Wojciech Bielas · Szymon Plewik

On RC-spaces

Received: 29 August 2018 / Accepted: 4 December 2018 / Published online: 14 December 2018
© The Author(s) 2018

Abstract Following Frink's characterization of completely regular spaces, we say that a regular T_1 -space is an RC-space whenever the family of all regular open sets constitutes a regular normal base. Normal spaces are RC-spaces and there exist completely regular spaces which are not RC-spaces. So the question arises, which of the known examples of completely regular and not normal spaces are RC-spaces. We show that the Niemytzki plane and the Sorgenfrey plane are RC-spaces.

Mathematics Subject Classification 54D15 · 54G20

1 Introduction

The aim of this note is to examine a topological space in which all regular open sets (a set is called *regular open* if it is the interior of a closed set) form a normal base. For our needs, any base $\mathcal{B}$ fulfilling conditions:

- (1) If $x \in U \subset X$, where U is open, then there exists a set $V \in \mathcal{B}$ such that $x \in X \setminus V \subset U$,
- (2) If $U, V \in \mathcal{B}$ and $U \cup V = X$, then there exist sets $U^*, V^* \in \mathcal{B}$ such that $X \setminus V \subset U^* \subset X \setminus V^* \subset U$;

is called a *normal base*.

Frink's theorem A T_1 -space is completely regular if and only if it has a normal base.

For a normal space, the family of all open sets is a normal base. Therefore, we narrow our attention to a completely regular space which is not normal. In fact, O. Frink defined a normal base and gave the characterization of a completely regular space as it is presented in the theorem above. Let us outline a proof of Frink's theorem, compare [3] or [2, Exercise 1.5.G]. For a completely regular space, the base consisting of all co-zero sets satisfies conditions (1) and (2) of Frink's characterization. If a base $\mathcal{B}$ fulfils conditions (1) and (2), then the space is completely regular which can be checked, repeating a proof of Urysohn's lemma.

Note that A. Zame [12] formally defined a regular normal base as a ring of regular closed sets (a complement of a regular open set is called *regular closed*) such that the family of all complements of members of this ring constitutes a normal base, see also [8]. We adapt his concept of a normal base in terms of open sets, omitting the assumption that a base must be a ring of sets. Consequently, a normal base consisting of regular open sets is called a *regular normal base*.

W. Bielas (✉) · Sz. Plewik
Institute of Mathematics, University of Silesia, ul. Bankowa 14, 40-007 Katowice, Poland
E-mail: wojciech.bielas@us.edu.pl

Sz. Plewik
E-mail: plewik@math.us.edu.pl

Our notation is mostly standard and follows [2]. However, names of considered topological spaces follows [11]. We introduce the RC-space concept, which we think has not been studied and described in the literature so far.¹ Therefore, we have limited our results to issues that require the geometric properties of the plane.

2 On RC-spaces

If X is a normal space, then the family of all open sets in X fulfils both conditions from Frink's characterization, i.e., the topology constitutes a normal base. Indeed, if $x \in U \subseteq X$, then the open set $X \setminus \{x\} = V$ is enough for (1) to be fulfilled. The condition (2) is just a form of the definition of normality. It appears to us that there is a gap in the literature, since we could not find any information concerning a space for which the family of all regular open sets is a normal base. Note that a union of two regular open sets may not be regular open, so omitting the assumption that a normal (regular) base has to be a ring is a significant modification, which we introduce for issues discussed here. We say that a regular space is an RC-space, if every two disjoint regular closed subsets have disjoint open neighbourhoods: in other words, a regular space is an RC-space, if the family of all regular open sets is a regular normal base. Obviously, any normal space is an RC-space. We assume that an RC-space is a regular space, so if $x \in U$, where the set U is regular open, then there exists a regular open set W such that $x \in W \subseteq \text{cl } W \subseteq U$, putting $V = X \setminus \text{cl } W$ we have verified (1). By Frink's characterization, we get that any RC-space X is completely regular. There exist examples of T_1 -spaces with bases consisting of closed-open sets, i.e., examples of completely regular spaces with regular normal bases, which are not RC-spaces. These examples are completely regular spaces with a one-point extension to a regular space, which is not completely regular, for example, spaces considering in [9] or [7], also counterexamples constructed by the method initiated in [4]. The difference between these examples and RC-spaces can be seen also in terms of one-point extensions.

Proposition 2.1 *Every regular one-point extension of an RC-space is completely regular.*

Proof Let $Y = X \cup \{\infty\}$ be a regular T_1 -space such that its subspace X is an RC-space. Suppose $F \subseteq Y$ is a closed set and let $p \in Y \setminus F$, then choose an open set V such that

$$p \in V \subseteq \text{cl } V \subseteq Y \setminus F.$$

If $p \neq \infty$, then choose an open set W such that $\infty \in W$ and $p \notin \text{cl } W$. If $f: X \rightarrow [0, 1]$ is a continuous function such that $f(p) = 0$ and $X \cap (F \cup \text{cl } W) \subseteq f^{-1}(1)$, then we get a continuous extension of f , putting $f(\infty) = 1$. But if $p = \infty$, then choose an open set U such that

$$p \in U \subseteq \text{cl } U \subseteq V \subseteq \text{cl } V \subseteq Y \setminus F.$$

Next, repeat the usual proof of Urysohn's lemma—compare [2, Ex. 1.5.G], starting in the first step from the disjoint regular closed sets $\text{cl } U$ and $\text{cl}(X \setminus \text{cl } V)$. $\square$

3 The Niemytzki plane is an RC-space

Recall, compare [2, p. 39], that the Niemytzki plane $\mathbb{L} = \mathbb{R} \times [0, \infty)$ is a closed half-plane which is endowed with the topology generated by open discs disjoint with the real axis $\mathbb{L}_1 = \{(x, 0) : x \in \mathbb{R}\}$ and all sets of the form $\{a\} \cup D$ where $D \subseteq \mathbb{L}$ is an open disc which is tangent to $\mathbb{L}_1$ at the point $a \in \mathbb{L}_1$.

For methods relevant to normal spaces, compare [2]. Interesting discussion of the lack of normality of the Niemytzki plane is presented in [1]. We shall use the following notation. If $(x, y) \in \mathbb{L}_2 = \mathbb{L} \setminus \mathbb{L}_1$ and $\alpha > 0$, then let $\mathbb{K}((x, y), \alpha)$ denote the intersection of $\mathbb{L}$ and the open disc centred at (x, y) and of radius α . By $\mathbb{K}((x, 0), \alpha)$ we denote the union of the one-point set $\{(x, 0)\}$ and the open disc centred at (x, α) and of radius α . Using elementary properties of the plane, one immediately checks the validity of the following fact.

¹ The referee has informed us that the notion of RC-space was defined by E. V. Shchepin [10]. Shchepin has called this property κ -normality. Some results concerning κ -normality were given in [5] and [6], too. Nevertheless, elementary proofs of Theorems 3.4 and 4.3 appear to be new.

Fact 3.1 Let $\{(x_n, y_n)\}_{n < \omega}$, where $y_n \geq 0$ and $x_n \in \mathbb{R}$, be a sequence which converges to a point (x, y) with respect to the Euclidean topology. If $\alpha > 0$, then

$$\mathbb{K}((x, y), \frac{\alpha}{2}) \setminus \{(x, y)\} \subseteq \bigcup \{\mathbb{K}((x_n, y_n), \alpha) : n < \omega\}.$$

□

The below picture illustrates a proof of Fact 3.1 for a case $y = 0$.

But if $y > 0$, then the set $\mathbb{K}((x, y), \frac{\alpha}{2}) \setminus \{(x, y)\}$ can be enlarged to $\mathbb{K}((x, y), \alpha)$.

If subsets $F, G \subseteq \mathbb{L}$ are fixed, then for every $\alpha > 0$ we set

$$F_\alpha = \{(x, y) \in F : \mathbb{K}((x, y), \alpha) \cap G = \emptyset\}.$$

Lemma 3.2 Suppose F and G are closed and disjoint subsets of the Niemytzki plane. If G is regular closed, then for any $\alpha > 0$ the closure of F_α , with respect to the Euclidean topology, is disjoint from G .

Proof Consider a point (x, y) which belongs to the closure of F_α with respect to Euclidean topology. Since G is regular closed, using Fact 3.1, we obtain

$$G \cap \mathbb{K}((x, y), \frac{\alpha}{2}) \setminus \{(x, y)\} = \emptyset.$$

But $\mathbb{K}((x, y), \frac{\alpha}{2})$ is an open neighbourhood of the point $(x, y) \in \mathbb{L}$, hence $(x, y) \notin G$. □

In the next lemma we use the following notation. If $(x, y) \in \mathbb{L}$ and $\mu > 0$, then let $\mathbb{C}((x, y), \mu)$ be the circle centred at (x, y) of radius μ .

Lemma 3.3 Suppose F and G are closed and disjoint subsets of the Niemytzki plane and let $0 < \varepsilon < 1$ and $0 < \alpha$. If G is regular closed, then the closure of

$$\bigcup \{\mathbb{K}((x, y), \varepsilon\alpha) : (x, y) \in F_\alpha\},$$

with respect to the Niemytzki plane, is disjoint from G .

Proof Fix numbers α, ε and sets F, F_α, G as in the assumptions. For any point $(p, q) \in G$, we shall find a number $\gamma > 0$ such that $(x, y) \in F_\alpha$ implies

$$\mathbb{K}((x, y), \varepsilon\alpha) \cap \mathbb{K}((p, q), \gamma) = \emptyset.$$

To find the appropriate $\gamma > 0$, just make γ meet the following restrictions. If $y > 0$, then one checks that the inequality $2\gamma \leq \alpha - \varepsilon\alpha$ is sufficient, since $(p, q) \notin \mathbb{K}((x, y), \alpha)$. When $(x, 0) \in F_\alpha$, choose a number $\beta > 0$ such that $\beta < \text{dist}(F_\alpha, (p, q))$ and $\beta < \varepsilon\alpha$, using Lemma 3.2. Fix $(a, b) \in \mathbb{C}((x, 0), \beta) \cap \mathbb{C}((x, \alpha), \alpha)$. Let $\mathbb{E}$ be the line passing through the points $(x, \varepsilon\alpha)$ and (a, b) . Fixing $(c, d) \in \mathbb{E} \cap \mathbb{C}((x, \varepsilon\alpha), \varepsilon\alpha)$, one checks that the inequality $2\gamma < |a - c|$ is sufficient. □

Theorem 3.4 The Niemytzki plane is an RC-space.

Proof Let F and G be regular closed and disjoint subsets of the Niemytzki plane and let $0 < \varepsilon < 1$. For any integer $n > 0$, consider open sets

$$W_n = \bigcup \left\{ \mathbb{K} \left((x, y), \frac{\varepsilon}{n} \right) : (x, y) \in F_{\frac{1}{n}} \right\}$$

and

$$V_n = \bigcup \left\{ \mathbb{K} \left((x, y), \frac{\varepsilon}{n} \right) : (x, y) \in G_{\frac{1}{n}} \right\}.$$

By Lemma 3.3, we have $G \cap \text{cl}_{\mathbb{L}} W_n = \emptyset$ and $F \cap \text{cl}_{\mathbb{L}} V_n = \emptyset$. Just like in the proof of [2, Lemma 1.5.14], we obtain open and disjoint sets

$$W^* = \bigcup \{W_n \setminus (\text{cl}_{\mathbb{L}} V_1 \cup \text{cl}_{\mathbb{L}} V_2 \cup \dots \cup \text{cl}_{\mathbb{L}} V_n) : n > 0\} \supseteq F$$

and

$$V^* = \bigcup \{V_n \setminus (\text{cl}_{\mathbb{L}} W_1 \cup \text{cl}_{\mathbb{L}} W_2 \cup \dots \cup \text{cl}_{\mathbb{L}} W_n) : n > 0\} \supseteq G.$$

We get

$$F \subset W^* \subset \text{int cl } W^* \text{ and } G \subset V^* \subset \text{int cl } V^*,$$

and

$$\text{int cl } W^* \cap \text{int cl } V^* = \emptyset.$$

It suffices to show that the family of all regular open subsets of the Niemytzki plane is a regular normal base. $\square$

4 The Sorgenfrey plane is an RC-space

Recall that any RC-space has to be regular, so we assume that all considered spaces are regular. The Sorgenfrey line $\mathbb{S}$ is the real line $\mathbb{R}$ with the topology generated by half-closed intervals of the form $[x, y)$: in other words, one can consider $\mathbb{S}$ as the reals $\mathbb{R}$ with the arrow topology. The Cartesian product $\mathbb{S} \times \mathbb{S} = \mathbb{S}^2$ equipped with the product topology is usually called the Sorgenfrey plane, compare [11, p. 103]. In this section we show that for any $\mathfrak{m}$, where $\mathfrak{m}$ is a finite cardinal or $\mathfrak{m} = \aleph_0$, the product space $\mathbb{S}^{\mathfrak{m}}$ is an RC-space, despite the fact that for $\mathfrak{m} > 1$ it is not a normal space. Our argumentation, although it is a modified discussion from the previous section, requires some adjustments and interpretations. Namely, fix a cardinal $\mathfrak{m}$, finite or $\mathfrak{m} = \aleph_0$. Let $\mathbb{R}^{\mathfrak{m}}$ be equipped with the product topology. Thus $\mathbb{R}^{\mathfrak{m}}$ is a metric space, since $\mathfrak{m}$ is countable. We now proceed to use the short-cut $\mathbf{x} = \{x_k\}_{0 \leq k < \mathfrak{m}}$, for any point $\mathbf{x} \in \mathbb{S}^{\mathfrak{m}}$. If $n > 0$ and $\mathbf{q} = \{q_k\}_{0 \leq k < \mathfrak{m}} \in \mathbb{S}^{\mathfrak{m}}$, then put $\mathbf{x} + \frac{1}{n} = \{x_k + \frac{1}{n}\}_{0 \leq k < \mathfrak{m}}$ and

$$[\mathbf{x}, \mathbf{q})_n = \{\mathbf{y} \in \mathbb{S}^{\mathfrak{m}} : x_i \leq y_i < q_i, \text{ whenever } 0 \leq i < \min\{n, \mathfrak{m}\}\},$$

and then put $\mathbb{P}(\mathbf{x}, n) = [\mathbf{x}, \mathbf{x} + \frac{1}{n})_n$. Thus, the sets $\mathbb{P}(\mathbf{x}, n)$ constitute a base for $\mathbb{S}^{\mathfrak{m}}$.

Fact 4.1 Let $\{\mathbf{x}_k\}_{k>0}$ be a sequence which converges to a point $\mathbf{x}$ with respect to $\mathbb{R}^{\mathfrak{m}}$. If $n > 0$, then

$$\text{int}_{\mathbb{R}^{\mathfrak{m}}} \mathbb{P}(\mathbf{x}, n) \subseteq \bigcup \{\mathbb{P}(\mathbf{x}_k, n) : k > 0\}.$$

$\square$

Given sets $F, G \subseteq \mathbb{S}^{\mathfrak{m}}$ and a natural number $n > 0$, put

$$F_{G,n} = \{\mathbf{x} \in F : \mathbb{P}(\mathbf{x}, n) \cap G = \emptyset\}.$$

Lemma 4.2 Suppose F and G are closed and disjoint subsets of $\mathbb{S}^{\mathfrak{m}}$. If G is regular closed, with respect to $\mathbb{S}^{\mathfrak{m}}$, then the closure of $F_{G,n}$, with respect to $\mathbb{R}^{\mathfrak{m}}$, is disjoint from G .

Proof Consider a point $\mathbf{x} \in G$. We shall find a natural number $m > 0$ such that

$$\mathbb{P}(\mathbf{x}, m) \cap \mathbb{P}(\mathbf{y}, 2n) = \emptyset,$$

for any $\mathbf{y} \in F_{G,n}$. Since G is regular closed, we can find a base set

$$\mathbb{P}(\mathbf{p}, i) \subset G \cap \mathbb{P}(\mathbf{x}, 2n)$$

such that $i \geq 2n$ and $x_k < p_k$ for $0 \leq k < \min\{2n, m\}$. Thus we get that if $\mathbf{y} \in [\mathbf{x} - \frac{1}{2n}, \mathbf{p} + \frac{1}{i}]_{2n}$, then $\mathbb{P}(\mathbf{y}, n) \cap \mathbb{P}(\mathbf{p}, i) \neq \emptyset$. Therefore, if $\mathbf{y} \in [\mathbf{x} - \frac{1}{2n}, \mathbf{p} + \frac{1}{i}]_{2n}$, then $\mathbf{y} \notin F_{G,n}$. Choosing $m > 0$ such that

$$\frac{1}{m} < p_k - x_k, \text{ for } 0 \leq k < \min\{2n, m\},$$

we obtain

$$\mathbb{P}(\mathbf{y}, 2n) \cap \mathbb{P}(\mathbf{x}, m) = \emptyset,$$

for any $\mathbf{y} \in F_{G,n}$. □

Theorem 4.3 *If m is finite or $m = \aleph_0$, then the space $\mathbb{S}^m$ is an RC-space.*

Proof Let F and G be regular closed and disjoint subsets of $\mathbb{S}^m$. For each integer $n > 0$, consider open sets

$$W_n = \bigcup \{\mathbb{P}(\mathbf{x}, 2n) : \mathbf{x} \in F_{G,n}\}$$

and

$$V_n = \bigcup \{\mathbb{P}(\mathbf{x}, 2n) : \mathbf{x} \in G_{F,n}\}.$$

By Lemma 4.2, we get $G \cap \text{cl}_{\mathbb{S}^m} W_n = \emptyset$ and $F \cap \text{cl}_{\mathbb{S}^m} V_n = \emptyset$. Just like in the proof of Theorem 3.4 or [2, Lemma 1.5.14], we are done. □

Corollary 4.4 *The Sorgenfrey plane is an RC-space.*

Proof The corollary is a special case of Theorem 4.3, as illustrated in the following figure.

If $\mathbf{z} \in [x_1 - \frac{1}{2n}, p_1 + \frac{1}{i}] \times [x_2 - \frac{1}{2n}, p_2 + \frac{1}{i}]$, then $G \cap \mathbb{P}(\mathbf{z}, n) \neq \emptyset$, hence $\mathbf{z} \notin F_{G,n}$. But if

$$\mathbf{y} \notin \left(x_1 - \frac{1}{2n}, x_1 + \frac{1}{m}\right) \times \left(x_2 - \frac{1}{2n}, x_2 + \frac{1}{m}\right),$$

then $\mathbb{P}(\mathbf{y}, 2n) \cap \mathbb{P}(\mathbf{x}, m) = \emptyset$. □

5 More examples of completely regular spaces which are not RC-spaces

Let $\mathbb{L}^1 \oplus \mathbb{L}^2 \oplus \mathbb{L}^3$ be the sum of three copies of the Niemytzki plane, for the definition of the sum of spaces see [2, p. 103]. Consider a quotient X of this sum, obtained by gluing copies of the rationals $\mathbb{Q} \subset \mathbb{L}_1 \subset \mathbb{L}^1$ and $\mathbb{Q} \subset \mathbb{L}_1 \subset \mathbb{L}^2$ and copies of irrationals $\mathbb{I} \subset \mathbb{L}_1 \subset \mathbb{L}^2$ and $\mathbb{I} \subset \mathbb{L}_1 \subset \mathbb{L}^3$. This quotient is a completely regular space which is not an RC-space. Indeed, subspaces $\mathbb{L}^1 \subset X$ and $\mathbb{L}^3 \subset X$ are regular closed and if $V \subset X$ and $U \subset X$ are open sets such that $\mathbb{L}^1 \subset V$ and $\mathbb{L}^3 \subset U$, then the intersection $\mathbb{L}^2 \cap U \cap V$ is non-empty, compare [1] or [11, pp. 101–102].

Let us note that the above construction of a quotient space X relies on a simplification of the constructions started in [4]. Of course, by analogy one can get many examples which are not RC-spaces, using other completely regular spaces which are not normal. For example, let $\mathbb{S}_1 \oplus \mathbb{S}_2 \oplus \mathbb{S}_3$ be the sum of three copies of the plane with the half-open square topology, compare [11, p. 103]. Consider a quotient Y of this sum, obtained by gluing copies of the rationals $\mathbb{Q} = \{(\alpha, -\alpha) : \alpha \text{ is a rational number}\} \subset \mathbb{S}_1$ and $\mathbb{Q} \subset \mathbb{S}_2$ and copies of irrationals $\mathbb{I} = \{(\alpha, -\alpha) : \alpha \text{ is an irrational number}\} \subset \mathbb{S}_2$ and $\mathbb{I} \subset \mathbb{S}_3$. According to a similar argument as for X above, the quotient space Y is completely regular and is not an RC-space.

We would like to thank both referees for comments which improved the readability of the paper. We hope that the remarks mentioned in the footnote will be helpful in conducting further research.

Open Access This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

References

1. Choudounský, D.: Non-normality and relative normality of Niemytzki plane. *Acta Univ. Carolin. Math. Phys.* **48**(2), 37–41 (2007)
2. Engelking, R.: *General Topology*. Mathematical Monographs, vol. 60. PWN—Polish Scientific Publishers, Warsaw (1977)
3. Frink, O.: Compactifications and semi-normal spaces. *Amer. J. Math.* **86**, 602–607 (1964)
4. Jones, F.B.: Hereditarily separable, non-completely regular spaces. In: Dickman, R.F., Fletcher, P. (eds.) *Topology Conference*. Lecture Notes in Mathematics, vol. 375. Springer, Berlin, Heidelberg (1974)
5. Kalantan, L.: Results about κ -normality. *Topol. Appl.* **125**(1), 47–62 (2002)
6. Kalantan, L., Szeptycki, P.J.: κ -normality and products of ordinals. *Topol. Appl.* **123**(3), 537–545 (2002)
7. Kalembe, P.; Plewik, Sz.: On regular but not completely regular spaces. <https://doi.org/10.1016/j.topol.2018.11.006>. compare [arXiv:1701.04322](https://arxiv.org/abs/1701.04322)
8. Misra, A.K.: Some regular Wallman βX . *Nederl. Akad. Wetensch. Proc. Ser. A 76=Indag. Math.* **35**, 237–242 (1973)
9. Mysior, A.: A regular space which is not completely regular. *Proc. Amer. Math. Soc.* **81**(4), 652–653 (1981)
10. Shchepin, E.V.: Real functions, and spaces that are nearly normal. (Russian) *Sibirsk. Mat. Ž.* **13**, 1182–1196 (1972)
11. Steen, L.A.; Seebach Jr., J.A.: *Counterexamples in topology*. Holt, Rinehart and Winston Inc, New York, pp XIII (1970)
12. Zame, A.: A note on Wallman spaces. *Proc. Amer. Math. Soc.* **22**, 141–144 (1969)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.