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Nonexistence of $\mathcal{PR}$-semi-slant warped product submanifolds in paracosymplectic manifolds

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Abstract In the present paper, we prove that there does not exist any $\mathcal{PR}$-semi-slant warped product submanifolds in paracosymplectic manifolds. In addition, by presenting a non-trivial example we find that there is no proper $\mathcal{PR}$-semi-slant warped product submanifold other than $\mathcal{PR}$-semi-invariant warped products.

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1 Introduction

The concept of warped products was initiated by Bishop et al. [5], as one of the most effective generalizations of Riemannian product manifold. Thereafter, O'Neill [27] generalized the study of warped products for semi-Riemannian manifolds. However, the theory progressed faster after Chen introduced a new class of CR-submanifolds called CR-warped products in Kaehlerian manifold and gave some fundamental results on the existence of such submanifolds [7,8]. By analogy, Hasegawa et al. [19] and Munteanu [25] continued with the study of the theory for a Sasakian manifold that can be viewed as an odd-dimensional counterpart of Kähler manifold. Since then several differential geometers contributed to the investigation of the existence and nonexistence of warped product submanifolds in complex and contact Riemannian geometries where the metric is necessarily positive definite (for instance, see [14–16,23,28,32,34–41]). Moreover in recent years, the premise of warped product with metric other than Riemannian metric has acknowledged several important contributions in semi-Riemannian geometry, and has been successfully applied in the study of general theory of relativity and black holes (cf. [12,20,21]). Due to the non-definitive nature of the Riemannian metric, the geometry of warped product submanifolds with more general non-degenerate metric (i.e., positive as well as negative definite metric) became a topic of investigation. In light of that, many authors discussed the existence and nonexistence of such warped product submanifolds in Lorentzian settings [33,42]. Recently, Chen and Munteanu [13], the authors of [30,31] and Aydin and Cöken [2] initiated the study of the geometry of pseudo-Riemannian warped products submanifolds in para-Kähler manifolds, paracontact manifolds and slant submanifold in semi-Riemannian manifolds, respectively. Motivated by the above studies, in the

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present paper, we investigate the existence or nonexistence of $\mathcal{PR}$-semi-slant warped product submanifolds in paracosymplectic manifolds.

The organization of the article is as follows. In Sect. 2, we recall some basic information about paracontact manifolds and their submanifolds. Moreover we review some known facts about warped product submanifolds and give some preparatory results for the existence or nonexistence of warped product submanifolds in a paracosymplectic manifold. In Sect. 3, we first define $\mathcal{PR}$-semi-slant submanifolds and then derive necessary and sufficient conditions for the distributions equipped with the definition of such submanifold to be involutive and totally geodesic. Section 4 deals with the nonexistence of the non-trivial $\mathcal{PR}$-semi-slant warped product submanifolds of the forms $M_T \times_f M_\lambda$ and $M_\lambda \times_f M_T$ in a paracosymplectic manifold $\overline{M}$ such that the characteristic vector field ξ is tangent to first factor or second factor in both cases, where M_T and M_λ are invariant submanifold and proper slant submanifold of $\overline{M}$. Finally, in Sect. 5, we present a non-trivial example of $\mathcal{PR}$-semi-invariant warped product submanifold which can be viewed as the non-trivial $\mathcal{PR}$-semi-slant warped product submanifold with an improper slant coefficient, i.e., $\lambda = 0$.

2 Preliminaries

Let $\overline{M}$ be an odd-dimensional smooth manifold. An almost paracontact structure on $\overline{M}$ is a triplet (ϕ, ξ, η) [22], such that ϕ is a tensor field of $(1, 1)$-type, and ξ is a vector field and, η is a 1-form satisfying the following conditions:

$$\phi^2 = I - \eta \otimes \xi, \quad \eta(\xi) = 1 \tag{2.1}$$

where I is the identity transformation and the tensor field ϕ induces on $2m$-dimensional horizontal distribution $D = \ker(\eta)$ an almost paracomplex structure J; that is, $J^2 = I$ and the eigen subbundles $D^\pm$ corresponding to the eigenvalues ± 1 of J, respectively, have equal dimension m; hence $D = D^+ \oplus D^-$. The direct consequence of Eq. (2.1) is that the structure endomorphism ϕ has rank $2m$, $\phi\xi = 0$ and $\eta \circ \phi = 0$. If a manifold $\overline{M}$ with (ϕ, ξ, η)-structure admits a pseudo-Riemannian metric $\overline{g}$ of signature $(m + 1, m)$ such that

$$\overline{g} = -\overline{g}(\phi\cdot, \phi\cdot) + \eta \otimes \eta, \tag{2.2}$$

then $\overline{M}$ is said to have an almost paracontact metric structure $(\phi, \xi, \eta, \overline{g})$ and the manifold $\overline{M}$ equipped with $(\phi, \xi, \eta, \overline{g})$-structure is called an almost paracontact metric manifold, here $\overline{g}$ is known as compatible metric (see [29,43]). With respect to $\overline{g}$, η is metrically dual to the unitary vector field ξ, *i.e.*, $\eta = \overline{g}(\cdot, \xi)$. With the consequences of Eqs. (2.1) and (2.2) we deduce that ϕ is a $\overline{g}$-skew-symmetric operator, that is,

$$\overline{g}(\phi X, Y) = -\overline{g}(X, \phi Y), \tag{2.3}$$

for any $X, Y \in \Gamma(T\overline{M})$; $\Gamma(T\overline{M})$ denotes the section of the tangent bundle $(T\overline{M})$ of $\overline{M}$. The fundamental 2-form $\Phi = \overline{g}(\cdot, \phi\cdot)$ is a non-degenerate on the horizontal distribution D and $\eta \wedge \Phi^m \neq 0$. If $\Phi = d\eta$, then η is a paracontact 1-form and the almost paracontact metric manifold $\overline{M}(\phi, \xi, \eta, \overline{g})$ is called a paracontact metric manifold.

Definition 2.1 An almost paracontact metric manifold $\overline{M}(\phi, \xi, \eta, \overline{g})$ is said to be

(a) almost paracosymplectic if the forms η and Φ are closed, *i.e.*, $d\eta = 0$ and $d\Phi = 0$ [17].
(b) paracosymplectic [17] if the forms η and Φ are parallel with respect to the Levi-Civita connection $\overline{\nabla}$ on $\overline{M}(\phi, \xi, \eta, \overline{g})$, *i.e.*,

$$\overline{\nabla}\eta = 0 \quad \text{and} \quad \overline{\nabla}\Phi = 0. \tag{2.4}$$

(c) normal if the eigen distributions $D^\pm$ of $\varphi|_D$ corresponding to the eigenvalues ± 1, respectively, are involutive and ξ is foliate with respect to both $D^\pm$ [26].

Let us consider that M is an isometrically immersed submanifold of a paracosymplectic manifold $\overline{M}$ in the sense of B. O'Neill [27]. Let g denote the induced metric on M such that $g = \overline{g}|_M$ [18], $\Gamma(TM^\perp)$ indicates the set of all vector fields normal to M and $\Gamma(TM)$ the sections of tangent bundle TM of M. Then the Gauss–Weingarten formulas are given, respectively, by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.5}$$
$$\overline{\nabla}_X \zeta = -A_\zeta X + \nabla^\perp_X \zeta, \tag{2.6}$$

for any $X, Y \in \Gamma(TM)$ and $\zeta \in \Gamma(TM^\perp)$, where ∇ is the induced connection, $\nabla^\perp$ is the normal connection on the normal bundle $\Gamma(TM^\perp)$, h is the second fundamental form, and the shape operator A_ζ associated with the normal section ζ is given in [11] by

$$g(A_\zeta X, Y) = \overline{g}\,(h(X, Y), \zeta)\,. \tag{2.7}$$

Now, for any $X \in \Gamma(TM)$ and $\zeta \in \Gamma(TM^\perp)$, we write

$$\phi X = tX + nX, \tag{2.8}$$
$$\phi\zeta = t'\zeta + n'\zeta, \tag{2.9}$$

where tX (resp., nX) is tangential (resp., normal) part of ϕX and $t'\zeta$ (resp., $n'\zeta$) is tangential (resp., normal) part of $\phi\zeta$. Then the submanifold M is said to be invariant if n is identically zero and anti-invariant if t is identically zero. From Eqs. (2.3) and (2.8), we obtain that for any $X, Y \in \Gamma(TM)$

$$g(X, tY) = -g(tX, Y). \tag{2.10}$$

By virtue of Gauss formula and the fact that the structure is paracosymplectic, we can give the following result for later use:

Lemma 2.2 *Let M be an isometrically immersed submanifold of a paracosymplectic manifold $\overline{M}(\phi, \xi, \eta, \overline{g})$ such that $\xi \in \Gamma(TM)$. Then for any $X \in \Gamma(TM)$, we have*

$$\nabla_X \xi = 0, \tag{2.11}$$
$$h(X, \xi) = 0. \tag{2.12}$$

Let (B, g_B) and (F, g_F) be two pseudo-Riemannian manifolds and f be a positive smooth function on B. Consider the product manifold $B \times F$ with canonical projections

$$\pi : B \times F \to B \quad \text{and} \quad \sigma : B \times F \to F. \tag{2.13}$$

Then the manifold $M = B \times_f F$ is said to be a warped product if it is equipped with the following warped metric

$$g(X, Y) = g_B\,(\pi_*(X), \pi_*(Y)) + (f \circ \pi)^2 g_F\,(\sigma_*(X), \sigma_*(Y)) \tag{2.14}$$

for all $X, Y \in \Gamma(TM)$ and '$*$' stands for derivation map, or equivalently,

$$g = g_B \oplus f^2 g_F. \tag{2.15}$$

The function f is called the warping function and a warped product manifold M is said to be trivial if f is constant. For the sake of simplicity, we will determine a vector field X on B with its lift $\overline{X}$ and a vector field Z on F with its lift $\overline{Z}$ on $M = B \times_f F$ (see also [5,13]).

Proposition 2.3 [5] *For $X, Y \in \Gamma(TB)$ and $Z, W \in \Gamma(TF)$, we obtain for the warped product manifold $M = B \times_f F$ that*

(i) $\nabla_X Y \in \Gamma(TB)$,
(ii) $\nabla_X Z = \nabla_Z X = \left(\frac{Xf}{f}\right) Z$,
(iii) $\nabla_Z W = \nabla'_Z W - \frac{-g(Z,W)}{f} \nabla f$,

where ∇ denotes the Levi-Civita connection on M and ∇f is the gradient of f defined by $g(\nabla f, X) = Xf$ and ∇' is the connection on F.

Remark 2.4 It is also important to note that for a warped product $M = B \times_f F$; B is totally geodesic and F is totally umbilical in M [5].

Furthermore, we prove an important theorem for later use;

Theorem 2.5 *Let $\overline{M}(\phi, \xi, \eta, \overline{g})$ be a paracosymplectic manifold. Then there does not exist a non-trivial warped product submanifold $M = B \times_f F$ of a paracosymplectic manifold, if $\xi \in \Gamma(TF)$.*

Proof In view of Lemma 2.2 and Proposition 2.3, we obtain that $X(\ln f)\xi = 0$, for any non-degenerate vector fields $X \in \Gamma(TB)$ and $\xi \in \Gamma(TF)$. This implies that f is constant function, since X, ξ are non-degenerate vector fields in M. Hence, the proof is complete. □

Here, we recall the following important results from [30] for later use when $\xi \in \Gamma(TB)$:

Lemma 2.6 *Let $M = B \times_f F$ be a non-trivial warped product submanifold in a paracosymplectic manifold $\overline{M}(\phi, \xi, \eta, \overline{g})$ such that $\xi \in \Gamma(TB)$. Then, we have*

$$\xi(\ln f) = 0, \tag{2.16}$$

$$A_{nZ}X = -t'h(X, Z), \tag{2.17}$$

$$g(A_{nZ}X, W) = g(A_{nW}X, Z) = -tX(\ln f)g(Z, W), \tag{2.18}$$

for any $X, Y \in \Gamma(TB)$ and $Z, W \in \Gamma(TF)$.

In [30], we have defined $\mathcal{PR}$-semi-invariant submanifolds in paracontact manifold as follows:

Definition 2.7 Let M be an isometrically immersed submanifold of an almost paracontact metric manifold $\overline{M}(\phi, \xi, \eta, \overline{g})$ such that the characteristic vector field $\xi \in \Gamma(TM)$. Then the submanifold M is called $\mathcal{PR}$-semi-invariant if it is furnished with a pair of non-degenerate orthogonal distribution $(\mathfrak{D}_T, \mathfrak{D}^{\perp})$ which satisfies the following conditions:

(i) $TM = \mathfrak{D}_T \oplus \mathfrak{D}^{\perp} \oplus \{\xi\}$,
(ii) the distribution $\mathfrak{D}_T$ is invariant under ϕ, i.e., $\phi(\mathfrak{D}_T) \subset \mathfrak{D}_T$ and
(iii) the distribution $\mathfrak{D}^{\perp}$ is anti-invariant under ϕ, i.e., $\phi(\mathfrak{D}^{\perp}) \subset \Gamma(TM)^{\perp}$.

A $\mathcal{PR}$-semi-invariant submanifold is said to be *proper*, if $\mathfrak{D}_T \neq \{0\}$ and $\mathfrak{D}^{\perp} \neq \{0\}$; and $\mathcal{PR}$- semi-invariant warped product if it is a warped product of the form: $M_T \times_f M_{\perp}$ and $M_{\perp} \times_f M_T$, where M_T and $M_{\perp}$ are invariant and anti-invariant submanifold of $\overline{M}$, respectively and f is a non-constant positive smooth function on the first factor. If the warping function f is constant then a $\mathcal{PR}$-semi-invariant warped product submanifold is said to be a $\mathcal{PR}$-semi-invariant product or trivial product.

3 $\mathcal{PR}$-semi-slant submanifolds

In this section, by following [1–3,6,31], we introduce $\mathcal{PR}$-semi-slant submanifolds in $\overline{M}$ which generalizes [30]. Since, submanifold M is non-degenerate submanifold so this class of submanifolds can be viewed as the generalization of submanifolds defined in [4,9,10,19,24] which includes the space-like vector fields only.
Let M be a non-degenerate submanifold of an almost paracontact metric manifold $\overline{M}$ such that $t^2X = \lambda(X - \eta(X)\xi) = \lambda\phi^2 X$, $\quad g(tX, Y) = -g(X, tY)$ for any $X, Y \in \Gamma(TM - <\xi>)$, where λ is a constant then with the help of Eq. (2.3), we have

$$\frac{g(\phi X, tY)}{|\phi X||tY|} = -\frac{g(X, \phi tY)}{|\phi X||tY|} = -\frac{g(X, t^2Y)}{|\phi X||tY|} = -\lambda\frac{g(X, \phi^2 Y)}{|\phi X||tY|} = \lambda\frac{g(\phi X, \phi Y)}{|\phi X||tY|}. \tag{3.1}$$

On the other hand,

$$\frac{g(\phi X, tY)}{|\phi X||tY|} = \frac{g(tX, tY)}{|\phi X||tY|}. \tag{3.2}$$

In particular, from Eqs. (3.1) and (3.2), we obtain for $X = Y$, that $\frac{g(\phi X, tX)}{|\phi X||tX|} = \sqrt{\lambda}$. Here we call λ a slant constant coefficient or simply slant coefficient and consequently M a slant submanifold. Conversely, assume that M is a slant submanifold then $\lambda \frac{|\phi X|}{|tX|} = \frac{|tX|}{|\varphi X|}$, where X is a non-light-like vector field. We obtain by the consequence of previous equation for any non-light like vector fields $X, Y \in \Gamma(TM- <\xi>)$, that $-\lambda \frac{g(X,\phi^2 Y)}{|\phi X||tY|} = \frac{g(\phi X,tY)}{|\phi X||tY|}$, which yields $g(X, t^2Y) = \lambda g(X, \phi^2 Y)$, $g(tX, Y) = -g(X, tY)$. Hence, $t^2 = \lambda(I - \eta \otimes \xi)$, $g(tX, Y) = -g(X, tY)$ by virtue of the fact that structure is paracosymplectic and X is non-degenerate vector fields.

Remark 3.1 The slant coefficient λ is sometimes $\cos^2\theta$ or $\cosh^2\theta$ or $-\sinh^2\theta$ for all vector fields tangent to M, where θ is a slant angle [1] and [3].

Now as a consequence of the above theory we have the following definitions.

Definition 3.2 Let M be an isometrically immersed submanifold of an almost paracontact metric manifold $\overline{M}(\phi, \xi, \eta, \overline{g})$ such that the characteristic vector field $\xi \in \Gamma(TM)$. Then the submanifold M is said to be a

- (**a**) slant submanifold if for any nonzero vectors $X, Y \in \Gamma(\mathfrak{D}_\lambda)$ at $p \in M$ which are not proportional to ξ_p, there exists a constant λ satisfying:

$$t^2 = \lambda(I - \eta \otimes \xi) \quad \text{and} \quad g(tX, Y) = -g(X, tY),$$

  where λ is a slant coefficient and slant distribution $\mathfrak{D}_\lambda$ indicate the non-degenerate distribution on M [31].
- (**b**) $\mathcal{PR}$-semi-slant if it is furnished with a pair of non-degenerate orthogonal distribution $(\mathfrak{D}_T, \mathfrak{D}_\lambda)$ satisfies the following conditions:
  - (i) $TM = \mathfrak{D}_T \oplus \mathfrak{D}_\lambda \oplus <\xi>$,
  - (ii) the distribution $\mathfrak{D}_T$ is invariant under ϕ, i.e., $\phi(\mathfrak{D}_T) \subset \mathfrak{D}_T$ and
  - (iii) the distribution $\mathfrak{D}_\lambda$ is slant distribution with slant coefficient λ.

  We define a $\mathcal{PR}$-semi-slant submanifold is proper if $\mathfrak{D}_T \neq \{0\}$, $\mathfrak{D}_\lambda \neq \{0\}$ and $\lambda \neq 0, 1$.

Remark 3.3 It is important to note that the invariant and anti-invariant submanifolds are slant submanifolds with slant coefficients $\lambda = 1$ and $\lambda = 0$, respectively. Hence, a slant submanifold which is neither invariant nor anti-invariant is called a proper slant submanifold.

Remark 3.4 A $\mathcal{PR}$-semi-slant submanifold is $\mathcal{PR}$-semi-invariant if $\mathfrak{D}_T \neq \{0\}$, $\mathfrak{D}_\lambda \neq \{0\}$ with $\lambda = 0$ [30].

Here, we give a numerical example for illustrating proper $\mathcal{PR}$-semi-slant submanifold M in a paracosymplectic manifold $\overline{M}$.

Example 3.5 Let $\overline{M} = \mathbb{R}^8 \times \mathbb{R}_+ \subset \mathbb{R}^9$ be a nine-dimensional manifold with the standard Cartesian coordinates $(\overline{x}_1, \overline{x}_2, \overline{x}_3, \overline{x}_4, \overline{x}_5, \overline{x}_6, \overline{x}_7, \overline{x}_8, \overline{x}_9)$, where $\overline{x}_9$ is the global coordinate on $\mathbb{R}_+$. Define the paracontact pseudo-Riemannian metric structure $(\phi, \xi, \eta, \overline{g})$ on $\overline{M}$ by

$$\begin{aligned} &\phi e_1 = e_5,\ \phi e_2 = e_6,\ \phi e_3 = e_7,\ \phi e_4 = e_8,\ \phi e_8 = e_4,\ \phi e_7 = e_3,\ \phi e_6 = e_2 \\ &\phi e_5 = e_1,\ \phi e_9 = 0, \xi = e_9,\ \eta = d\overline{x}_9,\ \overline{g} = \sum_{i=1}^{4}(d\overline{x}_i)^2 - \sum_{j=5}^{8}(d\overline{x}_j)^2 + \eta \otimes \eta. \end{aligned} \tag{3.3}$$

Here, $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9\}$ is a local orthonormal frame for $\Gamma(T\overline{M})$, given by $e_i = \frac{\partial}{\partial \overline{x}_i}$, $e_j = \frac{\partial}{\partial \overline{x}_j}$ and $e_9 = \frac{\partial}{\partial \overline{x}_9}$.

Let M be an isometrically immersed pseudo-Riemannian submanifold in a paracosymplectic manifold $\overline{M}$ defined by

$$\begin{aligned} \Omega(x_1, x_2, x_3, x_4, x_5) = (&x_1\cos(C), x_3\cosh(C), x_4\sinh(C), x_1\sin(C), \\ &x_2, x_4\cosh(C), x_3\sinh(C), k, x_5), \end{aligned}$$

where C and k are constants. Then the tangent bundle $\Gamma(TM)$ of M is spanned by the vector fields

$$\begin{aligned} &X_1 = \cos(C)e_1 + \sin(C)e_4, \quad X_2 = e_5, \\ &X_3 = \cosh(C)e_2 + \sinh(C)e_7, \\ &X_4 = \sinh(C)e_3 + \cosh(C)e_6, \quad X_5 = e_9, \end{aligned} \tag{3.4}$$

where $X_1, X_2, X_3, X_4, X_5 \in \Gamma(TM)$. Then the space $\phi(TM)$ with respect to the paracosymplectic pseudo-Riemannian metric structure $(\phi, \xi, \eta, \overline{g})$ of $\overline{M}$ becomes

$$\begin{aligned} \phi(X_1) &= \cos(C)e_5 + \sin(C)e_8, \quad \phi(X_2) = e_1, \\ \phi(X_3) &= \cosh(C)e_6 + \sinh(C)e_3, \\ \phi(X_4) &= \sinh(C)e_7 + \cosh(C)e_2, \quad \phi(X_5) = 0. \end{aligned} \tag{3.5}$$

Therefore from Eqs. (5.2), (5.3) and (5.1), we obtain that $\mathfrak{D}_T$, $<\xi>$ and $\mathfrak{D}_\lambda$, are the distributions defined by span$\{X_3, X_4\}$, span$\{X_5\}$ and span$\{X_1, X_2\}$, respectively, where $\mathfrak{D}_T$ is an invariant distribution and $\mathfrak{D}_\lambda$ is a slant distribution with slant coefficient $\lambda = |\cos(C)|$. Thus M becomes a proper $\mathcal{PR}$-semi-slant submanifold of $\overline{M}$.

Now, by virtue of the definition of slant submanifold of an almost paracontact manifolds, we deduce the following useful result:

Proposition 3.6 *Let M be a slant submanifold of an almost paracontact metric manifold $\overline{M}(\phi, \xi, \eta, \overline{g})$ with $\xi \in \Gamma(TM)$. Then*

$$g(tX, tY) = \lambda\{-g(X, Y) + \eta(X)\eta(Y)\} = \lambda g(\phi X, \phi Y), \tag{3.6}$$

$$g(nX, nY) = (1-\lambda)\{-g(X, Y) + \eta(X)\eta(Y)\} = (1-\lambda)g(\phi X, \phi Y), \tag{3.7}$$

for any $X, Y \in \Gamma(TM)$.

Proof We have from Eqs. (2.1), (2.3) and (2.10), that $g(tX, tY) = -g(X, t^2Y) = -\lambda g(X, \phi^2 Y) = \lambda g(\phi X, \phi Y)$. Therefore, from Eq. (2.2) and definition 3.2(a), we get Eq. (3.6). Equation (3.7) follows from Eqs. (2.8) and (3.6). This completes the proof. □

Now, we obtain necessary and sufficient conditions for the foliation determined by distributions allied with the definition of $\mathcal{PR}$-semi-slant submanifolds of a paracosymplectic manifold $\overline{M}$ to be involutive and totally geodesic.

Theorem 3.7 *Let M be a proper $\mathcal{PR}$-semi-slant submanifold of a paracosymplectic manifold $\overline{M}$. Then the distribution $(\mathfrak{D}_T \oplus \{\xi\})$*

(i) *is involutive if and only if $h(X, tY) = h(tX, Y)$;*
(ii) *defines a totally geodesic foliation if and only if $A_{nZ}tY = A_{ntZ}Y$;*

for any $X, Y \in \Gamma(\mathfrak{D}_T \oplus \{\xi\})$ and $Z \in \Gamma(\mathfrak{D}_\lambda)$.

Proof For M to be a proper $\mathcal{PR}$-semi-slant submanifold M of a paracosymplectic manifold, we have

$$g([X, Y], Z) = g(\nabla_X Y, Z) - g(\nabla_Y X, Z). \tag{3.8}$$

Then, by virtue of Eqs. (2.2), (2.5) and fact that ξ and Z are orthogonal, we achieve that

$$g([X, Y], Z) = -\overline{g}(\phi\overline{\nabla}_X Y, \phi Z) + \overline{g}(\phi\overline{\nabla}_Y X, \phi Z). \tag{3.9}$$

Employing Eq. (2.4), (2.7), (2.8), (3.2) and Gauss–Weingarten formulas in Eq. (3.9), we obtain that

$$g([X, Y], Z) = \lambda g([X, Y], Z) + \overline{g}(h(X, \phi Y), nZ) - \overline{g}(h(Y, \phi X), nZ). \tag{3.10}$$

From above equation, we can write that

$$(1-\lambda)g([X, Y], Z) = \overline{g}(h(X, \phi Y), nZ) - \overline{g}(h(Y, \phi X), nZ). \tag{3.11}$$

Since $\lambda = 1$ is impossible and X, Y, Z are all non-degenerate vector fields. Hence, from Eq. (3.11), we can easily deduce that the distribution $(\mathfrak{D}_T \oplus \{\xi\})$ is involutive if and only if $h(X, tY) = h(tX, Y)$, which is (i). Furthermore, for the totally geodesic condition of M in $\overline{M}$, we know that $g(\nabla_X Y, Z) = \overline{g}(\overline{\nabla}_X Y, Z)$. Now in light of Eq. (2.2), (2.4), (2.8), (3.2) and the fact that the distributions $(\mathfrak{D}_T \oplus \{\xi\}, \mathfrak{D}_\lambda)$ are orthogonal, we derive the required condition for $(\mathfrak{D}_T \oplus \{\xi\})$ to be totally geodesic, which proves the theorem completely. □

Theorem 3.8 *Let M be a proper $\mathcal{PR}$-semi-slant submanifold M of a paracosymplectic manifold $\overline{M}$. Then the distribution $(\mathfrak{D}_\lambda \oplus \{\xi\})$*

(i) *is involutive if and only if $g(A_{nW}Z - A_{nZ}W, tX) = g(A_{ntZ}W - A_{ntW}Z, X)$;*
(ii) *defines totally geodesic if and only if $A_{ntW}X = A_{nW}tX$;*

for any $Z, W \in \Gamma(\mathfrak{D}_\lambda \oplus \{\xi\})$ and $X, Y \in \Gamma(\mathfrak{D}_T)$.

Proof The proof follows the steps of Theorem 3.7, we can prove the theorem. □

4 $\mathcal{PR}$-semi-slant warped product submanifolds

In this section, we investigate the nonexistence of $\mathcal{PR}$-semi-slant warped product submanifolds of the forms $M_T \times_f M_\lambda$ and $M_\lambda \times_f M_T$ in a paracosymplectic manifold $\overline{M}$, whether the structure vector field ξ is tangent to first factor or second factor, where M_T and M_λ are invariant and proper slant submanifolds of $\overline{M}$, respectively.

As a straight forward consequence of Theorem 2.5, we conclude the following results when ξ is tangent to the fiber in each case.

Theorem 4.1 *There does not exist any $\mathcal{PR}$-semi-slant warped product submanifold $M = M_T \times_f M_\lambda$ of a paracosymplectic manifold $\overline{M}$ such that M_T is an invariant submanifold and M_λ is a proper slant submanifold tangent to ξ of $\overline{M}$.*

Theorem 4.2 *There does not exist a $\mathcal{PR}$-semi-slant warped product submanifold $M = M_\lambda \times_f M_T$ in a paracosymplectic manifold $\overline{M}$ such that M_T is an invariant submanifold tangent to ξ and M_λ is a proper slant submanifold of $\overline{M}$.*

Now, we derive the following major results when ξ is tangent to the base manifold.

Theorem 4.3 *There does not exist a $\mathcal{PR}$-semi-slant warped product submanifold of the form $M = M_T \times_f M_\lambda$ in a paracosymplectic manifold $\overline{M}(\phi, \xi, \eta, \overline{g})$ such that M_T is an invariant submanifold tangent to ξ and M_λ is a proper slant submanifold in $\overline{M}$.*

Proof Let us assume that $M = M_T \times_f M_\lambda$ is a $\mathcal{PR}$-semi-slant warped product submanifold of a paracosymplectic manifold $\overline{M}(\phi, \xi, \eta, \overline{g})$ with $\xi \in \Gamma(TM_T)$. Then using Eqs. (2.4) and (2.5), we obtain that

$$\overline{\nabla}_Z \phi X = \phi \overline{\nabla}_Z X \tag{4.1}$$

for any non-degenerate vector fields $X \in \Gamma(TM_T)$ and $Z \in \Gamma(TM_\lambda)$. Employing Eqs. (2.8), (2.9) and Proposition (2.3) in (4.1), we compute that

$$\nabla_Z tX + h(Z, tX) = X(\ln f)tZ + X(\ln f)nZ + t'h(X, Z) + n'h(X, Z). \tag{4.2}$$

By equating the tangential and normal parts of Eq. (4.2), we get

$$tX(\ln f)Z = X(\ln f)tZ + t'h(X, Z), \tag{4.3}$$

$$h(Z, tX) = X(\ln f)nZ + n'h(X, Z). \tag{4.4}$$

On the other hand, by virtue of Eq. (2.18) and the fact that M_T is invariant, we have

$$\overline{g}(h(X, Z), nZ) = \overline{g}(h(Z, Z), nX) = 0, \tag{4.5}$$

for all $X \in \Gamma(TM_T)$ and $Z \in \Gamma(TM_\lambda)$. Moreover, we can write from Eq. (4.5), that

$$g(h(X, Z), \phi Z) = g(t'h(X, Z), Z) - g(n'h(X, Z), Z) = 0, \tag{4.6}$$

this implies $g(t'h(X, Z), Z) = 0, \forall Z \in \Gamma(TM_\lambda)$. Thus, we conclude that

$$t'h(X, Z) \in \Gamma(TM_T). \tag{4.7}$$

Now, by taking inner product of tZ with Eq. (4.3), we obtain

$$tX(\ln f)g(Z, tZ) = X(\ln f)g(tZ, tZ) + g(t'h(X, Z), tZ). \tag{4.8}$$

Using the orthogonality of vector fields and the fact that $t'h(X, Z) \in \Gamma(TM_T)$, we derive

$$\lambda X(\ln f)g(Z, Z) = 0, \tag{4.9}$$

for any $X \in \Gamma(TM_T)$ and $Z \in \Gamma(TM_\lambda)$. Since g is pseudo-Riemannian, then from (4.9), we find that either M is $\mathcal{PR}$-semi-invariant warped products of the form $M = M_T \times_f M_\perp$ or f is constant on M. This completes the proof of the theorem. □

Remark 4.4 From Theorems 4.1 and 4.3, we conclude that there does not exist any proper $\mathcal{PR}$-semi-slant warped product submanifold of a paracosymplectic manifold other than $\mathcal{PR}$-semi-invariant warped products.

Theorem 4.5 *There does not exist any $\mathcal{PR}$-semi-slant warped product submanifold of the form $M = M_\lambda \times_f M_T$ in a paracosymplectic manifold $\overline{M}(\phi, \xi, \eta, \overline{g})$ such that M_T is an invariant submanifold and M_λ is a proper slant submanifold tangent to ξ of $\overline{M}$.*

Proof Let us assume that $M = M_\lambda \times_f M_T$ is a $\mathcal{PR}$-semi-slant warped product submanifold of a paracosymplectic manifold $\overline{M}(\phi, \xi, \eta, \overline{g})$ such that $\xi \in \Gamma(TM_\lambda)$. Then we can write from Proposition 2.3, Gauss formula and the Connection property for $\overline{\nabla}$, that

$$\overline{g}(\overline{\nabla}_X X, Z) = -g(X, \nabla_X Z) = -Z(\ln f)g(X, X), \tag{4.10}$$

for any $X \in \Gamma(TM_T)$ and $Z \in \Gamma(TM_\lambda)$. We also, from Eqs. (2.2), (2.5), (2.8), Lemma 2.2 and the fact that structure is paracosymplectic, achieve that

$$\overline{g}(\overline{\nabla}_X X, Z) = -g(\nabla_X \phi X, tZ) - \overline{g}(h(X, \phi X), nZ). \tag{4.11}$$

Above equation by the use of Connection property for ∇ and Proposition 2.3 is reduced to

$$\overline{g}(\overline{\nabla}_X X, Z) = tZ(\ln f)g(\phi X, X) - \overline{g}(h(X, \phi X), nZ). \tag{4.12}$$

Hence by virtue of Eqs. (4.10) and (4.12) and the orthogonality of vector fields, we obtain

$$Z(\ln f)g(X, X) = \overline{g}(h(X, \phi X), nZ). \tag{4.13}$$

Interchanging X by ϕX in (4.13) and using the fact that $\xi \in \Gamma(TM_\lambda)$, then using (2.2), we derive

$$\overline{g}(h(X, \phi X), nZ) = -Z(\ln f)g(X, X). \tag{4.14}$$

Thus, the result follows from Eqs. (4.13) and (4.14), which proves the theorem completely. □

Remark 4.6 From Theorems 4.2 and 4.5, we observe that there does not exist any $\mathcal{PR}$-semi-slant warped product submanifold $M = M_\lambda \times_f M_T$ of a paracosymplectic manifold $\overline{M}$, whether ξ is tangent to the base manifold or tangent to the fiber of warped products.

5 Example

In addition to the results in Section 4, we conclude here by giving an explicit example to illustrate that there does not exist any proper $\mathcal{PR}$-semi-slant warped product submanifold other than a $\mathcal{PR}$-semi-invariant warped product submanifold in a paracosymplectic manifold.

Example 5.1 Let $\overline{M} = \mathbb{R}^8 \times \mathbb{R}_+ \subset \mathbb{R}^9$ be a nine-dimensional manifold with the standard Cartesian coordinates $(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, z)$, where z being the global coordinate on $\mathbb{R}_+$. Define the paracosymplectic pseudo-Riemannian metric structure $(\phi, \xi, \eta, \overline{g})$ on $\overline{M}$ by

$$\begin{aligned} &\phi e_1 = e_5,\ \phi e_2 = e_6,\ \phi e_3 = e_7,\ \phi e_4 = e_8,\ \phi e_8 = e_4,\ \phi e_7 = e_3,\ \phi e_6 = e_2 \\ &\phi e_5 = e_1,\ \phi e_9 = 0, \xi = e_9,\ \ \eta = dz,\ \ \overline{g} = \sum_{i=1}^{4}(dx_i)^2 - \sum_{j=1}^{4}(dy_j)^2 + \eta \otimes \eta. \end{aligned} \tag{5.1}$$

Here, $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9\}$ is a local orthonormal frame for $T\overline{M}$, given by $e_i = \dfrac{\partial}{\partial x_i}$, $e_j = \dfrac{\partial}{\partial y_j}$ and $e_9 = \dfrac{\partial}{\partial z}$. Let M be an isometrically immersed pseudo-Riemannian submanifold in a paracosymplectic manifold $\overline{M}$ defined by

$\Omega(r,s,\alpha,\beta,z) = \begin{pmatrix} r\cosh(\alpha), r\cosh(\beta), s\sinh(\alpha), s\sinh(\beta), \\ s\cosh(\alpha), s\cosh(\beta), r\sinh(\alpha), r\sinh(\beta), z \end{pmatrix}$, where $\alpha, \beta \in (0, \pi/2)$ and either $r < s$ or $s < -r$. Then the tangent bundle TM of M is spanned by the vectors

$$\begin{aligned} X_r &= \cosh(\alpha)e_1 + \cosh(\beta)e_2 + \sinh(\alpha)e_7 + \sinh(\beta)e_8, \\ X_s &= \sinh(\alpha)e_3 + \sinh(\beta)e_4 + \cosh(\alpha)e_5 + \cosh(\beta)e_6, \\ X_\alpha &= r\sinh(\alpha)e_1 + s\cosh(\alpha)e_3 + s\sinh(\alpha)e_5 + r\cosh(\alpha)e_7, \\ X_\beta &= r\sinh(\beta)e_2 + s\cosh(\beta)e_4 + s\sinh(\beta)e_6 + r\cosh(\beta)e_8, \\ X_z &= e_9. \end{aligned} \tag{5.2}$$

The space $\phi(TM)$ with respect to the paracosymplectic pseudo-Riemannian metric structure $(\phi, \xi, \eta, \overline{g})$ of $\overline{M}$ becomes

$$\begin{aligned} \phi(X_r) &= \sinh(\alpha)e_3 + \sinh(\beta)e_4 + \cosh(\alpha)e_5 + \cosh(\beta)e_6, \\ \phi(X_s) &= \cosh(\alpha)e_1 + \cosh(\beta)e_2 + \sinh(\alpha)e_7 + \sinh(\beta)e_8, \\ \phi(X_\alpha) &= s\sinh(\alpha)e_1 + r\cosh(\alpha)e_3 + r\sinh(\alpha)e_5 + s\cosh(\alpha)e_7, \\ \phi(X_\beta) &= s\sinh(\beta)e_2 + r\cosh(\beta)e_4 + r\sinh(\beta)e_6 + s\cosh(\beta)e_8, \\ \phi(X_z) &= 0. \end{aligned} \tag{5.3}$$

From Eqs. (5.2) and (5.3), we obtain that $\phi(X_\alpha)$, $\phi(X_\beta)$ are orthogonal to TM, and $\phi(X_r)$, $\phi(X_s)$, and $\phi(X_z)$ are tangent to M. So $\mathfrak{D}_T$ and $\mathfrak{D}_\lambda$ can be taken as a subspace span$\{X_r, X_s, X_z\}$ and a subspace span$\{X_\alpha, X_\beta\}$, respectively, where $\xi = X_z$ for $\phi(X_z) = 0$ and $\eta(X_z) = 1$. Therefore, M becomes a $\mathcal{PR}$-semi-invariant submanifold. Furthermore, it is easy to observe that $\mathfrak{D}_T$ and $\mathfrak{D}_\lambda$ are integrable. Therefore by taking the integral manifolds of $\mathfrak{D}_T$ and $\mathfrak{D}_\lambda$ by M_T and M_λ, respectively, then the induced pseudo-Riemannian metric tensor g of M is given by

$$g = dz^2 + 2dr^2 - 2ds^2 + (s^2 - r^2)\{d\alpha^2 + d\beta^2\} = g_{M_T} \oplus (s^2 - r^2)g_{M_\lambda}.$$

Hence, M is a five-dimensional $\mathcal{PR}$-semi-invariant warped product submanifold of $\overline{M}$ with warping function $f = \sqrt{(s^2 - r^2)}$. Thus, M is a $\mathcal{PR}$-semi-slant warped product submanifold with slant coefficient $\lambda = \frac{\pi}{2}$ in M of $\overline{M}$, in other words, M is a $\mathcal{PR}$-semi-invariant warped product submanifold in a paracosymplectic manifold. It is clear that there is no proper $\mathcal{PR}$-semi-slant warped product other than $\mathcal{PR}$-semi-invariant warped product.

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