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Aluthge transform of operators on the Bergman space

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Abstract The aim of this paper is to explore some sufficient conditions for Aluthge transform of Toeplitz operators on the Bergman space to be unitary, average of unitaries and normal.

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1 Introduction

Let H, K be two separable, infinite dimensional, complex Hilbert spaces and let $\mathcal{L}(H)$ denote the algebra of all bounded linear operators on H . An arbitrary operator T in $\mathcal{L}(H)$ has a polar decomposition $T = U|T|$, where $|T| = (T^*T)^{\frac{1}{2}}$ and U is the appropriate partial isometry (with $\ker(U) = \ker(T)$ and $\ker(U^*) = \ker(T^*)$). The Aluthge transformation $\Delta(T) = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ was first introduced by Aluthge [2]. An operator T is said to be normal if $T^*T = TT^*$. The operator T is called quasinormal if $(TT^*)T = (T^*T)T$. An operator T is called hyponormal if $T^*T \geq TT^*$. An operator T is paranormal if $\|Tx\|^2 \leq \|T^2x\|\|x\|$. Here, we denote $\Delta^*(T) = (\Delta(T))^*$ for $T \in \mathcal{L}(H)$. Let $sp(T)$ denotes the spectrum of T . Define $m(T) = \inf\{\|Tx\| : \|x\| = 1\}$. An operator T is called w-hyponormal if $|\Delta(T)| \geq |T| \geq |\Delta^*(T)|$. An operator T is said to be binormal if $[|T|, |T^*|] = 0$, where $[T, S] = TS - ST$. The tensor product of $x \in H$ and $y \in K$ is a conjugate bilinear functional $x \otimes y : H \times K \rightarrow \mathbb{C}$ defined by $(x \otimes y)(u, v) = \langle x, u \rangle \langle y, v \rangle$ for every $(u, v) \in H \times K$. The collection of all (finite) sums of tensors $x_i \otimes y_i$ with $x_i \in H$ and $y_i \in K$, denoted by $H \otimes K$, is a complex linear space equipped with an inner product $\langle \cdot, \cdot \rangle : (H \otimes K) \times (H \otimes K) \rightarrow \mathbb{C}$ defined for arbitrary $\sum_{i=1}^N x_i \otimes y_i$, and $\sum_{j=1}^M w_j \otimes z_j$ in $H \otimes K$, by $\langle \sum_{i=1}^N x_i \otimes y_i, \sum_{j=1}^M w_j \otimes z_j \rangle = \sum_{i=1}^N \sum_{j=1}^M \langle x_i, w_j \rangle \langle y_i, z_j \rangle$ (the same notation for the inner products on H, K and $H \otimes K$). The tensor product on $H \otimes K$ of two operators T in $\mathcal{L}(H)$ and S in $\mathcal{L}(K)$ is the operator $T \otimes S : H \otimes K \rightarrow H \otimes K$ defined by $(T \otimes S) \sum_{i=1}^N x_i \otimes y_i = \sum_{i=1}^N Tx_i \otimes Sy_i$ for every $\sum_{i=1}^N x_i \otimes y_i \in H \otimes K$, which lies in $\mathcal{L}(H \otimes K)$. The complete inner product space $H \otimes K$ is denoted by $\hat{H \otimes K}$, which is the tensor product space of H and K . The extension of $T \otimes S$ over the Hilbert space

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$H \hat{\otimes} K$ denoted by $T \hat{\otimes} S$ is the tensor product of T and S on the tensor product space, which lies in $\mathcal{L}(H \hat{\otimes} K)$ (see [6], [12] and [17]).

Definition 1.1 If the sequence

$$\{\dots, T^3(T^3)^*, T^2(T^2)^*, TT^*, T^*T, (T^2)^*T^2, (T^3)^*T^3, (T^4)^*T^4, \dots\}$$

is commutative, the operator $T \in \mathcal{L}(H)$ is called centered operator (see [10]).

Definition 1.2 A bounded linear operator T is said to be **Fredholm** operator if and only if its range is closed, $\dim \ker T$ and $\dim \ker T^*$ are finite.

The index of an operator $T \in \mathcal{L}(H)$ is a function from $\mathcal{F}(H)$ to $\mathbb{Z}$ which is denoted by $i(T)$ and defined by $i(T) = \dim \ker T - \dim \ker T^*$, where $\mathcal{F}(H)$ is a collection of Fredholm operators on H (see [14]). An operator T is normaloid if $\|T\| = \sup\{|\langle Tx, x \rangle|; \|x\| = 1\}$.

The Bergman space $A^2(\mathbb{D})$ is the Hilbert space consisting of all analytic functions f on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ for which

$$\|f\|_{A^2(\mathbb{D})} = \left(\int_{\mathbb{D}} |f(z)|^2 dA(z) \right)^{\frac{1}{2}} < \infty,$$

where $dA(z)$ is the Lebesgue area measure on the open unit disk $\mathbb{D}$, normalized so that the measure of the disk $\mathbb{D}$ equals 1. If $h(z) = \sum_{n=0}^{\infty} a_n z^n$ and $k(z) = \sum_{n=0}^{\infty} b_n z^n$ are two functions in $A^2(\mathbb{D})$, then the inner product of h and k is given by

$$\langle h, k \rangle = \int_{\mathbb{D}} h(z) \overline{k(z)} dA(z) = \sum_{n=0}^{\infty} \frac{a_n \overline{b_n}}{n+1}.$$

For $z \in \mathbb{D}$, the Bergman reproducing kernel is the function $K_z \in A^2(\mathbb{D})$ such that $f(z) = \langle f, K_z \rangle$ for every $f \in A^2(\mathbb{D})$. The normalized reproducing kernel k_z is the function $\frac{K_z}{\|K_z\|_2}$. For any $n \geq 0$, $n \in \mathbb{Z}$, let $e_n(z) = \sqrt{n+1} z^n$. Then $\{e_n\}$ forms an orthonormal basis for $A^2(\mathbb{D})$. For $\phi \in L^\infty(\mathbb{D})$, the Toeplitz operator T_ϕ with symbol ϕ is the operator on $A^2(\mathbb{D})$ defined by $T_\phi f = P(\phi f)$; here, P is the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto $A^2(\mathbb{D})$.

Let $\overline{A^2(\mathbb{D})}$ be the space of conjugate analytic functions in $L^2(\mathbb{D}, dA)$. Then, $\overline{A^2(\mathbb{D})} = \{\overline{g} : g \in A^2(\mathbb{D})\}$ is closed in $L^2(\mathbb{D}, dA)$. Let $\phi \in L^\infty(\mathbb{D})$, the little Hankel operator $h_\phi : A^2(\mathbb{D}) \rightarrow \overline{A^2(\mathbb{D})}$ be defined by $h_\phi f = \overline{P}(\phi f)$, $f \in A^2(\mathbb{D})$ where $\overline{P}$ is the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto $\overline{A^2(\mathbb{D})}$. There are also numerous equivalent ways of defining little Hankel operators on $A^2(\mathbb{D})$. For illustration, define the map $S_\phi : A^2(\mathbb{D}) \rightarrow A^2(\mathbb{D})$ by $S_\phi f = PJ(\phi f)$, where J is self-adjoint, unitary mapping from $L^2(\mathbb{D}, dA)$ into itself given by $Jh(z) = h(\overline{z})$. Observe that $JS_\phi = h_\phi$. So S_ϕ is unitarily equivalent to h_ϕ .

Let $\text{Aut}(\mathbb{D})$ be the Lie group of all automorphisms (biholomorphic mappings) of $\mathbb{D}$. We can define for each $a \in \mathbb{D}$, an automorphism ϕ_a in $\text{Aut}(\mathbb{D})$ such that

- (i) $(\phi_a \circ \phi_a)(z) \equiv z$;
- (ii) $\phi_a(0) = a, \phi_a(a) = 0$;
- (iii) ϕ_a has a unique fixed point in $\mathbb{D}$.

In fact, $\phi_a(z) = \frac{a-z}{1-\overline{a}z}$ for all a and z in $\mathbb{D}$. It is easy to verify that the derivative of ϕ_a at z is equal to $-k_a(z)$.

It implies the real Jacobian determinant of ϕ_a at z is $J_{\phi_a(z)} = |k_a(z)|^2 = \frac{(1-|a|^2)^2}{|1-\overline{a}z|^4}$. Given $a \in \mathbb{D}$ and f (any measurable function on $\mathbb{D}$), let us define a function $U_a f$ on $\mathbb{D}$ by $U_a f(w) = k_a(w) f(\phi_a(w))$. Notice that U_a is a bounded linear operator on $L^2(\mathbb{D}, dA)$ and $A^2(\mathbb{D})$ for all $a \in \mathbb{D}$. Further, it can be verified that $U_a^2 = I$, the identity operator, $U_a^* = U_a$, $U_a(A^2(\mathbb{D})) \subset A^2(\mathbb{D})$ and $U_a((A^2(\mathbb{D}))^\perp) \subset (A^2(\mathbb{D}))^\perp$ for all $a \in \mathbb{D}$. Thus, $U_a P = P U_a$ for all $a \in \mathbb{D}$ (see [20]). Let $H^\infty(\mathbb{D})$ denote the space of bounded analytic functions on $\mathbb{D}$.

In this article, we establish some sufficient conditions for Aluthge transform of Toeplitz operators and elementary operators on Bergman space to be unitary and average of unitaries. Furthermore, we explore some sufficient conditions for Aluthge transform of tensor product of Toeplitz operators to be average of unitaries on the tensor product of Bergman spaces. Again, we explain some sufficient conditions on T_ϕ and S_ψ which imply $\Delta(T_\phi \hat{\otimes} S_\psi)$ is normal.

2 Unitary operator

This section contains a few sufficient conditions by which Aluthge transform of Toeplitz operators on $A^2(\mathbb{D})$ express as an average of unitaries.

Theorem 2.1 *Let $\phi \geq 0$ with $\|\phi\|_\infty \leq 1$ for $\phi \in L^\infty(\mathbb{D})$. If $\|T_{1+\phi}\| < 1$. Then, the Aluthge transform $\Delta(T_\phi)$ of T_ϕ on $A^2(\mathbb{D})$ can be represented as an average of two unitary operators.*

Proof Since ϕ is positive, $T_\phi \geq 0$ on $A^2(\mathbb{D})$. Then, by [1], we obtain $\|V - T_\phi\| \leq \|I + T_\phi\| = \|T_{1+\phi}\| < 1$, where V is any unitary operator on $A^2(\mathbb{D})$. Since $\|V - T_\phi\| < 1$, so it implies $\|I - V^*T_\phi\| < 1$. Hence, V^*T_ϕ and T_ϕ are invertible. Then, by ([13], Lemma 1.1), $\Delta(T_\phi)$ is invertible. Let $\Delta(T_\phi) = U|\Delta(T_\phi)|$ be the polar decomposition of $\Delta(T_\phi)$. As $\Delta(T_\phi)$ is invertible, the partial isometry U becomes a unitary operator. But from ([13], Proposition 1.6), we obtain $\|\Delta(T_\phi)\| \leq \|T_\phi\| \leq 1$. This implies $\| |\Delta(T_\phi)| \| \leq 1$. Therefore, $I - |\Delta(T_\phi)|^2 \geq 0$ and $\|I - |\Delta(T_\phi)|^2\| \leq 1$. Now, we define two operators as $W_1 = |\Delta(T_\phi)| + i(I - |\Delta(T_\phi)|^2)^{\frac{1}{2}}$ and $W_2 = |\Delta(T_\phi)| - i(I - |\Delta(T_\phi)|^2)^{\frac{1}{2}}$ such that $W_1^* = W_2$. Therefore, it is clear that $W_1 W_1^* = I$ and $W_1^* W_1 = I$. Thus, $W_1 W_1^* = W_1^* W_1 = I$ and $W_2 W_2^* = W_2^* W_2 = I$. This implies that W_1 and W_2 are two unitary operators on $A^2(\mathbb{D})$. Hence, $\Delta(T_\phi) = U|\Delta(T_\phi)| = U(\frac{W_1 + W_2}{2}) = \frac{1}{2}(UW_1 + UW_2) = \frac{U_1 + U_2}{2}$, where $U_1 = UW_1$ and $U_2 = UW_2$ are the two unitary operators on the Bergman space $A^2(\mathbb{D})$. The proof is thus completed. $\square$

Corollary 2.2 *Let $\|\phi\|_\infty \leq 1$, where $\phi \geq 0$ for $\phi \in L^\infty(\mathbb{D})$. If $\|U_a - T_\phi\| < 1$, $\forall a \in \mathbb{D}$. Then, the Aluthge transform $\Delta(T_{\phi \circ \phi_a})$ of $T_{\phi \circ \phi_a}$ can be written as an average of the two unitaries.*

Proof Since $\|U_a - T_\phi\| < 1$, by ([8], Corollary-1), T_ϕ is invertible. By ([4], Proposition 3.3), we have $\Delta(T_{\phi \circ \phi_a}) = \Delta(U_a T_\phi U_a) = U_a \Delta(T_\phi) U_a$. Hence, the result holds from Theorem 2.1. $\square$

Corollary 2.3 *Let ϕ be an essentially bounded Lebesgue measurable function on $\mathbb{D}$ such that $\phi \geq 0$ and $\|\phi\|_\infty \leq 1$. If $\|U_a - T_\phi\| < 1$, the Aluthge transform $\Delta(T_\phi)$ of T_ϕ can be represented as $\Delta(T_\phi) = \sum_{j=1}^4 \frac{(-1)^{j+1}}{4} V_j$, where V_j is a unitary operator.*

Proof Let $\Delta(T_\phi) = U|\Delta(T_\phi)|$ be the polar decomposition of $\Delta(T_\phi)$. As $\|\phi\|_\infty \leq 1$, so $\|T_\phi\| \leq 1$. Given that $\|U_a - T_\phi\| < 1$, from ([8], Corollary-1), T_ϕ is invertible. Therefore, $\Delta(T_\phi)$ is also invertible. So U is a unitary operator. It is known that $\|\Delta(T_\phi)\| \leq \|T_\phi\| \leq 1$. This implies $\| |\Delta(T_\phi)| \| \leq 1$. Hence, $I - |\Delta(T_\phi)|^2$ is a positive operator and $\|I - |\Delta(T_\phi)|^2\| \leq 1$. So we can define the four operators as $W_1 = |\Delta(T_\phi)| + i(I - |\Delta(T_\phi)|^2)^{\frac{1}{2}}$, $W_3 = |\Delta(T_\phi)| - i(I - |\Delta(T_\phi)|^2)^{\frac{1}{2}}$, $W_2 = -|\Delta(T_\phi)| + i(I - |\Delta(T_\phi)|^2)^{\frac{1}{2}}$ and $W_4 = -|\Delta(T_\phi)| - i(I - |\Delta(T_\phi)|^2)^{\frac{1}{2}}$. Hence, W_1, W_2, W_3 and W_4 are unitary operators on the Bergman space $A^2(\mathbb{D})$. Thus, $\Delta(T_\phi) = U|\Delta(T_\phi)| = U(\frac{W_1 - W_2 + W_3 - W_4}{4}) = \frac{1}{4}(UW_1 - UW_2 + UW_3 - UW_4) = \frac{V_1 - V_2 + V_3 - V_4}{4}$, where $V_1 = UW_1, V_2 = UW_2, V_3 = UW_3$ and $V_4 = UW_4$ are four unitary operators on $A^2(\mathbb{D})$. $\square$

Corollary 2.4 *Let $\phi_1, \phi_2 \in L^\infty(\mathbb{D})$ with $\|\phi_1\|_\infty, \|\phi_2\|_\infty \leq 1$ and let $T_{\phi_1} = Q_1|T_{\phi_1}|$ and $T_{\phi_2} = Q_2|T_{\phi_2}|$ be the polar decompositions of T_{ϕ_1} and T_{ϕ_2} , respectively. If T_{ϕ_1} doubly commutes with T_{ϕ_2} (that is, $[T_{\phi_1}, T_{\phi_2}] = 0$ and $[T_{\phi_1}, T_{\phi_2}^*] = 0$) and $\dim \ker(T_{\phi_1} T_{\phi_2}) = \dim \ker(T_{\phi_2} T_{\phi_1})$. Then, $\Delta(T_{\phi_1} T_{\phi_2})$ can be represented as $\Delta(T_{\phi_1} T_{\phi_2}) = \sum_{j=1}^4 \frac{(-1)^{j+1}}{4} V_j$, where V_j is a unitary operator.*

Proof Let $T_{\phi_1} = Q_1|T_{\phi_1}|$ and $T_{\phi_2} = Q_2|T_{\phi_2}|$ be the polar decompositions of T_{ϕ_1} and T_{ϕ_2} , respectively, and T_{ϕ_1} doubly commutes with T_{ϕ_2} then by [7], $T_{\phi_1} T_{\phi_2} = Q_1 Q_2 |T_{\phi_1} T_{\phi_2}|$ is also the polar decomposition of $T_{\phi_1} T_{\phi_2}$. Further, since $\dim \ker(T_{\phi_1} T_{\phi_2}) = \dim \ker(T_{\phi_2} T_{\phi_1})$, $Q_1 Q_2$ is unitary. Thus, the result is obvious by Corollary 2.3. $\square$

Corollary 2.5 *Let $T_\phi = U|T_\phi|$ be the polar decomposition of T_ϕ with the symbol $\phi \in L^\infty(\mathbb{D})$ and $\|\phi\|_\infty \leq 1$. If $\dim \ker(T_\phi^2) = \dim \ker(T_\phi^2)$, then $\Delta(T_\phi^2)$ can be represented as $\frac{1}{4}$ of the alternating finite series of four unitary operators.*

Proof Since $T_\phi^2 = U^2|T_\phi^2|$ is the polar decomposition of T_ϕ^2 and $\Delta(T_\phi^2) = U^2|\Delta(T_\phi^2)|$ is also the polar decomposition of $\Delta(T_\phi^2)$, the result follows from Corollary 2.4. $\square$

Corollary 2.6 Let $\|\phi\|_\infty \leq 1$ for $\phi \in L^\infty(\mathbb{D})$ and $T_\phi = U|T_\phi|$ be the polar decomposition of T_ϕ . If T_ϕ , centered with index of T_ϕ^n is equal to zero. Then, $\Delta(T_\phi^n)$ can be represented as $\frac{1}{4}$ times the alternating finite series of four unitary operators.

Proof If T_ϕ is centered then by [10], we obtain $T_\phi^n = U^n|T_\phi^n|$ is the polar decomposition for all $n \in \mathbb{N}$. Again $\Delta(T_\phi^n) = U^n|\Delta(T_\phi^n)|$ is also the polar decomposition of $\Delta(T_\phi^n)$. Thus, the proof is evident from Corollary 2.4. $\square$

Here, we introduce the tensor product of Toeplitz operators $T_{\phi_1} \hat{\otimes} T_{\phi_2} \hat{\otimes} \cdots \hat{\otimes} T_{\phi_n}$ on tensor product of Bergman spaces $A^2(\mathbb{D}) \hat{\otimes} \cdots \hat{\otimes} A^2(\mathbb{D})$.

Let $T_{\phi_j} = Q_j|T_{\phi_j}|$ be the polar decompositions of T_{ϕ_j} for each $j = 1, 2, \dots, n$. Then, $T_{\phi_1} \hat{\otimes} T_{\phi_2} \hat{\otimes} \cdots \hat{\otimes} T_{\phi_n} = (Q_1 \hat{\otimes} Q_2 \hat{\otimes} \cdots \hat{\otimes} Q_n)|T_{\phi_1} \hat{\otimes} T_{\phi_2} \hat{\otimes} \cdots \hat{\otimes} T_{\phi_n}|$ is the polar decomposition of $T_{\phi_1} \hat{\otimes} T_{\phi_2} \hat{\otimes} \cdots \hat{\otimes} T_{\phi_n}$. For more details, see [17].

Corollary 2.7 Let $\phi_j \in L^\infty(\mathbb{D})$ for $j = 1, 2, \dots, n$ such that $\phi_j \geq 0$ and $\|\phi_j\|_\infty \leq 1$. If $\|T_{1+\phi_j}\| < 1$. Then, the Aluthge transform $\Delta(T_{\phi_1} \hat{\otimes} T_{\phi_2} \hat{\otimes} \cdots \hat{\otimes} T_{\phi_n})$ of $T_{\phi_1} \hat{\otimes} T_{\phi_2} \hat{\otimes} \cdots \hat{\otimes} T_{\phi_n}$ can be represented as the average of 2^n unitary operators on $A^2(\mathbb{D}) \hat{\otimes} \cdots \hat{\otimes} A^2(\mathbb{D})$.

Proof Let there be $\phi_j \in L^\infty(\mathbb{D})$, $\phi_j \geq 0$ with $\|\phi_j\|_\infty \leq 1$, $\|T_{1+\phi_j}\| < 1$ for $j = 1, 2, \dots, n$, then by Theorem - 2.1, we find that $\Delta(T_{\phi_j})$ can be represented as an average of two unitary operators for each j . It is known from the properties of Aluthge transform [17] of operators that $\Delta(T_{\phi_1} \hat{\otimes} T_{\phi_2} \hat{\otimes} \cdots \hat{\otimes} T_{\phi_n}) = \Delta(T_{\phi_1}) \hat{\otimes} \Delta(T_{\phi_2}) \hat{\otimes} \cdots \hat{\otimes} \Delta(T_{\phi_n})$. Hence, by induction the result is obvious from Theorem -2.1. $\square$

For single operator $A \in \mathcal{L}(H)$, we define two elementary operators L_A and R_A on $\mathcal{L}(H)$ by $L_A(X) = AX$ and $R_A(X) = XA$ for every $X \in \mathcal{L}(H)$. For more details, see [15]. Here, our immediate corollary is about the Aluthge transform of tensor product of two elementary operators.

Corollary 2.8 Let $T_{\phi_1} = Q_1|T_{\phi_1}|$, $T_{\phi_2} = Q_2|T_{\phi_2}|$, $T_{\psi_1} = R_1|T_{\psi_1}|$ and $T_{\psi_2} = R_2|T_{\psi_2}|$ be the polar decompositions of T_{ϕ_1} , T_{ϕ_2} , T_{ψ_1} and T_{ψ_2} , respectively, and let $\|\phi_j\|_\infty \leq 1$, $\|\psi_j\|_\infty \leq 1$ for $j = 1, 2$ and T_{ϕ_1} doubly commutes with T_{ϕ_2} and T_{ψ_1} doubly commutes with T_{ψ_2} . If $\dim \ker(T_{\phi_1} T_{\phi_2}) = \dim \ker(T_{\phi_2} T_{\phi_1})$ and $\dim \ker(T_{\psi_1} T_{\psi_2}) = \dim \ker(T_{\psi_2} T_{\psi_1})$. Then, the Aluthge transform

$$\Delta(L_{T_{\phi_1}}(T_{\phi_2}) \hat{\otimes} R_{T_{\psi_2}}(T_{\psi_1})) \text{ of } L_{T_{\phi_1}}(T_{\phi_2}) \hat{\otimes} R_{T_{\psi_2}}(T_{\psi_1})$$

is the average of four unitary operators.

Proof Since T_{ϕ_1} doubly commutes with T_{ϕ_2} and T_{ψ_1} doubly commutes with T_{ψ_2} then by [7], $T_{\phi_1} T_{\phi_2} = Q_1 Q_2 |T_{\phi_1} T_{\phi_2}|$ and $T_{\psi_1} T_{\psi_2} = R_1 R_2 |T_{\psi_1} T_{\psi_2}|$ are the polar decompositions of $T_{\phi_1} T_{\phi_2}$ and $T_{\psi_1} T_{\psi_2}$, respectively. Further, as $\Delta(T_{\phi_1} T_{\phi_2}) = Q_1 Q_2 |\Delta(T_{\phi_1} T_{\phi_2})|$ and $\Delta(T_{\psi_1} T_{\psi_2}) = R_1 R_2 |\Delta(T_{\psi_1} T_{\psi_2})|$ are the polar decompositions of $T_{\phi_1} T_{\phi_2}$ and $T_{\psi_1} T_{\psi_2}$, respectively. Thus, $T_{\phi_1} T_{\phi_2} \hat{\otimes} T_{\psi_1} T_{\psi_2} = (Q_1 Q_2 \hat{\otimes} R_1 R_2) |T_{\phi_1} T_{\phi_2} \hat{\otimes} T_{\psi_1} T_{\psi_2}|$ is also the polar decomposition of $T_{\phi_1} T_{\phi_2} \hat{\otimes} T_{\psi_1} T_{\psi_2}$. It is clear from the properties of Aluthge transform of an operator that $\Delta(T_{\phi_1} T_{\phi_2} \hat{\otimes} T_{\psi_1} T_{\psi_2}) = \Delta(T_{\phi_1} T_{\phi_2}) \hat{\otimes} \Delta(T_{\psi_1} T_{\psi_2})$. Given that $\dim \ker(T_{\phi_1} T_{\phi_2}) = \dim \ker(T_{\phi_2} T_{\phi_1})$ and $\dim \ker(T_{\psi_1} T_{\psi_2}) = \dim \ker(T_{\psi_2} T_{\psi_1})$, this implies $Q_1 Q_2$ and $R_1 R_2$ are unitary operators. Now,

$$\begin{aligned} \|\Delta(T_{\phi_1} T_{\phi_2} \hat{\otimes} T_{\psi_1} T_{\psi_2})\| &= \|\Delta(T_{\phi_1} T_{\phi_2}) \hat{\otimes} \Delta(T_{\psi_1} T_{\psi_2})\| \\ &\leq \|T_{\phi_1} T_{\phi_2}\| \|T_{\psi_1} T_{\psi_2}\| \\ &\leq 1. \end{aligned}$$

Hence, the result is evident from Theorem 2.1. $\square$

3 Normal operator

In this section, we state some sufficient conditions for Aluthge transform of tensor product of Toeplitz and little Hankel operators to be normal.

Theorem 3.1 Let $\phi \geq 0$, $\psi \geq 0 \in L^\infty(\mathbb{D})$ be such that $\|1 + \phi\|_\infty < 1$ and $\|1 + \psi\|_\infty < 1$. Suppose that $T_\phi = U|T_\phi|$ and $S_\psi = V|S_\psi|$ be the polar decompositions of T_ϕ and S_ψ on $A^2(\mathbb{D})$, respectively, with $sp(U)$ and $sp(V)$ are contained in some open semi-circles. Then, $\Delta(T_\phi \hat{\otimes} S_\psi)$ is normal iff T_ϕ , S_ψ are normal.

Proof Since $\phi, \psi \geq 0$ then T_ϕ and S_ψ are positive on $A^2(\mathbb{D})$. Thus by ([1], theorem-3.1), for every unitary operators W_1 and W_2 on $A^2(\mathbb{D})$ we have $\|W_1 - T_\phi\| \leq \|I + T_\phi\| = \|T_{1+\phi}\| \leq \|1 + \phi\|_\infty < 1$ and $\|W_2 - S_\psi\| \leq \|I + S_\psi\| = \|S_{1+\psi}\| \leq \|1 + \psi\|_\infty < 1$. Hence, T_ϕ and S_ψ are invertible. Since $sp(U)$ contained in some open semi-circle then by [11], $\Delta(T_\phi)$ is normal iff T_ϕ is normal. Similarly using the same argument for S_ψ , We get $\Delta(S_\psi)$ is normal iff S_ψ is normal. As $\Delta(T_\phi \hat{\otimes} S_\psi) = \Delta(T_\phi) \hat{\otimes} \Delta(S_\psi)$ for $T_\phi, S_\psi \in \mathcal{L}^\infty(\mathbb{D})$. Therefore, by ([12], Proposition-5), $\Delta(T_\phi \hat{\otimes} S_\psi)$ is normal iff T_ϕ and S_ψ are normal. $\square$

Corollary 3.2 *Let $\phi, \psi \in L^\infty(\mathbb{D})$ be such that $\phi, \psi \geq 0$ and $\|1 + \phi\|_\infty < 1$ and $\|1 + \psi\|_\infty < 1$. If T_ϕ and S_ψ are hyponormal operators with spectrum of T_ϕ and S_ψ are arcs, respectively, then $\Delta(T_\phi \hat{\otimes} S_\psi)$ is normal.*

Proof As T_ϕ and S_ψ are hyponormal with their spectrums as arcs, T_ϕ and S_ψ are normal. Since T_ϕ is invertible, so by [3], T_ϕ and $\Delta(T_\phi)$ are similar operators. Similarly, S_ψ and $\Delta(S_\psi)$ are similar operators. Again by [3], $|T_\phi|$ and $|S_\psi|$ are invertible as T_ϕ and S_ψ are invertible. Therefore, T_ϕ and S_ψ can be expressed as $|T_\phi|^{-\frac{1}{2}} \Delta(T_\phi) |T_\phi|^{\frac{1}{2}}$ and $|S_\psi|^{-\frac{1}{2}} \Delta(S_\psi) |S_\psi|^{\frac{1}{2}}$, respectively. Hence

$$T_\phi^* T_\phi = |T_\phi|^{-\frac{1}{2}} \Delta^*(T_\phi) \Delta(T_\phi) |T_\phi|^{\frac{1}{2}}. \quad (3.1)$$

Similarly

$$T_\phi T_\phi^* = |T_\phi|^{-\frac{1}{2}} \Delta(T_\phi) \Delta^*(T_\phi) |T_\phi|^{\frac{1}{2}}. \quad (3.2)$$

As T_ϕ is normal, it is clear from equations (3.1) and (3.2) that $\Delta(T_\phi)$ is normal. Similarly, $\Delta(S_\psi)$ is normal. Thus, by Theorem 3.1, $\Delta(T_\phi \hat{\otimes} S_\psi)$ is normal. $\square$

Corollary 3.3 *Let $T_\phi = U|T_\phi|$ and $S_\psi = V|S_\psi|$ be the polar decompositions of T_ϕ and S_ψ , respectively. If U and V are hyponormal with $sp(U)$ and $sp(V)$ as the set of real numbers, then $\Delta(T_\phi \hat{\otimes} S_\psi)$ is normal.*

Proof Let $T_\phi = U|T_\phi|$ and $S_\psi = V|S_\psi|$ be the polar decomposition of T_ϕ and S_ψ , respectively. Now

$$\begin{aligned} \Delta^*(T_\phi \hat{\otimes} S_\psi) &= |T_\phi \hat{\otimes} S_\psi|^{\frac{1}{2}} (U^* \hat{\otimes} V^*) |T_\phi \hat{\otimes} S_\psi|^{\frac{1}{2}} \\ &= (|T_\phi|^{\frac{1}{2}} U^* |T_\phi|^{\frac{1}{2}}) \hat{\otimes} (|S_\psi|^{\frac{1}{2}} V^* |S_\psi|^{\frac{1}{2}}) \end{aligned}$$

As U and V are hyponormal and $sp(U), sp(V)$ are the set of real numbers. So by [16], U and V are self-adjoint. Therefore

$$\begin{aligned} \Delta^*(T_\phi \hat{\otimes} S_\psi) &= (|T_\phi|^{\frac{1}{2}} U |T_\phi|^{\frac{1}{2}}) \hat{\otimes} (|S_\psi|^{\frac{1}{2}} V |S_\psi|^{\frac{1}{2}}) \\ &= \Delta(T_\phi \hat{\otimes} S_\psi) \end{aligned}$$

So $\Delta(T_\phi \hat{\otimes} S_\psi)$ is self-adjoint. Since self-adjoint operators are normal, so $\Delta(T_\phi \hat{\otimes} S_\psi)$ is normal. $\square$

Theorem 3.4 *Let $\phi, \psi \in L^\infty(\mathbb{D})$ and $X \in \mathcal{L}(A^2(\mathbb{D}))$. If ϕ and $\psi \geq 0$ with $T_\phi X = I$ and $S_\psi X = I$ then $\Delta(T_\phi \hat{\otimes} S_\psi)$ is normal.*

Proof As ϕ and ψ are positive, T_ϕ and S_ψ are positive on $A^2(\mathbb{D})$. Since $T_\phi X = I$, T_ϕ is right invertible. Now $X^* T_\phi = (T_\phi X)^* = I^* = I$. This implies that T_ϕ is left invertible. Hence, T_ϕ is invertible. Similarly, S_ψ is invertible. Then by [5], $\Delta(T_\phi), \Delta(S_\psi)$ are positive. Since positive operators are normal, then $\Delta(T_\phi), \Delta(S_\psi)$ are normal. Then by [12], $\Delta(T_\phi \hat{\otimes} S_\psi)$ is normal. $\square$

Theorem 3.5 *Let $T_\phi, S_\psi \in \mathcal{L}(A^2(\mathbb{D}))$ with $m(T_\phi), m(T_\phi^*) > 0$ and $m(S_\psi), m(S_\psi^*) > 0$. If T_ϕ and S_ψ are partial isometry satisfying $T_\phi^{*2} T_\phi^2 + 2k T_\phi^* T_\phi + k^2 \geq 0$ and $S_\psi^{*2} S_\psi^2 + 2k S_\psi^* S_\psi + k^2 \geq 0$ for all real k then $\Delta(T_\phi \hat{\otimes} S_\psi)$ is normal.*

Proof Let $m(T_\phi) = \inf\{\|T_\phi x\| : \|x\| = 1\}$ and $m(T_\phi^*) = \inf\{\|T_\phi^* x\| : \|x\| = 1\}$. Since $m(T_\phi), m(T_\phi^*) > 0$ then by [18], T_ϕ is invertible. Similarly, S_ψ is invertible. Again since $T_\phi^{*2} T_\phi^2 + 2k T_\phi^* T_\phi + k^2 \geq 0$ and $S_\psi^{*2} S_\psi^2 + 2k S_\psi^* S_\psi + k^2 \geq 0$ for any real number k , so T_ϕ and S_ψ are paranormal. As partial isometry and

paranormal operators are quasinormal. So T_ϕ and S_ψ are quasinormal. Therefore by [13], $\Delta(T_\phi) = T_\phi$ and $\Delta(S_\psi) = S_\psi$. Thus,

$$\begin{aligned}\Delta(T_\phi \hat{\otimes} S_\psi) &= \Delta(T_\phi) \hat{\otimes} \Delta(S_\psi) \\ &= T_\phi \hat{\otimes} S_\psi.\end{aligned}$$

Since T_ϕ and S_ψ are quasinormal and invertible operators, this implies that T_ϕ and S_ψ are normal. Hence by [12], $\Delta(T_\phi \hat{\otimes} S_\psi)$ is normal. $\square$

Corollary 3.6 *Let $T_\phi = T_{\phi_1} T_{\phi_2}$, $S_\psi = S_{\psi_1} S_{\psi_2}$. If T_{ϕ_1} , T_{ϕ_2} , S_{ψ_1} and S_{ψ_2} are invertible quasinormal operators with $[T_{\phi_1}, T_{\phi_2}] = [T_{\phi_1}, T_{\phi_2}^*] = 0$ and $[S_{\psi_1}, S_{\psi_2}] = [S_{\psi_1}, S_{\psi_2}^*] = 0$ then $\Delta(T_\phi \hat{\otimes} S_\psi)$ is normal.*

Proof Since T_{ϕ_1} and T_{ϕ_2} are invertible quasinormal operators and also $[T_{\phi_1}, T_{\phi_2}] = [T_{\phi_1}, T_{\phi_2}^*] = 0$ then by [9], $T_{\phi_1} T_{\phi_2}$ is normal. Similarly, $S_{\psi_1} S_{\psi_2}$ is normal. Now

$$\Delta((T_{\phi_1} T_{\phi_2}) \hat{\otimes} (S_{\psi_1} S_{\psi_2})) = \Delta(T_{\phi_1} T_{\phi_2}) \hat{\otimes} \Delta(S_{\psi_1} S_{\psi_2}). \quad (3.3)$$

As T_{ϕ_1} and T_{ϕ_2} are quasinormal with $T_{\phi_1} T_{\phi_2} = T_{\phi_2} T_{\phi_1}$ and $T_{\phi_1} T_{\phi_2}^* = T_{\phi_2}^* T_{\phi_1}$ then $T_{\phi_1} T_{\phi_2}$ is quasinormal. Similarly, $S_{\psi_1} S_{\psi_2}$ is quasinormal. Hence by [13], $\Delta(T_{\phi_1} T_{\phi_2}) = T_{\phi_1} T_{\phi_2}$ and $\Delta(S_{\psi_1} S_{\psi_2}) = S_{\psi_1} S_{\psi_2}$. Therefore, from Eq. (3.3), we obtained

$$\begin{aligned}\Delta((T_{\phi_1} T_{\phi_2}) \hat{\otimes} (S_{\psi_1} S_{\psi_2})) &= \Delta(T_{\phi_1} T_{\phi_2}) \hat{\otimes} \Delta(S_{\psi_1} S_{\psi_2}) \\ &= T_{\phi_1} T_{\phi_2} \hat{\otimes} S_{\psi_1} S_{\psi_2}.\end{aligned}$$

Thus, the proof is evident from [12]. $\square$

Theorem 3.7 *Let $\phi = \phi_1 + \phi_2 \in L^\infty(\mathbb{D})$ and $\psi = \psi_1 + \psi_2 \in L^\infty(\mathbb{D})$ are positive measurable functions. Suppose T_ϕ , S_ψ are invertible operators with $[\Delta(T_{\phi_1}), \Delta(T_{\phi_2}^*)] = 0$ and $[\Delta(S_{\psi_1}), \Delta(S_{\psi_2}^*)] = 0$ then $\Delta(T_\phi \hat{\otimes} S_\psi)$ is normal.*

Proof Since ϕ and $\psi \geq 0$ then T_ϕ , S_ψ are positive operators. Therefore

$$T_\phi = T_{\phi_1} + T_{\phi_2} \quad \text{and} \quad S_\psi = S_{\psi_1} + S_{\psi_2} \quad (3.4)$$

are positive. So T_{ϕ_1} , T_{ϕ_2} are positive and as T_{ϕ_1} , T_{ϕ_2} are invertible operators, By [5], $\Delta(T_{\phi_1})$ and $\Delta(T_{\phi_2})$ are positive. Since positive operators are normal. So $\Delta(T_{\phi_1})$ and $\Delta(T_{\phi_2})$ are normal. Similarly, we can easily seen that $\Delta(S_{\psi_1})$ and $\Delta(S_{\psi_2})$ are normal. Therefore

$$\begin{aligned}\Delta(T_\phi \hat{\otimes} S_\psi) &= \Delta(T_\phi) \hat{\otimes} \Delta(S_\psi) \\ &= \Delta(T_{\phi_1} + T_{\phi_2}) \hat{\otimes} \Delta(S_{\psi_1} + S_{\psi_2}) \\ &= (\Delta(T_{\phi_1}) + \Delta(T_{\phi_2})) \hat{\otimes} (\Delta(S_{\psi_1}) + \Delta(S_{\psi_2})).\end{aligned}$$

Further, since $[\Delta(T_{\phi_1}), \Delta(T_{\phi_2}^*)] = 0$ and $[\Delta(S_{\psi_1}), \Delta(S_{\psi_2}^*)] = 0$, $\Delta(T_{\phi_1}) + \Delta(T_{\phi_2})$ and $\Delta(S_{\psi_1}) + \Delta(S_{\psi_2})$ are normal. Hence, $\Delta(T_\phi \hat{\otimes} S_\psi)$ is normal. $\square$

Theorem 3.8 *Let $T_\phi = T_{\phi_1} + iT_{\phi_2}$ and $S_\psi = S_{\psi_1} + iS_{\psi_2}$ be the cartesian decompositions of T_ϕ and S_ψ , respectively. Suppose T_ϕ and S_ψ are invertible operators. If T_ϕ , S_ψ are hyponormal operators and T_{ϕ_2} , S_{ψ_2} are compact, then $\Delta(T_\phi \hat{\otimes} S_\psi)$ is normal.*

Proof Let $T_\phi = T_{\phi_1} + iT_{\phi_2}$ and $S_\psi = S_{\psi_1} + iS_{\psi_2}$ be the cartesian decompositions of T_ϕ and S_ψ , respectively. Since T_ϕ , S_ψ are hyponormal operators whose imaginary parts are compact, so by [19], T_ϕ , S_ψ are normal. Again, since T_ϕ and S_ψ are invertible, therefore by [3], T_ϕ and $\Delta(T_\phi)$ are similar operators. Similarly, S_ψ and $\Delta(S_\psi)$ are similar operators and also $|T_\phi|$ and $|S_\psi|$ are invertible as T_ϕ and S_ψ are invertible. Therefore, we can express T_ϕ and S_ψ as $T_\phi = |T_\phi|^{\frac{-1}{2}} \Delta(T_\phi) |T_\phi|^{\frac{1}{2}}$ and $S_\psi = |S_\psi|^{\frac{-1}{2}} \Delta(S_\psi) |S_\psi|^{\frac{1}{2}}$, respectively. So, as T_ϕ and S_ψ are normal it can be easily shown that $\Delta(T_\phi)$ and $\Delta(S_\psi)$ are normal. Hence, $\Delta(T_\phi \hat{\otimes} S_\psi)$ is normal. $\square$

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References

1. Aiken, J.G.; Erdos, J.A.; Goldstein, J.A.: Unitary approximation of positive operators. *Illinois J. Math.* **24**, 61–72 (1980)
2. Aluthge, A.: On p -hyponormal operators for $0 < p < 1$. *Integr. Equ. Oper. Theory* **13**, 307–315 (1990)
3. Aluthge, A.; Wang, D.: W -hyponormal operators. *Integr. Equ. Oper. Theory* **36**, 1–10 (2000)
4. Antezana, J.; Massey, P.; Stojanoff, D.: λ -Aluthge transforms and Schatten ideals. *Linear Algebra Appl.* **405**, 177–199 (2005)
5. Chabbabi, F.: product commuting maps with the λ aluthge transform (2016). [arXiv:1606.06165v2](https://arxiv.org/abs/1606.06165v2) [math.FA]
6. Das, N.; Jena, P.K.: On the range and kernel Of Toeplitz and little Hankel operators. *Methods Funct. Anal. Topol.* **19**, 55–67 (2013)
7. Furuta, T.: On the polar decomposition of an operator. *Acta Sci. Math. (Szeged)* **46**, 261–268 (1983)
8. Lin, C.S.: The unilateral shift and a norm equality for bounded linear operators. *Proc. Am. Math. Soc.* **127**, 1693–1696 (1999)
9. Imagiri, S.K.; Khalagai, J.M.; Pokariyal, G.P.: On the normality of the product of n -power quasinormal operators. *Far East J. Appl. Math.* **82**(1), 41–53 (2013)
10. Morrel, B.B.; Muhly, P.S.: Centered operators. *Studia Math.* **51**, 251–263 (1974)
11. Moslehian, M.S., Nabavi Sales, S.M.S.: Some conditions implying normality of operators, *C. R. Math.* 349(5-6), 251-254 (2011)
12. Kubrusly, C.S.: Concise introduction to tensor product. *Far East J. Math. Sci.* **22**, 137–171 (2006)
13. Jung, I.B.; Ko, E.; Percy, C.: Aluthge transforms of operators. *Integr. Equ. Oper. Theory* **37**, 437–448 (2000)
14. Olsen, C.L.: Unitary approximation. *J. Funct. Anal.* **85**, 392–419 (1989)
15. Seddik, A.: Rank one operators and norm of elementary operators. *Linear Algebra Appl.* **424**, 177–183 (2007)
16. Stampfli, J.G.: Hyponormal operators and Spectral density. *Trans. Am. Math. Soc.* **117**, 496–476 (1965)
17. Tanahashi, K.; Cho, M.: Tensor products of log-hyponormal and of class $A(s, t)$ operators. *Glasg. Math. J.* **46**, 91–95 (2004)
18. Wu, P.Y.: Approximation by invertible and noninvertible operators. *J. Approx. Theory* **56**, 267–276 (1989)
19. Whitley, R.: A note on hyponormal operators. *Proc. Am. Math. Soc.* **49**(2), 399–400 (1975)
20. Zhu, K.: *Operator Theory in Function Spaces*. Marcel Dekker, New York (1990)

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