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On compact and bounded embedding in variable exponent Sobolev spaces and its applications

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Abstract For a weighted variable exponent Sobolev space, the compact and bounded embedding results are proved. For that, new boundedness and compact action properties are established for Hardy's operator and its conjugate in weighted variable exponent Lebesgue spaces. Furthermore, the obtained results are applied to the existence of positive eigenfunctions for a concrete class of nonlinear ode with nonstandard growth condition.

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1 Introduction

Dirichlet's problem for a class of nonlinear differential equations with nonstandard growth condition is a subject of a study of boundedness and compactness results in variable exponent Lebesgue and Sobolev spaces. In this paper, the exponent functions are characterized for the weighted Hardy's operator to be bounded and compact, to get its application to the solvability problem of the first boundary value problem for a concrete class of nonlinear ode coming from the physics.

Mostly log-regularity condition near origin and infinity is considered in a study the boundedness and compactness results for Hardy's operator in weighted variable exponent Lebesgue spaces (see, e.g., [4–6,13–15]). The originality of the present study placed also in that, we do not use traditional logarithmic regularity condition for the exponent functions. In place, the conditions of almost decreasing (a.d.) and (or) almost increasing (a.i.) are assumed near the origin and l. The idea of use a.i. (or a.d.) condition is new and essentially comes from [9,11,12]. The cited studies show effectiveness of this conditions (a.i. or a.d.) in study the boundedness and compactness properties of Hardy's operator in variable exponent Lebesgue space.

The equations with nonstandard growth condition appear, e.g., in modeling the so-called "Winslow effect" phenomena for smart materials [20]. For solvability of the arising nonlinear differential equations, Ambrosetti–Rabinovich's mountain pass theorem approach turns out fruitful (see, e.g., [3,18,19]). In addition, the variable

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exponent and variable order approaches find application in the theory of nonlinear pde and modeling of different physical phenomena of modern applied science (see, e.g., [1,2,10,16,17,21,22]).

In light of the mentioned results on problem (1), Theorem 3.6 turns out to be actual, since it states $\lambda_1 = 0$ for the eigenvalue problem (2) (since for any $\lambda > 0$, there exist a solution of the eigenvalue problem). According to [7], if $p^- > 1$, then there are a sequence of discreet eigenvalues λ_n with $\lambda_\infty = \limsup \lambda_n = \infty$ and $\lambda_1 = \liminf \lambda_n \geq 0$ of the eigenvalue problem:

$$\begin{cases} \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) + \lambda |u|^{p(x)-2}u = 0 \quad \text{in} \quad \Omega \subset \mathbb{R}^N, \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

which implies that the list eigenvalue may be equal to zero. In [7], it was proved that this problem may has $\lambda_1 = 0$ provided that there exists an open set $U \subset \Omega$ and a point $x_0 \in U$, such that $p(x_0) <$ (or $>$) $p(x)$ for all $x \in \partial\Omega$.

Note, the list eigenvalue of the problem (1) is positive in the case of constant exponent (or according to [8], for one-dimensional case with monotony variable exponent $p(x)$).

2 Notation, definitions

For $1 < p < \infty$, the p' denotes conjugate number, $\frac{1}{p} + \frac{1}{p'} = 1$; for $p = \infty$, the $p' = 1$, and for $p = 1$, the $p' = \infty$. The notations $p^+ = \sup_{t\in(0,l)} p(t)$ and $p^- = \inf_{t\in(0,l)} p(t)$ are used to denote essential maximum and minimum values of a measurable function $p(\cdot)$. χ_E-denotes the characteristic function of set E.

$C, C_1, C_2, \ldots$ denote different constants, the values of which are not essential and may be varied in each appearance.

Denote $Hf(x) = \int_0^x f(t)\mathrm{d}t$—the Hardy operator and $H^* f(x) = \int_x^l f(t)\,\mathrm{d}t$—its conjugate.

We say that the function $g : (0,l) \to (0,\infty)$ is almost increasing (decreasing) if there exists a constant $C > 0$, such that for any $0 < t_1 < t_2 < l$, the inequality $g(t_1) \leq Cg(t_2)$ $(g(t_1) \geq Cg(t_2))$ holds.

Define the following variable exponent spaces that will be used in this paper. For a function $f(x)$ and the exponent $p(x)$, define the modular

$$I_{p(\cdot)}(f) = \int_0^l |f(x)|^{p(x)}\,\mathrm{d}x.$$

The variable exponent Lebesgue space $L^{p(\cdot)}(0,l)$ is a space of measurable functions $f : (0,l) \to \mathbb{R}^n$ with finite norm:

$$\|f(x)\|_{L^{p(\cdot)}(0,l)} = \inf\left\{\lambda > 0 : I_{p(\cdot)}\left(\frac{f(x)}{\lambda}\right) \leq 1\right\}.$$

Denote $L^{p(\cdot),\beta}(0,l)$ the space of measurable functions in $(0,l)$ with finite norm $\|x^\beta f(x)\|_{L^{p(\cdot)}(0,l)}$.

$W_\beta^{1,p(\cdot)}(0,l)$ denotes a Sobolev space of absolutely continuous functions $f : (0,l) \to \mathbb{R}$, $f(0) = 0$ endowed with a norm:

$$\|f\|_{W_\beta^{1,p(\cdot)}(0,l)} = \|x^\beta f'(x)\|_{L^{p(\cdot)}(0,l)}.$$

Denote $\tilde{L}^{p,\beta}(0,l)$ a space of measurable functions with finite norm $\|(xl - x^2)^\beta f(x)\|_{L^{p(\cdot)}(0,l)}$. Denote $\tilde{W}_\beta^{1,p(\cdot)}(0,l)$ a Sobolev space of absolutely continuous functions on $(0,l)$ with $y(0) = y(l) = 0$ and having a finite norm:

$$\|y\|_{\tilde{W}_\beta^{1,p(\cdot)}(0,l)} = \left\|d(x)^\beta \frac{\mathrm{d}y}{\mathrm{d}x}\right\|_{L^{p(\cdot)}(0,l)},$$

where $d(x) = \min\{x, l-x\}$. Since $xl - x^2$ for $0 < x < l$ is equivalently to $ld(x)$, sometimes, we may use expression $lx - x^2$ in place of $ld(x)$.

Denote $\bar{W}_\beta^{1,p(\cdot)}(0,l)$ a closure of $C_0^\infty(0,l)$ functions in the norm of space $\tilde{W}_\beta^{1,p(\cdot)}(0,l)$.

Definition Consider the eigenvalue problem:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}x}\left(d(x)^{\beta p(x)}\left|\frac{\mathrm{d}y}{\mathrm{d}x}\right|^{p(x)-2}\frac{\mathrm{d}y}{\mathrm{d}x}\right) + \lambda\, b(x)|y(x)|^{q(x)-2}y(x)d(x)^{q(x)(\beta-\frac{1}{p'(x)}-\frac{1-\varepsilon}{q(x)})} = 0, \\ 0 < x < l,\ \ y(0) = y(l) = 0,\ \ \lambda \in \mathbb{R}, \end{cases} \tag{2}$$

where $b(x)$ is a positive bounded measurable function on $(0, l)$.

We say that the function $y = y(x)$ is a solution of the preceding problem if $y \in \bar{W}_\beta^{1,p(\cdot)}(0, l)$ and for any $v \in \bar{W}_\beta^{1,p(\cdot)}(0, l)$, it holds the identity

$$\int_0^l d(x)^{\beta p(x)}\left|y'\right|^{p(x)-2}y'(x)v'(x)\mathrm{d}x + \lambda \int_0^l b(x)|y(x)|^{q(x)-2}y(x)v(x)d(x)^{q(x)\left(\beta-1/p'(x)-(1+\varepsilon)/q(x)\right)}\,\mathrm{d}x = 0. \tag{3}$$

3 Main results

Following main results are obtained in this paper.

Theorem 3.1 *Let $q, p : (0, l) \to (1, \infty)$ be measurable functions on $(0, l)$, such that*

$$1 < p^- \le p(x) \le q(x) \le q^+ < \infty \tag{4}$$

and

$$\beta < 1 - \frac{1}{p^-}. \tag{5}$$

Assume that p be monotony increasing near origin and there exists $\varepsilon > 0$, such that the function $x^{\beta-\frac{1}{p'(x)}+\varepsilon}$ a.d. on a little δ-neighborhood of origin.

Then, operator H acts boundedly from $L^{p(\cdot),\beta}(0, l)$ into $L^{q(\cdot),\beta-\frac{1}{p'(\cdot)}-\frac{1}{q(\cdot)}}(0, l)$. Moreover, the norm of mapping depends on p^-, p^+, ε, β, δ.

For any absolutely continues function, $y : (0, l) \to \mathbb{R}$ with $y(0) = 0$ Theorem 3.1 immediately gives the inequality:

$$\left\| x^{\beta-\frac{1}{p'(x)}-\frac{1}{q(x)}}y(x)\right\|_{L^{q(x)}(0,l)} \le C\left\|x^\beta y'(x)\right\|_{L^{p(x)}(0,l)}, \tag{6}$$

i.e., the following assertion takes place.

Theorem 3.2 *Let $q, p : (0, l) \to (1, \infty)$ be measurable functions satisfying (4) and (5). Let p be monotony increasing near origin and there exists $\varepsilon > 0$, such that the function $x^{\beta-\frac{1}{p'}+\varepsilon}$ a.d. on a little δ neighborhood of origin.*

Then, the identity operator maps boundedly space of functions $y \in \bar{W}_\beta^{1,p(\cdot)}(0, l)$ with $y(0) = 0$ into $L^{q(\cdot),\beta-\frac{1}{p'(\cdot)}-\frac{1}{q(\cdot)}}(0, l)$. Moreover, the norm of mapping is estimated by a constant depending on $p(\cdot), q(\cdot), \varepsilon, \delta, \beta$.

Theorem 3.3 *Let $q, p : (0, l) \to (1, \infty)$ be measurable functions satisfying (4) and (5). Let $p(\cdot)$ be increasing near origin and there exists $\varepsilon > 0$ such that $x^{\beta-\frac{1}{p'(x)}+\varepsilon}$ a.d. a little δ-neighborhood of origin.*

Then, operator H acts compactly $L^{p(\cdot),\beta}(0, l)$ into $L^{q(\cdot),\beta-\frac{1}{p'(\cdot)}-\frac{1-\varepsilon}{q(\cdot)}}(0, l)$.

Below using Theorem 3.1, 3.2 we prove the next assertion.

Theorem 3.4 *Let $q, p : (0, l) \to (1, \infty)$ be measurable functions satisfying (4) and (5). Let p be monotone increasing near origin and decreasing near l. In addition, assume that there exists $\varepsilon > 0$, such that $x^{\beta - \frac{1}{p'(\cdot)} + \varepsilon}$ is a.d. near origin and a.i. near l on a little δ-neighborhood.*

Then, for all absolutely continuous functions $y : (0, l) \to \mathbb{R}$ with $y(0) = y(l) = 0$, it holds

$$\left\| d(x)^{\beta - \frac{1}{p'(x)} - \frac{1}{q(x)}} y \right\|_{L^{q(\cdot)}(0,l)} \le C \left\| d(x)^{\beta} y' \right\|_{L^{p(\cdot)}(0,l)},$$

where the constant $C > 0$ depends on $p(\cdot), q(\cdot), \beta, \delta, \varepsilon$.

Theorem 3.5 *Let $q, p : (0, l) \to (1, \infty)$ be measurable functions satisfying (4) and (5). Let $p(\cdot)$ be a monotone increasing near origin, and decreasing near l. Assume that there exists $\varepsilon > 0$, such that the function $x^{\beta - \frac{1}{p'(\cdot)} + \varepsilon}$ be a.d. near origin, and a.i. near l on a little δ-neighborhood. Then, the identity operator maps compactly $\tilde{W}_{\beta}^{1,p(\cdot)}(0, l)$ to $\tilde{L}^{q(\cdot), \beta - \frac{1}{p'(\cdot)} - \frac{1-\varepsilon}{q(\cdot)}}(0, l)$.*

The proof of Theorem 3.5 is similarly to the proof of Theorem 3.4.

The following assertion takes place for the eigenvalue problem (2).

Theorem 3.6 *Let $q, p : (0, l) \to (1, \infty)$ be measurable functions satisfying*

$$1 < p^- \le p(x) \le p^+ < q^- \le q(x) \le q^+ < \infty \tag{7}$$

and the real number β satisfies (5). Assume that $p(x)$ increases near origin and decreases near l. Furthermore, there exist $\varepsilon > 0$, such that the function $x^{\beta - \frac{1}{p'(x)} + \varepsilon}$ a.d. near origin and a.i. near l on a δ -neighborhood.

Then, for any $\lambda > 0$, there exist a nontrivial positive solution of the eigenvalue problem (2) in space $\bar{W}_{\beta}^{1,p(\cdot)}(0, l)$.

4 Proof of the results

To start the proof of Theorem 3.1, we need on the next assertion.

Lemma 4.1 *Let the conditions of* Theorem 3.1 *be satisfied, that is, $p(x)$ increases in $(0, l)$ and be a.d. near origin. There exists $\varepsilon > 0$, such that the function $x^{\beta - \frac{1}{p'(x)} + \varepsilon}$ a.d. on a little δ-neighborhood of origin. Let $t \in A_n(x) = (2^{-n-1}x, 2^{-n}x]$ and $x \in (0, l)$.*

Then, it holds

$$t^{-\frac{1}{p'(t)}} \le C t^{-\frac{1}{(p_{x,n}^-)'}},$$

where $p_{x,n}^- = \inf\limits_{t \in A_n(x)} p(t)$.

Proof of Lemma 4.1 Let $y \in A_n(x)$ be a point, where $t^{-\frac{1}{p'(y)}} \le 2t^{-\frac{1}{(p_{x,n}^-)'}}$. Let $y < t$ and both points t, y lie in $A_n(x)$. Applying a.d. of the function $x^{\beta - \frac{1}{p'(x)} + \varepsilon}$, it follows:

$$t^{\beta - \frac{1}{p'(t)} + \varepsilon} \le C\, y^{\beta - \frac{1}{p'(y)} + \varepsilon}.$$

In addition, using $t, y \in A_n(x)$ and $(p_{x,n}^-)' > 1$ it follows

$$t^{-\frac{1}{p'(t)}} \le 2^{\beta+\varepsilon} C\, y^{-\frac{1}{p'(y)}} \le 2^{\beta+2+\varepsilon} C\, t^{-\frac{1}{(p_{x,n}^-)'}}.$$

If $y > t$ using increasing p, $\frac{1}{p'}$ also will be increasing. Since $\frac{1}{p'(t)} < \frac{1}{p'(y)}$, it follows

$$\left(\frac{1}{t}\right)^{\frac{1}{p'(t)}} \le C \left(\frac{1}{t}\right)^{\frac{1}{p'(y)}} \le 2C t^{-\frac{1}{(p_{x,n}^-)'}},$$

where $C = l^{\frac{1}{p^{-\prime}}} + l^{\frac{1}{p^{+\prime}}}$.

Lemma 4.1 has been proved.

Proof of Theorem 3.1 Let $f : (0, l) \to (0, \infty)$ be a positive measurable function. It holds the identity

$$Hf(x) = \sum_{n=1}^{\infty} \int_{2^{-n-1}x}^{2^{-n}x} f(t)\mathrm{d}t,\ x \in (0, l). \tag{8}$$

Assume $\|t^{\beta} f(t)\|_{L^{p(\cdot)}(0,l)} = 1$. Using the triangle property of $p(\cdot)$ norms

$$\left\| x^{\beta - \frac{1}{p'(x)} - \frac{1}{q(x)}} Hf \right\|_{L^{q(\cdot)}(0,l)} \le \sum_{n=1}^{\infty} \left\| x^{\beta - \frac{1}{p'(x)} - \frac{1}{q(x)}} \int_{A_n(x)} f(t)\mathrm{d}t \right\|_{L^{q(\cdot)}(0,l)}. \tag{9}$$

Derive estimation for every summand in (9). For this purpose, get an estimation for the proper modular:

$$I_{q(\cdot)}\left(x^{\beta - \frac{1}{p'(x)} - \frac{1}{q(x)}} \int_{A_n(x)} f(t)\mathrm{d}t \right) = \int_0^l \left(x^{\beta - \frac{1}{p'(x)} - \frac{1}{q(x)}} \int_{A_n(x)} f(t)\mathrm{d}t \right)^{q(x)} \mathrm{d}x.$$

Using the assumption on β and almost decreasing of $x^{\beta - \frac{1}{p'} + \varepsilon}$, we have

$$\begin{aligned} I_{q(x)}\left(x^{\beta - \frac{1}{p'(x)} - \frac{1}{q(x)}} \int_{A_n(x)} f(t)\mathrm{d}t \right) &= \int_0^l \left(x^{\beta - \frac{1}{p'(x)} + \varepsilon} \int_{A_n(x)} f(t)\mathrm{d}t \right)^{q(x)} \frac{\mathrm{d}x}{x^{1+\varepsilon q(x)}} \\ &\le C^{q+} 2^{-n\varepsilon q^-} \int_0^l \frac{\mathrm{d}x}{x} \left(\int_{A_n(x)} t^{\beta} f(t) t^{-\frac{1}{p'(t)}} \mathrm{d}t \right). \end{aligned} \tag{10}$$

Notice, applying a.d. of $x^{\beta - \frac{1}{p'(x)} + \varepsilon}$, and Lemma 4.1 it has been used that $x^{\beta - \frac{1}{p'(x)} + \varepsilon} \le C t^{\beta - \frac{1}{p'(t)} + \varepsilon}$ for $2^{-n-1}x < t \le 2^{-n}x$ and $0 < x < l$. Therefore, and applying Hölder's inequality from (10), it follows

$$\begin{aligned} & I_{q(\cdot)}\left(x^{\beta - \frac{1}{p'(x)} - \frac{1}{q(x)}} \int_{A_n(x)} f(t)\mathrm{d}t \right) \\ & \le C^{q^+} 2^{-n\varepsilon q^-} \int_0^l \left(\int_{A_n(x)} \left(t^{\beta} f(t)\right)^{p_{\bar{x},n}^-} \mathrm{d}t \right)^{\frac{q(x)}{p_{\bar{x},n}^-}} \left(\int_{A_n(x)} t^{-\frac{(p_{\bar{x},n}^-)'}{p'(t)}} \mathrm{d}t \right)^{\frac{q(x)}{(p_{\bar{x},n}^-)'}} \frac{\mathrm{d}x}{x}. \end{aligned} \tag{11}$$

Applying Lemma 4.1 and estimate (11), it follows from (10) that

$$I_{q(\cdot)}\left(x^{\beta - \frac{1}{p'(x)} - \frac{1}{q(x)}} \int_{A_n(x)} f(t)\mathrm{d}t \right) \le (C_1 \ln 2)^{\frac{q^+}{p^-}} 2^{-n\varepsilon q^-} C^{q^+} \int_0^l \left(\int_{A_n(x)} \left(t^{\beta} f(t)\right)^{p_{\bar{x},n}^-} \mathrm{d}t \right)^{\frac{q(x)}{(p_{\bar{x},n}^-)}} \frac{\mathrm{d}x}{x}.$$

Since

$$\int_{A_n(x)} \left(t^{\beta} f(t)\right)^{p_{\bar{x},n}^-} \mathrm{d}t \le \int_{A_n(x)} \left(t^{\beta} f(t)\right)^{p(t)} \mathrm{d}t + \int_{A_n(x)} \mathrm{d}t \le 1 + 2^{-n}x \le 1 + 2^{-n}l \le l + 1$$

it follows

$$
\begin{aligned}
& I_{q(\cdot)}\left(x^{\beta-\frac{1}{p'(x)}-\frac{1}{q(x)}}\int\limits_{A_n(x)} f(t)\mathrm{d}t\right)\\
&\le (C_1\ln 2)^{q^+}2^{-n\varepsilon q^-}(l+1)^{q^+}\int\limits_0^l\left(\frac{1}{l+1}\int\limits_{A_n(x)}\left(t^\beta f(t)\right)^{p^-_{x,n}}\mathrm{d}t\right)^{\frac{q(x)}{(p^-_{x,n})}}\frac{\mathrm{d}x}{x}\\
&\le (C\ln 2(l+1))^{q^+}2^{-n\varepsilon q^-}\int\limits_0^l\left(\frac{1}{l+1}\int\limits_{A_n(x)}\left[\left(t^\beta f(t)\right)^{p(t)}+1\right]\mathrm{d}t\right)^{\frac{p(x)}{(p^-_{x,n})}}\frac{\mathrm{d}x}{x}\\
&\le 2^{-n\varepsilon q^-}C_1^{q^+}(l+1)^{q^+-1}\int\limits_0^l\left(\int\limits_{A_n(x)}\left[\left(t^\beta f(t)\right)^{p(t)}+1\right]\mathrm{d}t\right)\frac{\mathrm{d}x}{x}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
& I_{q(\cdot)}\left(x^{\beta-\frac{1}{p'(x)}-\frac{1}{q(x)}}\int\limits_{A_n(x)} f(t)\mathrm{d}t\right)\\
&\le 2^{-n\varepsilon q^-}C_3C^{q^+}\int\limits_0^l\left(\int\limits_{A_n(x)}\left[\left(t^\beta f(t)\right)^{p(t)}+1\right]\mathrm{d}t\right)\frac{\mathrm{d}x}{x}\\
&\le 2^{-n\varepsilon q^-}C_3\int\limits_0^{2^{-n}l}\left[\left(t^\beta f(t)\right)^{p(t)}+1\right]\left(\int\limits_{2^n t}^{2^{n+1}t}\frac{\mathrm{d}x}{x}\right)\mathrm{d}t\\
&= 2^{-n\varepsilon q^-}C^{q^+}C_3\ln 2\int\limits_0^{2^{-n}l}\left[\left(t^\beta f(t)\right)^{p(t)}+1)\right]\mathrm{d}t\\
&\le 2^{-n\varepsilon q^-}C^{q^+}C_3\ln 2(1+2^{-n}l)=C_4 2^{-n\varepsilon q^-}.
\end{aligned}
$$

It has been proved that

$$
I_{q(\cdot)}\left(x^{\beta-\frac{1}{p'(x)}-\frac{1}{q(x)}}\int\limits_{A_n(x)} f(t)\mathrm{d}t\right)\le C_4\,2^{-n\varepsilon q^-},
$$

this implies

$$
\left\|x^{\beta-\frac{1}{p'(x)}-\frac{1}{q(x)}}\int\limits_{A_n(x)} f(t)\mathrm{d}t\right\|_{L^{q(x)}(0,l)}\le C_4^{\frac{1}{q^+}}2^{-n\varepsilon\frac{q^-}{q^+}}. \tag{12}
$$

Inserting (12) in (9), we get

$$
\left\|x^{\beta-\frac{1}{p'(x)}-\frac{1}{q(x)}}\int\limits_{A_n(x)} Hf\right\|_{L^{q(x)}(0,l)}\le C_4^{\frac{1}{q^+}}\sum_{n=1}^{\infty}2^{-n\varepsilon\frac{q^-}{q^+}}=C_5.
$$

Theorem 3.1 has been proved.

Proof of Theorem 3.3 To proof Theorem 3.3, we apply the approaches, e.g., in [5,6]. Insert the operators:

$$P_1 f(x) = \chi_{(0,a)}(x)\, x^{\beta - \frac{1}{p'(x)} - \frac{1-\varepsilon}{q(x)}} \int_0^x f(t)\,\mathrm{d}t;$$

$$P_2 f(x) = \chi_{(a,l)}(x)\, x^{\beta - \frac{1}{p'(x)} - \frac{1-\varepsilon}{q(x)}} \int_0^a f(t)\mathrm{d}t;$$

$$P_3 f(x) = \chi_{(a,l)}(x)\, x^{\beta - \frac{1}{p'(x)} - \frac{1-\varepsilon}{q(x)}} \int_x^a f(t)\,\mathrm{d}t.$$

As it was stated in [5], P_3 is a limit of finite rank operators, while P_2 is a finite rank operator. From Theorem 3.1, it follows that

$$\|Hf - P_2 f - P_3 f\|_{L^{q(\cdot)}(0,l)} \le \|P_1 f\|_{L^{q(\cdot)}(0,l)} \le C a^{\frac{\varepsilon}{p^+}} \left\|x^\beta f(x)\right\|_{L^{p(\cdot)}(0,l)}$$

or

$$\|H - P_2 - P_3\|_{L^{p(\cdot),\beta} \to L^{q(\cdot),\beta - \frac{1}{p'(\cdot)} - \frac{1-\varepsilon}{q(\cdot)}}}$$
$$\le \|P_1\|_{L^{p(\cdot),\beta} \to L^{p(\cdot),\beta - \frac{1}{p'(\cdot)}\frac{1-\varepsilon}{q(\cdot)}}} \le C a^{\frac{\varepsilon}{p^+}} \to 0 \quad \text{as} \quad a \to 0.$$

This completes the proof of Theorem 3.3. □

Proof of Theorem 3.4 Notice, the inequality

$$\frac{y(x)}{\left(xl - x^2\right)^{\frac{1}{p'(x)} + \frac{1}{q(x)} - \beta}} \le C\left[\frac{y(x)}{x^{\frac{1}{p'(x)} + \frac{1}{q(x)} - \beta}} + \frac{y(x)}{(l-x)^{\frac{1}{p'(x)} + \frac{1}{q(x)} - \beta}}\right], 0 < x < l,$$

where $C > 0$ depends on $l, \beta, p(\cdot), q(\cdot)$. The boundedness in $L^{q(\cdot)}(0,l)$ for the first summand in the right hand side follows from Theorem 3.2, while the boundedness of the second summand easily can be derived using the assertion of Theorem 3.1, i.e., we need to show the inequality:

$$\left\|(l-x)^{\beta - \frac{1}{p'(x)} - \frac{1}{q(x)}}\, y(x)\right\|_{L^{q(\cdot)}(0,l)} \le C \left\|(l-x)^\beta\, y'(x)\right\|_{L^{p(\cdot)}(0,l)}, \ \ y(l) = 0.$$

To prove this inequality is the same to show that

$$\left\|(l-x)^{\beta - \frac{1}{p'(x)} - \frac{1}{q(x)}} \int_x^l g(t)\mathrm{d}t\right\|_{L^{q(\cdot)}(0,l)} \le C \left\|(l-x)^\beta\, g(x)\right\|_{L^{p(\cdot)}(0,l)}$$

for any positive measurable function $g : (0,l) \to (0,\infty)$.

Using the definition of variable exponent norm, we have

$$\left\|(l-x)^\beta g(x)\right\|_{L^{p(\cdot)}(0,l)} = \inf\left\{\lambda > 0 : \int_0^l \left|\frac{(l-x)^\beta g(x)}{\lambda}\right|^{p(x)} \mathrm{d}x \le 1\right\}$$

(inserting $g(x) = f(l-x)$)

$$= \inf\left\{\lambda > 0 : \int_0^l \left|\frac{(l-x)^\beta f(l-x)}{\lambda}\right|^{p(x)} \mathrm{d}x \le 1\right\}$$

(changing the variable of integration $y = l - x$)

$$= \inf\left\{\lambda > 0 : \int_0^l \left|\frac{y^\beta f(y)}{\lambda}\right|^{p(l-y)} \mathrm{d}y \le 1\right\} = \left\|y^\beta f(y)\right\|_{L^{\tilde{p}(y)};(0,l)},$$

where $\tilde{p}(x) = p(l - x)$.

On the other hand

$$\left\|(l-z)^{\beta - \frac{1}{p'(z)} - \frac{1}{q(z)}} \int_z^l g(t)\mathrm{d}t\right\|_{L^{q(\cdot)}(0,l)}$$

$$= \inf\left\{\lambda > 0 : \int_0^l \left|\lambda^{-1}(l-z)^{\beta - \frac{1}{p'(z)} - \frac{1}{q(z)}} \int_z^l g(t)\mathrm{d}t\right|^{q(z)} \mathrm{d}x \le 1\right\},$$

inserting $g(t) = f(l - t)$ in the interior integral:

$$= \inf\left\{\lambda > 0 : \int_0^l \left|\lambda^{-1}(l-z)^{\beta - \frac{1}{p'(z)} - \frac{1}{q(z)}} \int_z^l f(l-t)\mathrm{d}t\right|^{q(z)} \mathrm{d}z \le 1\right\},$$

changing the variable $y = l - t$:

$$= \inf\left\{\lambda > 0 : \int_0^l \left|\lambda^{-1}(l-z)^{\beta - \frac{1}{p'(z)} - \frac{1}{q(z)}} \int_0^{l-z} f(y)\mathrm{d}y\right|^{q(z)} \mathrm{d}z \le 1\right\},$$

changing the variable $z = l - x$,

$$= \inf\left\{\lambda > 0 : \int_0^l \left|\lambda^{-1}x^{\beta - \frac{1}{p'(l-x)} - \frac{1}{q(l-x)}} \int_0^x f(y)\mathrm{d}y\right|^{q(l-x)} \mathrm{d}x \le 1\right\}$$

$$= \inf\left\{\lambda > 0 : \int_0^l \left|\lambda^{-1}x^{\beta - \frac{1}{p'(l-x)} - \frac{1}{q(l-x)}} \int_0^l f(y)\mathrm{d}y\right|^{\tilde{q}(x)} \mathrm{d}x \le 1\right\}$$

$$= \left\|x^{\beta - \frac{1}{\tilde{p}'(x)} - \frac{1}{\tilde{q}(x)}} \int_0^x f(y)\mathrm{d}y\right\|_{L^{\tilde{q}(\cdot)}(0,l)}.$$

Now, since the functions $\tilde{p}, \tilde{q}$ satisfies all conditions of Theorem 3.1, we get

$$\left\|(l-x)^{\beta - \frac{1}{p'(x)} - \frac{1}{q(x)}} \int_x^l g(t)\mathrm{d}t\right\|_{L^{q(\cdot)}(0,l)} = \left\|t^{\beta - \frac{1}{\tilde{p}'(x)} - \frac{1}{\tilde{q}(x)}} \int_0^t f(y)\mathrm{d}y\right\|_{L^{\tilde{q}(\cdot)}(0,l)}$$

$$\le C\left\|x^\beta f\right\|_{\tilde{p};(0,l)} = C\left\|(l-x)^\beta g\right\|_{L^{q(\cdot)}(0,l)}.$$

Note, we have used that the condition $\beta < 1 - \frac{1}{\tilde{p}^-}$ is the same condition $\beta < 1 - \frac{1}{p^-}$. This completes the proof of inequality:

$$\left\|(l-x)^{\beta - \frac{1}{p'(x)} - \frac{1}{q(x)}} \int_x^l g(t)\mathrm{d}t\right\|_{L^{q(\cdot)}(0,l)} \le C\left\|(l-x)^\beta g(x)\right\|_{L^{q(\cdot)}(0,l)}.$$

□

Proof of Theorem 3.6 To prove this assertion, we shall use the well-known mountain pass theorem approaches. Set $E = \bar{W}_{\beta}^{1,p(\cdot)}(0, l)$. Define the functional

$$I_\lambda(y) = \int_0^l \frac{1}{p(x)}\Big(\left|y'(x)\right| (xl - x^2)^\beta\Big)^{p(x)} \mathrm{d}x - \lambda \int_0^l \frac{b(x)}{q(x)}\Big((xl - x^2)^{\beta - \frac{1}{p'(x)} - \frac{1-\varepsilon}{q(x)}} y_+(x)\Big)^{q(x)} \mathrm{d}x.$$

Using the standard argues (see, e.g., [19]), it is not difficult to see that the functional has Gateaux derivative and $I_\lambda \in C^1(E, R)$. It means $I'_\lambda \in E^*$, and $I'_\lambda : E \to E^*$ continuous. Furthermore, for $\forall v \in E$

$$\begin{aligned}\left\langle I'_\lambda(y), v\right\rangle &= \int_0^l (xl - x^2)^{\beta p(x)} \left|y'(x)\right|^{p(x)-2} y'(x)v'(x) \\ &\quad -\lambda \int_0^l b(x)(xl - x^2)^{\left(\beta - \frac{1}{p'(x)} - \frac{1-\varepsilon}{q(x)}\right)q(x)} y_+^{q(x)-1} v(x)\,\mathrm{d}x.\end{aligned}$$

Palais–Smale condition. Show that Palais–Smale (PS) condition is satisfied for the problem (2). Let $\{y_n\} \in E$ be a sequence satisfying the conditions:

1. $|I_\lambda(y_n)| \le M$;
2. $\left\|I'(y_n)\right\|_{E^*} \to 0$ as $n \to \infty$.

To show PS condition, we should prove the sequence $\{y_n\} \in E$ is compact, i.e., contains a converging in E subsequence $y_{n_k} \to y \in E$.

To show it, first, establish the boundedness of $\{y_n\}$ in E. Using 1), it follows

$$\int_0^l \frac{\left|(xl - x^2)^\beta \left|y'_n\right|\right|^{p(x)}}{p(x)} \mathrm{d}x - \int_0^l \frac{\lambda b(x)}{q(x)} \left[(xl - x^2)^{\beta - \frac{1}{p'(x)} - \frac{1-\varepsilon}{q(x)}} (y_n)_+\right]^{q(x)} \mathrm{d}x \le M.$$

Then

$$\frac{1}{p^+} \int_0^l (xl - x^2)^{\beta p(x)} \left|y'_n\right|^{p(x)} \mathrm{d}x \le \frac{\lambda}{q^-} \int_0^l b(x) \left((xl - x^2)^{\beta - \frac{1}{p'(x)} - \frac{1-\varepsilon}{q(x)}} (y_n)_+\right)^{q(x)} \mathrm{d}x + M. \tag{13}$$

On the other hand, using condition 2), $\|I'_\lambda(y_n)\|_{E^*} = o(1)$ as $n \to \infty$. It means

$$\left\langle I'_\lambda(y_n), v\right\rangle = o(1)\ \|v\|_E\,,\ \ \forall v \in E. \tag{14}$$

In particular, inserting $v = y_n$, we get

$$\left\langle I'_\lambda(y_n), y_n\right\rangle = o(1)\ \|y_n\|_E\,,$$

that is

$$\begin{aligned}&\lambda \int_0^l b(x) \left((xl - x^2)^{\beta - \frac{1}{p'(x)} - \frac{1-\varepsilon}{q(x)}}\right)^{q(x)} (y_n)_+^{q(x)-1} y_n \mathrm{d}x \\ &\quad = o(1)\, \|y_n\|_E + \int_0^l \left((xl - x^2)^\beta \left|y'_n\right|\right)^{p(x)} \mathrm{d}x\end{aligned}$$

Inserting this, it follows

$$\left(\frac{1}{p^+} - \frac{1}{q^-}\right) \int_0^l \left((xl - x^2)^\beta \left|y'_n\right|\right)^{p(x)} \mathrm{d}x \le M + \frac{o(1)}{q^-}\, \|y_n\|_E$$

From this, since $q^- > p^+$, it follows

$$\int_0^l \left((xl-x^2)^\beta \left|y_n'\right|\right)^{p(x)} \mathrm{d}x \le \frac{2Mp^+q^-}{q^- - p^+} + \frac{o(1)}{q^-}\left\|y_n\right\|_E$$

or

$$\left\|(xl-x^2)^\beta y_n'(x)\right\|_{L^{p(\cdot)}(0,l)}^{p^-} \le \frac{2Mp^+q^-}{q^- - p^+} + o(1)\left\|y_n\right\|_E .$$

Using Young's inequality and $p^- > 1$ from here, it follows

$$\|(xl-x^2)^\beta y_n'(x)\|_{L^{p(\cdot)}(0,l)} \le C(M). \tag{15}$$

This completes the boundedness of $\{y_n\}$ in E. □

Applying well-known fact, there exists a weak convergent subsequence $y_{n_k} \rightharpoonup y$ in E. Denote it again y_n. It follows from the compact embedding Theorem 3.3 that a strong convergence $y_n \to y$ in $L^{q(\cdot),\beta-\frac{1}{p'(\cdot)}-\frac{1-\varepsilon}{q(\cdot)}}(0,l)$ holds, that is

$$\|y_n - y\|_{L^{q(\cdot),\beta-\frac{1}{p'(\cdot)}-\frac{1-\varepsilon}{q(\cdot)}}(0,l)} \to 0.$$

Now, we are ready to show the strong convergence $y_n \to y$ in E . For this, insert $v = y_n - y$in (14):

$$\int_0^l (xl-x^2)^{\beta p(x)}\left|y_n'\right|^{p(x)-2} y_n'(y_n'-y')$$
$$-\lambda\int_0^l b(x)(xl-x^2)^{\left(\beta-\frac{1}{p'(x)}-\frac{1-\delta}{q(x)}\right)q(x)}(y_n)_+^{q(x)-1}(y_n-y) = o(1)\left\|y_n-y\right\|_E .$$

From this, since $y_n \to y$ in $L^{q(\cdot),\beta-\frac{1}{p'(\cdot)}-\frac{1-\varepsilon}{q(\cdot)}}(0,l)$, and using Holder's inequality, it follows

$$\left|\int_0^l b(x)(xl-x^2)^{\left(\beta-\frac{1}{p'(x)}-\frac{1-\varepsilon}{q(x)}\right)q(x)}(y_n)_+^{q(x)-1}(y_n-y)\right|$$
$$\le C\left\|\left((lx-x^2)^{\beta-\frac{1}{p'(\cdot)}-\frac{1-\varepsilon}{q(\cdot)}}(y_n)_+\right)^{q(x)-1}\right\|_{L^{q'(x)}(0,l)} \|y_n-y\|_{L^{q(\cdot),\beta-\frac{1}{p'(\cdot)}-\frac{1-\varepsilon}{q(\cdot)}}(0,l)}$$
$$= o(1)\left\|(y_n)_+(lx-x^2)^{\beta-\frac{1}{p'(\cdot)}-\frac{1-\varepsilon}{q(\cdot)}}\right\|_{L^{q(\cdot)}(0,l)}^{q^+-1} \to 0,$$

where also has been used Theorem 3.4 and the estimate (15), to assert the bounded ness $\{y_n\}$ in $L^{q(\cdot),\beta-\frac{1}{p'(\cdot)}-\frac{1-\varepsilon}{q(\cdot)}}(0,l)$.

Therefore

$$\int_0^l (lx-x^2)^{\beta p(x)}\left|y_n'\right|^{p(x)-2} y_n'(y_n'-y) = o(1) + o(1)\left\|y_n-y\right\|_E .$$

From this, we infer

$$\int_0^l (lx - x^2)^{\beta p(x)} \left(\left| y_n' \right|^{p(x)-2} y_n' - \left| y_n' \right|^{p(x)-2} y' \right) (y_n' - y')\mathrm{d}x$$

$$+ \int_0^l (lx - x^2)^{\beta p(x)} \left| y_n' \right|^{p(x)-2} y'(y_n' - y')\mathrm{d}x = o(1) + o(1) \left\| y_n - y \right\|_E .$$

Since $y_n \to y$ weakly in E, it holds

$$\int_0^l (lx - x^2)^{\beta p(x)} \left| y_n' \right|^{p(x)-2} y'(y_n' - y')\mathrm{d}x = o(1) \quad as \quad n \to \infty.$$

This ensures that

$$\int_0^l (xl - x^2)^{\beta p(x)} \left(\left| y_n' \right|^{p(x)-2} y_n' - \left| y_n' \right|^{p(x)-2} y' \right) (y_n' - y')\mathrm{d}x$$

$$= o(1) + o(1) \left\| y_n' - y' \right\|_E .$$

In the next, we will apply the following two inequalities:

$$\left(\left| y_n' \right|^{p(x)-2} y_n' - \left| y' \right|^{p(x)-2} y' \right) \left(y_n' - y' \right) \geq \gamma_1(p) \left| y_n' - y' \right|^{p(x)}$$

for $p(x) \geq 2$ and

$$\left(\left| y_n' \right|^{p(x)-2} y_n' - \left| y_n' \right|^{p(x)-2} y' \right) \left(y' - y' \right) \geq \gamma_2(p) \frac{\left| y_n' - y' \right|^2}{\left| y_n' \right|^{2-p} + \left| y_n' \right|^{2-p}}$$

for $1 < p(x) \leq 2$. Then, for the case $p(x) \geq 2$, we get

$$\int_0^l (xl - x^2)^{\beta p(x)} \left| y_n' - y' \right|^{p(x)} \mathrm{d}x = o(1) + o(1) \left\| y_n - y \right\|_E . \tag{16}$$

As to the case $1 < p(x) \leq 2$, we have

$$\int_0^l \frac{(xl - x^2)^{\beta p(x)} \left(y_n' - y' \right)^2}{\left| y_n' \right|^{2-p} + |y'|^{2-p}} \mathrm{d}x \leq o(1) + o(1) \left\| y_n - y \right\|_E .$$

Using Young's inequality from here, it follows that

$$\int_0^l (lx - x^2)^{\beta p(x)} \left| y_n' - y' \right|^{p(x)} \mathrm{d}x \leq \varepsilon_1 \int_0^l \left(\left| y_n' \right| + \left| y' \right| \right)^{p(x)} (xl - x^2)^{\beta p(x)} \mathrm{d}x +$$

$$+C(\varepsilon_1) \int_0^l \frac{\left| y_n' - y' \right|^2 (lx - x^2)^{\beta p}}{\left(\left| y_n' \right| + |y'| \right)^{2-p}} \mathrm{d}x$$

$$\leq M\varepsilon_1 + C(\varepsilon_1) \left(o(1) + o(1) \left\| y_n - y \right\|_E \right), \quad \forall \varepsilon_1 > 0.$$

Therefore

$$\left\| y_n - y \right\|_E^{p^-} \leq M\varepsilon_1 + c(\varepsilon_1) \left(o(1) + o(1) \left\| y_n - y \right\|_E \right),$$

where M does not depend on $n \in N$. This and the above estimation together with Young's inequality yield:

$$\|y_n - y\|_E^{p^-} \leq M\varepsilon_1 + o(1).$$

Therefore, $y_n \to y$ in E.

Now, we are ready to apply the mountain pass theorem. Notice our argues before based on the contrary assumption $\|y_n - y\|_E \nrightarrow 0$. Under it, the estimate was established:

$$\left\|(xl - x^2)^\beta \left(y_n' - y'\right)\right\|_{L^{p(\cdot)}(0,l)}^{p^-} = o(1) + o(1)\left\|(xl - x^2)^\beta (y_n' - y')\right\|_{L^{p(\cdot)}(0,l)}.$$

Therefore, using assumption $p^- > 1$ and Young's inequality, we come to the conclusion:

$$\left\|(xl - x^2)^\beta \left(y_n' - y'\right)\right\|_{L^{p(\cdot)}(0,l)} = o(1),$$

i.e., $y_n \to y$ in E strongly.

This completes the proof of PS-property.

Mountain pass theorem. Apply the Mountain pass theorem to show the existence of solution for the problem (2).

For $\|y\|_E \leq 1$, we have

$$\begin{aligned} I_\lambda(y) \geq \frac{1}{p^+}\int_0^l \left(\frac{\left((xl - x^2)^\beta \left|y'\right|\right)}{\|y\|_E}\right)^{p(x)} \|y\|_E^{p^+} \mathrm{d}x \\ - \frac{\lambda}{q^-}\int_0^l b(x)\left(\frac{(xl - x^2)^{\beta - \frac{1}{p'(x)} - \frac{1-\varepsilon}{q(x)}} y_+}{\|y\|_E}\right)^{q(x)} \|y\|_E^{q^-}. \end{aligned} \quad (17)$$

Using Theorem 3.1,

$$\|y\|_{\tilde{L}^{q(\cdot),\beta - \frac{1}{p'(\cdot)} - \frac{1}{q(\cdot)}}(0,l)} \leq C\left\|y'\right\|_{\tilde{L}^{q(\cdot),\beta}(0,l)}. \quad (18)$$

Then, (18) implies

$$\begin{aligned} I_\lambda(y) &\geq \frac{1}{p^+}\|y\|_E^{p^+} - \frac{\lambda}{q^-}\int_0^l b(x)\left(\frac{(xl - x^2)^{\beta - \frac{1}{p'(x)} - \frac{1}{q(x)}} C(l)\, y_+}{\|y\|_{\tilde{L}^{q(\cdot),\beta - \frac{1}{p'(\cdot)} - \frac{1}{q(\cdot)}}(0,l)}}\right)^{q(x)} \|y\|_E^{q^-} \\ &\geq \frac{1}{p^+}\|y\|_E^{p^+} - \frac{\lambda C_1 C(l)^{q^+}}{q^-}\|y\|_E^{q^-}, \end{aligned}$$

where $C(l) = \max\left(l^{\frac{2\varepsilon}{q^-}}, l^{\frac{2\varepsilon}{q^+}}\right)$. Hence, for $\|y\|_E \leq 1$, it follows

$$\begin{aligned} I_\lambda(y) &\geq \frac{1}{p^+}\|y\|_E^{p^+} - \frac{\lambda C_1 C(l)^{q^+}}{q^-}\|y\|_E^{q^-} \\ &= \|y\|_E^{p^+}\left(\frac{1}{p^+} - \frac{\lambda C_1 C(l)^{q^+} p^+}{q^-}\|y\|_E^{q^- - p^+}\right). \end{aligned}$$

Therefore

$$I_\lambda(y) \geq \|y\|_E^{p^+}\frac{\lambda C_1 C(l)^{q^+}}{q^-}\left(\frac{q^-}{\lambda C_1 C(l)^{q^+} p^+} - \|y\|_E^{q^- - p^+}\right).$$

If we choose the sphere in E as $\|y\|_E = \min\left\{1, \left(\frac{q^-}{2\lambda C_1 C(l)^{q^+} p^+}\right)^{\frac{1}{q^- - p^+}}\right\}$, it follows

$$I_\lambda(y) \geq \left(\frac{q^-}{2\lambda C_1 C(l)^{q^+} p^+}\right)^{\frac{1}{q^- - p^+}} \frac{1}{2}\frac{q^-}{2\lambda C_1 C(l)^{q^+} p^+}.$$

Choose a sphere with radii $R = \left(\frac{1}{2\lambda C_1 C(l)^{q^+} p^+}\right)^{\frac{1}{q^- - p^+}}$ in E to apply the mountain pass theorem.

Now, it remains to find a point $y_0 \in E$ lied out of the ball $B(0, R)$ in E, where $I_\lambda(y_0) < 0$. To show it, apply the fibering method: for $y \in E$ be fixed and sufficiently large $t > 1$, it holds

$$\begin{aligned} I_\lambda(ty) &= \int_0^l \frac{t^{p(x)}}{p(x)} \left|(xl - x^2)^\beta y'\right|^{p(x)} \mathrm{d}x \\ &\quad -\lambda \int_0^l \frac{t^{q(x)}}{q(x)} b(x) \left((xl - x^2)^{\beta - \frac{1}{p'(x)} - \frac{1-\varepsilon}{q(x)}} y_+\right)^{q(x)} \mathrm{d}x \\ &\le \frac{t^{p^+}}{p^-} \int_0^l \left((xl - x^2)^\beta |y'|\right)^{p(x)} \mathrm{d}x \\ &\quad - \frac{t^{q^-}\lambda}{q^+} \int_0^l b(x) \left((xl - x^2)^{\beta - \frac{1}{p'(x)} - \frac{1-\varepsilon}{q(x)}} y_+\right)^{q(x)} \mathrm{d}x < 0. \end{aligned}$$

Applying mountain pass theorem, there exists a point $\tilde{y} \in E$ with $I_\lambda(\tilde{y}) = c$ and $I'_\lambda(\tilde{y}) = 0$. Here

$$c = \inf_{\gamma(t)\in\Gamma} \sup \ \{I_\lambda(\gamma(t))\},$$

where the infimum is taken all over the curves

$$\gamma : [0, 1] \to E, \gamma \in C^1[0, 1; E], \quad \text{and such that} \quad \gamma(0) = 0, \gamma(1) = \tilde{y}.$$

Therefore, $I_\lambda(\tilde{y}) > 0$, $I'(\tilde{y}) = 0$. To show that $\tilde{y}$ is a positive solution of (2), insert $v = \tilde{y}_-$ in $\langle I'\tilde{y}, v\rangle = 0$.

$$\begin{aligned} &\int_0^l (xl - x^2)^{\beta p(x)} \left|\tilde{y}'\right|^{p(x)-2} \tilde{y}'\tilde{y}'_- \\ &\quad -\lambda \int_0^l b(x)(xl - x^2)^{\left(\beta - \frac{1}{p'(x)} - \frac{1-\varepsilon}{q(x)}\right)q(x)} \tilde{y}_+^{q(x)-1} \tilde{y}_- \mathrm{d}x = 0. \end{aligned}$$

Since the second integral is zero ($\tilde{y}_+^{q(x)-1} \tilde{y}_- \equiv 0$), we have

$$0 = \int_0^l (xl - x^2)^{\beta p(x)} \left|\tilde{y}'_-\right|^{p(x)} \mathrm{d}x.$$

Using Theorem 3.1, it follows $\tilde{y}_-(x) \equiv 0$; therefore, $\tilde{y}(x) > 0$.

This completes the proof of Theorem 3.6, and which proves the existence of positive solution for problem (2) for any $\lambda > 0$.

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