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Two-valenced association schemes and the Desargues theorem

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Abstract The main goal of the paper is to establish a sufficient condition for a two-valenced association scheme to be schurian and separable. To this end, an analog of the Desargues theorem is introduced for a noncommutative geometry defined by the scheme in question. It turns out that if the geometry has enough many Desarguesian configurations, then under a technical condition, the scheme is schurian and separable. This result enables us to give short proofs for known statements on the schurity and separability of quasi-thin and pseudocyclic schemes. Moreover, by the same technique, we prove a new result: given a prime p , any $\{1, p\}$ -scheme with thin residue isomorphic to an elementary abelian p -group of rank greater than two, is schurian and separable.

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1 Introduction

An association scheme can be thought as a partition of the arcs of a complete directed graph into digraphs connected via special regularity conditions. Numerous examples of associative schemes include the orbital schemes of transitive permutation groups, the Cayley schemes corresponding to Schur rings, the schemes of distance-regular graphs, etc., see [3]. It is now widely believed that the theory of association schemes is one of the most important branches of algebraic combinatorics.

One of the fundamental problems in theory of association schemes is to determine whether a given scheme is schurian, i.e., comes from a permutation group, and/or separable, i.e., uniquely determined by its intersection number array (for the exact definitions, see Sect. 2). In the last 2 decades, these two problems are intensively studied for the two-valenced schemes, see, e.g., [2, 6, 11, 13]; here, an association scheme is said to be two-valenced if the valencies of its basic relations take exactly two values, and if they are 1 and k , the term $\{1, k\}$ -scheme is also used.

An analysis of the known proofs that certain two-valenced schemes are schurian or separable shows that in all cases, the following two properties are significant. The first one is that there are sufficiently many intersection numbers of the scheme in question that are equal to 1; to define this property precisely, we introduce, in Sect. 3,

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the saturation condition (a special case of it appeared in [2]). The second property expresses the fact that in a noncommutative “affine” geometry determined by the two-valenced scheme, there are sufficiently many Desarguesian configurations, see Sect. 4.

A two-valenced scheme having the first and second properties is said to be saturated and Desarguesian, respectively. A model example illustrating these two properties is the scheme of a finite affine space that is two-valenced, saturated (except for few cases), and Desarguesian if the dimension of the affine space is at least 3 (see Examples in Sects. 3 and 4). Numerous examples of other saturated Desarguesian schemes can be found among pseudocyclic and quasi-thin schemes, see the proofs of Corollaries 1.2 and 1.3.

Theorem 1.1 *Let $\mathcal{X}$ be a two-valenced scheme. Assume that $\mathcal{X}$ is saturated and Desarguesian. Then, $\mathcal{X}$ is schurian and separable.*

For the scheme of a finite affine space of dimension at least 3, Theorem 1.1 expresses a well-known fact that in the Desarguesian case, this scheme is reconstructed from its automorphism group and is uniquely determined by its order.

The power of Theorem 1.1 is illustrated by three statements. The first two are known results, but using the theorem, we are able to give much shorter and clear proofs than in the original papers. The third one (Theorem 1.4) is a new result, which is discussed later.

Corollary 1.2 [11] *There exists a function f , such that any pseudocyclic scheme of valency $k > 1$ and degree at least $f(k)$ is schurian and separable.*

From the proof of Corollary 1.2 given in Sect. 6, it follows that the function f satisfies the inequality $f(k) \leq 1 + 3k^6$. More subtle arguments used in [2] show that $f(k) \leq 1 + 6k(k-1)^2$.

Corollary 1.3 [12, 13] *Any quasi-thin scheme satisfying the condition $n_{s*s} \neq 2$ for all basis relations, s , is schurian and separable.*

Except for the separability statement, the result of Corollary 1.3 is contained in [13]. On the other hand, the schurian and separable quasi-thin schemes were characterized in [12]. This result can also be deduced from an analog of Theorem 1.1, in which the Desarguesian condition is replaced by a weaker one (namely, the amount of the required Desarguesian configuration is reduced). To keep the text more compact, we do not go into detailed explanation of this topic.

Our second main result (also deduced from Theorem 1.1) concerns the schurity and separability of a class of meta-thin schemes introduced in [7]. A meta-thin scheme can be thought as an extension of a regular scheme $\mathcal{X}$ by another regular scheme. Even if the meta-thin scheme is a $\{1, p\}$ -scheme, where p is a prime, the schurity and separability problems seem to be very complicated. In a sense, the answer depends on the scheme $\mathcal{X}$, which can be chosen as the thin residue of the scheme in question. For example, if the group associated with $\mathcal{X}$ has distributive lattice of normal subgroups, then the scheme in question is schurian and separable [9]. However, there are many non-schurian and non-separable meta-thin $\{1, p\}$ -schemes for which that group is elementary abelian of order p^2 [8]. The following theorem shows that if the group is an elementary abelian p -group of rank greater than two, then the situation is smooth.

Theorem 1.4 *Given a prime p , any $\{1, p\}$ -scheme with thin residue isomorphic to an elementary abelian p -group of rank greater than two, is schurian and separable.*

Except for the above-mentioned examples from [8], not so much is known on the schurity and separability of a $\{1, p\}$ -scheme with thin residue isomorphic to an elementary abelian p -group of rank two. For $p = 2$, this case was studied in [12]; it seems that a complete picture can be obtained with the help of recent classification of separable coherent configurations with fibers of size at most 4 [5].

The paper is organized as follows. For the reader convenience a background on association schemes and related concepts is given in Sect. 2. In Sects. 3 and 4, we introduce and study the saturated and Desarguesian two-valenced schemes, respectively. The proof of Theorem 1.1 is given in Sect. 5. In Sect. 6, we prove the corollaries and Theorem 1.4.

Notation.

Throughout the paper, Ω denotes a finite set.

The diagonal of the Cartesian product $\Omega \times \Omega$ is denoted by 1_Ω or 1 .

For a relation $r \subseteq \Omega \times \Omega$, we set $r^* = \{(\beta, \alpha) : (\alpha, \beta) \in r\}$ and $\alpha r = \{\beta \in \Omega : (\alpha, \beta) \in r\}$ for all $\alpha \in \Omega$.

For a relation $r \subseteq \Omega \times \Omega$ and sets $\Delta, \Gamma \subseteq \Omega$, we set $r_{\Delta, \Gamma} = r \cap (\Delta \times \Gamma)$. If S is a set of relations, we put $S_{\Delta, \Gamma} = \{r_{\Delta, \Gamma} : r \in S\}$ and denote $S_{\Delta, \Delta}$ by S_Δ .

For relations $r, s \subseteq \Omega \times \Omega$, we set $r \cdot s = \{(\alpha, \beta) : (\alpha, \gamma) \in r, (\gamma, \beta) \in s \text{ for some } \gamma \in \Omega\}$. If S and T are sets of relations, we set $S \cdot T = \{s \cdot t : s \in S, t \in T\}$.

For a set S of relations on Ω , we denote by $S^\cup$ the set of all unions of the elements of S , and put $S^* = \{r^* : r \in S\}$ and $\alpha S = \cup_{r \in S} \alpha r$, where $\alpha \in \Omega$.

An elementary abelian p -group of order p^m is denoted by E_{p^m} .

2 Association schemes

In our presentation of association schemes, we follow papers [4, 12] and monographs [3, 14]. All the facts which we use can be found in these sources and references therein.

2.1 Definitions

A pair $\mathcal{X} = (\Omega, S)$, where Ω is a finite set and S is a partition of $\Omega \times \Omega$, is called an association scheme or scheme on Ω if the following conditions are satisfied: $1_\Omega \in S$, $S^* = S$, and given $r, s, t \in S$, the number

$$c_{rs}^t := |\alpha r \cap \beta s^*|$$

does not depend on the choice of $(\alpha, \beta) \in t$.

The elements of Ω , S , $S^\cup$, and the numbers c_{rs}^t are called the points, basis relations, relations, and intersection numbers of $\mathcal{X}$, respectively. The numbers $|\Omega|$ and $|S|$ are called the degree and the rank of $\mathcal{X}$. The unique basic relation containing a pair $(\alpha, \beta) \in \Omega \times \Omega$ is denoted by $r(\alpha, \beta)$. Since the mapping $r : \Omega \times \Omega \rightarrow S$ depends only on $\mathcal{X}$, it should be denoted by $r_{\mathcal{X}}$, but we usually omit the subindex if this does not lead to confusion.

2.2 Complex product

The set $S^\cup$ contains the relation $r \cdot s$ for all $r, s \in S^\cup$, see [3, Proposition 2.1.4]. It follows that this relation is the union (possibly empty) of basis relations of $\mathcal{X}$; the set of these relations is called the complex product of r and s and denoted by rs . Thus:

$$rs \subseteq S, \quad r, s \in S.$$

In what follows, for any $X, Y \subseteq S$, we denote by XY the union of all sets rs with $r \in X$ and $s \in Y$. Obviously, $(XY)Z = X(YZ)$ for all $X, Y, Z \subseteq S$.

2.3 Valencies

For any basic relation $s \in S$, the number $|\alpha s|$ with $\alpha \in \Omega$ equals the intersection number $c_{ss^*}^1$, and, hence, does not depend on the choice of the point α . It is called the valency of s and denoted by n_s ; we say that s is thin if $n_s = 1$.

For the intersection numbers, we have the following well-known identities:

$$c_{r^*s^*}^{t^*} = c_{sr}^t \quad \text{and} \quad n_t c_{rs}^{t^*} = n_r c_{st}^{r^*} = n_s c_{tr}^{s^*}, \quad r, s, t \in S. \quad (1)$$

The scheme $\mathcal{X}$ is said to be regular (respectively, $\{1, k\}$ -scheme) if $n_s = 1$ (respectively, $k > 1$ and $n_s = 1$ or k) for all $s \in S$.

2.4 Isomorphisms and schurity

A bijection from the point set of a scheme $\mathcal{X}$ to the point set of a scheme $\mathcal{X}'$ is called an isomorphism from $\mathcal{X}$ to $\mathcal{X}'$ if it induces a bijection between their sets of basis relations. The schemes $\mathcal{X}$ and $\mathcal{X}'$ are said to be isomorphic if there exists an isomorphism from $\mathcal{X}$ to $\mathcal{X}'$.

An isomorphism from a scheme $\mathcal{X}$ to itself is called an automorphism if the induced bijection on the basis relations of $\mathcal{X}$ is the identity. The set of all automorphisms of a scheme $\mathcal{X}$ is a group with respect to composition and will be denoted by $\text{Aut}(\mathcal{X})$.¹

Conversely, let $K \leq \text{Sym}(\Omega)$ be a transitive permutation group, and let S denote the set of orbits in the induced action of K on Ω^2 . Then, $\mathcal{X} = (\Omega, S)$ is a scheme; we say that $\mathcal{X}$ is associated with K . A scheme on Ω is said to be schurian if it is associated with some transitive permutation group on Ω . A scheme $\mathcal{X}$ is schurian if and only if it is associated with the group $\text{Aut}(\mathcal{X})$.

2.5 Algebraic isomorphisms and separability

Let $\mathcal{X}$ and $\mathcal{X}'$ be schemes. A bijection $\varphi : S \rightarrow S'$, $r \mapsto r'$ is called an algebraic isomorphism from $\mathcal{X}$ to $\mathcal{X}'$ if

$$c_{rs}^t = c_{r's'}^{t'}, \quad r, s, t \in S. \quad (2)$$

In this case, $\mathcal{X}$ and $\mathcal{X}'$ are said to be algebraically isomorphic.

Each isomorphism f from $\mathcal{X}$ onto $\mathcal{X}'$ induces an algebraic isomorphism:

$$\varphi_f : r \mapsto r^f$$

between these schemes. The set of all isomorphisms inducing the algebraic isomorphism φ is denoted by $\text{Iso}(\mathcal{X}, \mathcal{X}', \varphi)$. In particular:

$$\text{Iso}(\mathcal{X}, \mathcal{X}, \text{id}_S) = \text{Aut}(\mathcal{X}), \quad (3)$$

where id_S is the identity mapping on S . A scheme $\mathcal{X}$ is said to be separable if, for any algebraic isomorphism $\varphi : S \rightarrow S'$, the set $\text{Iso}(\mathcal{X}, \mathcal{X}', \varphi)$ is not empty.

The algebraic isomorphism φ induces a bijection from $S^\cup$ onto $(S')^\cup$: the union $r \cup s \cup \dots$ of basis relations of $\mathcal{X}$ is taken to $r' \cup s' \cup \dots$. This bijection is also denoted by φ .

2.6 Faithful maps

Let $\mathcal{X} = (\Omega, S)$ and $\mathcal{X}' = (\Omega', S')$ be schemes, and let $\varphi : S \rightarrow S'$ be an algebraic isomorphism. A bijection f from a subset of Ω to a subset of Ω' is said to be φ -faithful if

$$r(\alpha, \beta)^\varphi = r'(\alpha^f, \beta^f) \quad \text{for all } \alpha, \beta \in \text{Dom}(f),$$

where $\text{Dom}(f)$ is the domain of f . Clearly, if f is a φ -faithful map, then the restriction of f to any subset of $\text{Dom}(f)$ is also φ -faithful.

A φ -faithful map f is said to be φ -extendable to a point $\gamma \in \Omega$ if there exists a φ -faithful map with domain $\text{Dom}(f) \cup \{\gamma\}$, or, equivalently, if

$$\bigcap_{\alpha \in \text{Dom}(f)} \alpha^f r(\alpha, \gamma)^\varphi \neq \emptyset. \quad (4)$$

A φ -faithful map, which is φ -extendable to every point of Ω , is said to be φ -extendable. From the definitions of schemes and algebraic isomorphisms, it follows that every φ -faithful map f with $|\text{Dom}(f)| \leq 2$ is φ -extendable. In these terms, one can give a sufficient condition for schurity and separability of a scheme; this condition is used in the proof of Theorem 1.1.

Theorem 2.1 [9, Corollary 2.2] *Let $\mathcal{X}$ be a scheme. Suppose that for every algebraic isomorphism φ to another scheme, each φ -faithful map is φ -extendable. Then, $\mathcal{X}$ is schurian and separable.*

¹ It is more natural to define an automorphism of a scheme as an isomorphism of it to itself; however, we should follow a long tradition in which a scheme is treated as a colored graph.

3 Saturation condition

Throughout this section, $k > 1$ is an integer and $\mathcal{X} = (\Omega, S)$ is a scheme. By technical reasons, the basis relations of $\mathcal{X}$ are mainly denoted below by x, y, z rather than r, s, t . Our primary goal is to define a graph with vertex set:

$$S_k = \{x \in S : n_x = k\}$$

that accumulates an information about intersection numbers equal to 1 (this graph was used implicitly in [11] and explicitly in [2]). The following simple lemma (not formulated but proved in [2]) indicates a way how we do this.

Lemma 3.1 *Given $x, y \in S_k$,*

$$|x^*y| = k \iff c_{xs}^y = 1 \text{ for all } s \in x^*y. \quad (5)$$

Proof We have $n_x = n_{x^*} = n_y = n_{y^*} = k$. By formulas (1), this implies that

$$k^2 = n_{x^*}n_y = \sum_{s \in x^*y} n_s c_{x^*y}^s = \sum_{s \in x^*y} n_{y^*} c_{s^*x^*}^{y^*} = k \sum_{s \in x^*y} c_{xs}^y.$$

Since $c_{xs}^y \geq 1$ for all $s \in x^*y$, we are done. $\square$

Let us define a relation $\sim$ on S_k by setting $x \sim y$ if the right- or left-hand side in formula (5) holds true. This relation is symmetric, because

$$c_{xs}^y = \frac{n_{x^*}}{n_y} c_{sy^*}^{x^*} = c_{ys^*}^x$$

for all $x, y \in S_k$. The (undirected) graph $\mathfrak{X} = \mathfrak{X}_k$ associated with the scheme $\mathcal{X}$ has vertex set S_k and adjacency relation $\sim$. Note that this graph can contain loops.

Definition 3.2 The scheme $\mathcal{X}$ is said to be k -saturated if, for any set $T \subseteq S_k$ with at most four elements, the set

$$N(T) = \{y \in S_k : y \sim x \text{ for all } x \in T\} \quad (6)$$

is not empty.

In this case, any two vertices of the graph $\mathfrak{X}$ are connected by a path of length at most two. A k -saturated $\{1, k\}$ -scheme is said to be saturated and the mention of k is omitted. The following statement immediately follows from the definitions.

Lemma 3.3 *Any algebraic isomorphism from $\mathcal{X}$ to $\mathcal{X}'$ induces an isomorphism from the graph $\mathfrak{X}$ to the graph $\mathfrak{X}'$ associated with $\mathcal{X}'$. In particular, $\mathcal{X}$ is k -saturated if and only if so is $\mathcal{X}'$.*

Example: schemes of affine spaces. Let $\mathcal{A}$ be a finite affine space with point set Ω and line set L , see [1]. Denote by $\mathcal{P}$ the set of parallel classes of lines. The lines belonging to a class $P \in \mathcal{P}$ form a partition of Ω ; the corresponding equivalence relation with removed diagonal is denoted by e_P . Using the axioms of affine spaces, one can easily verify that the set $S = S_{\mathcal{A}}$ of all the e_P together with 1_{Ω} forms a commutative scheme, such that

$$c_{e_P e_P}^s = \begin{cases} q - 1 & \text{if } s = 1, \\ q - 2 & \text{if } s = e_P, \\ 0 & \text{otherwise,} \end{cases} \quad (7)$$

and

$$c_{e_P e_Q}^s = \begin{cases} 1 & \text{if } s \subseteq e_P \cdot e_Q \text{ and } s \notin \{1, e_P, e_Q\}, \\ 0 & \text{otherwise.} \end{cases} \quad (8)$$

where q is the cardinality of a line. We say that $\mathcal{X} = (\Omega, S)$ is the scheme associated with the affine space $\mathcal{A}$. Formulas (7) and (8) imply that $\mathcal{X}$ is a $\{1, q - 1\}$ -scheme. Moreover, $\mathfrak{X}$ is a complete graph, i.e., any two

Fig. 1 The Desarguesian configuration

distinct vertices form an edge. In particular, the scheme $\mathcal{X}$ is $(q - 1)$ -saturated whenever the rank of $\mathcal{X}$ is at least 6.

We complete the section by establishing a sufficient condition for a scheme $\mathcal{X}$ to be k -saturated in terms of the indistinguishing number:

$$c(s) = \sum_{t \in S} c_{tt^*}^s$$

of the relation s , see [11]. One can see that this number is equal to the cardinality of the set:

$$\Omega_{\alpha, \beta} = \{\gamma \in \Omega : r(\gamma, \alpha) = r(\gamma, \beta)\} \quad (9)$$

for any $(\alpha, \beta) \in s$. The following result is used in the proof of Corollary 1.2.

Theorem 3.4 [2, Lemma 5.2] *Let c be the maximum of $c(s)$, $s \in S^\#$. Then, the scheme $\mathcal{X}$ is k -saturated whenever $|S_k| > 4c(k - 1)$.*

4 Desarguesian two-valenced schemes

The concept of a Desarguesian scheme comes from a property of a geometry to be Desarguesian. Throughout this section, $k > 1$ is an integer and $\mathcal{X} = (\Omega, S)$ is a $\{1, k\}$ -scheme. We also keep notation of Sect. 3.

Let us define a noncommutative geometry associated with the scheme $\mathcal{X}$ as follows: the points are elements of S , the lines are the sets x^*y , $x, y \in S$, and the incidence relation is given by inclusion. Thus, the point $z \in S$ belongs to the line x^*y if and only if $z \in x^*y$. The geometry is extremely unusual: the line x^*y does not necessarily contain the points x, y , and can be different from y^*x . However, in the terms of this geometry, one can define Desarguesian configurations, see below.

Assume that we are given two triangles with vertices $x, y, z \in S$ and $u, v, w \in S$, respectively, that are perspective with respect to a point q , that is:

$$u \in x^*q, \quad v \in y^*q, \quad w \in z^*q, \quad (10)$$

see the configuration depicted in Fig. 1; note that the intersections of lines do not necessarily consist of a unique point, and even may be empty. However, if

$$x^*z \cap uw^* = \{r\}, \quad z^*y \cap vw^* = \{s\}, \quad x^*y \cap uv^* = \{t\} \quad (11)$$

for some $r, s, t \in S$; then, as in the case of Desargues' theorem, we would like that the point t would lie on the line rs . When this is true, this configuration is said to be Desarguesian. More precisely, the ten relations in Fig. 1 form a Desarguesian configuration if conditions (10) and (11) are satisfied and $t \in rs$. In what follows we are going to study $\{1, k\}$ -schemes with sufficiently many Desarguesian configurations.

Fig. 2 The relations r and s are linked with respect to (x, y, z)

Let $x, y, z \in S_k$ and $r, s \in S$ be basis relations of the scheme $\mathcal{X}$. We say that they form an initial configuration if

$$x \sim z \sim y \quad \text{and} \quad r \in x^*z, \quad s \in z^*y. \quad (12)$$

In geometric language, this means that the points r and s belong to the lines x^*z and z^*y , respectively, and each of these lines consists of exactly k points.

Definition 4.1 The relations r and s are said to be linked with respect to (x, y, z) if the initial configuration is contained in a Desarguesian configuration, namely, there exist

$$q \in N(x, y, z), \quad u, v, w \in S, \quad t \in rs,$$

for which conditions (10) and (11) are satisfied, where $N(x, y, z) = N(\{x, y, z\})$ (a more compact picture of the linked relations is given in Fig. 2).

Assume that the relations r and s are linked with respect to (x, y, z) . Then, the relation t is uniquely determined by the third of equalities (11). The following statement shows that in this case, t does not depend on the choice of q and u, v, w . Below, we fix a point $\alpha \in \Omega$ and set:

$$r_{x,y} = r \cap (\alpha x \times \alpha y) \quad (13)$$

for all $r \in S$ and $x, y \in S_k$.

Lemma 4.2 Assume that r and s are linked with respect to (x, y, z) . Then:

$$r_{x,z} \cdot s_{z,y} \subseteq t_{x,y}, \quad (14)$$

with equality if $x \sim y$.

Proof By formulas (11), we have:

$$u_{x,q} \cdot w_{q,z}^* \subseteq r_{x,z}, \quad w_{z,q} \cdot v_{q,y}^* \subseteq s_{z,y}, \quad u_{x,q} \cdot v_{q,y}^* \subseteq t_{x,y}. \quad (15)$$

The relations $u_{x,q} \cdot w_{q,z}^*$ and $r_{x,z}$ are matchings, because $x \sim q \sim z$, and $x \sim z$. By the first inclusion in (15), this implies the first of the following two equalities, the second one is proved similarly:

$$u_{x,q} \cdot w_{q,z}^* = r_{x,z} \quad \text{and} \quad w_{z,q} \cdot v_{q,y}^* = s_{z,y}.$$

Now, the third inclusion in (15) yields:

$$r_{x,z} \cdot s_{z,y} = (u_{x,q} \cdot w_{q,z}^*) \cdot (w_{z,q} \cdot v_{q,y}^*) = u_{x,q} \cdot v_{q,y}^* \subseteq t_{x,y},$$

which proves (14). If $x \sim y$, then the relation $t_{x,y}$ is a matching and we are done by the above argument. $\square$

At this point, we need an auxiliary statement. In the geometric language, the conclusion of this statement means that the lines rs and x^*y have a unique common point.

Lemma 4.3 [11, Theorem 5.1] Let $x, y, z \in S_k$ be such that:

$$(xx^*yy^*) \cap zz^* = \{1\}. \quad (16)$$

Then, $|rs \cap x^*y| = 1$ for all $r \in x^*z$ and $s \in z^*y$.

The statement below establishes two sufficient conditions for relations r and s to be linked with respect to (x, y, z) .

Corollary 4.4 *Let $x, y, z \in S_k$ and $r, s \in S$ form an initial configuration. Then, r and s are linked with respect to (x, y, z) whenever at least one of the following statements holds:*

(L1) z is a loop of the graph $\mathfrak{X}$ and condition (16) is satisfied.

(L2) there exists $q \in S_k$, such that

$$qq^* \cap (xx^*yy^* \cup xx^*zz^* \cup zz^*yy^*) = \{1\}. \quad (17)$$

Proof Assume that (L1) holds. Then, by Lemma 4.3, we have $rs \cap x^*y = \{t\}$ for some $t \in S$. Then, condition (11) is obviously satisfied for $q = z$ and $u = r, v = s^*,$ and $w = 1$.

Now, assume that (L2) holds. Fix a point α . Since $r \in x^*z$ and $s \in z^*y$, one can find points $\beta \in \alpha x, \gamma \in \alpha z,$ and $\delta \in \alpha y$, such that

$$(\beta, \gamma) \in r \quad \text{and} \quad (\gamma, \delta) \in s. \quad (18)$$

Now, take $\epsilon \in \alpha q$ and set

$$u := r(\beta, \epsilon), \quad v := r(\delta, \epsilon), \quad w := r(\gamma, \epsilon).$$

Then, formula (10) holds. Moreover by (L2), the hypothesis of Lemma 4.3 is satisfied in the following three cases:

$$\begin{aligned} (x, y, z) &= (x, z, q), \quad (r, s) = (u, w^*), \\ (x, y, z) &= (z, y, q), \quad (r, s) = (w, v^*), \\ (x, y, z) &= (x, y, q), \quad (r, s) = (u, v^*). \end{aligned}$$

Therefore, formula (11) also holds with $t = r(\beta, \gamma)$. Finally, $t \in rs$ in view of (18). It follows that the configuration formed by $x, y, z, u, v, w, r, s, t,$ and q is Desarguesian. Thus, r and s are linked with respect to (x, y, z) . $\square$

Now, we arrive to the main definition in this section. Namely, the scheme $\mathcal{X}$ is said to be Desarguesian with respect to S_k if for all $x, y, z \in S_k$ and all $r, s \in S$ satisfying (12), the elements r and s are linked with respect to (x, y, z) . When the scheme is two-valenced, the mention of S_k is omitted. The following statement immediately follows from the definitions.

Lemma 4.5 *Let $\mathcal{X}$ and $\mathcal{X}'$ be algebraically isomorphic two-valenced schemes. Then, $\mathcal{X}$ is Desarguesian if and only if so is $\mathcal{X}'$.*

Example: schemes of affine spaces (continuation). Let $\mathcal{X}$ be the scheme associated with an affine space $\mathcal{A}$ of order q and dimension at least 3. Then, from formulas (7) and (8), it follows that for any three parallel classes $P, Q,$ and R there exists a parallel class T , such that

$$e_T \notin (epe_Q \cup epe_R \cup ere_Q).$$

It follows that the statement (L2) of Corollary 4.4 holds for $q = e_T, x = e_P, y = e_Q,$ and $z = e_R$. Therefore, any relations $r \in epe_R$ and $s \in ere_Q$ are linked with respect to (e_P, e_Q, e_R) . Thus, the scheme $\mathcal{X}$ is Desarguesian. If the space $\mathcal{A}$ is an affine plane, i.e., an affine space of dimension 2, then $\mathcal{X}$ is not Desarguesian.

5 Proof of Theorem 1.1

Let $\mathcal{X} = (\Omega, S), \mathcal{X}' = (\Omega', S'),$ and φ is an algebraic isomorphism from $\mathcal{X}$ onto $\mathcal{X}'$. Then, obviously, $\mathcal{X}'$ is two-valenced. Moreover, $\mathcal{X}'$ is saturated and Desarguesian by Lemmas 3.3 and 4.5, respectively. By Theorem 2.1, it suffices to verify that given $\alpha, \beta \in \Omega,$ any φ -faithful map

$$f : \{\alpha, \beta\} \rightarrow \Omega'$$

is extended to a combinatorial isomorphism $\widehat{f} : \mathcal{X} \rightarrow \mathcal{X}'$. Without loss of generality, we may assume that $r(\alpha, \beta) \in S_k,$ where $k > 1$ is the valency of $\mathcal{X}$ (and $\mathcal{X}'$). Indeed, in this case, the isomorphism $\widehat{f}$ extends any faithful map $f : \{\alpha, \beta\} \rightarrow \Omega',$ where $r(\alpha, \beta) \in S_1$.

By the definition of the graph $\mathfrak{X} = \mathfrak{X}_k$, the union of the sets:

$$\begin{aligned}\Delta_0 &:= \alpha S_1 \\ \Delta_1 &:= \{\delta \in \alpha S_k : r(\alpha, \beta) \sim r(\alpha, \delta)\} \\ \Delta_2 &:= \{\delta \in \alpha S_k : r(\alpha, \beta) \approx r(\alpha, \delta)\}\end{aligned}$$

equals Ω . Clearly, $\beta \in \Delta_1$ if and only if $r(\alpha, \beta)$ forms a loop of the graph $\mathfrak{X}$. The statement below immediately follows from the definition of the relation $\sim$, Lemma 3.3, and the fact that $\mathfrak{X}$ is of diameter at most 2 (the saturation condition).

Lemma 5.1 *For any point $\delta \in \Omega$, the following statements hold:*

- (1) *if $\delta \in \Delta_0$, then the relation $r(\alpha, \delta)$ is thin,*
- (2) *if $\delta \in \Delta_1$, then $c_{r(\alpha, \delta), r(\delta, \beta)}^{r(\alpha, \beta)} = 1$,*
- (3) *if $\delta \in \Delta_2$, then there exists $\gamma \in \Delta_1$, such that $c_{r(\alpha, \gamma), r(\gamma, \delta)}^{r(\alpha, \delta)} = 1$.*

By Lemma 5.1, one can define a mapping $\hat{f} : \Omega \rightarrow \Omega'$ by the following conditions:

$$\{\delta \hat{f}\} = \begin{cases} \alpha^f r(\alpha, \delta)^\varphi & \text{if } \delta \in \Delta_0, \\ \alpha^f r(\alpha, \delta)^\varphi \cap \beta^f r(\beta, \delta)^\varphi & \text{if } \delta \in \Delta_1, \\ \alpha^f r(\alpha, \delta)^\varphi \cap \gamma^f r(\gamma, \delta)^\varphi & \text{if } \delta \in \Delta_2, \end{cases}$$

where, in the last line, γ is as in statement (3) of Lemma 5.1, that is

$$r(\alpha, \beta) \sim r(\alpha, \gamma) \sim r(\alpha, \delta).$$

From the definition of the graph $\mathfrak{X}$, it follows that the $\hat{f}$ -image is uniquely determined for all $\delta \in \Delta_0 \cup \Delta_1$. However, if $\delta \in \Delta_2$, then the image depends on the choice of the point γ .

We complete the proof by verifying that the mapping $\hat{f}$ is φ -faithful, that is:

$$r(\delta, \epsilon)^\varphi = r(\delta \hat{f}, \epsilon \hat{f}) \quad (19)$$

for all $\delta, \epsilon \in \Omega$. Note that this is true by the definition of $\hat{f}$ whenever $\alpha \in \{\delta, \epsilon\}$. In what follows we need the following lemma.

Lemma 5.2 *Let $\delta, \gamma, \epsilon \in \Omega$ be such that*

- (1) *$r(\alpha, \delta) \sim r(\alpha, \gamma) \sim r(\alpha, \epsilon)$;*
- (2) *$\hat{f}|_{\{\delta, \gamma\}}$ and $\hat{f}|_{\{\gamma, \epsilon\}}$ are φ -faithful.*

Then, $\hat{f}|_{\{\delta, \epsilon\}}$ is φ -faithful.

Proof Let $x = r(\alpha, \delta)$, $y = r(\alpha, \epsilon)$, and $z = r(\alpha, \gamma)$. Then, by condition (1), the triple (x, y, z) together with relations $r = r(\delta, \gamma)$, $s = r(\gamma, \epsilon)$ forms an initial configuration. The Desarguesian condition implies that r and s are linked with respect to (x, y, z) . Therefore, the relation $t = r(\delta, \epsilon)$ is uniquely determined by the relations $r_{x,z}$ and $r_{z,y}$ (Lemma 4.2). By condition (2), we have:

$$(r_{x,z})^{\hat{f}} = (r^\varphi)_{x^\varphi, z^\varphi} \quad \text{and} \quad (s_{z,y})^{\hat{f}} = (s^\varphi)_{z^\varphi, y^\varphi}$$

and, hence:

$$r(\delta \hat{f}, \epsilon \hat{f}) \in r(\delta \hat{f}, \gamma \hat{f}) r(\gamma \hat{f}, \epsilon \hat{f}) = r(\delta, \gamma)^\varphi r(\gamma, \epsilon)^\varphi.$$

Since also

$$r(\delta, \epsilon)^\varphi \in (r(\delta, \gamma) r(\gamma, \epsilon))^\varphi = r(\delta, \gamma)^\varphi r(\gamma, \epsilon)^\varphi,$$

we conclude that

$$r(\delta \hat{f}, \epsilon \hat{f}) = r(\delta, \epsilon)^\varphi,$$

Fig. 3 Configuration for Case 3

as required.

The rest of the proof of formula (19) is divided for several cases considered separately.

Case 1: $\delta \in \Delta_0$ and $\epsilon \in \Omega$. Here, the relations $r(\delta, \alpha)$ and $r(\delta^{\hat{f}}, \alpha^{\hat{f}})$ are thin. Therefore:

$$\{r(\delta, \epsilon)\} = r(\delta, \alpha)r(\alpha, \epsilon) \quad \text{and} \quad r(\delta^{\hat{f}}, \alpha^{\hat{f}})r(\alpha^{\hat{f}}, \epsilon^{\hat{f}}) = \{r(\delta^{\hat{f}}, \epsilon^{\hat{f}})\}.$$

This implies that

$$\{r(\delta, \epsilon)^{\varphi}\} = r(\delta, \alpha)^{\varphi}r(\alpha, \epsilon)^{\varphi} = r(\delta^{\hat{f}}, \alpha^{\hat{f}})^{\varphi}r(\alpha^{\hat{f}}, \epsilon^{\hat{f}})^{\varphi} = \{r(\delta^{\hat{f}}, \epsilon^{\hat{f}})^{\varphi}\},$$

as required.

Case 2: $\delta, \epsilon \in \Delta_1$. The required statement follows from Lemma 5.2 for $\gamma = \beta$.

Case 3: $\delta \in \Delta_1$ and $\epsilon \in \Delta_2$. By the definition of Δ_2 , there exists $\delta_1 \in \Delta_1$, such that

$$r_{\beta} \sim r_{\delta_1} \sim r_{\epsilon} \quad \text{and} \quad \{\epsilon^{\hat{f}}\} = \alpha^f(r_{\epsilon})^{\varphi} \cap \delta_1^{\hat{f}}r(\delta_1, \epsilon)^{\varphi},$$

where $r_{\beta} = r(\alpha, \beta)$, $r_{\delta_1} = r(\alpha, \delta_1)$, and $r_{\epsilon} = r(\alpha, \epsilon)$. By the saturated condition, there exists $\gamma \in \Delta_1$, such that

$$r_{\gamma} \in N(r_{\beta}, r_{\delta}, r_{\delta_1}, r_{\epsilon}),$$

the obtained configuration is depicted in Fig. 3. From Case 2, it follows that the conditions of Lemma 5.2 are satisfied for

$$(\delta, \gamma, \epsilon) = (\gamma, \delta_1, \epsilon).$$

Therefore, $\hat{f}|_{\{\epsilon, \gamma\}}$ is φ -faithful. This shows that the conditions of Lemma 5.2 are satisfied and, hence, $\hat{f}|_{\{\delta, \epsilon\}}$ is φ -faithful.

Case 4: $\delta, \epsilon \in \Delta_2$. By the definition of Δ_2 , there exists $\delta_1 \in \Delta_1$, such that

$$r_{\beta} \sim r_{\delta_1} \sim r_{\delta} \quad \text{and} \quad \{\delta^{\hat{f}}\} = \alpha^f(r_{\delta})^{\varphi} \cap \delta_1^{\hat{f}}r(\delta_1, \delta)^{\varphi}.$$

By the saturation condition, there exists $\gamma \in \Delta_1$, such that

$$r_{\gamma} \in N(r_{\beta}, r_{\delta}, r_{\delta_1}, r_{\epsilon}).$$

By the similar argument as in Case 3, we obtain $r(\delta, \epsilon)^{\varphi} = r(\delta^{\hat{f}}, \epsilon^{\hat{f}})^{\varphi}$, where Case 3 is used to deal with $r(\delta, \delta_1)$, $r(\delta, \gamma)$, and $r(\epsilon, \gamma)$. $\square$

6 Proofs of corollaries and Theorem 1.4

Proof of Corollary 1.2 Set $f(x) = 3x^6 + 1$. By Theorem 1.1, it suffices to verify that any pseudocyclic scheme $\mathcal{X}$ of degree n and valency k , is saturated and Desarguesian, whenever

$$n > 3k^6. \quad (20)$$

By [11, Theorem 3.2], we have $n_s = k$ and $c(s) = k - 1$ for any irreflexive basis relation s of $\mathcal{X}$. It follows that for $k \geq 2$:

$$|S_k| = \frac{n-1}{k} > \frac{3k^6-1}{k} > 4k(k-1) = 4ck.$$

Thus, $\mathcal{X}$ is saturated by Theorem 3.4.

To prove that $\mathcal{X}$ is Desarguesian, let $x, y, z \in S_k$ (here $S_k = S^\#$ and we do not assume that $x \sim z \sim y$). By Corollary 4.4, it suffices to find $q \in S_k$, such that condition (17) is satisfied. Assume on the contrary that no q satisfies this condition. Then, for a fixed $\alpha \in \Omega$ and each $q \in S_k$, there exists:

$$\beta_q \in \alpha \underbrace{(xx^*yy^* \cup xx^*zz^* \cup zz^*yy^*)}_T$$

other than α and such that $r(\alpha, \beta_q) \in qq^*$. It follows that:

$$\Omega_{\alpha, \beta_q} \cap \alpha q^* \neq \emptyset, \quad (21)$$

where the set Ω_{α, β_q} is as in (9). Since

$$|\alpha T| \leq n_x^2 n_y^2 + n_y^2 n_z^2 + n_z^2 n_x^2 = 3k^4 \quad \text{and} \quad |S_k| = \frac{n-1}{k} \geq \frac{3k^6}{k} = 3k^5,$$

there exist a point $\beta \neq \alpha$, such that $\beta = \beta_q$ for at least k relations $q \in S_2$. In view of (21), this implies that for $s = r(\alpha, \beta)$, we have

$$k-1 = c(s) \geq |\Omega_{\alpha, \beta}| \geq k,$$

a contradiction. $\square$

Proof of Corollary 1.3 Let $\mathcal{X} = (\Omega, S)$ be a quasi-thin scheme satisfying the condition $n_{s^*s} \neq 2$ for all $s \in S$. Without loss of generality, we may assume that $|\Omega| > 24$. $\square$

Claim. The graph $\mathfrak{X}$ is complete and has at least 9 vertices. Moreover, for any $x, y \in S_2$:

$$xx^* = yy^* \Leftrightarrow x = y. \quad (22)$$

Proof From [6, Lemma 4.1], it follows that given $x, y \in S_2$ and $s \in S$, the number c_{xs}^y is at most 2, and by [10, Lemma 3.1] also:

$$c_{xs}^y = 2 \Leftrightarrow s \in S_2, ss^* \subset S_1,$$

Since $n_{ss^*} \neq 2$, this implies (22) and shows that in our case, $c_{xs}^y \leq 1$ for all $x, y \in S_2$; in particular, $\mathfrak{X}$ is a complete graph with loops. Furthermore, again by [10, Lemma 3.1], in our case, $xs \not\subset S_1$ for all $x, s \in S_2$. It follows that:

$$|\Omega| \geq 3|S_1|.$$

Assume on the contrary that $\mathfrak{X}$ has at most 8 vertices, i.e., $|S_2| \leq 8$. Then:

$$3|S_1| \leq |\Omega| = |S_1| + 2|S_2| \leq |S_1| + 16,$$

whence $|S_1| \leq 8$. It follows that $|\Omega| \leq 24$, a contradiction. $\square$

By Theorem 1.1, it suffices to verify that $\mathcal{X}$ is saturated and Desarguesian. The former follows from the Claim. To prove that $\mathcal{X}$ is Desarguesian let $x, y, z \in S_k$ and $r, s \in S$ be basis relations of $\mathcal{X}$ that form an initial configuration. We have to verify that r and s are linked with respect to (x, y, z) . Note that since $n_z = 2$, the vertex z forms a loop of the graph $\mathfrak{X}$. Therefore, if $(xx^*yy^*) \cap zz^* = \{1\}$, then we are done by Corollary 4.4 (see the statement (L1)).

Let $zz^* \subseteq xx^* \cap yy^*$. Then, the intersection of S_2 with the set:

$$xx^*yy^* \cup xx^*zz^* \cup zz^*yy^* = \{1, x^\perp, y^\perp\} \cup x^\perp y^\perp \cup x^\perp z^\perp \cup z^\perp y^\perp$$

contains at most 8 elements, where for any $s \in S_2$, we set $s^\perp$ to be the unique non-thin element of ss^* . On the other hand, in view of the Claim, the graph $\mathfrak{X}$ contains at least 9 vertices. Thus, there exists $q' \in S_2$, such that

$$q' \notin xx^*yy^* \cup xx^*zz^* \cup zz^*yy^*.$$

In view of formula (22) and the assumption $n_{ss^*} \neq 2$, one can find $q \in S_2$, for which $q' = q^\perp$. Now, the required statement follows from Corollary 4.4 (see the statement (L2)). $\square$

Proof of Theorem 1.4 Let $\mathcal{X} = (\Omega, S)$ be a meta-thin $\{1, p\}$ -scheme. Then, the group formed by the thin basis relations of $\mathcal{X}$ (with respect to the composition) contains a subgroup (the thin residue) generated by the sets ss^* , $s \in S$. Assume that this subgroup is isomorphic to an elementary abelian p -group of rank greater than two. By Theorem 1.1, it suffices to verify that $\mathcal{X}$ is saturated and Desarguesian.

To prove that $\mathcal{X}$ is saturated, let $s_i \in S_p$, $1 \leq i \leq 4$. By the assumption, there exist $u_1, u_2, u_3 \in S$, such that

$$\langle u_1 u_1^*, u_2 u_2^*, u_3 u_3^* \rangle \simeq E_{p^3}.$$

Denote by T the set $\{s_1 s_1^*, \dots, s_4 s_4^*\}$. First, assume that $u_j u_j^* \notin T$ for some j . Then:

$$u_j u_j^* \cap s_i s_i^* = \{1\}, \quad 1 \leq i \leq 4.$$

In view of [11, Lemma 2.3], this implies that u_j is adjacent in the graph $\mathfrak{X}$ with each of the s_i .

Now, without loss of generality, we may assume that $u_j u_j^* \in T$ for all j . Since $u_j u_j^* \cap u_k u_k^* = \{1\}$ for all distinct j, k , there exist $i \in \{1, 2, 3, 4\}$ and $j \in \{1, 2, 3\}$, such that

$$s_i s_i^* = u_j u_j^* \neq s_k s_k^*, \quad k \neq i.$$

Taking into account that every element in S_p has a loop, we conclude that s_i is adjacent in the graph $\mathfrak{X}$ with each of the s_i with $i \in \{1, 2, 3, 4\}$. Thus, the scheme $\mathcal{X}$ is saturated.

To prove that the scheme $\mathcal{X}$ is Desarguesian, let $x, y, z \in S_k$ and $r, s \in S$ be basis relations of $\mathcal{X}$ that form an initial configuration. We have to verify that r and s are linked with respect to (x, y, z) . If

$$\langle x x^*, y y^*, z z^* \rangle \simeq E_{p^3},$$

then this follows from Corollary 4.4 (see the statement (L1)). Assume that this condition is not satisfied, i.e., the group on the left side is isomorphic to E_p or E_{p^2} . Then, by the assumption of the theorem, there exists $q \in S_p$, such that

$$q q^* \cap \langle x x^*, y y^*, z z^* \rangle = \{1\}.$$

It follows that condition (17) is satisfied. Thus, the required statement follows from Corollary 4.4 (see the statement (L2)). $\square$

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