Arab. J. Math. (2020) 9:715–726
https://doi.org/10.1007/s40065-019-00269-7
Arabian Journal of Mathematics

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Almost $*$-Ricci soliton on paraKenmotsu manifolds

Received: 2 April 2019 / Accepted: 28 August 2019 / Published online: 14 September 2019

Abstract We consider almost $*$-Ricci solitons in the context of paracontact geometry, precisely, on a paraKenmotsu manifold. First, we prove that if the metric g of η-Einstein paraKenmotsu manifold is $*$Ricci soliton, then M is Einstein. Next, we show that if η-Einstein paraKenmotsu manifold admits a gradient almost $*$-Ricci soliton, then either M is Einstein or the potential vector field collinear with Reeb vector field ξ. Finally, for three-dimensional case we show that paraKenmotsu manifold is of constant curvature -1. An illustrative example is given to support the obtained results.

Mathematics Subject Classification 53C15 · 53C25 · 53B20 · 53D15

1 Introduction

On the analogy of almost contact manifolds, Sato [27] introduced the notion of almost paracontact manifolds. An almost contact manifold is always odd dimensional, but an almost paracontact manifold could be of even dimension as well. Takahashi [31] defined almost contact manifolds, in particular, Sasakian manifolds equipped with an associated pseudo-Riemannian metric. Later, Kaneyuki and Williams [17] introduced the notion of an almost paracontact pseudo-Riemannian structure, as a natural odd dimensional counterpart to paraHermitian structure. In [37], Zamkovoy showed that any almost paracontact structure admits a pseudo-Riemannian metric with signature $(n+1, n)$. In recent years, almost paracontact structure has been studied by many authors, particularly since the appearance of [37]. The curvature identity for different classes of almost paracontact geometry was obtained in [9,35,37]. The notion of paraKenmotsu manifold was introduced by Welyczko [34]. This structure is an analogy of Kenmotsu manifold [18] in paracontact geometry. ParaKenmotsu (briefly p-Kenmotsu) and special paraKenmotsu (briefly sp-Kenmotsu) manifolds were studied by Sinha and Prasad [29], Blaga [2], Sai Prasad and Satyanarayana [25], Prakasha and Vikas [22], and many others.

A Ricci soliton is a generalization of an Einstein metric. We reminisce the notion of Ricci soliton according to [15]. On the manifold M, a Ricci soliton is a triple (g, V, λ) with g, a psuedo-Riemannian metric, V, a vector field called potential vector field and λ, a real scalar, such that

The third author (D.M.N) was financially supported by University Grants Commission, New Delhi (Ref. No.:20/12/2015 (ii)EU-V) in the form of Junior Research Fellowship.

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$$(L_V g)(X, Y) + 2\mathrm{Ric}(X, Y) = 2\lambda g(X, Y), \tag{1}$$

where L denotes Lie derivative along V and Ric denotes the Ricci tensor. The Ricci soliton is a special self similar solution of the Hamilton's Ricci flow: $\frac{\partial}{\partial t} g(t) = -\mathrm{Ric}(t)$ with initial condition $g(0) = g$; and is said to be shrinking, steady, and expanding accordingly, as λ is positive, zero, and negative, respectively. If the vector field V is the gradient of a smooth function f on M, that is, $V = \nabla f$, then we say that Ricci soliton is gradient and f is potential function. For a gradient Ricci soliton, Eq. (1) takes the form:

$$\mathrm{Hess} f + \mathrm{Ric} = \lambda g,$$

where Hess denotes the Hessian operator ∇^2 (∇ denotes the Riemannian connection of g). We recommend the reference [8] for more details about the Ricci flow and Ricci soliton. In the context of paracontact geometry, Ricci solitons were first initiated by Calvaruso and Perrone in [6]. Then, these are extensively studied by [1,2,5,10,26] and many others. In this junction, it is suitable to mention that η-Ricci solitons on paraSasakian manifolds were studied in the paper [19,23]

In 2014, Kaimakamis and Panagiotidou [16] introduced the concept of $*$-Ricci solitons within the framework of real hypersurfaces of a complex space form, where they essentially modified the definition of Ricci soliton by replacing the Ricci tensor Ric in (1) with the $*$-Ricci tensor Ric^*. A pseudo-Riemannian metric g on a manifold M is called a $*$-Ricci soliton if there exist a constant λ and a vector field V, such that

$$(L_V g)(X, Y) + 2\mathrm{Ric}^*(X, Y) = 2\lambda g(X, Y), \tag{2}$$

for all vector fields X, Y on M. Moreover, if the vector field V is a gradient of a smooth function f, then we say that $*$-Ricci soliton is gradient and equation (2) takes the form

$$\mathrm{Hess} f + \mathrm{Ric}^* = \lambda g. \tag{3}$$

Note that a $*$-Ricci soliton is trivial if the vector field V is Killing, and in this case, the manifold becomes $*$-Einstein. Here, it is suitable to mention that the notion of $*$-Ricci tensor was first introduced by Tachibana [30] on almost Hermitian manifolds and further studied by Hamada [13] on real hypersurfaces of non-flat complex space forms. If λ appearing in (2) and (3) is a variable smooth function on M, then g is called almost $*$-Ricci soliton and gradient almost $*$-Ricci soliton, respectively.

Very recently in 2018, Ghosh and Patra [12] first undertook the study of $*$-Ricci solitons on almost contact metric manifolds. The case of $*$-Ricci soliton in paraSasakian manifold was treated by Prakasha and Veeresha in [24]. Here, they proved that if the metric of paraSasakian manifold is a $*$-Ricci Soliton, then it is η-Einstein ([24], Lemma 5). In this connection, it is suitable to mention that the present authors [33] studied $*$-Ricci soliton on η-Einstein Kenmotsu and three-dimensional Kenmotsu manifolds, and proved that if metric of a η-Einstein Kenmotsu manifold is $*$-Ricci soliton, then it is Einstein (see [33], Theorem 3.2). For three-dimensional case, it is proved that if M admits a $*$-Ricci soliton, then it is of constant sectional curvature -1 (see [33], Theorem 3.3). It is mentioned that any three-dimensional paraKenmotsu manifold is η-Einstein (i.e., the Ricci tensor Ric is of the form $\mathrm{Ric} = ag + b\eta \otimes \eta$, where a, b are known as associated functions). However, in higher dimensions this is not true. We also know (see [38], Proposition 4.1) that for dimension > 3, the associated functions of an η-Einstein paraKenmotsu manifold are not constant, like paraSasakian manifolds [37].

Inspired by above-mentioned works, here, we consider $*$-Ricci soliton in the framework of paraKenmotsu manifold. The present paper is organized as follows: In Sect. 2, we reminisce some fundamental formulas and properties of paraKenmotsu manifolds. In Sect. 3, we prove that if η-Einstein paraKenmotsu manifold admits $*$-Ricci soliton, then M is Einstein. Next, we consider a gradient almost $*$-Ricci soliton and show that either M is Einstein or potential vector field collinear with Reeb vector field. Also for three-dimensional case, we prove that if three-dimensional paraKenmotsu manifold admits $*$-Ricci soliton, then it is of constant negative curvature -1. In Sect. 4, we given an example to verify our main results.

2 Preliminaries

In this section, we reminisce some basic notions of almost paracontact metric manifold and refer to [4,17,21, 32,37] for more information and details.

A 2n+1-dimensional smooth manifold M is said to have an almost paracontact structure if it admits a (1,1)-tensor field φ, a vector field ξ, and a 1-form η satisfying the following conditions:

(i) $\varphi^2 = I - \eta \otimes \xi$, $\eta(\xi) = 1$.
(ii) The tensor field φ induces an almost paracomplex structure on each fiber of $\mathcal{D} = \ker(\eta)$, i.e., the ± 1-eigen distributions $\mathcal{D}^{\pm} := \mathcal{D}\varphi(\pm)$ of φ have equal dimension n.

From the definition, it pursues that $\varphi\xi = 0$, $\eta \circ \varphi = 0$ and $\mathrm{rank}(\varphi) = 2n$. An almost paracontact structure is said to be normal [17] if and only if the (1,2) type torsion tensor $N_\varphi := [\varphi, \varphi] - 2d\eta \otimes \xi$ vanishes identically, where $[\varphi, \varphi](X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, \varphi Y] - \varphi[X, \varphi Y]$. If an almost paracontact manifold is endowed with a pseudo-Riemannian metric g, such that

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{4}$$

where signature of g is necessarily $(n + 1, n)$ for all $X, Y \in TM$, then $(M, \varphi, \xi, \eta, g)$ is called an almost paracontact metric manifold. By Q and r, we will indicate the Ricci operator determined by $S(X, Y) = g(QX, Y)$ and the scalar curvature of the metric g, respectively. The fundamental 2-form Φ of an almost paracontact metric structure (φ, ξ, η, g) is defined by $\Phi(X, Y) = g(X, \varphi Y)$. If $\Phi = \mathrm{d}\eta$, then the manifold $(M, \varphi, \xi, \eta, g)$ is called a paracontact metric manifold and g is an associated metric.

An almost paracontact metric manifold $(M, \varphi, \xi, \eta, g)$ is called paraKenmotsu manifold if it satisfies (see [28,38]):

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X, \tag{5}$$

for any vector fields $X, Y \in TM$. In [38], Zamkovoy proved that $(M, \varphi, \xi, \eta, g)$ is normal but not quasi-paraSasakian and hence not paraSasakian. Also in a paraKenmotsu manifold, we have the following formulas (see [38]):

$$\nabla_X \xi = X - \eta(X)\xi, \tag{6}$$
$$(\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y), \tag{7}$$
$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{8}$$
$$R(X, \xi)Y = g(X, Y)\xi - \eta(Y)X, \tag{9}$$
$$\mathrm{Ric}(X, \xi) = -2n\eta(X), \tag{10}$$
$$(L_\xi g)(X, Y) = -2\{g(X, Y) - \eta(X)\eta(Y)\}. \tag{11}$$

Note that (11) implies that ξ is not Killing in paraKenmotsu manifold. An almost paracontact metric manifold M is said to be η-Einstein if there exist smooth functions a and b, such that

$$\mathrm{Ric}(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \tag{12}$$

for all $X, Y \in TM$. If $b = 0$, then M becomes an Einstein manifold. Following Zamkovoy ([38], Proposition 4.1), it is showed that if M is an η-Einstein paraKenmotsu manifold of dimension > 3, then we have:

$$Z(b) - 2b\eta(Z) = 0,$$

for any $Z \in TM$.

Now, we recall the notion of $*$-Ricci tensor. This is defined by (see [3,11]):

$$\mathrm{Ric}^*(X, Y) = \sum_i g(R(X, e_i)\varphi e_i, \varphi Y) = \frac{1}{2}\sum_i g(\varphi R(X, \varphi Y)e_i, e_i), \tag{13}$$

where e_i is a local orthonormal frame and the last equality follows from the first Bianchi identity. It should be remarked that Ric^* is not symmetric, in general. Thus, the condition $*$-Einstein (that is, Ric^* is a constant multiple of the metric g) automatically requires a symmetric property of the $*$-Ricci tensor [14].

3 Main results

Before entering our main results, first, we find the expression of $*$-Ricci tensor in paraKenmotsu manifolds

Lemma 3.1 *A $*$-Ricci tensor on a $(2n+1)$-dimensional paraKenmotsu manifold $(M, \varphi, \xi, \eta, g)$ is given by:*

$$\text{Ric}^*(X, Y) = -\text{Ric}(X, Y) - (2n-1)g(X, Y) - \eta(X)\eta(Y), \tag{14}$$

for any vector fields X, Y on M.

Proof In a paraKenmotsu manifold, the following formula is known (see [38])

$$\begin{aligned} R(X, Y)\varphi Z - \varphi R(X, Y)Z = g(Y, Z)\varphi X - g(X, Z)\varphi Y \\ - g(Y, \varphi Z)X + g(X, \varphi Z)Y. \end{aligned} \tag{15}$$

From (15), it follows that:

$$\begin{aligned} R(X, Y, \varphi Z, \varphi W) - g(\varphi R(X, Y)Z, \varphi W) = g(Y, Z)g(\varphi X, \varphi W) - g(X, Z)g(\varphi Y, \varphi W) \\ - g(Y, \varphi Z)g(X, \varphi W) + g(X, \varphi Z)g(Y, \varphi W). \end{aligned}$$

Making use of (4) with the above equation takes the form:

$$\begin{aligned} R(X, Y, \varphi Z, \varphi W) = -R(X, Y, Z, W) - g(Y, Z)g(X, W) + g(X, Z)g(Y, W) \\ - g(Y, \varphi Z)g(X, \varphi W) + g(X, \varphi Z)g(Y, \varphi W). \end{aligned} \tag{16}$$

Contracting (16) over Y and Z and by definition of $*$-Ricci tensor, we obtain (14). This completes the proof. □

In view of recent results on paraSasakian manifold [24] and η-Einstein Kenmotsu manifold [33], a natural question arises whether there exists paraKenmotsu manifold admits a $*$-Ricci soliton. For this, we consider an η-Einstein paraKenmotsu manifold; such a manifold is in general not paraSasakian. Now, we prove the following.

Theorem 3.2 *If the metric of η-Einstein paraKenmotsu manifold of dimension > 3 is a $*$-Ricci soliton, then it is Einstein manifold.*

Proof Since M is η-Einstein, taking $Y = \xi$ in (12) and making use of (10), we have:

$$a + b = -2n. \tag{17}$$

Contracting (12) gives the scalar curvature $r = (2n+1)a + b$. Combining this with (17) yields $a = (1 + \frac{r}{2n})$ and $b = -\{(2n+1) + \frac{r}{2n}\}$. Thus, Eq. (12) takes the form:

$$\text{Ric}(X, Y) = \left(1 + \frac{r}{2n}\right) g(X, Y) - \left\{(2n+1) + \frac{r}{2n}\right\} \eta(X)\eta(Y). \tag{18}$$

In view of (14) and (18), Eq. (2) can be written as:

$$(L_V g)(Y, Z) = \left\{2(2n + \lambda) + \frac{r}{n}\right\} g(Y, Z) - \left\{4n + \frac{r}{n}\right\} \eta(Y)\eta(Z). \tag{19}$$

Differentiating (19) along an arbitrary vector field X and using (7), we obtain:

$$\begin{aligned} (\nabla_X L_V g)(Y, Z) = \frac{Xr}{n} g(Y, Z) - \frac{Xr}{n}\eta(Y)\eta(Z) - \left(4n + \frac{r}{n}\right)\{g(X, Y)\eta(Z) \\ + g(X, Z)\eta(Y) - 2\eta(X)\eta(Y)\eta(Z)\}. \end{aligned} \tag{20}$$

We know the following commutation formula (see [36]):

$$\begin{aligned} &(L_V \nabla_X g - \nabla_X L_V g - \nabla_{[V,X]} g)(Y, Z) \\ &= -g((L_V \nabla)(X, Y), Z) - g((L_V \nabla)(X, Z), Y), \end{aligned}$$

for all vector fields X, Y, Z on M. Since g is parallel with respect to Levi-Civita connection ∇, the above relation becomes:

$$(\nabla_X L_V g)(Y, Z) = g((L_V\nabla)(X, Y), Z) + g((L_V\nabla)(X, Z), Y). \tag{21}$$

We know that $L_V\nabla$ is a symmetric tensor of type (1, 2) and so it follows from (21) that

$$g((L_V\nabla)(X, Y), Z) = \frac{1}{2}\{(\nabla_X L_V g)(Y, Z) + (\nabla_Y L_V g)(Z, X) - (\nabla_Z L_V g)(X, Y)\}. \tag{22}$$

By a straightforward combinatorial computation, and keeping in mind that $L_V\nabla$ is a symmetric operator, the foregoing equation gives:

$$\begin{aligned}2n(L_V\nabla)(X, Y) = {} & (Xr)Y - (Xr)\eta(Y)\xi + (Yr)X \\ & - (Yr)\eta(X)\xi - g(X, Y)Dr + \eta(X)\eta(Y)Dr \\ & - 2(4n^2 + r)\{g(X, Y)\xi - \eta(X)\eta(Y)\xi\},\end{aligned} \tag{23}$$

for all vector fields Z and D is the gradient operator of g. Setting $X = Y = e_i$ (where $\{e_i : i = 1, 2, \ldots, 2n+1\}$ is an orthonormal frame) in (23) and summing over i, we find:

$$n\sum_{i=1}^{2n+1}\varepsilon_i(L_V\nabla)(e_i, e_i) = (1 - n)Dr - (\xi r)\xi - 2n(4n^2 + r)\xi, \tag{24}$$

where $\varepsilon_i = g(e_i, e_i)$. Now, taking covariant differentiation of $*$-Ricci soliton Eq. (2) along a vector field X, we obtain $(\nabla_X L_V g)(Y, Z) = -2(\nabla_X \mathrm{Ric}^*)(Y, Z)$. Substituting this in (22), we have:

$$g((L_V\nabla)(X, Y), Z) = (\nabla_Z \mathrm{Ric}^*)(X, Y) - (\nabla_X \mathrm{Ric}^*)(Y, Z) - (\nabla_Y \mathrm{Ric}^*)(X, Z). \tag{25}$$

Again, taking covariant differentiation of (14) with respect to Z and then using (7), we get:

$$\begin{aligned}(\nabla_Z \mathrm{Ric}^*)(X, Y) = {} & -(\nabla_Z \mathrm{Ric})(X, Y) - \{g(Z, X)\eta(Y) \\ & + g(Z, Y)\eta(X) - 2\eta(X)\eta(Y)\eta(Z)\}.\end{aligned} \tag{26}$$

Combining (26) with (25) yields:

$$\begin{aligned}g((L_V\nabla)(X, Y), Z) = {} & -(\nabla_Z \mathrm{Ric})(X, Y) + (\nabla_X \mathrm{Ric})(Y, Z) + (\nabla_Y \mathrm{Ric})(Z, X) \\ & + 2\{g(X, Y)\eta(Z) - \eta(X)\eta(Y)\eta(Z)\}.\end{aligned} \tag{27}$$

Replacing X and Y by e_i in (27) and summing over i, we find:

$$\sum_{i=1}^{2n+1}\varepsilon_i(L_V\nabla)(e_i, e_i) = 4n\xi. \tag{28}$$

In view of (28) and (24), we at once obtain:

$$(n - 1)\mathrm{D}r + (\xi r)\xi + 2n\{2n(2n + 1) + r\}\xi = 0. \tag{29}$$

Taking inner product of (29) with ξ yields $(\xi r) + 2(2n(2n + 1) + r) = 0$. Making use of this in (29) provides $Dr = (\xi r)\xi$, as $n > 1$. Next, substituting $Y = \xi$ in (23), it follows that

$$2n(L_V\nabla)(X, \xi) = (\xi r)\varphi^2 X. \tag{30}$$

Differentiating (30) along an arbitrary vector field Y and using (6) and (30), we find:

$$\begin{aligned}2n(\nabla_Y L_V\nabla)(X, \xi) + 2n(L_V\nabla)(X, Y) = {} & (Y(\xi r))\varphi^2 X \\ & - (\xi r)\{g(X, Y)\xi + \eta(X)Y - \eta(Y)X - \eta(X)\eta(Y)\xi\}.\end{aligned} \tag{31}$$

According to Yano [36], we have the following well-known commutation formula:

$$(L_V R)(X, Y)Z = (\nabla_X L_V\nabla)(Y, Z) - (\nabla_Y L_V\nabla)(X, Z). \tag{32}$$

Replacing Z by ξ in (32) and taking into account of (31), we obtain:

$$2n(L_V R)(X, Y)\xi = (X(\xi r))\varphi^2 Y - (Y(\xi r))\varphi^2 X + 2(\xi r)\{\eta(X)Y - \eta(Y)X\}. \tag{33}$$

Contracting this over X and noting that $\mathrm{D}r = (\xi r)\xi$, we have $(L_V \mathrm{Ric})(Y, \xi) = 0$. Next, taking Lie derivative of (10) along V, making use of last equation and (18), we have:

$$\begin{aligned}&\left(1 + \frac{r}{2n}\right) g(Y, L_V\xi) - \left\{(2n+1) + \frac{r}{2n}\right\} \eta(Y)\eta(L_V\xi)\\ &= -4n\lambda\eta(Y) - 2ng(Y, L_V\xi).\end{aligned} \tag{34}$$

Taking $Y = \xi$ in (34), we have $\lambda = 0$. Furthermore, substituting ξ for Y and Z in (19) gives $\eta(L_V\xi) = 0$. Thus, making use of $\lambda = 0$ and $\eta(L_V\xi) = 0$, Eq. (34) becomes:

$$\{2n(2n+1) + r\}L_V\xi = 0. \tag{35}$$

Now, if $r = -2n(2n+1)$, then it follows from (18) that M is Einstein.

Suppose that we assume that $r \neq -2n(2n+1)$ in some open set $\mathcal{O}$ of M. Then, on $\mathcal{O}$, $L_V\xi = 0$. This together with (6) yields:

$$\nabla_\xi V = V - \eta(V)\xi.$$

Taking $Y = \xi$ in (19) and using $\lambda = 0$, we have $(L_V g)(X, \xi) = 0$. From this, we have $L_V\eta = 0$. Replacing Y by ξ in the well-known formula (see [36]):

$$(L_V\nabla)(X, Y) = L_V\nabla_X Y - \nabla_X L_V Y - \nabla_{[V,X]}Y,$$

and by virtue of (6), (30), $L_V\xi = 0$ and $L_V\eta = 0$, we obtain $(\xi r) = 0$. Since $Dr = (\xi r)\xi$, we see that r is constant. Thus, (29) implies that $r = -2n(2n+1)$ on $\mathcal{O}$. This contradicts our assumption. This establish the proof. □

Now, we consider gradient almost $*$-Ricci soliton in η-Einstein paraKenmotsu manifolds and prove the following;

Theorem 3.3 *Let M be a $(2n+1)$-dimensional η-Einstein paraKenmotsu manifold. If g represents a gradient almost $*$-Ricci soliton, then either M is Einstein or the potential vector field is pointwise colinear with the Reeb vector field ξ.*

Proof If the metric g of a η-Einstein paraKenmotsu manifold is gradient almost $*$-Ricci soliton, then from (14) and (3), we obtain:

$$\nabla_X \mathrm{D}f = QX + (2n - 1 + \lambda)X + \eta(X)\xi, \tag{36}$$

for any vector field X on M. Taking covariant differentiation of (36) in the direction of an arbitrary vector field Y on M yields:

$$\begin{aligned}\nabla_Y\nabla_X \mathrm{D}f &= (\nabla_Y Q)X + Q\nabla_Y X + (2n - 1 + \lambda)\nabla_Y X + (Y\lambda)X\\ &\quad + (\nabla_Y\eta)(X)\xi + \eta(\nabla_Y X)\xi + \eta(X)\nabla_Y\xi.\end{aligned} \tag{37}$$

Making use of (36) and (37) in the well-known expression of curvature tensor $R(X, Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$, we deduce:

$$R(X, Y)\mathrm{D}f = (\nabla_X Q)Y - (\nabla_Y Q)X + (X\lambda)Y - (Y\lambda)X + \eta(Y)X - \eta(X)Y. \tag{38}$$

In view of (18), we have:

$$QX = \left(1 + \frac{r}{2n}\right) X - \left\{(2n+1) + \frac{r}{2n}\right\} \eta(X)\xi. \tag{39}$$

Differentiating the foregoing equation along an arbitrary vector field Y and using (7), we obtain:

$$(\nabla_Y Q)X = \frac{Yr}{2n}\{X - \eta(X)\xi\} - \left\{(2n+1) + \frac{r}{2n}\right\}\{g(X, Y)\xi - \eta(X)Y\}. \tag{40}$$

In view of (40), we get from (38) that

$$R(X,Y)\mathrm{D}f = \frac{Xr}{2n}(Y-\eta(Y)\xi) - \frac{Yr}{2n}(X-\eta(X)\xi) + \Big\{(2n+2) + \frac{r}{2n}\Big\}(\eta(Y)X - \eta(X)Y) + (X\lambda)Y - (Y\lambda)X. \tag{41}$$

By virtue of above equation, we can easily see that

$$g(R(X,Y)\mathrm{D}f,\xi) = (X\lambda)\eta(Y) - (Y\lambda)\eta(X). \tag{42}$$

Also, we have from (8) that

$$g(R(X,Y)\xi,\mathrm{D}f) = (Yf)\eta(X) - (Xf)\eta(Y). \tag{43}$$

Comparing (42) with (43) and substituting Y by ξ in the resulting equation, we obtain:

$$d(\lambda - f) = \xi(\lambda - f)\eta, \tag{44}$$

where d is the exterior derivative. This means that $\lambda - f$ is invariant along the distribution $\mathcal{D}$ (where $\mathcal{D}$ is $Ker\eta$); that is, $\lambda - f$ is constant for all vector field $X \in \mathcal{D}$.

Contracting (38) over Y, we obtain:

$$S(X,\mathrm{D}f) = -\frac{1}{2}Xr - 2n(X\lambda) + 2n\eta(Y). \tag{45}$$

In view of (39), the foregoing equation gives:

$$\Big(1+\frac{r}{2n}\Big)(Xf) - \Big\{(2n+1)+\frac{r}{2n}\Big\}\eta(X)(\xi f) + \frac{1}{2}(Xr) + 2n(X\lambda) - 2n\eta(X) = 0, \tag{46}$$

for all $X \in TM$. Replacing X by ξ in the above equation, we have:

$$2n\xi(\lambda - f) + \frac{1}{2}(\xi r) - 2n = 0. \tag{47}$$

Now, plugging Y by ξ in (41) and taking inner product with Y yield:

$$g(R(X,\xi)\mathrm{D}f,Y) = -\frac{(\xi r)}{2n}(g(X,Y) - \eta(X)\eta(Y)) + \Big\{(2n+2)+\frac{r}{2n}\Big\}(g(X,Y) - \eta(X)\eta(Y)) + (X\lambda)\eta(Y) - (\xi\lambda)g(X,Y). \tag{48}$$

By virtue of (9), we have:

$$g(R(X,\xi)Y,\mathrm{D}f) = (\xi f)g(X,Y) - (Xf)\eta(Y). \tag{49}$$

Comparing (48) with (49), one can get:

$$-\frac{(\xi r)}{2n}(g(X,Y) - \eta(X)\eta(Y)) + \Big\{(2n+2)+\frac{r}{2n}\Big\}(g(X,Y) - \eta(X)\eta(Y)) + X(\lambda - f)\eta(Y) - \xi(\lambda - f)g(X,Y) = 0. \tag{50}$$

Contracting the above equation, we get:

$$-(\xi r) + 2n\Big(2n+2+\frac{r}{2n}\Big) - 2n\xi(\lambda - f) = 0. \tag{51}$$

Making use of (51) in (47), we easily obtain:

$$(\xi r) = 2\{r + 2n(2n+1)\}. \tag{52}$$

By virtue of (52), we can easily find that

$$\xi(\lambda - f) = -\left(\frac{r}{2n} + 2n\right). \tag{53}$$

Making use of (53) in (44), we obtain that

$$d(\lambda - f) = -\left(\frac{r}{2n} + 2n\right)\eta. \tag{54}$$

Applying the well-known Poincare lemma and using the fact $d\eta = 0$ on the above equation, we obtain $-dr \wedge \eta = 0$, and making use of (52), we have:

$$Dr = 2\{r + 2n(2n+1)\}\xi. \tag{55}$$

Suppose that X in (46) is orthogonal to ξ. Keeping in mind that $\lambda - f$ is constant along $\mathcal{D}$ and making use of (54) and (55), one gets $\{r + 2n(2n+1)\}(Xf) = 0$, for all $X \in \mathcal{D}$. This implies that

$$\{r + 2n(2n+1)\}(\mathrm{D}f - (\xi f)\xi) = 0. \tag{56}$$

Suppose $r = -2n(2n+1)$, then using this relation in (39), we see that $QX = -2nX$, and hence. M is Einstein. If $r \neq -2n(2n+1)$, then we have $\mathrm{D}f = (\xi f)\xi$. This shows that potential vector field is collinear with ξ, and this completes the proof. □

Now, we study $*$-Ricci soliton in three-dimensional paraKenmotsu manifold and prove the following;

Theorem 3.4 *If the metric g of three-dimensional paraKenmotsu manifold is a $*$-Ricci soliton, then it is of constant curvature -1.*

Proof It is known that for any three-dimensional pseudo-Riemannian manifold, we have the following well-known expression:

$$\begin{aligned} R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + \mathrm{Ric}(Y,Z)X - \mathrm{Ric}(X,Z)Y \\ - \frac{r}{2}\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{57}$$

Setting $Y = Z = \xi$ in the above relation and making use of (8) and (10) give:

$$QX = \left(1 + \frac{r}{2}\right)X - \left(3 + \frac{r}{2}\right)\eta(X)\xi,$$

which is equivalent to

$$\mathrm{Ric}(X,Y) = \left(1 + \frac{r}{2}\right)g(X,Y) - \left(3 + \frac{r}{2}\right)\eta(X)\eta(Y). \tag{58}$$

Proceeding in the similar manner as in proof of Theorem 3.2. In dimension 3, that is, for $n = 1$ all Eqs. (19)–(33) holds true. Thus, (33) becomes:

$$\begin{aligned} 2(L_V R)(X,Y)\xi = (X(\xi r))\varphi^2 Y - (Y(\xi r))\varphi^2 X \\ + 2(\xi r)\{\eta(X)Y - \eta(Y)X\}. \end{aligned} \tag{59}$$

Lie differentiating (8) along V and making use of (19) give:

$$\begin{aligned} (L_V R)(X,Y)\xi + R(X,Y)L_V\xi = 2\lambda\{\eta(X)Y - \eta(Y)X\} \\ + g(X, L_V\xi)Y - g(Y, L_V\xi)X. \end{aligned} \tag{60}$$

In view of (59) and (60), we have:

$$\begin{aligned} (X(\xi r))\varphi^2 Y - (Y(\xi r))\varphi^2 X + 2(\xi r)\{\eta(X)Y - \eta(Y)X\} + 2R(X,Y)L_V\xi \\ = 4\lambda\{\eta(X)Y - \eta(Y)X\} + 2\{g(X, L_V\xi)Y - g(Y, L_V\xi)X\}. \end{aligned}$$

Contracting the foregoing equation over X and using (58), we have:

$$(r+6)g(Y, L_V\xi) - (r+6)\eta(Y)\eta(L_V\xi) \\ = Y(\xi r) + \{\xi(\xi r) + 4(\xi r) - 8\lambda\}\eta(Y). \tag{61}$$

Setting $Y = \xi$ in (61) and using (29), we have $\lambda = 0$. In view of (19), we obtain $(L_V g)(Y, \xi) = 0$, which implies $\eta(L_V\xi) = 0$. Thus, using $\lambda = 0$, $\eta(L_V\xi) = 0$ and (29), Eq. (61) reduces: to

$$(r+6)g(Y, L_V\xi) = -2\{Yr - (\xi r)\eta(Y)\}. \tag{62}$$

Suppose that $r = -6$, then from (58), we can see that it is Einstein and $QX = -2X$. This together with (57) gives:

$$R(X, Y)Z = g(X, Z)Y - g(Y, Z)X, \tag{63}$$

showing that M is of constant curvature -1.
On the other hand, suppose that $r \neq -6$ in some open set $\mathcal{O}$ of M. Then, Eq. (62) can be written as:

$$L_V\xi = f\{Dr - (\xi r)\xi\}, \tag{64}$$

where $f = -\frac{2}{r+6}$. Replacing Y by ξ in the well-known commutation formula [36]:

$$(L_V\nabla)(X, Y) = L_V\nabla_X Y - \nabla_X L_V Y - \nabla_{[V,X]}Y,$$

and using (6), (30), and (64), one can easily get:

$$f\{(Xr)\eta(Y) + (Yr)\eta(X) - 2(\xi r)\eta(X)\eta(Y) + g(\nabla_X Dr, Y) - (X(\xi r))\eta(Y)\} \\ + (Xf)\{Yr - (\xi r)\eta(Y)\} + \left\{f(\xi r) - \frac{1}{2}(\xi r)\right\} g(\varphi X, \varphi Y) = 0.$$

Interchanging X, Y in the foregoing equation and recalling the Poincare lemma: $g(\nabla_X Dr, Y) = g(X, \nabla_Y Dr)$, we find:

$$f\{(Y(\xi r))\eta(X) - (X(\xi r))\eta(Y)\} + (Xf)\{Yr - (\xi r)\eta(Y)\} \\ - (Yf)\{Xr - (\xi r)\eta(X)\} = 0. \tag{65}$$

Substituting Y by ξ in (65) and making use of (29), we obtain:

$$(2f - \xi f)(Xr - (\xi r)\eta(X)) = 0,$$

which implies that

$$(2f - \xi f)(Dr - (\xi r)\xi) = 0. \tag{66}$$

From (66), we have either $Dr = (\xi r)\xi$ or $\xi f = 2f$.
Case 1: First, we assume that $Dr = (\xi r)\xi$. By virtue of this, Eq. (64) can be written as $L_V\xi = 0$ on $\mathcal{O}$. This together with (6) gives:

$$\nabla_\xi V = V - \eta(V)\xi. \tag{67}$$

By virtue of $(L_V g)(X, \xi) = 0$ and (67), one gets:

$$g(\nabla_X, V) = -g(X, \nabla_\xi V) = -g(X, V) + \eta(X)\eta(V). \tag{68}$$

Replacing Y by ξ in the well-known formula [36]

$$(L_V\nabla)(X, Y) = \nabla_X\nabla_Y V - \nabla_{\nabla_X Y}V + R(V, X)Y, \tag{69}$$

and making use of (6), (8), (30), (67), and (68), we find $\xi r = 0$. Hence from (29), it follows that $r = -6$ on $\mathcal{O}$, which yields a contradiction.

Case 2: Now, assume that $\xi f = 2f$. This together with $f = -\frac{2}{r+6}$, we have $\xi r = -2(r+6)$, and consequently, we get $g(Dr, \xi) = -2(r+6)$. Since $r \neq -6$ in some open set $\mathcal{O}$, so the last equation implies that $Dr = f\xi$, for some smooth function f. In fact, we have $Dr = (\xi r)\xi$, and so, by Case 1, we get a contradiction. This establishes the proof of the theorem. □

As we know, in differential geometry, symmetric spaces play an important role. In the late 20s, Cartan [7] initiated Riemannian symmetric spaces and obtained a classification of those spaces. If the Riemannian curvature tensor of a Riemannian manifold satisfies the condition $\nabla R = 0$, then this manifold is called locally symmetric [7]. For every point of this manifold, this symmetry condition is equivalent to the fact that the local geodesic symmetry is an isometry [20]. The class of Riemannian symmetric manifolds is very natural generalization of the class of manifolds of constant curvature.

Definition 3.5 An almost paracontact metric manifold is said to be locally φ-symmetric if

$$\varphi^2(\nabla_W R)(X, Y)Z = 0,$$

for all vector fields W,X,Y,Z orthogonal to ξ.

It is known that a three-dimensional paraKenmotsu manifold is locally φ-symmetric if and only if the scalar curvature is constant [38]. Therefore, by Theorem 3.4, we state the following.

Corollary 3.6 *A three-dimensional paraKenmotsu manifold admitting $*$-Ricci soliton is locally φ-symmetric.*

Remark 3.7 Corollary 3.6 builds the connection between $*$-Ricci soliton and symmetry of the manifold. The symmetry of a manifold is vital, because it is connected with the curvature of the manifold. The curvature has important physical significance in the theory of gravitation.

4 Example

In this section, we give an example of $*$-Ricci solitons in three-dimensional paraKenmotsu manifold which verifies Theorem 3.4 and Corollary 3.6.

Example 4.1 We consider three-dimensional manifold $M = \{(x, y, z) \in \mathcal{R}^3, z \neq 0\}$ with the Cartesian coordinates (x, y, z) and the vector fields:

$$\partial_1 = \varphi\partial_2, \quad \partial_2 = \varphi\partial_1, \quad \varphi\partial_3 = 0,$$

where

$$\partial_1 = \frac{\partial}{\partial x}, \quad \partial_2 = \frac{\partial}{\partial y}, \quad \partial_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

The 1-form $\eta = dz$ defines an almost paracontact structure on M with characteristic vector field $\xi = \partial_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$. Let g be a pseudo-Riemannian metric defined by:

$$[g(\partial_i, \partial_j)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

with respect to the basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$.

Using Koszul formula, we have:

$$\begin{aligned} &\nabla_{\partial_1}\partial_1 = -\partial_3, \quad \nabla_{\partial_1}\partial_2 = 0, \quad \nabla_{\partial_1}\partial_3 = \partial_1, \\ &\nabla_{\partial_2}\partial_1 = 0, \quad \nabla_{\partial_2}\partial_2 = \partial_3, \quad \nabla_{\partial_2}\partial_3 = \partial_2, \\ &\nabla_{\partial_3}\partial_1 = 0, \quad \nabla_{\partial_3}\partial_2 = 0, \quad \nabla_{\partial_3}\partial_3 = 0. \end{aligned}$$

It is not hard to verify that the conditions (5) and (6) for paraKenmotsu manifold are satisfied. Hence, the manifold under consideration is a paraKenmotsu manifold. The components of the curvature tensor are:

$$\begin{aligned} &R(\partial_1, \partial_2)\partial_1 = \partial_2, \quad R(\partial_1, \partial_2)\partial_2 = \partial_1, \quad R(\partial_1, \partial_2)\partial_3 = 0, \\ &R(\partial_1, \partial_3)\partial_1 = \partial_3, \quad R(\partial_1, \partial_3)\partial_2 = 0, \quad R(\partial_1, \partial_3)\partial_3 = -\partial_1, \\ &R(\partial_2, \partial_3)\partial_1 = 0, \quad R(\partial_2, \partial_3)\partial_2 = -\partial_3, \quad R(\partial_2, \partial_3)\partial_3 = -\partial_2. \end{aligned}$$

The components of Ricci tensor and $*$-Ricci tensor are:

$$\begin{aligned} &\mathrm{Ric}(\partial_1, \partial_1) = -2, \quad \mathrm{Ric}(\partial_2, \partial_2) = 2, \quad \mathrm{Ric}(\partial_3, \partial_3) = -2, \\ &\mathrm{Ric}^*(\partial_1, \partial_1) = 1, \quad \mathrm{Ric}^*(\partial_2, \partial_2) = -1, \quad \mathrm{Ric}^*(\partial_3, \partial_3) = 0. \end{aligned} \tag{70}$$

If we choose $V = \partial_1 - \partial_3$, then we see that

$$\begin{aligned} (L_V g)(\partial_1, \partial_1) &= V g(\partial_1, \partial_1) - 2g(L_V \partial_1, \partial_1) \\ &= -2g([\partial_1 - \partial_3, \partial_1], \partial_1) = -2. \end{aligned} \tag{71}$$

Thus, V is not a Killing vector field. Now, the $*$-Ricci soliton equation

$$(L_V g)(\partial_1, \partial_1) + 2\mathrm{Ric}^*(\partial_1, \partial_1) = 2\lambda g(\partial_1, \partial_1),$$

gives $\lambda = 0$ (we know that if paraKenmotsu manifold admits $*$-Ricci soliton, then $\lambda = 0$). Similarly, we can check the other components and verify that M satisfies:

$$(L_V g)(X, Y) + 2\mathrm{Ric}^*(X, Y) = 2\lambda g(X, Y).$$

Hence, the metric g is a $*$-Ricci soliton. Using (70), we have constant scalar curvature as follows:

$$r = \mathrm{Ric}(\partial_1, \partial_1) - \mathrm{Ric}(\partial_2, \partial_2) + \mathrm{Ric}(\partial_3, \partial_3) = -6. \tag{72}$$

Because of scalar curvature $r = -6$, from Theorem 3.4, we can conclude that M is an Einstein manifold. It is easy to verify that the manifold is locally φ-symmetric. Hence, the results of Theorem 3.4 and Corollary 3.6 are verified.

Acknowledgements The authors would like to thank the anonymous referee for his or her valuable suggestions that have improved the original paper.

References

1. Bejan, C.L.; Crasmareanu, M.: Second order parallel tensors and Ricci solitons in 3-dimensional normal paracontact geometry. Ann. Glob. Anal. Geom. **46**, 117–127 (2014)
2. Blaga, A.M.: Eta-Ricci solitons on para-Kenmotsu manifolds. Balk. J. Geom. Appl. **20**(1), 1–13 (2015)
3. Blair, D.E.: Riemannian geometry of contact and symplectic manifolds, vol. 203. Birkhauser, Boston (2002)
4. Calvaruso, G.: Homogeneous paracontact metric three-manifolds. Ill. J. Math. **55**, 697–718 (2011)
5. Calvaruso, G.; Perrone, A.: Ricci solitons in three-dimensional paracontact geometry. J. Geom. Phys. **98**, 1–12 (2015)
6. Calvaruso, G.; Perrone, D.: Geometry of H-paracontact metric manifolds Publ. Math. Debr. **86**, 325–346 (2015)
7. Cartan, E.: Surune classe remarquable d' espaces de riemannian. Bulletin de la Societe Mathematique de France **54**, 214–264 (1926)
8. Chow, B.; Knopf, D.: The Ricci Flow: An Introduction, Mathematical Surveys and Monographs, vol. 110. American Mathematical Society, Providence (2004)
9. Dacko, P.: On almost para-cosymplectic manifolds. Tsukuba J. Math. **28**, 193–213 (2004)
10. De, U.C.: Ricci soliton and gradient Ricci soliton on P-Sasakian manifolds. Aligarh Bull. Math. **29**, 29–34 (2010)
11. Ghosh, A.: Certain results on three-dimensional contact metric manifolds. Asian-Eur. J. Math. **3**(4), 577–591 (2010)
12. Ghosh, A.; Patra, D.S.: $*$-Ricci Soliton within the framework of Sasakian and (κ, μ)-contact manifold. Int. J. Geom. Methods Mod. Phys. https://doi.org/10.1142/S0219887818501207
13. Hamada, T.: Real hypersurfaces of complex space forms in terms of Ricci $*$-tensor. Tokyo J. Math. **25**, 473–483 (2002)
14. Hamada, T.; Inoguchi, J.I.: Real hypersurfaces of complex space forms with symmetric Ricci $*$-tensor. Mem. Fac. Sci. Eng. Shimane Univ. Ser. B Math. Sci. **38**, 1–5 (2005)
15. Hamilton, R.S.: The Ricci flow on surfaces. In: Mathematics and General Relativity (Santa Cruz, CA, 1986), Contemp. Math., vol. 7. American Mathematical Society, Providence (1988)
16. Kaimakamis, G.; Panagiotidou, K.: $*$-Ricci solitons of real hypersurface in non-flat complex space forms. J. Geom. Phy. **76**, 408–413 (2014)
17. Kaneyuki, S.; Williams, F.L.: Almost paracontact and parahodge structures on manifolds. Nagoya Math. J. **99**, 173–187 (1985)
18. Kenmotsu, K.: A class of almost contact Riemaniian manifolds. Tohoku Math. J. **24**, 93–103 (1972)

19. Naik, D.M.; Venkatesha, V.: η-Ricci soliton and almost η-Ricci soliton on paraSasakian manifolds. Int. J. Geom. Methods Mod. Phys. (2019). https://doi.org/10.1142/S0219887819501342
20. O'Neill, B.: Semi-Riemannian Geometry with Applications to the Relativity. Academic Press, New York (1983)
21. Perrone, A.: Some results on almost paracontact metric manifolds. Mediterr. J. Math. **13**, 3311–3326 (2016)
22. Prakasha, D.G.; Vikas, K.: On ϕ-recurrent para-Kenmotsu manifolds. Int. J. Pure. Eng. Math. **3**(2), 17–26 (2015)
23. Prakasha, D.G.; Hadimani, B.S.: η-Ricci solitons on para-Sasakian manifolds. J. Geom. **108**, 383–392 (2017)
24. Prakasha, D.G.; Veeresha, P.: ParaSasakian manifolds and $*$-Ricci solitons. arXiv:1801.01727. (2018)
25. Sai Prasad, K.L.; Satyanarayana, T.: On para-Kenmotsu manifold. Int. J. Pure Appl. Math. **90**(1), 35–41 (2014)
26. Sarkar, A.; Paul, A.K.; Mondal, R.: On α-para Kenmotsu 3-manifolds with Ricci solitons. Balk. J. Geom. Appl. **23**(1), 100–112 (2018)
27. Sato I.: On a structure similar to the almost contact structure I. Tensor N.S. **30**, 219–224 (1976)
28. Singh, A.; Kishor, S.: Certain results on para-Kenmotsu manifolds equipped with M-projective curvature tensor. Tbilisi Math. J. **11**(3), 125–132 (2018)
29. Sinha, B.B.; Prasad, K.L.: A class of almost paracontact metric manifold. Bull. Culcutta Math. Soc. **87**, 307–312 (1995)
30. Tachibana, S.: On almost-analytic vectors in almost Kahlerian manifolds. Tohoku Math. J. **11**, 247–265 (1959)
31. Takahashi, T.: Sasakian manifold with pseudo-Riemannian metric. Tohoku Math. J. **21**(2), 644–653 (1969)
32. Venkatesha, V.; Naik, D.M.: Certain results on K-paracontact and paraSasakian manifolds. J. Geom. **108**, 939–952 (2017)
33. Venkatesha, V.; Naik, D.M.; H.A. Kumara.: $*$-Ricci solitons and gradient almost $*$-Ricci soliton on Kenmotsu manifolds. Math. Slovaca **(accepted)**. arXiv:1901.05222
34. Welyczko, J.: Slant curves in 3-dimensional normal almost paracontact metric manifolds. Mediterr. J. Math. **11**(965), (2014) https://doi.org/10.1007/s00009-013-0361-2
35. Wełyczko, J.: On basic curvature identities for almost (para)contact metric manifolds. Available in arxiv:1209.4731v1 [math.DG]
36. Yano, K.: Integral Formulas in Riemannian Geometry. Marcel Dekker, New York (1970)
37. Zamkovoy, S.: Canonical connections on paracontact manifolds. Ann. Glob. Anal. Geom. **36**, 37–60 (2009)
38. Zamkovoy, S.: On paraKenmotsu manifolds. Filomat **32**(14), (2018)