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On some recent progress in the classification of $(P$ and Q)-polynomial association schemes

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Abstract The problem of classification of $(P$ and Q)-polynomial association schemes, as a finite analogue of E. Cartan’s classification of compact symmetric spaces, was posed in the monograph “Association schemes” by E. Bannai and T. Ito in the early 1980s. In this expository paper, we report on some recent results towards its solution.

1 Introduction

We consider finite graphs without loops or multiple edges. Let Γ be such a graph with vertex set V , and ∂ denote the path-length distance function on Γ . Recall that an *automorphism* of Γ is a bijection $V \mapsto V$ that preserves the edges of Γ . Clearly, the automorphisms of Γ form a group denoted by $\text{Aut}(\Gamma)$ under composition of bijections. In algebraic combinatorics, we are mainly interested in “nice” graphs, for example, in the sense of richness of their groups of automorphisms. From this point of view, *distance-transitive* graphs form one of the most interesting classes of graphs. Such a graph Γ can be defined as follows: for any vertices $x_1, x_2, y_1, y_2 \in V$ with $\partial(x_1, x_2) = \partial(y_1, y_2)$ there exists an automorphism $\gamma \in \text{Aut}(\Gamma)$ such that $\gamma(x_1) = y_1$ and $\gamma(x_2) = y_2$.

The skeletons of the platonic solids provide natural examples of distance-transitive graphs. Less straightforward distance-transitive graphs arise from the so-called multiplicity-free permutation representations of almost simple groups or affine groups. Much attention has been paid to the problem of classification of distance-transitive graphs, and we refer the reader to [2–4, 7] for its current state and recent progress. We only mention that most of distance-transitive graphs are related to classical algebraic objects such as dual polar spaces and forms over finite fields.

Let Γ be a distance-transitive graph of *diameter* $D = \max\{\partial(x, y) \mid x, y \in V\}$. For every $i, 0 \leq i \leq D$, define the *distance- i adjacency* matrix A_i of Γ as a square matrix from $\mathbb{R}^{V \times V}$ with entries given by

$$(A_i)_{x,y} := \begin{cases} 1 & \text{if } \partial(x, y) = i, \\ 0 & \text{if } \partial(x, y) \neq i. \end{cases}$$

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Then for all $0 \leq i \leq D$,

$$A_i = A_i^\top, \quad (1.1)$$

$$A_0 = I, \quad A_0 + A_1 + \cdots + A_D = J, \quad (1.2)$$

where J is the square all-one matrix and, moreover, for all $0 \leq i, j \leq D$, the following equations hold:

$$A_i A_j = \sum_{h=0}^D p_{ij}^h A_h \quad (1.3)$$

with some non-negative integer coefficients p_{ij}^h . To see Eq. (1.3), we count $(A_i A_j)_{x,y}$ with $\partial(x, y) = h$, which equals the number of vertices z such that $\partial(x, z) = i$ and $\partial(y, z) = j$. Since Γ is distance transitive, this number does not depend on the particular choice of vertices x, y with $\partial(x, y) = h$ and, therefore, we denote it by p_{ij}^h .

The notion of distance-transitive graphs can be weakened by taking Eqs. (1.1), (1.2), and (1.3) as the defining properties. A graph whose distance matrices satisfy Eqs. (1.1), (1.2), and (1.3) is said to be *distance regular*. Note that a distance-regular graph may have, in principle, even trivial group of automorphisms.

We further axiomatize these properties to define our main object of interest, a *symmetric association scheme* $\mathfrak{X}$ (of D classes) as a set of $(0, 1)$ -matrices $A_0, A_1, \dots, A_D$ satisfying Eqs. (1.1), (1.2), and (1.3). Alternatively, one can think of $\mathfrak{X}$ as a finite set V together with binary symmetric relations $R_0, R_1, \dots, R_D$ which partition $V \times V$. Moreover, we assume that $R_0 = \text{id}$ and, for every $(x, y) \in R_h$, the number of elements $z \in V$ such that $(x, z) \in R_i$ and $(y, z) \in R_j$ does not depend on the particular choice of pair $(x, y) \in R_h$. The corresponding numbers p_{ij}^h are called the *intersection numbers* of $\mathfrak{X}$.

As we have just seen, distance-transitive and, more generally, distance-regular graphs give rise to association schemes. In this case, one can observe the following remarkable fact [6], for every $0 \leq i \leq D$, there exists a polynomial v_i of degree i such that

$$A_i = v_i(A_1).$$

This property is called *P-polynomiality* of an association scheme, and an association scheme having this property is said to be *P-polynomial* (or *metric*). Moreover, the converse statement holds [6, Chapter 3]: a *P-polynomial* association scheme gives rise to a distance-regular graph.

The concept of *P-polynomial* association schemes was introduced by Delsarte in his seminal work [18] as an algebraic definition of association schemes generated by distance-regular graphs. His motivation was to use *P-polynomial* association schemes as an algebraic framework for the theory of error-correcting codes to prove various bounds on the parameters of codes in a uniform way.

Delsarte further introduced the concept of *Q-polynomial* (also known as *cometric*) association schemes by dualizing that of *P-polynomial* association schemes (see the precise definition in Section 2), and showed that such schemes provide an algebraic framework for the theory of combinatorial designs. This provided a uniform point of view on coding theory and design theory, for which *P*- and *Q*-polynomial association schemes serve as underlying spaces, respectively, and led to the so-called Delsarte theory, a linear-algebraic approach to their problems.

It was further observed in the introduction of the monograph by Bannai and Ito [6] that these concepts remind the situation in Riemannian geometry. Let $G = G(\mathcal{M})$ be the group of isometries of a metric space $\mathcal{M}$ with a metric μ , i.e., $\mu(gx_1, gx_2) = \mu(x_1, x_2)$ for all $x_1, x_2 \in \mathcal{M}$ and $g \in G$. The space $\mathcal{M}$ is called *two-point homogeneous*, if for any two pairs of points x_1, x_2 and y_1, y_2 with $\mu(x_1, x_2) = \mu(y_1, y_2)$ there exists an isometry $g \in G$, such that $y_1 = gx_1, y_2 = gx_2$. One can immediately see that, from this point of view, (connected) distance-transitive graphs or, more generally, (primitive) *P*-polynomial association schemes can be seen as a finite-analogue of (compact) two-point homogeneous spaces.

Further, a (Riemannian) *symmetric space* is a Riemannian manifold S with the property that the geodesic reflection at any point is an isometry of S , i.e., for any point $x \in S$, there is some $\sigma_x \in G$, where G is the isometry group of S with the properties $\sigma_x(x) = x, (d\sigma_x)x = -\text{id}$. While a combinatorial (or finite) analogue of this notion is not so obvious, it was observed by Bannai and Ito [6] that the theory of designs in *Q*-polynomial association schemes goes in parallel with that of combinatorial configurations of compact symmetric spaces of rank one (for example, spherical designs). From this point of view, *Q*-polynomial association schemes can be seen as a finite analogue of compact symmetric spaces of rank one.

Compact symmetric spaces of rank one were classified by Cartan [12], while Wang showed [53] that a compact symmetric space of rank one is a compact two-point homogeneous space and vice versa. These fundamental results from Riemannian geometry lead to the following conjecture in [6]: “*Primitive P -polynomial association schemes of sufficiently large diameter are Q -polynomial, and vice versa*”, and to the problem of classification of primitive (both P and Q)-polynomial association schemes, which can be seen as a finite-analogue of Cartan’s classification. The list of currently known primitive (both P and Q)-polynomial association schemes can be found in [5, 17]. See also [38] for the non-symmetric analogue of this problem.

As suggested in [6], the classification problem naturally falls into two steps:

- (1) Show that (both P and Q)-polynomial association schemes with diameter sufficiently large have the same parameters (i.e., the intersection numbers) as those in the list of known examples.
- (2) Characterize (both P and Q)-polynomial association schemes in the list by their parameters.

Part (1) turned out to be related to the theory of orthogonal polynomials ([6, 31]), and further to the theory of tridiagonal pairs and the representation theory of quantum affine groups as it was elaborated by Ito and Terwilliger in a series of papers [28]. We recall that Bannai & Ito’s interpretation of the Leonard theorem [6, Theorem 5.1] says that the parameters of a (both P and Q)-polynomial association scheme should have at least one of seven possible types denoted by 1, 1A, 2, 2A, 2B, 2C, or 3, and type 1 remains the most complicated case for the classification [48].

As for Part (2), there have appeared many works in the 1980s and the 1990s done by researchers from all over the world, many but not all schemes were characterized by their parameters: the Hamming schemes [20], the Johnson schemes [41, 50], and their quotients [10, 22, 33, 36, 37], the schemes of Hermitian forms and Hermitian dual polar spaces of unitary type (in even dimension) [29, 30], the association schemes of bilinear forms [15, 26, 35] (some case left open). In this paper, we focus on Part (2) of the classification problem in regard to the Grassmann schemes and the association schemes of bilinear forms.

2 Basic theory of association schemes

In this section, we recall some basic facts about association schemes. For more comprehensive background on distance-regular graphs and association schemes, we refer the reader to [6, 8, 17].

Let Γ be a connected graph. The distance $\partial(x, y) := \partial_\Gamma(x, y)$ between any two vertices x, y of Γ is the length of a shortest path connecting x and y in Γ . For a subset X of the vertex set of Γ , we will also write X for the subgraph of Γ induced by X . For a vertex $x \in \Gamma$, define $\Gamma_i(x)$ to be the set of vertices that are at distance precisely i from x ($0 \leq i \leq D$), where $D := \max\{\partial(x, y) \mid x, y \in \Gamma\}$ is the *diameter* of Γ . In addition, define $\Gamma_{-1}(x) = \Gamma_{D+1}(x) = \emptyset$. The subgraph induced by $\Gamma_1(x)$ is called the *neighborhood* or the *local graph* of a vertex x . We often write $\Gamma(x)$ instead of $\Gamma_1(x)$ for short, and $x \sim_\Gamma y$ or simply $x \sim y$ if two vertices x and y are adjacent in Γ . A graph Γ is *regular* with *valency* k if the local graph $\Gamma(x)$ contains precisely k vertices for all $x \in \Gamma$.

For a set $\{x_1, x_2, \dots, x_s\}$ of vertices of Γ , let $\Gamma(x_1, x_2, \dots, x_s)$ denote $\bigcap_{i=1}^s \Gamma(x_i)$. In particular, for a pair x, y of vertices of Γ with $\partial(x, y) = 2$, the subgraph induced on $\Gamma(x, y)$ is commonly known as the μ -*graph* (of x and y).

The *eigenvalues* of a graph Γ are the eigenvalues of its adjacency matrix $A := A(\Gamma)$. If for an eigenvalue η of Γ , its eigenspace contains a vector orthogonal to the all-one vector, we say that η is *non-principal*. If Γ is regular with valency k , then all its eigenvalues are non-principal unless the graph is connected and then the only eigenvalue that is principal is its valency k . For a graph Γ , we write its spectrum in the form $[\theta_0]^{m_0}, [\theta_1]^{m_1}, \dots, [\theta_d]^{m_d}$, where $\theta_0 > \theta_1 > \dots > \theta_d$ are all distinct eigenvalues of Γ , and $m_0, m_1, \dots, m_d$ are their respective multiplicities.

Recall that an s -*clique* of a graph is its complete subgraph with exactly s vertices. We call an s -clique simply a *clique* if we do not refer to its cardinality. By the $(s \times t)$ -*grid*, we mean the Cartesian product of two complete graphs on s and t vertices. We say that two graphs are *cospectral* if they have the same spectrum.

A graph Γ is the q -*clique extension* of a graph Δ , if there exists a mapping ε of the vertex set of Γ onto the vertex set of Δ such that $|\varepsilon^{-1}(x)| = q$ for every $x \in \Delta$ and two distinct vertices $u, w \in \Gamma$ are adjacent whenever their images $\varepsilon(u)$ and $\varepsilon(w)$ are either equal or adjacent in Δ .

Let X be a finite set of vertices and $\{R_0, R_1, \dots, R_D\}$ be a set of non-empty symmetric binary relations on X which partition $X \times X$. Let A_i denote the adjacency matrix of the graph with vertex set X and edge set R_i ($0 \leq i \leq D$). The pair $(X, \{R_i\}_{i=0}^D)$ is called a *symmetric association scheme* of D classes if the following conditions hold:

- (1) $A_0 = I_{|X|}$, which is the identity matrix of size $|X|$,
- (2) $\sum_{i=0}^D A_i = J_{|X|}$, the square all-one matrix of size $|X|$,
- (3) $A_i^\top = A_i$ ($1 \leq i \leq D$),
- (4) $A_i A_j = \sum_{h=0}^D p_{ij}^h A_h$, where p_{ij}^h are nonnegative integers ($0 \leq i, j \leq D$), the intersection numbers of the scheme.

The matrices $\{A_i\}_{i=0}^D$ generate the matrix algebra $\mathcal{A}$ over $\mathbb{R}$. Since $\mathcal{A}$ is commutative and semisimple, there exists a unique basis of $\mathcal{A}$ consisting of *primitive idempotents* $E_0 = \frac{1}{|X|} J_{|X|}$, $E_1, \dots, E_D$, which, in fact, are projectors onto the common eigenspaces of $A_0, \dots, A_D$. One can see that the algebra $\mathcal{A}$ is closed under the entry-wise multiplication denoted by $\circ$. This property allows us to define the *Krein parameters* q_{ij}^k ($0 \leq i, j, k \leq D$) by

$$E_i \circ E_j = \frac{1}{|X|} \sum_{k=0}^D q_{ij}^k E_k. \quad (2.1)$$

Note that the Krein parameters are nonnegative real numbers (see [18, Lemma 2.4]). Since both $\{A_0, A_1, \dots, A_D\}$ and $\{E_0, E_1, \dots, E_D\}$ form bases of $\mathcal{A}$, there exist matrices $P = (P_{ij})_{i,j=0}^D$ and $Q = (Q_{ij})_{i,j=0}^D$ with entries given by

$$A_i = \sum_{j=0}^D P_{ji} E_j \quad \text{and} \quad E_i = \frac{1}{|X|} \sum_{j=0}^D Q_{ji} A_j. \quad (2.2)$$

The matrices P and Q are called the *first* and *second eigenmatrix* of the scheme $(X, \{R_i\}_{i=0}^D)$, respectively.

For an association scheme $(X, \{R_i\}_{i=0}^D)$, an ordering of $A_1, \dots, A_D$ such that for each i ($0 \leq i \leq D$) there exists a polynomial $v_i(x)$ of degree i with $A_i = v_i(A_1)$ is called a *P-polynomial ordering* of relations. An association scheme is said to be *P-polynomial* if it admits a *P-polynomial ordering* of relations. The notion of an association scheme together with a *P-polynomial ordering* of its relations is equivalent to that of a *distance-regular graph*. Such a graph Γ has adjacency matrix A_1 , and A_i ($0 \leq i \leq D$) defines the adjacency matrix of its distance- i graph (i.e., $(x, y) \in R_i$ whenever vertices x and y are at distance i in Γ), and the number of classes D equals the diameter of the graph. It is known that an ordering of relations is *P-polynomial* if and only if $p_{ij}^k = 0$ holds whenever the indices i, j, k do not satisfy the *triangle inequality*, i.e., when $|i - j| < k$ or $i + j > k$. For a *P-polynomial ordering* of relations of an association scheme, denote $a_i = p_{1,i}^1$, $b_i = p_{1,i+1}^1$, and $c_i = p_{1,i-1}^1$. These intersection numbers are usually gathered to form the *intersection array* $\{b_0, b_1, \dots, b_{D-1}; c_1, c_2, \dots, c_D\}$, as the remaining intersection numbers can be computed from them. Note that $a_i = b_0 - b_i - c_i$ for all i , where we set $b_D = c_0 = 0$. For an association scheme with a *P-polynomial ordering* of relations, an ordering $E_1, \dots, E_D$ is called the *natural ordering* of eigenspaces if the numbers $(P_{i1})_{i=0}^D$ (which are exactly the distinct eigenvalues of A_1) form a decreasing sequence.

Dually, for an association scheme $(X, \{R_i\}_{i=0}^D)$, an ordering of $E_1, \dots, E_D$ such that for each i ($0 \leq i \leq D$), there exists a polynomial $v_i^*(x)$ of degree i with $E_i = v_i^*(E_1)$, is called a *Q-polynomial ordering* of eigenspaces. An association scheme is said to be *Q-polynomial* if it admits a *Q-polynomial ordering* of its eigenspaces. Similarly, it is known that an ordering of eigenspaces is *Q-polynomial* if and only if $q_{ij}^k = 0$ holds whenever the indices i, j, k do not satisfy the triangle inequality. For a *Q-polynomial ordering* of the eigenspaces, denote $a_i^* = q_{1,i}^1$, $b_i^* = q_{1,i+1}^1$, and $c_i^* = q_{1,i-1}^1$. Again, these Krein parameters are gathered in the *Krein array* $\{b_0^*, b_1^*, \dots, b_{D-1}^*; c_1^*, c_2^*, \dots, c_D^*\}$ as the remaining Krein parameters can be computed from them. For an association scheme with a *Q-polynomial ordering* of eigenspaces, an ordering $A_1, \dots, A_D$ is called the *natural ordering* of relations if the numbers $(Q_{i1})_{i=0}^D$ form a decreasing sequence.

An association scheme is *primitive* if all of $A_1, \dots, A_D$ are adjacency matrices of connected graphs. A distance-regular graph is imprimitive precisely when it is a cycle of composite length, an antipodal graph, or a bipartite graph (perhaps more than one of these), see [8, Theorem 4.2.1]. The last two properties can be recognised from the intersection array as $b_i = c_{D-i}$ ($0 \leq i \leq D$, $i \neq \lfloor D/2 \rfloor$) and $a_i = 0$ ($0 \leq i \leq D$), respectively. We define dual properties of a *Q-polynomial association scheme*—it is *Q-antipodal* if $b_i^* = c_{D-i}^*$ ($0 \leq i \leq D$, $i \neq \lfloor D/2 \rfloor$), and *Q-bipartite* if $a_i^* = 0$ ($0 \leq i \leq D$). All *known* imprimitive *Q-polynomial association schemes* are schemes of cycles of composite length, *Q-antipodal* or *Q-bipartite* (again, possibly

more than one of these). There might exist imprimitive Q -polynomial association schemes of two more types, as shown by the classification theorem by Suzuki [45]. A (both P and Q)-polynomial association scheme is Q -antipodal if and only if it is bipartite, and is Q -bipartite if and only if it is antipodal.

Any association scheme with two classes is always (both P and Q)-polynomial, for any of the two orderings of relations and eigenspaces. The graph with adjacency matrix A_1 of such a scheme is said to be *strongly regular* with parameters (n, k, λ, μ) , where $n = |X|$ is the number of vertices, $k = p_{11}^0$ is the valency of each vertex, and each two distinct vertices have precisely $\lambda = p_{11}^1$ common neighbours if they are adjacent, and $\mu = p_{11}^2$ common neighbours if they are not adjacent.

For a triple of vertices $u, v, w \in X$ and integers i, j, k ($0 \leq i, j, k \leq D$), we denote by $\begin{bmatrix} u & v & w \\ i & j & k \end{bmatrix}$ (or simply $[i \ j \ k]$ when it is clear which triple (u, v, w) we mean) the number of vertices $x \in X$ such that $(u, x) \in R_i$, $(v, x) \in R_j$ and $(w, x) \in R_k$. These numbers are called the *triple intersection numbers*.

Unlike the intersection numbers of association schemes, their triple intersection numbers depend, in general, on the choice of (u, v, w) . For a fixed triple (u, v, w) , we may write down a system of linear Diophantine equations with triple intersection numbers as variables, which relate them to the intersection numbers, cf. [27]. Moreover, it is known that vanishing of some of the Krein parameters often leads to non-trivial additional equations with respect to triple intersection numbers, see, for example, [8, Theorem 2.3.2], [11, 13, 27, 51, 52] and [17, Section 6.3]. Thus, this may provide some extra information on a possible combinatorial structure of an association scheme. Unfortunately, in general, these equations are rather complicated, especially, for a family of schemes with unbounded number of classes D , as the numbers of equations and variables depend on D .

Many of the Krein parameters vanish when a distance-regular graph Γ is Q -polynomial. This suggests that the triple intersection numbers should play an important role in the problem of classification of Q -polynomial distance-regular graphs. For example, Ivanov and Shpectorov [30] proved that a distance-regular graph Γ with the same intersection numbers as the graph $Her(n, q)$ of Hermitian $n \times n$ -forms over $\mathbb{F}_q$ is indeed isomorphic to $Her(n, q)$ if $[2, 1, 1] = 0$ holds for any three pairwise adjacent vertices x, y, z of Γ . Terwilliger [49, Corollary 2.13] completed the characterization of $Her(n, q)$ by its intersection numbers by using that $[i, i-1, i-1] = 0$ with $2 \leq i \leq D$ holds for any three pairwise adjacent vertices x, y, z of a distance-regular graph with classical parameters (D, b, α, β) with $b < -1$.

3 Geometric approach for the schemes of bilinear forms and the Grassmann schemes

There were many attempts to characterize the schemes of bilinear forms and the Grassmann schemes by their parameters. The strongest results were obtained by Metsch in [32, 35], where he showed that $J_q(n, D)$, $n \geq 2D \geq 6$, is determined by its parameters if $n \neq 2D$, $n \neq 2D+1$, ($n \neq 2D+2$ if $q \in \{2, 3\}$), or ($n \neq 2D+3$ if $q = 2$), and $Bil_q(d \times e)$, $e \geq d \geq 3$, is determined by its parameters if $e > d + \max(2, 5-q)$.

These results by Metsch rely on a characterization of some geometries (incidence structures) corresponding to these schemes. We first recall that an *incidence structure* is a triple (P, L, I) where P and L are sets (whose elements are called *points* and *lines*, respectively) and $I \subseteq P \times L$ is the *incidence relation*. We always assume that every line is incident with at least two points. An incidence structure is said to be *semilinear*, or a *partial linear space*, if there exists at most one line through any two points. The *point graph* of the incidence structure (P, L, I) is a graph defined on P as the vertex set, with two points being adjacent if they are collinear, i.e., they belong to the same line.

Semilinear incidence structures can be naturally derived from the bilinear forms graph $Bil_q(d \times e)$ or the Grassmann graph $J_q(n, D)$ by taking their vertices as points and maximum cliques as lines. Recall that the bilinear forms graph $Bil_q(d \times e)$ [8, Chapter 9.5.A] can be defined on a vector space V of dimension $e + d$ over $\mathbb{F}_q$ as follows. Let W be a fixed e -subspace of V . For $i \in \{d-1, d\}$, define

$$\mathcal{A}_i = \{U \subseteq V \mid \dim(U) = i, \dim(U \cap W) = 0\},$$

then $(\mathcal{A}_d, \mathcal{A}_{d-1}, \supset)$ is a semilinear incidence structure called the (e, q, d) -attenuated space, while its point graph is isomorphic to $Bil_q(d \times e)$. In other words, the vertices of $Bil_q(d \times e)$ are the subspaces of $\mathcal{A}_d$, with two such subspaces adjacent if and only if their intersection has dimension $d-1$. From this point of view, $Bil_q(d \times e)$ is a subgraph of $J_q(d+e, d)$.

It is easily seen that $Bil_q(d \times e)$ has two types of maximal cliques. The maximal cliques of the first type are the collections of subspaces of $\mathcal{A}_d$ containing a fixed subspace of dimension $d-1$, and each of them contains

$\begin{bmatrix} e+1 \\ 1 \end{bmatrix}_q - \begin{bmatrix} e \\ 1 \end{bmatrix}_q = q^e$ vertices, while the maximal cliques of the other type are the collections of subspaces of $\mathcal{A}_d$ contained in a fixed subspace of dimension $d+1$, and each of them contains $\begin{bmatrix} d+1 \\ 1 \end{bmatrix}_q - \begin{bmatrix} d \\ 1 \end{bmatrix}_q = q^d$ vertices, where $\begin{bmatrix} n \\ m \end{bmatrix}_q$ denotes the q -ary Gaussian binomial coefficient. Note that the maximal cliques of the first type correspond to the lines of the semilinear incidence structure $(\mathcal{A}_d, \mathcal{A}_{d-1}, \supset)$.

Similarly, define $\mathcal{G}_i$ to be the set of i -dimensional subspaces of an n -dimensional vector space V over $\mathbb{F}_q$. Then $J_q(n, D)$ has two families of maximal cliques corresponding to the sets $\mathcal{G}_{D-1}$ and $\mathcal{G}_{D+1}$: the maximal cliques of the former family are the collections of D -subspaces of V containing a fixed subspace of dimension $D-1$, and each of them is of size $(q^{n-(D-1)} - 1)/(q - 1)$, while the maximal cliques of the latter family are the collections of D -subspaces of V contained in a fixed subspace of dimension $D+1$, and each of them is of size $(q^{D+1} - 1)/(q - 1)$. Any edge of $J_q(n, D)$ belongs to a unique clique of each family, thus one can see that $(\mathcal{G}_D, \mathcal{G}_{D-1}, \supset)$ is a partial linear space with the point graph isomorphic to $J_q(n, D)$.

Suppose now that a graph Γ is distance regular with the same intersection array as $Bil_q(d \times e)$ or $J_q(n, D)$. A key idea of the works by Huang [26] and Metsch [32, 35] was as follows. Under certain conditions on e, d or n, D and q , it is possible to show that every edge of Γ is contained in a unique “grand” clique (sufficiently large but not necessarily maximum). Hence, $(V(\Gamma), \mathcal{L}, \in)$ is a semilinear incidence structure, where $\mathcal{L}$ is the set of all grand cliques of Γ , and Γ is its point graph. In case of $Bil_q(d \times e)$, to show the existence of grand cliques, Huang used the so-called Bose–Laskar argument, which was valid for $e \geq 2d \geq 6$, and Metsch applied its improved version [34] (also in case of $J_q(n, D)$), which was valid under weaker assumptions on e, q , and d .

The partial linear spaces constructed in this way (in fact, one also needs to show that they satisfy some additional regularity properties) were earlier characterized in finite geometry. In case of $Bil_q(d \times e)$, $(V(\Gamma), \mathcal{L}, \in)$ is a so-called d -net (see [26]), and, the result by Sprague [44] shows, for an integer $d \geq 3$, every finite d -net is the (e, q, d) -attenuated space for some prime power q and positive integer e and, therefore, Γ is isomorphic to $Bil_q(d \times e)$. In case of $J_q(n, D)$, the result follows by Chaudhuri and Sprague [42], as they showed that corresponding incidence structure must be isomorphic to $(\mathcal{G}_D, \mathcal{G}_{D-1}, \supset)$.

For the cases remained open after the Metsch results, it seems that such a geometric approach cannot be applied. Indeed, for example, when $e = d$ or $n = 2D$, the maximal cliques of both families (in $Bil_q(d \times e)$ or $J_q(n, D)$, respectively) have the same size. Therefore, even if one can show that Γ contains such cliques, every edge is contained in *two* grand cliques. Thus, to construct a partial linear space, one has to prove that it is possible to pick just one family of grand cliques that cover all points (when $e \neq d$ or $n \neq 2D$, respectively, we can distinguish between families of maximal cliques by their sizes). Unfortunately, it is not possible in general; for example, the quotient of the Johnson graph $J(2d, d)$ has two families of maximal cliques of the same size, but this graph is not the point graph of any semilinear incidence structure, see [14, Proposition 2.7, Remark 2.8]. Furthermore, Van Dam and Koolen [16] discovered a new family of distance-regular graphs, the so-called *twisted Grassmann* graphs, which have the same intersection numbers as $J_q(2D+1, D)$ for any prime power q and which are not point graphs of any partial linear space. The existence of such graphs is very intriguing and shows that geometric approach likely fails even when $n \neq 2D$, so the classification problem of the remaining open cases is very challenging.

4 Recent results

We describe an approach exploiting the Q -polynomiality of the bilinear forms graphs and the Grassmann graphs. Suppose that Γ is a Q -polynomial distance-regular graph with diameter $D \geq 3$. In 1993, Terwilliger (see ‘Lecture note on Terwilliger algebra’ edited by Suzuki, [46]) showed that, for $i = 2, 3, \dots, D-1$, there exists a polynomial $T_i(\lambda) \in \mathbb{C}[\lambda]$ of degree 4 such that for any i , any vertex $x \in \Gamma$, and any non-principal eigenvalue η of the local graph $\Gamma(x)$, one has

$$T_i(\eta) \geq 0. \quad (4.1)$$

We call $T_i(\lambda)$ the *Terwilliger polynomial* of Γ . In [22], the authors gave an explicit formula for this polynomial, and applied it to complete the classification of pseudo-partition graphs.

Since the Terwilliger polynomial depends only on the intersection array of Γ and its Q -polynomial ordering (recall that the property “being Q -polynomial” is determined by the intersection array), any two Q -polynomial

distance-regular graphs with the same intersection array and Q -polynomial ordering have the same Terwilliger polynomials.

Our strategy is as follows. Suppose that a graph Γ is distance regular with the same intersection array as $Bil_q(d \times e)$ or $J_q(n, D)$. In the first step, using Condition (4.1) on the Terwilliger polynomial, we can obtain some information about the eigenvalues of the local graphs in Γ .

Proposition 4.1 [23] *Let Γ be a distance-regular graph with the same intersection array as $Bil_q(d \times e)$, $e \geq d \geq 3$. Let η be a non-principal eigenvalue of the local graph of a vertex of Γ . Then η satisfies*

$$-q - 1 \leq \eta \leq -1, \quad \text{or} \quad q^d - q - 1 \leq \eta \leq q^e - q - 1.$$

In the cases, we will consider below (namely when $d = e$ and $q = 2$ or $n = 2D$, respectively) this information suffices to determine the spectrum of the local graph exactly.

Lemma 4.2 [23] *Let Γ be a distance-regular graph with the same intersection array as $Bil_2(d \times d)$, $d \geq 3$. Then the local graph of a vertex of Γ has spectrum*

$$[2(2^d - 2)]^1, [2^d - 3]^{2(2^d - 2)}, [-2]^{(2^d - 2)^2},$$

and Δ is the $(2^d - 1) \times (2^d - 1)$ -grid.

Note that the assumption $q = 2$ is crucial here, as we use the fact that all roots of the characteristic polynomial of a local graph of Γ lie in a short interval (according to Proposition 4.1). Already for $q = 3$ we obtain too many candidates for the characteristic polynomials of the local graphs.

Recall that a local graph in the Grassmann graph $J_q(2D, D)$ is the q -clique extension of the $(\begin{bmatrix} D \\ 1 \end{bmatrix}_q \times \begin{bmatrix} D \\ 1 \end{bmatrix}_q)$ -grid, which has spectrum:

$$[\hat{\theta}_1]^{g(\hat{\theta}_1)}, [-1]^{g(-1)}, [\hat{\theta}_D]^{g(\hat{\theta}_D)}, [a_1]^1,$$

where the valency $a_1 := q\left(2\begin{bmatrix} D \\ 1 \end{bmatrix}_q - 1\right) - 1$, and

$$\begin{aligned} \hat{\theta}_1 &:= -q - 1, & \hat{\theta}_D &:= q\left(\begin{bmatrix} D \\ 1 \end{bmatrix}_q - 1\right) - 1, \\ g(\hat{\theta}_1) &:= \left(\begin{bmatrix} D \\ 1 \end{bmatrix}_q - 1\right)^2, & g(\hat{\theta}_D) &:= 2\left(\begin{bmatrix} D \\ 1 \end{bmatrix}_q - 1\right). \\ g(-1) &:= (q - 1)\begin{bmatrix} D \\ 1 \end{bmatrix}_q^2. \end{aligned}$$

Proposition 4.3 [24] *Let Γ be a distance-regular graph with the same intersection array as the Grassmann graph $J_q(2D, D)$ with $D \geq 3$. Then the local graph $\Delta = \Gamma(x)$ of any vertex $x \in \Gamma$ is cospectral to the q -clique extension of the $(\begin{bmatrix} D \\ 1 \end{bmatrix}_q \times \begin{bmatrix} D \\ 1 \end{bmatrix}_q)$ -grid.*

In the second step, we aim to determine the structure of the local graph from its spectrum. It is the problem of spectral characterization of graphs, which appears to be hard in general [9, Chapter 14]. Fortunately, it follows immediately in case of $Bil_2(d \times d)$, as the corresponding local graphs have only three distinct eigenvalues, and thus they are strongly regular with the same parameters as the $(2^d - 1) \times (2^d - 1)$ -grid. The $(m \times m)$ -grid is uniquely determined by its parameters whenever $m \neq 4$ (see [43]).

The case of $J_q(2D, D)$ is trickier, as the corresponding local graphs are clique extensions of strongly regular graphs. To recover the local structure of a graph Γ with the same intersection numbers as the Grassmann graph $J_q(2D, D)$, we use another consequence of the Terwilliger algebra, which gives equations relating certain triple intersection numbers. More precisely, we can show the following.

Proposition 4.4 [24] *Let Γ be a distance-regular graph with the same intersection array as the Grassmann graph $J_q(2D, D)$ with $D \geq 4$. The following holds for the local graph $\Delta = \Gamma(x)$ of any vertex $x \in \Gamma$.*

- (1) $|\Delta(y, z)| \equiv q - 2 \pmod{\begin{bmatrix} D-1 \\ 1 \end{bmatrix}_q}$ for any pair y, z of vertices of Δ with $y \sim z$.
- (2) $|\Delta(y, z)| = 2q$ for any pair y, z of distinct vertices of Δ with $y \approx z$.

Combining these restrictions on triple intersection numbers together with the spectrum of the local graphs, we proceed by proving that the local graphs of Γ are indeed isomorphic to those of $J_q(2D, D)$ provided that the diameter D is not too small, namely $D \geq \chi(q)$ holds, where for a natural number $q \geq 2$, $\chi(q)$ is defined by

$$\chi(q) = \begin{cases} 9 & \text{if } q = 2, \\ 8 & \text{if } q = 3, \\ 7 & \text{if } q \in \{4, 5, 6\}, \\ 6 & \text{if } q \geq 7. \end{cases} \quad (4.2)$$

The proof combines some tricks from algebraic graph theory (see [25]) with a counting argument to construct large cliques in the local graph.

In the third step, assuming that Γ is a distance-regular graph with the same intersection array as $Bil_2(d \times d)$ or $J_q(2D, D)$, we know that Γ contains two families of maximal cliques of the same size. By the remark from the preceding section, we cannot immediately derive a semilinear incidence structure from Γ .

In case of $Bil_2(d \times d)$, we apply a theorem by Munemasa and Shpectorov [39], and prove a more general result (Theorem 4.5), which requires distance regularity of Γ up to distance 2 only. In doing so, we first show that certain semi-partial geometries can be derived from Γ , and this yields that $m = 2^d - 1$, $n = 2^e - 1$ for some natural numbers d and e , and Γ has induced subgraphs isomorphic to the graphs $Bil_2(2 \times d)$ and $Bil_2(2 \times e)$. We then have an isomorphism between the local graphs of Γ and the local graphs of $Bil_2(d \times e)$. The Munemasa–Shpectorov theorem shows that an isomorphism between the local graphs can be extended to a covering map, i.e., Γ is covered by the bilinear forms graphs $Bil_2(d \times e)$.

Theorem 4.5 [23] *Suppose that Γ is a graph with diameter $D \geq 2$ and with the following intersection numbers well defined:*

$$b_0 = nm, \quad b_1 = (n-1)(m-1), \quad b_2 = (n-3)(m-3), \quad \text{and } c_2 = 6$$

for some integers $n \geq 3$, $m \geq 3$, and such that, for every vertex $x \in \Gamma$, its local graph $\Gamma(x)$ is the $(n \times m)$ -grid. Then there exist natural numbers d and e such that $\min(m, n) = 2^d - 1$, $\max(m, n) = 2^e - 1$, and Γ is covered by the graph of bilinear $(d \times e)$ -forms over $\mathbb{F}_2$.

We recall that the problem of characterization of all locally grid graphs is well known and is rather difficult, see [1]. In this context, Theorem 4.5 is of independent interest. All together, we obtain the following result.

Theorem 4.6 [23] *Suppose that Γ is a distance-regular graph with the same intersection array as $Bil_2(d \times d)$, $d \geq 3$. Then Γ is isomorphic to $Bil_2(d \times d)$.*

In case of $J_q(2D, D)$, we again use a geometric approach; however, we consider a characterization of another partial linear space derived from the Grassmann graph $J_q(n, D)$ whose points are the vertices and whose lines are the singular lines. Here by a *singular line* we mean the non-trivial intersection of two cliques from different families. A characterization of such partial linear spaces was obtained by Cooperstein, Cohen, and Numata [8, Theorem 9.3.8] as follows. Recall that an *s-coclique* of a graph is an induced subgraph on s vertices but without edges. We call an *s-coclique* simply a *coclique* if we do not refer to its cardinality.

Theorem 4.7 *Let Γ be a finite connected graph such that*

- (i) *for every pair of vertices $x, y \in \Gamma$ with $\partial(x, y) = 2$, the μ -graph of x, y is a non-degenerate grid, and*
- (ii) *if $x, y, z \in \Gamma$ induce a 3-coclique, then $\Gamma(x, y, z)$ is a coclique.*

Then Γ is either a clique, or a Johnson graph $J(n, k)$, or the quotient of the Johnson graph $J(2k, k)$ obtained by identifying a k -set with the image of its complement under the identity or an involution in $\text{Sym}(2k)$ with at least 10 fixed points (i.e., a folded Johnson graph), or a Grassmann graph $J_q(n, D)$ over a finite field $\mathbb{F}_q$.

It is not hard to see the following.

Corollary 4.8 *Let Γ be a distance-regular graph, the same intersection array as the Grassmann graph $J_q(n, D)$. Suppose that, for every vertex $x \in \Gamma$, its local graph $\Gamma(x)$ is isomorphic to the q -clique extension of the $\left(\begin{bmatrix} n-D \\ 1 \end{bmatrix}_q \times \begin{bmatrix} D \\ 1 \end{bmatrix}_q\right)$ -grid. Then q is a prime power and Γ is isomorphic to $J_q(n, D)$.*

This allows to recognize the Grassmann graphs by their local graphs. Thus, applying the corollary gives the following result.

Theorem 4.9 [24] *For a prime power q and a natural number $D \geq \chi(q)$, suppose that Γ is a distance-regular graph with the same intersection numbers as the Grassmann graph $J_q(2D, D)$. Then Γ is isomorphic to $J_q(2D, D)$.*

Thus, we settled the problem of characterization of the Grassmann graphs $J_q(n, D)$ in the case $n = 2D$ and the diameter D is large enough. For the cases $n = 2D + 2$ and $n = 2D + 3$, a characterization of the Grassmann graphs $J_2(n, D)$ will be shown in a forthcoming paper of the first author.

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