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Catalogue of the Star graph eigenvalue multiplicities

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Abstract The Star graph S_n, $n \geqslant 2$, is the Cayley graph over the symmetric group Sym_n generated by transpositions $(1\ i)$, $2 \leqslant i \leqslant n$. This set of transpositions plays an important role in the representation theory of the symmetric group. The spectrum of S_n contains all integers from $-(n-1)$ to $n-1$, and also zero for $n \geqslant 4$. In this paper we observe methods for getting explicit formulas of eigenvalue multiplicities in the Star graphs S_n, present such formulas for the eigenvalues $\pm(n-k)$, where $2 \leqslant k \leqslant 12$, and finally collect computational results of all eigenvalue multiplicities for $n \leqslant 50$ in the catalogue.

Mathematics Subject Classification 05C25 · 05E10 · 05E15 · 90B10

1 Introduction

The *Star graph* S_n, $n \geqslant 2$, is the Cayley graph over the symmetric group Sym_n of permutations $\pi = [\pi_1\pi_2 \ldots \pi_n]$ with the generating set $\{(1\ i) \in \mathrm{Sym}_n : 2 \leqslant i \leqslant n\}$ of all transpositions $(1\ i)$ swapping the 1st and the ith elements of a permutation π. It is a connected bipartite $(n-1)$-regular graph of order $n!$ and diameter $\mathrm{diam}(S_n) = \lfloor \frac{3(n-1)}{2} \rfloor$ [3].

A graph is *integral* if all eigenvalues of its adjacency matrix are integers. In 1974, Harary and Schwenk [10] posed a question on graphs having integral spectra. In general, most of the graphs have nonintegral eigenvalues [2].

In 2000, Friedman [8] investigated the second smallest non-negative eigenvalue λ_2 of Cayley graphs on the symmetric group generated by transpositions. He proved that among all sets of $n-1$ transpositions which generate the symmetric group, the set whose associated Cayley graph has $\lambda_2 = 1$ is the set $\{(1\,2), (1\,3), \ldots, (1\,n)\}$. This means that there are no other integral Cayley graphs over the symmetric group generated by sets of $n-1$ transpositions.

In 2009, Abdollahi and Vatandoost conjectured [1] that the spectrum of S_n is integral, and contains all integers in the range from $-(n-1)$ up to $n-1$ (with the sole exception that when $n \leqslant 3$, zero is not an

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eigenvalue of S_n). Partially this conjecture was based on the known fact about the spectrum of an r-regular graph which lies in the segment $[-r, r]$ [5]. They verified this conjecture numerically using GAP for $n \leqslant 6$.

In 2012, Krakovski and Mohar [14] proved the second part of the conjecture. More precisely, they proved that for $n \geqslant 2$ and for each integer $1 \leqslant k \leqslant n-1$, the values $\pm(n-k)$ are eigenvalues of S_n with multiplicity at least $\binom{n-2}{k-1}$. If $n \geqslant 4$, then 0 is an eigenvalue of S_n with multiplicity at least $\binom{n-1}{2}$. Since the Star graph is bipartite, $\mathrm{mul}(n-k) = \mathrm{mul}(-n+k)$ for each integer $1 \leqslant k \leqslant n$. Moreover, $\pm(n-1)$ are simple eigenvalues of S_n.

At the same time, Chapuy and Feray [6] showed that the integrality of the Star graphs was already solved in another context, since it is equivalent to studying the spectrum of Jucys–Murphy elements in the algebra of the symmetric group [11]. This connection between two kinds of spectra implies that the Star graph is integral. References on the topic can be also found in the introduction of the paper by Renteln [15].

A lower bound on multiplicities of eigenvalues of S_n given by Krakovski and Mohar was improved by Chapuy and Feray as follows:

$$\mathrm{mul}(n-k) \geqslant \binom{n-2}{n-k-1}\binom{n-1}{n-k}. \tag{1}$$

In 2016, Avgustinovich et al. [4,12] suggested a method for getting explicit formulas for multiplicities of eigenvalues $\pm(n-k)$ in the Star graphs S_n and presented such formulas for $2 \leqslant k \leqslant 5$. Moreover, a lower bound on multiplicity of eigenvalues of S_n for sufficiently large n was obtained. It was proved that for a fixed integer eigenvalue of the Star graph S_n, its multiplicity is at least $2^{\frac{1}{2}n\log n(1-o(1))}$ [4].

In 2018, Khomyakova [13] investigated the behavior of the eigenvalues multiplicity function of the Star graph S_n for eigenvalues $\pm(n-k)$ where $1 \leqslant k \leqslant \frac{n+1}{2}$. It was shown that the function has a polynomial behavior in n. Moreover, explicit formulas for calculating multiplicities of eigenvalues $\pm(n-k)$ where $2 \leqslant k \leqslant 12$ were also presented in the paper. Computational results showed that the same polynomial behavior of the eigenvalues multiplicity function occurs for any integers $n \geqslant 2$ and $1 \leqslant k \leqslant n$.

In this paper, we review methods used for getting explicit formulas for eigenvalue multiplicities in the Star graphs S_n, present these formulas for the eigenvalues $\pm(n-k)$, where $2 \leqslant k \leqslant 12$, and finally collect computational results of all eigenvalue multiplicities for $n \leqslant 50$ in the catalogue provided in the electronic supplementary material.

2 Theoretical results

To describe a combinatorial approach for calculating multiplicities of eigenvalues of the Star graphs $S_n, n \geqslant 2$, we need to give basic definitions and notation on representation theory of the symmetric group [16].

The symmetric group Sym_n consists of all bijections of $\{1, 2, \ldots, n\}$ to itself using compositions as the multiplication. For any permutation $\pi \in Sym_n$, we view its cycle type as a partition.

A *partition* of n is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, $l \leqslant n$, such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_l$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_l = n$. We denote a partition of n as $\lambda \vdash n$. If $\pi \in \mathrm{Sym}_n$ is decomposed into a product of disjoint cycles of length $\lambda_1, \ldots, \lambda_l$, where $\lambda_1 \geqslant \cdots \geqslant \lambda_l$ so that $\lambda_1 + \cdots + \lambda_l = n$, then a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ is called the *cycle type* of π. A partition λ is presented by its Young diagram. In this paper, we use the French notation for Young diagrams [9].

Let λ is a partition of n. A *Young tableau of shape* λ is obtained by filling in the boxes of a Young diagram of λ with the elements $\{1, 2, \ldots, n\}$, where each number occurring exactly once. Thus, the Young tableau of shape λ is the set $[\lambda] = \{(i, j) : 1 \leqslant j \leqslant \lambda_i,\ 1 \leqslant i \leqslant l\}$. Let us define values $c(m) = i - j$, where $m \in \{1, \ldots, n\}$ and i, j are the ordinate and the abscissa of the box containing m, correspondingly. A *standard Young tableau* is a Young tableau whose the entries are increasing across each row and each column.

We write λ' for the *conjugate partition* of λ defined by $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_{l'})$, where $l' = \lambda_1$, $\lambda'_j = max\{j : (i, j) \in [\lambda]\}$, $1 \leqslant i \leqslant l'$. So, $(i, j) \in [\lambda]$ if and only if $(j, i) \in [\lambda']$. Then, the *hook length* h_{ij} is defined by the following formula:

$$h_{ij} = \lambda_i - j + \lambda'_j - i + 1. \tag{2}$$

Now let us show relationships between standard Young tableaux and eigenvalue multiplicities of the Star graphs.

Let G be a group and V be a finite-dimensional vector space over the complex numbers. Let $GL(V)$ stands for the set of all invertible linear transformations of V to itself, called the general linear group of V. Then a *representation* of G on V is a group homomorphism $\rho : G \to GL(V)$, and V is a vector space of the representation with dimension $\dim(V)$. The representation is *irreducible* if it has no proper subspace closed under the action of ρ. Two representations $\rho_1 : G \to GL(V_1)$ and $\rho_2 : G \to GL(V_2)$ are *equivalent* if there exists a bijective linear map $\varphi : V_1 \to V_2$ such that $\varphi\rho_2(g) = \rho_1(g)\varphi$ for all $g \in G$.

The symmetric group Sym_n has order $n!$, its conjugacy classes are labeled by partitions of n, and according to the representation theory of a finite group, the set of inequivalent irreducible representations is defined by partitions of n. We denote by V_λ a vector space of the irreducible representation associated with the partition $\lambda \vdash n$. It is known [16] that

$$\sum_{\lambda \vdash n} (\dim(V_\lambda))^2 = |\mathrm{Sym}_n|, \tag{3}$$

and the following equality holds [6]

$$\mathrm{mul}(n-k) = \sum_{\lambda \vdash n} \dim(V_\lambda) I_\lambda(n-k), \tag{4}$$

where $I_\lambda(n-k)$ is the number of standard Young tableaux of shape λ satisfying $c(n) = n-k$. Since $\dim(V_\lambda)$ is equal to the number of all partitions of shape λ, i.e., the number of standard Young tableaux, it is calculated by the Hook Formula [7]:

$$\dim(V_\lambda) = \frac{n!}{\prod\limits_{(i,j)\in[\lambda]} h_{ij}}, \tag{5}$$

where $\lambda \vdash n$. Let $\mathcal{A}_k$ be the set of partitions of n of length $l = n-k+1$ with the last element 1. For any $\lambda = (\lambda_1, \ldots, \lambda_{n-k}, 1)$, let $\hat{\lambda}$ be a partition $(\lambda_1, \ldots, \lambda_{n-k})$ of $n-1$. Then the following result holds.

Lemma 2.1 [13] *For any integer k, $1 \leqslant k \leqslant \frac{n+1}{2}$, we have*

$$I_\lambda(n-k) = \begin{cases} \dim(V_{\hat{\lambda}}), & \text{if } \lambda \in \mathcal{A}_k; \\ 0, & \text{if } \lambda \notin \mathcal{A}_k. \end{cases} \tag{6}$$

We set $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \vdash n,\ l = n-k+1,\ \lambda \in \mathcal{A}_k$, and calculate the hook length $\widehat{h_{ij}}$ in $[\hat{\lambda}]$ by the following formulas:

$$\widehat{h_{ij}} = \begin{cases} \lambda_i - j + (\lambda'_j - 1) - i + 1, & j = 1,\ 1 \leqslant i \leqslant l-1; \\ \lambda_i - j + \lambda'_j - i + 1, & 1 < j \leqslant \lambda_i,\ 1 \leqslant i \leqslant l-1; \\ 0, & i = l = n-k+1. \end{cases} \tag{7}$$

Then by Lemma 2.1, for any k, $1 \leqslant k \leqslant \frac{n+1}{2}$, formula (4) can be rewritten as:

$$\mathrm{mul}(n-k) = \sum_{\lambda \in \mathcal{A}_k} \frac{n!}{\prod\limits_{(i,j)\in[\lambda]} h_{ij}} \cdot \frac{(n-1)!}{\prod\limits_{(i,j)\in[\hat{\lambda}]} \widehat{h_{ij}}}. \tag{8}$$

The main result is given by the following theorem.

Theorem 2.2 [13] *Let $n, k \in \mathbb{Z}$, $n \geqslant 2$ and $1 \leqslant k \leqslant \frac{n+1}{2}$, then the multiplicity* $\mathrm{mul}(n-k)$ *of eigenvalue $(n-k)$ of the Star graph S_n is calculated by the formula:*

$$\mathrm{mul}(n-k) = \frac{n^{2(k-1)}}{(k-1)!} + P(n),$$

where $P(n)$ is a polynomial of degree $2k-3$.

Computational results show that theorem holds for any $n \geqslant 2$ and $1 \leqslant k \leqslant n$.

3 Practical results

Explicit formulas of multiplicities for the positive eigenvalues $(n-k)$, where $2 \leqslant k \leqslant 12$, are obtained by (8) and given in "Appendix A". Since the Star graph S_n is bipartite, the same formulas hold for negative eigenvalues $-(n-k)$. The results of our computations are summarized in electronic supplementary material. To get these results an algorithm based on the method above was implemented in Golang. The calculations were performed on the Intel(R) Core(TM) i7-4790K CPU @ 4.00GHz with the following elapsed time:

n	50	55	60	65	70	75	80	85	90	95	100
sec	1.5	3.6	8.2	18.6	40.6	87	181.8	373.1	745.5	1461.9	2827.9

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Appendix A: Explicit formulas of multiplicities $(n-k)$ for $2 \leqslant k \leqslant 12$

$$\mathrm{mul}(n-2) = (n-1)(n-2).$$

$$\mathrm{mul}(n-3) = \frac{(n-1)(n-3)}{2!}(n^2-4n+2).$$

$$\mathrm{mul}(n-4) = \frac{(n-1)(n-2)}{3!}(n^4-12n^3+47n^2-62n+12).$$

$$\mathrm{mul}(n-5) = \frac{(n-1)(n-2)}{4!}(n^6-21n^5+169n^4-647n^3+1174n^2-820n+60).$$

$$\mathrm{mul}(n-6) = \frac{(n-1)(n-2)(n-3)}{5!}(n^7-29n^6+335n^5-1955n^4+6004n^3-9028n^2+5068n-120).$$

$$\mathrm{mul}(n-7) = \frac{(n-1)(n-2)(n-3)}{6!}(n^9-42n^8+744n^7-7228n^6+41889n^5$$
$$-147190n^4+302970n^3-328600n^2+140336n-840).$$

$$\mathrm{mul}(n-8) = \frac{(n-1)(n-2)(n-3)(n-4)}{7!}(n^{10}-53n^9+1206n^8-15400n^7+120995n^6-602539n^5$$
$$+1884770n^4-3526192n^3+3515644n^2-1385752n+1680).$$

$$\mathrm{mul}(n-9) = \frac{(n-1)(n-2)(n-3)(n-4)}{8!}(n^{12}-70n^{11}+2165n^{10}-38962n^9+452071n^8$$
$$-3538962n^7+18992119n^6-69486678n^5+168611532n^4-255823264n^3$$
$$+215298016n^2-74535168n+15120).$$

$$\mathrm{mul}(n-10) = \frac{(n-1)(n-2)(n-3)(n-4)(n-5)}{9!}(n^{13}-84n^{12}+3149n^{11}-69516n^{10}+1003947n^9$$
$$-9962604n^8+69399287n^7-340356732n^6+1159537744n^5-2655899000n^4$$
$$+3843130464n^3-3110922016n^2+1044298320n-30240).$$

$$\mathrm{mul}(n-11) = \frac{(n-1)(n-2)(n-3)(n-4)(n-5)}{10!}(n^{15}-105n^{14}+5005n^{13}-143355n^{12}+2752087n^{11}$$
$$-37392489n^{10}+370002755n^9-2701681905n^8+14581677068n^7-57662568122n^6$$

$$
\begin{aligned}
&+163637288684n^5-321068683400n^4+408207829104n^3\\
&-297483438704n^2+92156167776n-332640).\\
\mathrm{mul}(n-12)=\;&\frac{(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)}{11!}(n^{16}-122n^{15}+6800n^{14}-229410n^{13}\\
&+5231382n^{12}-85270438n^{11}+1024324268n^{10}-9213080470n^9+62383290673n^8\\
&-316945929608n^7+1193507219068n^6-3254674992768n^5+6181444807824n^4\\
&-7653794621472n^3+5460913410224n^2-1664568429792n+665280).
\end{aligned}
$$

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