{p_n,q_n}_{n,\alpha,\beta}(f; x) - f(x) = f'(x)D^{p_n,q_n}_{n,\alpha,\beta}((t - x); x) + \frac{1}{2}f''(x)D^{p_n,q_n}_{n,\alpha,\beta}((t - x)^2; x) + D^{p_n,q_n}_{n,\alpha,\beta}(v(t, x)(t - x)^2; x).$$

By Cauchy–Schwarz inequality, we have

$$D^{p_n,q_n}_{n,\alpha,\beta}(v(t, x)(t - x)^2; x) \leq \sqrt{D^{p_n,q_n}_{n,\alpha,\beta}(v^2(t, x); x)}\sqrt{D^{p_n,q_n}_{n,\alpha,\beta}((t - x)^4; x)}. \tag{13}$$

We can see that $v^2(x, x) = 0$ and $v^2 \in C_2^*(\mathbb{R}_+)$.

Then using Theorem 4.1, we say that

$$\lim_{n\to\infty} D^{p_n,q_n}_{n,\alpha,\beta}(v^2(t, x); x) = v^2(x, x) = 0, \tag{14}$$

uniformly with respect to $x \in [0, L]$.

Therefore, from (13) and (14), we obtain

$$\lim_{n\to\infty}[n]_{p_n,q_n}D^{p_n,q_n}_{n,\alpha,\beta}(v(t, x)(t - x)^2; x) = 0.$$

Thus

$$\lim_{n\to\infty}[n]_{p_n,q_n}\left(D^{p_n,q_n}_{n,\alpha,\beta}(f; x) - f(x)\right) = f'(x)\lim_{n\to\infty}[n]_{p_n,q_n}D^{p_n,q_n}_{n,\alpha,\beta}((t - x); x) + \frac{1}{2}f''(x)\lim_{n\to\infty}D^{p_n,q_n}_{n,\alpha,\beta}((t - x)^2; x). \tag{15}$$

Now

$$\lim_{n\to\infty}[n]_{p_n,q_n}\left(D^{p_n,q_n}_{n,\alpha,\beta}(f; x) - f(x)\right) = \lim_{n\to\infty}[n]_{p_n,q_n}\frac{\alpha - \beta x}{[n]_{p_n,q_n} + \beta} = \alpha - \beta x. \tag{16}$$

Using the equality $[k]_{p_n,q_n} = q_n^{k-1} + p_n[k-1]_{p_n,q_n}$, we have

$$\begin{aligned}\lim_{n\to\infty}[n]_{p_n,q_n} D^{p_n,q_n}_{n,\alpha,\beta}((t-x)^2;x) &= \lim_{n\to\infty}[n]_{p_n,q_n}\Bigg[\left(\frac{p_n[n]^3_{p_n,q_n}}{([n]_{p,qn}+\beta)^3[n-1]_{p_n,q_n}} - \frac{2[n]_{p_n,q_n}}{[n]_{p_n,q_n}+\beta}+1\right)x^2 \\ &\quad + \left(\frac{q_n[2]_{p_n,q_n}[n]^2_{p_n,q_n}}{([n]_{p_n,q_n}+\beta)^2 p_n[n-1]_{p_n,q_n}} + \frac{2\alpha[n]_{p_n,q_n}}{([n]_{p_n,q_n}+\beta)^2} - \frac{2\alpha}{[n]_{p_n,q_n}+\beta}\right)x \\ &\quad + \frac{(\alpha)^2}{([n]_{p_n,q_n}+\beta)^2}\Bigg] \\ &= 2x + x^2 + \lim_{n\to\infty}[n]_{p_n,q_n}\left(\frac{[n]^2_{p_n,q_n}p_n^2}{([n]_{p_n,q_n}+\beta)^2} - \frac{2[n]_{p_n,q_n}}{([n]_{p_n,q_n}+\beta)}+1\right)x^2 \\ &= 2x + x^2(1+A). \end{aligned} \tag{17}$$

Using (15), (16) and (17), we get the desired result,

$$\lim_{n\to\infty}[n]_{p_n,q_n}\left(D^{p_n,q_n}_{n,\alpha,\beta} - f(x)\right) = (\alpha-\beta x)f'(x) + \frac{f''(x)}{2}[2x + x^2(1+A)].$$

□

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