Arab. J. Math. (2020) 9:89–99
https://doi.org/10.1007/s40065-018-0236-2
Arabian Journal of Mathematics

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Forward–backward splitting algorithm for fixed point problems and zeros of the sum of monotone operators

Received: 7 February 2018 / Accepted: 29 December 2018 / Published online: 8 January 2019

Abstract In this paper, we construct a forward–backward splitting algorithm for approximating a zero of the sum of an α-inverse strongly monotone operator and a maximal monotone operator. The strong convergence theorem is then proved under mild conditions. Then, we add a nonexpansive mapping in the algorithm and prove that the generated sequence converges strongly to a common element of a fixed points set of a nonexpansive mapping and zero points set of the sum of monotone operators. We apply our main result both to equilibrium problems and convex programming.

Mathematics Subject Classification 47H05 · 47H09

1 Introduction

A very common problem in diverse areas of mathematics and physical sciences consists of finding zero points of some nonlinear operators. For instance, evolution equations, complementarity problems, mini-max problems, variational inequalities and optimization problems; please, see: Kinderlehrer and Stampacchia [19], Kamimura and Takahashi [16], Cho et al. [8], Qin and Su [31] Qin et al. [32] and the references therein. One of the methods of approximating zero points is the proximal point algorithm. This algorithm has been introduced by Martinet [23]. Rockafellar [35] studied the proximal point algorithm for maximal monotone operators to find a zero of the monotone operator. After that, many authors considered this method and studied it and its modified versions in Hilbert and Banach spaces: Burachik and Scheimberg [6], Rouhani and Khatibzadeh [36], Li and Song [20], Alber and Yao [1], Boikanyo and Morosanu [3], Matsushita and Xu [24], Khatibzadeh [18], Dadashi [10,11].

Moreover, Takahashi [39], Yao and Noor [45], Wang and Cui [44], Tian and Wang [41] and Wang and Cui [43] investigated the contraction proximal point algorithm and viscosity approximation method for finding zeros of maximal monotone operators. They proved the strong convergence of this method under some appropriate conditions.

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One of the problems of finding zero points of nonlinear operator is to find zero points of the sum of an α-inverse strongly monotone operator and a maximal monotone operator. There are various applications of the problem of finding zero points of the sum of two operators; see [12,13,28,33,42] for example and the references therein. Passty [30] introduced an iterative method called forward–backward method for finding zero points of the sum of two operators. Moudafi and Oliny [27] considered an algorithm and proved the generated sequence by their algorithm converges weakly to a zero point of the sum of two maximal monotone operators. See also [25].

In recent years, monotone operators have received a lot of attention for treating zero points of monotone operators and fixed point of mappings which are Lipschitz continuous; see [7,17,22,26,29,32,40,47,48] and the references therein.

Very recently, Boikanyo [4] used proximal point algorithm for finding zero points of the sum of two operators such that the sequence of error terms is square summable in norm.

In this paper, we use and generalize the proximal point algorithm in Yao and Shahzad [46] for finding a point in the intersection of fixed points set of the nonexpansive mapping and zero points set of the sum of monotone operators. For the case when the sequence of error terms converges strongly to zero in norm, we prove that the generated sequence by our algorithm converges strongly to zero in norm.

The paper is organized as follows. Section 2 gathers some definitions and lemmas of geometry of Banach spaces and monotone operators, which will be needed in the remaining sections. In Sect. 3, two iterative algorithms are proposed and strong convergence theorems for finding a point in the intersection of fixed point set of a nonexpansive mapping and zero set of sum of two operators are established. Finally, in Sect. 4, the ideas of Sect. 3 are applied to solve equilibrium problems and find the minimizer of a convex function.

2 Preliminaries

In this section, we provide some basic concepts, definitions and lemmas which will be used in the sequel. Let C be a nonempty, closed, and convex subset of a real Hilbert space H with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. For a mapping $T\colon C\to C$, the fixed points set of T is denoted by $F(T)=\{x\in C : Tx=x\}$. We write $x_n\to x$ to indicate that the sequence $\{x_n\}$ strongly converges to x.

Definition 2.1 A multifunction $B\colon H\rightrightarrows 2^H$ is called a *monotone operator* if for every $x,y\in H$,

$$\langle x^*-y^*, x-y\rangle \geq 0, \qquad \forall x^*\in B(x), \quad \forall y^*\in B(y).$$

A monotone operator $B\colon H\rightrightarrows 2^H$ is said to be *maximal monotone*, when its graph is not properly included in the graph of any other monotone operator on the same space. The zero points set of B is denoted by $B^{-1}(0)=\{x\in H:\ 0\in Bx\}$.

Definition 2.2 A single valued operator $A\colon H\to H$ is called *α-inverse strongly monotone* for a positive number α if

$$\langle Ax-Ay, x-y\rangle \geq \alpha\|Ax-Ay\|^2, \qquad \forall x,y\in H.$$

It is easy to see that every α-inverse strongly monotone is monotone and continuous.

Lemma 2.3 [29] *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let the mapping $A\colon C\to H$ be α-inverse strongly monotone and $\lambda>0$ be a constant. Then, we have*

$$\|(I-\lambda A)x-(I-\lambda A)y\|^2 \leq \|x-y\|^2+\lambda(\lambda-2\alpha)\|Ax-Ay\|^2$$

for all $x,y\in C$. In particular, if $0<\lambda\leq 2\alpha$, then $I-\lambda A$ is nonexpansive.

Remark 2.4 Let H be a Hilbert space and $B\colon H\rightrightarrows 2^H$ be a maximal monotone operator. By Theorem 3.4 of Chapter 5 of [9], the equation $0\in\lambda B\tilde{x}+(\tilde{x}-x)$ has a unique solution $x_\lambda\in D(B)$ for every $x\in H$. The operator $J_\lambda\colon H\to D(B)$ defined by $J_\lambda(x)=x_\lambda$ is called the resolvent of B of order λ, which x_λ satisfies $\frac{1}{\lambda}(x-x_\lambda)\in B(x_\lambda)$. Therefore, $\frac{1}{\lambda}(x-J_\lambda(x))\in B(J_\lambda(x))$. Since B is maximal monotone, it is easy to see that J_λ is firmly nonexpansive, and $F(J_\lambda)=(B)^{-1}(0)$.

Lemma 2.5 [2] *Let C be a nonempty, closed, and convex subset of a real Hilbert space H and $A: C \to H$ an operator. If $B: H \to 2^H$ is a maximal monotone operator, then*

$$F(J_\lambda(I - \lambda A)) = (A + B)^{-1}(0).$$

Lemma 2.6 [5] *For $\lambda > 0$, $\mu > 0$ and $x \in H$,*

$$J_\lambda x = J_\mu \left(\frac{\mu}{\lambda} x + \left(1 - \frac{\mu}{\lambda}\right) J_\lambda x \right).$$

Let C be a convex closed subset of H. The operator P_C is called a metric projection operator if it assigns to each $x \in H$ its nearest point $y \in C$ such that

$$\|x - y\| = \min\{\|x - z\| : z \in C\}.$$

The element y above is called the metric projection of H onto C and denoted by $P_C x$. It exists and is unique at any point of the reflexive strictly convex space.

Lemma 2.7 *Let H be a Hilbert space and C is a nonempty, closed and convex subset of H. Then, for all $x \in H$, the element $z = P_C x$ if and only if*

$$\langle x - z, z - y \rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.8 [38] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|x_n - y_n\| \leq 0$.*

Lemma 2.9 [14] *Let C be a nonempty, closed, and convex subset of H. Let $S: C \to C$ be a nonexpansive mapping. Then, the mapping $I - S$ is demiclosed at zero, that is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $\|x_n - Sx_n\| \to 0$, then $x \in F(S)$.*

Lemma 2.10 [37] *Suppose that H is a real Hilbert space and $0 < p \leq t_n \leq q < 1$ for all $n \geq 1$. Suppose further that $\{x_n\}$ and $\{y_n\}$ are sequences of H such that*

$$\limsup_{n\to\infty} \|x_n\| \leq r, \quad \limsup_{n\to\infty} \|y_n\| \leq r, \quad \text{and} \quad \lim_{n\to\infty} \|t_n x_n + (1 - t_n)y_n\| = r$$

hold for some $r \geq 0$. Then, $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.11 [21] *Assume that $\{s_n\}$ is a sequence of nonnegative numbers such that*

$$s_{n+1} \leq (1 - \gamma_n)s_n + \delta_n, \quad \forall n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$,
(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

3 Main results

In this section, using the forward–backward splitting algorithm we prove some strong convergence theorems for approximating a zero of the sum of an α-inverse strongly monotone operator and a maximal monotone operator.

To prove the first result, we use the technique developed by Yao and Shahzad [46].

Theorem 3.1 *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $A: C \to H$ be an α-inverse strongly monotone mapping and B be maximal monotone operator of H into 2^H such that the domain of B is included in C and $(A + B)^{-1}(0) \neq \emptyset$. Assume that J_λ is the resolvent of B for $\lambda > 0$. Let the sequence $\{z_n\}$ generated as:*

$$\begin{cases} w_n = J_{\lambda_n}(z_n - \lambda_n A z_n), \\ z_{n+1} = \alpha_n z_n + \beta_n w_n + \gamma_n e_n \end{cases} \tag{3.1}$$

where e_n is an error vector, $z_1 \in H$, $\alpha_n, \beta_n, \gamma_n \in (0, 1)$ and $\alpha_n + \beta_n + \gamma_n = 1$. Suppose the control sequences satisfy the following conditions:

(a) $\lim_{n\to\infty}\gamma_n = 0$, *and* $\sum_{n=1}^{\infty}\gamma_n = \infty$;
(b) $e_n \to 0$;
(c) $0 < a \le \alpha_n \le b < 1$ *and* $0 < c \le \beta_n \le d < 1$;
(d) $0 < \varepsilon \le \lambda_n < 2\alpha$ *and* $\lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = 0$.

Then, $\{z_n\}$ *converges strongly to the point* $q \in (A+B)^{-1}(0)$, *where* $q = P_{(A+B)^{-1}(0)}(0)$.

Proof First of all, we shall show that $\{z_n\}$ generated by algorithm (3.1) is bounded. Fix $p \in (A+B)^{-1}(0)$, by the fact that J_{λ_n} is nonexpansive and Lemma 2.3, we obtain

$$\begin{aligned}\|w_n - p\| &= \|J_{\lambda_n}(z_n - \lambda_n A z_n) - J_{\lambda_n}(p - \lambda_n A p)\| \\ &\le \|(I - \lambda_n A)z_n - (I - \lambda_n A)p\| \\ &\le \|z_n - p\|.\end{aligned}$$

It follows that

$$\begin{aligned}\|z_{n+1} - p\| &\le \alpha_n\|z_n - p\| + \beta_n\|w_n - p\| + \gamma_n\|e_n - p\| \\ &\le (1-\gamma_n)\|z_n - p\| + \gamma_n(\|e_n\| + \|p\|) \\ &\le (1-\gamma_n)\|z_n - p\| + \gamma_n M_1 \\ &\le \max\{\|z_n - p\|, M_1\},\end{aligned}$$

where $\sup\|e_n\| + \|p\| \le M_1$ for some $M_1 > 0$. By induction on n, we obtain that

$$\|z_{n+1} - p\| \le \max\{\|z_1 - p\|, M_1\}.$$

Hence, the sequence $\{z_n\}$ is bounded and so is $\{w_n\}$.

Define $z_{n+1} = \alpha_n z_n + (1-\alpha_n)v_n$ for all $n \ge 0$. Then, we obtain from (3.1),

$$\begin{aligned}v_{n+1} - v_n &= \frac{z_{n+2} - \alpha_{n+1}z_{n+1}}{1-\alpha_{n+1}} - \frac{z_{n+1} - \alpha_n z_n}{1-\alpha_n} \\ &= \frac{\beta_{n+1}w_{n+1} + \gamma_{n+1}e_{n+1}}{1-\alpha_{n+1}} - \frac{\beta_n w_n + \gamma_n e_n}{1-\alpha_n} \\ &= \frac{\beta_{n+1}}{1-\alpha_{n+1}}(w_{n+1} - w_n) + \left(\frac{\beta_{n+1}}{1-\alpha_{n+1}} - \frac{\beta_n}{1-\alpha_n}\right)w_n \\ &\quad + \frac{\gamma_{n+1}e_{n+1}}{1-\alpha_{n+1}} - \frac{\gamma_n e_n}{1-\alpha_n}.\end{aligned}$$

Set $u_n = (I - \lambda_n A)z_n$. It follows from Lemma 2.3 that

$$\|u_{n+1} - u_n\| \le \|z_{n+1} - z_n\|.$$

If $\lambda_n \le \lambda_{n+1}$, by Lemma 2.6, we have

$$w_{n+1} = J_{\lambda_{n+1}}(u_{n+1}) = J_{\lambda_n}\left(\frac{\lambda_n}{\lambda_{n+1}}u_{n+1} + \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\right)J_{\lambda_{n+1}}(u_{n+1})\right),$$

and hence, we get,

$$\begin{aligned}\|w_{n+1} - w_n\| &= \|J_{\lambda_{n+1}}(u_{n+1}) - J_{\lambda_n}(u_n)\| \\ &\le \frac{\lambda_n}{\lambda_{n+1}}\|u_{n+1} - u_n\| + \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\right)\|w_{n+1} - u_n\| \\ &\le \|z_{n+1} - z_n\| + \frac{1}{\varepsilon}|\lambda_{n+1} - \lambda_n|\|w_{n+1} - u_n\|. \qquad (3.2)\end{aligned}$$

If $\lambda_n > \lambda_{n+1}$, again using Lemma 2.6, we obtain

$$\begin{aligned}\|w_n - w_{n+1}\| &= \|J_{\lambda_n}(u_n) - J_{\lambda_{n+1}}(u_{n+1})\| \\ &\le \frac{\lambda_{n+1}}{\lambda_n}\|u_n - u_{n+1}\| + \left(1 - \frac{\lambda_{n+1}}{\lambda_n}\right)\|w_n - u_{n+1}\| \\ &\le \|z_{n+1} - z_n\| + \frac{1}{\varepsilon}|\lambda_{n+1} - \lambda_n|\|w_n - u_{n+1}\|. \end{aligned} \tag{3.3}$$

Therefore, from (3.2) and (3.3), we imply that

$$\|w_{n+1} - w_n\| \le \|z_{n+1} - z_n\| + \frac{M_2}{\varepsilon}|\lambda_{n+1} - \lambda_n|$$

where, M_2 satisfies

$$\sup\{\|w_{n+1} - u_n\|, \|w_n\| + \|u_{n+1}\|, \ \ n \ge 0\} \le M_2.$$

Hence, we get

$$\begin{aligned}\|v_{n+1} - v_n\| \le& \frac{\beta_{n+1}}{1-\alpha_{n+1}}\|w_{n+1} - w_n\| + \left|\frac{\beta_{n+1}}{1-\alpha_{n+1}} - \frac{\beta_n}{1-\alpha_n}\right|\|w_n\| \\ &+ \frac{\gamma_{n+1}\|e_{n+1}\|}{1-\alpha_{n+1}} + \frac{\gamma_n\|e_n\|}{1-\alpha_n} \\ \le& \frac{\beta_{n+1}}{1-\alpha_{n+1}}\|z_{n+1} - z_n\| + \frac{\beta_{n+1}}{1-\alpha_{n+1}}\frac{M_2}{\varepsilon}|\lambda_{n+1} - \lambda_n| \\ &+ \left|\frac{\beta_{n+1}}{1-\alpha_{n+1}} - \frac{\beta_n}{1-\alpha_n}\right| M_2 + \left(\frac{\gamma_{n+1}}{1-\alpha_{n+1}} + \frac{\gamma_n}{1-\alpha_n}\right) M_1,\end{aligned}$$

which implies that

$$\begin{aligned}&\limsup\left(\|v_{n+1} - v_n\| - \|z_{n+1} - z_n\|\right) \\ &\le \limsup\left[\frac{\beta_{n+1}}{1-\alpha_{n+1}}\frac{M_2}{\varepsilon}|\lambda_{n+1} - \lambda_n| + \left|\frac{\beta_{n+1}}{1-\alpha_{n+1}} - \frac{\beta_n}{1-\alpha_n}\right| M_2 + \frac{\gamma_{n+1}}{1-\alpha_{n+1}}M_1 + \frac{\gamma_n}{1-\alpha_n}M_1\right], \\ &\le \limsup\left[\frac{\beta_{n+1}}{1-\alpha_{n+1}}\frac{M_2}{\varepsilon}|\lambda_{n+1} - \lambda_n| + \left|\frac{\gamma_{n+1}}{1-\alpha_{n+1}} - \frac{\gamma_n}{1-\alpha_n}\right| M_2 + \frac{\gamma_{n+1}}{1-\alpha_{n+1}}M_1 + \frac{\gamma_n}{1-\alpha_n}M_1\right] \\ &= 0.\end{aligned}$$

By Lemma 2.8, we have

$$\lim_{n\to\infty}\|z_n - v_n\| = 0,$$

and, hence,

$$\lim_{n\to\infty}\|z_{n+1} - z_n\| = \lim_{n\to\infty}(1-\gamma_n)\|v_n - z_n\| = 0.$$

Also, we have

$$\begin{aligned}\|z_n - w_n\| &\le \|z_n - z_{n+1}\| + \|z_{n+1} - w_n\| \\ &\le \|z_n - z_{n+1}\| + \alpha_n\|z_n - w_n\| + \gamma_n\|e_n - w_n\|.\end{aligned}$$

Therefore

$$\|z_n - w_n\| \le \frac{1}{1-\alpha_n}\|z_n - z_{n+1}\| + \frac{\gamma_n}{1-\alpha_n}\|e_n - w_n\|,$$

which implies that

$$\lim_{n\to\infty}\|z_n - w_n\| = 0.$$

Next, we show that $w_w(z_n) \subset (A+B)^{-1}(0)$.

Let $p \in w_w(z_n)$ Then, there exists a subsequence $\{z_{n_j}\}$ converging weakly to p. Since $J_{\lambda_{n_j}}$ is resolvent of B, by definition of w_n, we have that

$$\frac{z_{n_j} - w_{n_j}}{\lambda_{n_j}} - Az_{n_j} \in Bw_{n_j}.$$

By monotonicity of B, we obtain

$$0 \le \left\langle \frac{z_{n_j} - w_{n_j}}{\lambda_{n_j}} - Az_{n_j} - w, w_{n_j} - u \right\rangle \tag{3.4}$$

for each $(u, w) \in B$. Taking limit in (3.4) and $j \to \infty$, it follows that $\langle 0 - Ap - w, p - u\rangle \ge 0$. Maximal monotonicity of B implies that $-Ap \in Bp$ and hence $p \in (A+B)^{-1}(0)$.

Set $q = P_{(A+B)^{-1}(0)}(0)$ and take a subsequence $\{z_{n_j}\}$ of $\{z_n\}$ such that $z_{n_j} \rightharpoonup z$ and

$$\limsup_{n\to\infty} \langle q, q - z_n\rangle = \lim_{j\to\infty} \langle q, q - z_{n_j}\rangle.$$

Thus, $z \in (A+B)^{-1}(0)$ and hence

$$\limsup_{n\to\infty} \langle q, q - z_n\rangle = \langle q, q - z\rangle \le 0.$$

Finally, we show that the sequence $\{z_n\}$ converges strongly to $q = P_{(A+B)^{-1}(0)}(0)$.

By (3.1), we have

$$\begin{aligned}
\|z_{n+1} - q\|^2 &\le \|\alpha_n(z_n - q) + \beta_n(w_n - q) - \gamma_n q + \gamma_n e_n\|^2 \\
&\le \left[\|\alpha_n(z_n - q) + \beta_n(w_n - q) - \gamma_n q\| + \gamma_n\|e_n\|\right]^2 \\
&= \|\alpha_n(z_n - q) + \beta_n(w_n - q) - \gamma_n q\|^2 \\
&\quad + \gamma_n\|e_n\| \left[\gamma_n\|e_n\| + 2\|\alpha_n(z_n - q) + \beta_n(w_n - q) - \gamma_n q\|\right] \\
&\le \|\alpha_n(z_n - q) + \beta_n(w_n - q)\|^2 - 2\gamma_n\langle q, z_{n+1} - q - \gamma_n e_n\rangle + M_3\gamma_n\|e_n\| \\
&\le (\alpha_n\|z_n - q\| + \beta_n\|w_n - q\|)^2 - 2\gamma_n\langle q, z_{n+1} - q\rangle + 2\gamma_n^2\|e_n\|\|q\| + M_3\gamma_n\|e_n\| \\
&\le (1-\gamma_n)\|z_n - q\|^2 + \gamma_n[2\langle q, q - z_{n+1}\rangle + (2\|q\| + M_3)\gamma_n\|e_n\|] \\
&= (1-\gamma_n)\|z_n - q\|^2 + \delta_n,
\end{aligned}$$

where M_3 satisfies in

$$\sup\{\|e_n\| + 2\|\alpha_n(z_n - q) + \beta_n(w_n - q) - \gamma_n q\|, \;\; n \ge 0\} \le M_3.$$

Then, Lemma 2.11 implies that $z_n \to q$ as $n \to \infty$, and this completes the proof. □

If we take $A = 0$ in Theorem 3.1, then we obtain the following result.

Corollary 3.2 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let B be maximal monotone operator of H into 2^H such that the domain of B is included in C and $B^{-1}(0) \neq \emptyset$. Assume that J_λ is the resolvent of B for $\lambda > 0$. Let the sequence $\{z_n\}$ generated by the following algorithm*:

$$z_{n+1} = \alpha_n z_n + \beta_n J_{\lambda_n} z_n + \gamma_n e_n,$$

where $z_1 \in H$ and $\alpha_n, \beta_n, \gamma_n \in (0, 1)$ and $\alpha_n + \beta_n + \gamma_n = 1$. Suppose that the control sequences satisfy in the conditions (a), (b), (c) *and* (d′) $0 < \varepsilon \le \lambda_n$ *and* $\lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = 0$.

Then, $\{z_n\}$ converges strongly to a point $q \in B^{-1}(0)$, where $q = P_{B^{-1}(0)}(0)$.

Theorem 3.3 *Let C be a nonempty closed convex subset of a real Hilbert space H. Suppose that $S: C \to C$ is a nonexpansive mapping, $A: C \to H$ an α-inverse strongly monotone mapping and B a maximal monotone operator of H into 2^H such that the domain of B is included in C and $F(S) \cap (A+B)^{-1}(0) \neq \emptyset$. Assume that J_λ is the resolvent of B for $\lambda > 0$. Let the sequence $\{x_n\}$ generated by the following algorithm:*

$$\begin{cases} y_n = J_{\lambda_n}(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n S x_n + \beta_n y_n + \gamma_n e_n \end{cases} \tag{3.5}$$

where $x_1 \in H$ and $\alpha_n, \beta_n, \gamma_n \in (0,1)$ and $\alpha_n + \beta_n + \gamma_n = 1$. Suppose that the control sequences satisfy in the conditions (a), (b), (c), (d) *and* (e) *either $\sum \alpha_n < \infty$ or $\frac{\alpha_n}{\gamma_n} \to 0$.*

Then, $\{x_n\}$ converges strongly to a point $q \in F(S) \cap (A+B)^{-1}(0)$, where $q = P_{F(S)\cap(A+B)^{-1}(0)}(0)$.

Proof At first, we show that $\{x_n\}$ generated by algorithm (3.5) is bounded. Similar to (3.2), we have

$$\|y_n - p\| \leq \|x_n - p\|, \tag{3.6}$$

for each $p \in (A+B)^{-1}(0)$. Fix $p \in F(S) \cap (A+B)^{-1}(0)$. By (3.5), we get

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n \|Sx_n - p\| + \beta_n \|y_n - p\| + \gamma_n \|e_n - p\| \\ &\leq (1-\gamma_n)\|x_n - p\| + \gamma_n(\|e_n\| + \|p\|) \\ &\leq (1-\gamma_n)\|x_n - p\| + \gamma_n M \\ &\leq \max\{\|x_n - p\|, M\}, \end{aligned}$$

where $\sup \|e_n\| + \|p\| \leq M$ for some $M > 0$. By induction on n, we obtain that

$$\|x_{n+1} - p\| \leq \max\{\|x_1 - p\|, M\}.$$

Hence, the sequence $\{x_n\}$ is bounded and so are $\{y_n\}$ and $\{Sx_n\}$.

Let the sequence $\{z_n\}$ generated by (3.1). We prove that $\{x_n\}$ and $\{z_n\}$ are equivalent. Since J_{λ_n} is nonexpansive and by Lemma 2.3, we obtain

$$\begin{aligned} \|y_n - w_n\| &= \|J_{\lambda_n}(x_n - \lambda_n A x_n) - J_{\lambda_n}(z_n - \lambda_n A z_n)\| \\ &\leq \|(I - \lambda_n A)x_n - (I - \lambda_n A)z_n\| \\ &\leq \|x_n - z_n\|, \end{aligned}$$

and hence,

$$\begin{aligned} \|x_{n+1} - z_{n+1}\| &\leq \alpha_n \|Sx_n - z_n\| + \beta_n \|y_n - w_n\| \\ &\leq \alpha_n \|Sx_n - y_n\| + \alpha_n \|y_n - w_n\| + \alpha_n \|w_n - z_n\| + \beta_n \|y_n - w_n\| \\ &\leq (\alpha_n + \beta_n)\|x_n - z_n\| + \alpha_n(\|Sx_n - y_n\| + \|w_n - z_n\|) \\ &\leq (1-\gamma_n)\|x_n - z_n\| + \alpha_n(\|Sx_n - y_n\| + \|w_n - z_n\|). \end{aligned}$$

Applying Lemma 2.11 with conditions (a) and (e), we conclude that $\|x_n - z_n\| \to 0$. Using Theorem 3.1, we imply that $x_n \to q = P_{(A+B)^{-1}(0)}(0)$ and so $y_n \to q$ by (3.6).

To finish our proof, it suffices to show that $q = P_{F(S)\cap(A+B)^{-1}(0)}(0)$. We notice that

$$\begin{aligned} \|Sx_n - p + \gamma_n(e_n - Sx_n)\| &\leq \|Sx_n - p\| + \gamma_n \|e_n - Sx_n\| \\ &\leq \|x_n - p\| + \gamma_n \|e_n - Sx_n\|. \end{aligned}$$

This implies from the conditions that

$$\limsup_{n\to\infty} \|Sx_n - p + \gamma_n(e_n - Sx_n)\| \leq \|q - p\|.$$

We also have

$$\begin{aligned} \|y_n - p + \gamma_n(e_n - Sx_n)\| &\leq \|y_n - p\| + \gamma_n \|e_n - Sx_n\| \\ &\leq \|x_n - p\| + \gamma_n \|e_n - Sx_n\|. \end{aligned}$$

Therefore

$$\limsup_{n\to\infty} \|y_n - p + \gamma_n(e_n - Sx_n)\| \le \|q - p\|.$$

On the other hand, we have

$$\begin{aligned}\lim_{n\to\infty} \|\beta_n(y_n - p + \gamma_n(e_n - Sx_n)) + (1-\beta_n)(Sx_n - p + \gamma_n(e_n - Sx_n))\| &= \lim_{n\to\infty} \|x_{n+1} - p\| \\ &= \|q - p\|.\end{aligned}$$

It follows from Lemma 2.10 that $\|Sx_n - y_n\| \to 0$. Then, we obtain

$$\|Sx_n - x_n\| \le \|Sx_n - y_n\| + \|y_n - x_n\| \to 0.$$

From Lemma 2.9, we conclude that $q \in F(S)$. This together with $q = P_{(A+B)^{-1}(0)}(0)$ implies that $q = P_{F(S)\cap(A+B)^{-1}(0)}(0)$ and hence $\{x_n\}$ converges strongly to $q = P_{F(S)\cap(A+B)^{-1}(0)}(0)$. □

If we take $A = 0$ in Theorem 3.3, then we obtain the following result.

Corollary 3.4 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S: C \to C$ be a nonexpansive mapping and B be a maximal monotone operator of H into 2^H such that the domain of B is included in C and $F(S) \cap B^{-1}(0) \neq \emptyset$. Assume that J_λ be the resolvent of B for $\lambda > 0$. Let the sequence $\{x_n\}$ generated by the following algorithm*:

$$x_{n+1} = \alpha_n Sx_n + \beta_n J_{\lambda_n} x_n + \gamma_n e_n$$

where $x_1 \in H$ and $\alpha_n, \beta_n, \gamma_n \in (0, 1)$ and $\alpha_n + \beta_n + \gamma_n = 1$. If the control sequences satisfy in the conditions (a), (b), (c), (d′) *and* (e), *then $\{x_n\}$ converges strongly to a point $q \in F(S)\cap B^{-1}(0)$, where $q = P_{F(S)\cap B^{-1}(0)}(0)$.*

Corollary 3.5 *Let H be a real Hilbert space. Suppose that B and M be maximal monotone operators of H into 2^H such that $M^{-1}(0) \cap B^{-1}(0) \neq \emptyset$. Assume that J_λ^B and J_r^M be the resolvent of B and M for $\lambda > 0$ and $r > 0$, respectively. Let the sequence $\{x_n\}$ generated by the following algorithm*:

$$x_{n+1} = \alpha_n J_r^M x_n + \beta_n J_{\lambda_n}^B x_n + \gamma_n e_n,$$

where $x_1 \in H$ and $\alpha_n, \beta_n, \gamma_n \in (0, 1)$ and $\alpha_n + \beta_n + \gamma_n = 1$. Suppose that the control sequences satisfy in the conditions (a), (b), (c), (d) *and* (e) *either $\sum \alpha_n < \infty$ or $\frac{\alpha_n}{\gamma_n} \to 0$.*
Then, $\{x_n\}$ converges strongly to a point $q \in M^{-1}(0) \cap B^{-1}(0)$, where $q = P_{M^{-1}(0)\cap B^{-1}(0)}(0)$.

4 Applications

Our aim in this section is to discuss an application of our results both to equilibrium problems and convex programming. In this respect, suppose C be a nonempty subset of H. A function $F: C \times C \to \mathbb{R}$ is called bifunction if $F(x, x) = 0$ for all $x \in C$. We consider the problem of finding a solution $z \in X$ of

$$F(z, y) \ge 0, \quad \forall y \in C. \tag{4.1}$$

The bifunction $F: C \times C \to \mathbb{R}$ is called monotone if $F(x, y) + F(y, x) \le 0$, for all $x, y \in C$. Hadjisavvas and Khatibzadeh [15] introduced a monotone operator B^F to each monotone bifunction F by

$$B^F(x) = \begin{cases} \{x^* \in X^* : F(x, y) \ge \langle x^*, y - x\rangle,\ \forall y \in C\}, & x \in C \\ \emptyset, & x \in X\backslash C. \end{cases}$$

The monotone bifunction F is said to be maximal monotone if B^F is maximal monotone. It is obvious that $\bar{x} \in C$ is a solution of an equilibrium problem (4.1) for F if and only if $0 \in B^F(\bar{x})$.

The following theorems have been proved in [15] for the maximality of bifunction F.

Theorem 4.1 *Let $C \subseteq X$ be nonempty, closed and convex. If F is monotone, $F(\cdot, y)$ is upper hemicontinuous (i.e., upper semicontinuous on line segments) for all $y \in C$ and $F(x, \cdot)$ is convex and lower semicontinuous for all $x \in C$, then F is maximal monotone.*

Theorem 4.2 *A monotone bifunction F is maximal monotone if and only if for each $\lambda > 0$ (equivalently, for some $\lambda > 0$) and each $x \in H$ there exists $x_\lambda \in C$ such that*

$$\lambda F(x_\lambda, y) + \langle y - x_\lambda, x_\lambda - x\rangle \geq 0, \quad \forall y \in C. \tag{4.2}$$

This element x_λ is uniquely defined.

Note that (4.2) implies that for every $x \in X$, $J_\lambda^{B^F}(x) = x_\lambda$ and $x_\lambda \in D(B^F)$. Therefore, if F satisfies the assumptions of Theorem 4.1. Then, by (4.2) and for each $n \in \mathbb{N}$ there exist $y_n,\ x_n \in X$ which satisfy

$$\begin{cases} \lambda_n F(y_n, y) + \langle y - y_n, y_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n S x_n + \beta_n y_n + \gamma_n e_n \end{cases} \tag{4.3}$$

which is equivalent to (3.5) with B^F instead of B and $A = 0$. Therefore, every convergence result for the sequence generated by (3.5) is true for the sequence generated by (4.3). Then, we get p is a common solution of fixed point set $F(S)$ and equilibrium problem (4.1), whenever $\{x_n\}$ converges strongly to $p \in F(S) \cap (B^F)^{-1}(0)$. In fact, we have the following theorem:

Theorem 4.3 *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Suppose that $S\colon C \to C$ is a nonexpansive mapping, and $F\colon C \times C \to \mathbb{R}$ is a monotone bifunction such that $F(\cdot, y)$ is upper hemicontinuous for all $y \in C$ and $F(x, \cdot)$ is convex and lower semicontinuous for all $x \in C$ such that $F(S) \cap EP(F) \neq \emptyset$. Let the sequence $\{x_n\}$ generated by the following algorithm*:

$$\begin{cases} \lambda_n F(y_n, y) + \langle y - y_n, y_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n S x_n + \beta_n y_n + \gamma_n e_n \end{cases}$$

where $x_1 \in H$ and $\alpha_n, \beta_n, \gamma_n \in (0, 1)$ and $\alpha_n + \beta_n + \gamma_n = 1$. If the control sequences satisfy the conditions (a), (b), (c), (d) *and* (e), *then $\{x_n\}$ converges strongly to the common solution of fixed point set and equilibrium problem.*

Suppose that f is a proper, convex and lower semicontinuous function. The subdifferential $\partial f\colon H \rightrightarrows 2^H$ of f is defined as follows:

$$\partial f(x) = \{z \in H \ : \ \langle y - x, z\rangle \leq f(y) - f(x), \ \ \forall y \in H\}.$$

Then, ∂f is maximal monotone (see [34]). It is obvious that $0 \in \partial f(x)$ if and only if $f(x) = \min_{y \in H} f(y)$.

Theorem 4.4 *Let H be a Hilbert space and $f\colon H \to (-\infty, +\infty]$ a proper, convex and lower semicontinuous function such that $(\partial f)^{-1}(0) \neq \emptyset$. Assume that the sequence $\{z_n\}$ generated by the following algorithm*:

$$\begin{cases} w_n = \operatorname{argmin}_{z \in X} \left\{ f(z) + \frac{1}{2\lambda_n}\|z - z_n\|^2 \right\}, \\ z_{n+1} = \alpha_n z_n + \beta_n w_n + \gamma_n e_n, \end{cases}$$

where $z_1 \in H$ and $\alpha_n, \beta_n, \gamma_n \in (0, 1)$ and $\alpha_n + \beta_n + \gamma_n = 1$. If the control sequences satisfy the conditions (a), (b), (c) *and* (d′), *then $\{z_n\}$ converges strongly to the minimizer of f.*

Proof If we take $B = \partial f$ in Corollary 3.2, then we obtain the desired conclusion immediately. □

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