Arnab Bhattacharjee

Reflexive-nilpotents-property skewed by ring endomorphisms

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Abstract The reflexive property for rings was introduced by Mason and play roles in noncommutative ring theory. A ring R is called reflexive if for $a, b \in R$, $aRb = 0$ implies $bRa = 0$. Recently, Kheradmand et al. introduced the notion of RNP (reflexive-nilpotents-property) rings by restricting the reflexive property to nilpotent elements. In this article, we study reflexive-nilpotents-property skewed by a ring endomorphism α and introduce the notion of α -skew RNP rings. We investigate various properties and extensions of these rings and also determine the structure of minimal noncommutative α -skew RNP rings.

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1 Introduction

Throughout this article, all rings are associative with unity, unless explicitly mentioned and all ring endomorphisms are nonzero. Given a ring R , the polynomial ring with an indeterminate x over R is denoted by $R[x]$, the n by n full (resp., upper triangular) matrix ring over R by $M_n(R)$ (resp., $U_n(R)$), $D_n(R)$ denotes the subring $\{(a_{ij}) \in U_n(R) : \text{diagonal entries of } (a_{ij}) \text{ are equal}\}$ of $U_n(R)$, $V_n(R)$ denotes the ring of all matrices $(a_{ij}) \in D_n(R)$ such that $a_{ij} = a_{(i+1)(j+1)}$ for $i = 1, \dots, n-2$ and $j = 2, \dots, n-1$, $N^*(R)$ and $N(R)$, respectively, denote the upper nilradical (i.e., the sum of all nil ideals) and the set of all nilpotent elements of R . Use E_{ij} to denote the matrix with (i, j) -entry 1 and other entries 0, and $\mathbb{Z}_n$ denotes the ring of integers modulo n .

Recall that a ring R is called reduced, if it has no nonzero nilpotent elements. Due to Mason [16], a right ideal I of a ring R is called reflexive if for $a, b \in R$, $aRb \subseteq I$ implies $bRa \subseteq I$ and a ring R is called reflexive, if the zero ideal of R is reflexive. It is well known that reflexive rings are generalizations of both commutative and reduced rings. Recently, Kheradmand et al. [10] generalized the notion of reflexive rings to RNP rings. A ring R is called RNP (reflexive-nilpotents-property) [10], if for $a, b \in N(R)$, $aRb = 0$ implies $bRa = 0$.

Generalized reduced rings were extended to ring endomorphisms by Krempa. Following [12], an endomorphism α of a ring R is called rigid if for $a \in R$, $a\alpha(a) = 0$ implies $a = 0$ and a ring R is called α -rigid, if there exists a rigid endomorphism α of R . Kwak et al. [14] extended the notion of reflexive rings to ring endomorphisms. Due to [14], an endomorphism α of a ring R is called right (resp., left) skew reflexive if for $a, b \in R$, $aRb = 0$ implies $bR\alpha(a) = 0$ (resp., $\alpha(b)Ra = 0$) and a ring R is called right (resp., left) α -skew reflexive, if there exists a right (resp., left) skew reflexive endomorphism α of R . A ring R is called α -skew reflexive [14], if it is both right and left α -skew reflexive. Note that α -rigid rings are right α -skew reflexive by [14, Theorem 2.6].

A. Bhattacharjee (✉)
Department of Mathematics, Assam University, Silchar, Assam 788011, India
E-mail: arnab.math.au@gmail.com

Motivated by above, we extend the notion of RNP rings to ring endomorphisms and introduce α -skew RNP rings as a generalization of α -skew reflexive rings. Our aim is to study the structure of these rings and related properties. We begin with the following definitions.

- Definition 1.1** (1) An endomorphism α of a ring R is called right (resp., left) skew RNP, if for $a, b \in N(R)$, $aRb = 0$ implies $bR\alpha(a) = 0$ (resp., $\alpha(b)Ra = 0$).
- (2) A ring R is called right (resp., left) α -skew RNP, if there exists a right (resp., left) skew RNP endomorphism α of R .
- (3) A ring R is called α -skew RNP, if it is both right and left α -skew RNP.

Remark 1.2 (1) Reduced rings are clearly α -skew RNP for any endomorphism α .

(2) Every right (resp., left) α -skew reflexive ring is right (resp., left) α -skew RNP. However, the converse is not true in general by means of [14, Example 2.7(1)] and (1) above.

Next, we show that the notion of α -skew RNP rings is not left–right symmetric.

Example 1.3 Consider a ring $R = U_2(\mathbb{Z}_8)$. Following [10, Example 2.5(1)], R is not RNP.

- (1) Let $\alpha : R \rightarrow R$ be an endomorphism defined by

$$\alpha \left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

Suppose that $ARB = 0$ for

$$A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, B = \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} \in N(R).$$

Clearly, $a, a' \in N(\mathbb{Z}_8)$. Then for any $r \in \mathbb{Z}_8$, we have

$$S = \begin{pmatrix} r & r \\ 0 & r \end{pmatrix} \in R$$

and $ASB = 0$, yielding $ara' = 0$ and so $a\mathbb{Z}_8a' = 0$. Since $\mathbb{Z}_8$ is commutative, so it is reflexive and consequently, $a'\mathbb{Z}_8a = 0$. This gives $BR\alpha(A) = 0$, entailing R is right α -skew RNP. For

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \in N(R),$$

we have $ARB = 0$ but

$$\alpha(B)A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \neq 0,$$

entailing $\alpha(B)RA \neq 0$. This shows that R is not left α -skew RNP.

- (2) Let $\alpha' : R \rightarrow R$ be an endomorphism defined by

$$\alpha' \left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}.$$

By applying arguments similar to (1), we can show that R is left α' -skew RNP which is not right α' -skew RNP.

Next, we give a sufficient condition under which the notions of right and left α -skew RNP rings are equivalent.

Proposition 1.4 Let R be a ring with an endomorphism α . Suppose that R is RNP. Then, R is right α -skew RNP if and only if R is left α -skew RNP.

Proof Let R be right α -skew RNP. Suppose that $a, b \in N(R)$ such that $aRb = 0$. Since R is RNP, so $bRa = 0$. Then $aR\alpha(b) = 0$ as R is right α -skew RNP. Since $\alpha(N(R)) \subseteq N(R)$ and R is RNP, therefore $\alpha(b)Ra = 0$. Hence, R is left α -skew RNP. Converse can be proved similarly. $\square$

Remark 1.5 The condition “ R is an RNP ring” in Proposition 1.4 is not superfluous by Example 1.3.

Next, we show that there exists an endomorphism α of a commutative ring R for which R is neither right nor left α -skew RNP.

Example 1.6 Consider $R = \mathbb{Z}_8 \oplus \mathbb{Z}_8$ with usual addition and multiplication. Let $\alpha : R \rightarrow R$ be an endomorphism defined by $\alpha((a, b)) = (b, a)$. For $a = (4, 2)$, $b = (2, 0) \in N(R)$, we have $aRb = 0$. However, $b\alpha(a) = (4, 0) \neq 0$ and $\alpha(b)a = (0, 4) \neq 0$, entailing neither $bR\alpha(a) = 0$ nor $\alpha(b)Ra = 0$. Hence R is neither right nor left α -skew RNP.

From Examples 1.3 and 1.6, it is clear that the classes of RNP and right (left) α -skew RNP rings are independent of each other.

Following [2], an endomorphism α of a ring R is called compatible if for $a, b \in R$, $ab = 0 \Leftrightarrow \alpha\alpha(b) = 0$ and a ring R is called α -compatible if there exists a compatible endomorphism α of R . By [1, Lemma 2.1], for a ring R with an endomorphism α , α is compatible if and only if for $a, b \in R$, $\alpha(a)b = 0 \Leftrightarrow ab = 0$; moreover, if α is compatible, then α is injective.

Proposition 1.7 Let R be a ring with an endomorphism α . Suppose that R is α -compatible. Then, the following are equivalent:

- (1) R is RNP.
- (2) R is right α -skew RNP.
- (3) R is left α -skew RNP.

Proof (1) $\Rightarrow$ (2) Let R be RNP and let $a, b \in N(R)$ such that $aRb = 0$. Then, $arb = 0$ for any $r \in R$ and so $\alpha(a)Rb = 0$ as R is α -compatible. Since $\alpha(N(R)) \subseteq N(R)$ and R is RNP, therefore $bR\alpha(a) = 0$. Hence, R is right α -skew RNP.

(2) $\Rightarrow$ (1) Let R be right α -skew RNP and let $a, b \in N(R)$ such that $aRb = 0$. Then $arb = 0$ for any $r \in R$ and so $aR\alpha(b) = 0$ as R is α -compatible. Since $\alpha(N(R)) \subseteq N(R)$ and R is right α -skew RNP, therefore $\alpha(b)R\alpha(a) = 0$. Then, for any $r \in R$, $\alpha(bra) = \alpha(b)\alpha(r)\alpha(a) = 0$. Since R is α -compatible, therefore α is injective and so $bRa = 0$. Hence R is RNP.

(1) $\Leftrightarrow$ (3) can be proved similarly. □

Proposition 1.8 Let R be a ring with an endomorphism α . If R is α -skew RNP, then for $a, b \in N(R)$, $aRb = 0$ implies $aR\alpha^{2k}(b) = 0$ and $\alpha^{2k-1}(b)Ra = 0$; moreover, $\alpha^{2k}(a)Rb = 0$ and $bR\alpha^{2k-1}(a) = 0$ for all $k \geq 1$.

Proof Let R be α -skew RNP and let $a, b \in N(R)$ such that $aRb = 0$. Since $\alpha(N(R)) \subseteq N(R)$, therefore

$$\begin{aligned} \alpha(b)Ra = 0 &\Rightarrow aR\alpha^2(b) = 0 \Rightarrow \alpha^3(b)Ra = 0 \Rightarrow aR\alpha^4(b) = 0, \dots; \\ bR\alpha(a) = 0 &\Rightarrow \alpha^2(a)Rb = 0 \Rightarrow bR\alpha^3(a) = 0 \Rightarrow \alpha^4(a)Rb = 0, \dots; \end{aligned}$$

and so we get $aR\alpha^{2k}(b) = 0$, $\alpha^{2k-1}(b)Ra = 0$, $\alpha^{2k}(a)Rb = 0$ and $bR\alpha^{2k-1}(a) = 0$ for $k \geq 1$ inductively. □

Next, we determine the structure of noncommutative α -skew RNP rings of minimal order.

Proposition 1.9 For an endomorphism α of a ring R , if R is a minimal noncommutative α -skew RNP ring, then R is of order 8 and is isomorphic to $U_2(\mathbb{Z}_2)$.

Proof By [6, Theorem], if the order of a finite ring R has a cube-free factorization, then R is commutative, and if R is of order p^3 , p a prime, then $R \cong U_2(GF(p))$ by [6, Proposition], where $GF(p)$ denotes the Galois field of order p . Therefore, order of the minimal noncommutative ring is 8 and is isomorphic to $U_2(\mathbb{Z}_2)$. Since E_{12} is the only nonzero nilpotent element of $U_2(\mathbb{Z}_2)$, it follows that $U_2(\mathbb{Z}_2)$ is α -skew RNP for any endomorphism α , and hence the result. □

2 Properties and extensions of right α -skew RNP rings

For a nonempty subset S of a ring R , the right annihilator of S in R is denoted and defined by $r_R(S) = \{c \in R : Sc = 0\}$. The left annihilator is defined analogously and denoted by $\ell_R(S)$.

Proposition 2.1 *For a ring R with an endomorphism α , the following are equivalent:*

- (1) R is right α -skew RNP.
- (2) $r_R(aR) \cap N(R) \subseteq l_R(R\alpha(a)) \cap N(R)$ for all $a \in N(R)$.
- (3) $ARB = 0$ implies $BR\alpha(A) = 0$ for all nonempty subsets A, B of $N(R)$.
- (4) $IJ = 0$ implies $J\alpha(I) = 0$ for all right (left) ideals I, J of R generated by subsets of $N(R)$.
- (5) $IJ = 0$ implies $J\alpha(I) = 0$ for all principal right (left) ideals I, J of R generated by nilpotent elements of R .
- (6) $IJ = 0$ implies $J\alpha(I) = 0$ for all ideals I, J of R generated by subsets of $N(R)$.
- (7) $IJ = 0$ implies $J\alpha(I) = 0$ for all principal ideals I, J of R generated by nilpotent elements of R .

Proof (1) $\Rightarrow$ (2) Let R be right α -skew RNP and let $a \in N(R)$. Suppose that $b \in r_R(aR) \cap N(R)$. Then $b \in N(R)$ and $aRb = 0$. Since R is right α -skew RNP, therefore $bR\alpha(a) = 0$ and so $b \in l_R(R\alpha(a)) \cap N(R)$. Thus $r_R(aR) \cap N(R) \subseteq l_R(R\alpha(a)) \cap N(R)$.

- (2) $\Rightarrow$ (1) Assume that (2) holds. Suppose that $a, b \in N(R)$ such that $aRb = 0$. Then $b \in r_R(aR) \cap N(R)$. By assumption, $b \in l_R(R\alpha(a)) \cap N(R)$ and so $bR\alpha(a) = 0$, entailing R is right α -skew RNP.
- (1) $\Rightarrow$ (3) Let R be right α -skew RNP and let A, B be nonempty subsets of $N(R)$ such that $ARB = 0$. Then for all $a \in A, b \in B$, we have $aRb = 0$ and $a, b \in N(R)$. Since R is right α -skew RNP, therefore $bR\alpha(a) = 0$ and so

$$BR\alpha(A) = \sum_{\substack{a \in A, b \in B \\ \text{(finite)}}} bR\alpha(a) = 0.$$

- (3) $\Rightarrow$ (4) Assume that (3) holds. Suppose that $IJ = 0$ for $I = AR$ and $J = BR$ where A, B are nonempty subsets of $N(R)$. Then $ARB \subseteq IJ = 0$ and so $BR\alpha(A) = 0$, entailing $J\alpha(I) = 0$. The remaining part can be proved similarly.
- (4) $\Rightarrow$ (5) $\Rightarrow$ (7) and (4) $\Rightarrow$ (6) $\Rightarrow$ (7) are straightforward.
- (7) $\Rightarrow$ (1) Assume that (7) holds. Let $a, b \in N(R)$ such that $aRb = 0$. Then, $(RaR)(RbR) = 0$ and so $(RbR)\alpha(RaR) = 0$ by assumption. Thus, $bR\alpha(a) = 0$ and hence R is right α -skew RNP. $\square$

Proposition 2.2 *Let R be a ring with an automorphism α . Then, the following are equivalent:*

- (1) R is right α -skew RNP.
- (2) For $a, b \in N(R)$, $aR\alpha(b) = 0$ implies $bRa = 0$.
- (3) For $a \in N(R)$, $r_R(aR) \cap N(R) = l_R(R\alpha(a)) \cap N(R)$.

Proof (1) $\Rightarrow$ (2) Let R be right α -skew RNP. Let $a, b \in N(R)$ such that $aR\alpha(b) = 0$. Since $\alpha(N(R)) \subseteq N(R)$ and R is right α -skew RNP, therefore $\alpha(b)R\alpha(a) = 0$. Then for any $r \in R$, $\alpha(bra) = \alpha(b)\alpha(r)\alpha(a) = 0$ and so $bRa = 0$ as α is injective.

- (2) $\Rightarrow$ (1) Assume that (2) holds. Let $a, b \in N(R)$ such that $aRb = 0$. Since α is surjective, therefore $\alpha(a)R\alpha(b) = 0$. By hypothesis, $bR\alpha(a) = 0$. Hence R is right α -skew RNP.
- (1) $\Rightarrow$ (3) Let R be right α -skew RNP and let $a \in N(R)$. Then $r_R(aR) \cap N(R) \subseteq l_R(R\alpha(a)) \cap N(R)$ by Proposition 2.1. Let $b \in l_R(R\alpha(a)) \cap N(R)$. Then $b \in N(R)$ and $bR\alpha(a) = 0$. By (2), $aRb = 0$ and so $b \in r_R(aR) \cap N(R)$. Thus $l_R(R\alpha(a)) \cap N(R) \subseteq r_R(aR) \cap N(R)$ and hence $r_R(aR) \cap N(R) = l_R(R\alpha(a)) \cap N(R)$.
- (3) $\Rightarrow$ (1) It follows directly from Proposition 2.1. $\square$

Proposition 2.3 *Let R be a ring with an endomorphism α . Suppose that S is a ring and $\sigma : R \rightarrow S$ an isomorphism. Then R is right α -skew RNP if and only if S is right α' -skew RNP where $\alpha' = \sigma \circ \alpha \circ \sigma^{-1}$.*

Proof Let R be right α -skew RNP and let $x, y \in N(S)$ such that $xSy = 0$. Since σ is bijective, therefore $N(S) = \sigma(N(R))$ and so there exist $a, b \in N(R)$ such that $\sigma(a) = x, \sigma(b) = y$, and $aRb = 0$. By hypothesis, $bR\alpha(a) = 0$. Using the surjectivity of σ , we obtain

$$yS\alpha'(x) = \sigma(b)\sigma(R)(\sigma \circ \alpha \circ \sigma^{-1})(\sigma(a)) = \sigma(bR\alpha(a)) = 0.$$

Therefore, S is right α' -skew RNP. Converse can be proved similarly. $\square$

Let $\{R_\lambda : \lambda \in \Lambda\}$ be a class of rings such that for each $\lambda \in \Lambda$, α_λ is an endomorphism of R_λ . For the direct sum $\bigoplus_{\lambda \in \Lambda} R_\lambda$, the mapping $\bar{\alpha} : \bigoplus_{\lambda \in \Lambda} R_\lambda \rightarrow \bigoplus_{\lambda \in \Lambda} R_\lambda$ defined by $\bar{\alpha}((a_\lambda)) = (\alpha_\lambda(a_\lambda))$ induces an endomorphism of $\bigoplus_{\lambda \in \Lambda} R_\lambda$.

Proposition 2.4 *The direct sum $\bigoplus_{\lambda \in \Lambda} R_\lambda$ is right $\bar{\alpha}$ -skew RNP if and only if each R_λ is right α_λ -skew RNP.*

Proof It follows from the fact that $N(\bigoplus_{\lambda \in \Lambda} R_\lambda) = \bigoplus_{\lambda \in \Lambda} N(R_\lambda)$. $\square$

Proposition 2.5 *Let R be a ring with an endomorphism α . Suppose that $\alpha(e) = e$ for any $e^2 = e \in R$.*

- (1) *If R is right α -skew RNP, then so is eRe for any $e^2 = e \in R$.*
- (2) *If e is central idempotent in R , then R is right α -skew RNP if and only if eR and $(1 - e)R$ are right α -skew RNP.*

Proof (1) Let R be right α -skew RNP and let $a, b \in N(eRe)$ such that $a(eRe)b = 0$. Then $ae = a = ea$ and $be = b = eb$. Since R is right α -skew RNP and $a, b \in N(R)$, therefore $bR\alpha(a) = 0$ and consequently, $b(eRe)\alpha(a) = 0$. Therefore, eRe is right α -skew RNP.

- (2) It follows directly from Proposition 2.4 and (1) above. $\square$

For a ring R with an endomorphism α and an ideal I of R , if I is an α -ideal (i.e., $\alpha(I) \subseteq I$) of R , then $\bar{\alpha} : R/I \rightarrow R/I$ defined by $\bar{\alpha}(a + I) = \alpha(a) + I$ for $a \in R$, induces an endomorphism of R/I . The class of right α -skew RNP rings is not closed under homomorphic images by the following example.

Example 2.6 Let K be a field and let $A = K\langle a, b \rangle$ be the free algebra with noncommuting indeterminates a, b over K . Consider an automorphism α of A defined by

$$a \mapsto b \text{ and } b \mapsto a.$$

Since A is a domain, therefore A is clearly right α -skew RNP. Let I be an ideal of A generated by

$$ab, ba, a^4, b^4.$$

Consider $R = A/I$. For simplicity, we identify the elements of A with their images in R . Clearly, $a, b \in N(R)$. With an easy computation, $aRb = 0$ but $b\bar{\alpha}(a) = b^2 \neq 0$, entailing $bR\bar{\alpha}(a) \neq 0$. Therefore, R is not right $\bar{\alpha}$ -skew RNP.

Due to Marks [15], a ring R is called NI, if $N^*(R) = N(R)$. Clearly, commutative rings are NI. A ring R is called nil-IFP [3], if for $a, b \in R$, $ab \in N(R)$ implies $aRb \subseteq N(R)$. NI rings are clearly nil-IFP [3]. Note that the classes of nil-IFP and right α -skew RNP rings are independent of each other by means of Examples 2.16 and 2.22 to follow. However, we have the following.

Proposition 2.7 *Let R be a ring with an endomorphism α . Suppose that R is nil-IFP such that $N(R)^2 = 0$, then R is right α -skew RNP.*

Proof Let $a, b \in N(R)$ such that $aRb = 0$. Since R is nil-IFP, therefore by [3, Lemma 1.2], $bR \subseteq N(R)$. Since $\alpha(N(R)) \subseteq N(R)$ and $N(R)^2 = 0$, therefore $bR\alpha(a) = 0$. Hence R is right α -skew RNP. $\square$

Corollary 2.8 *Let R be a reduced ring. Then, both $U_2(R)$ and $D_2(R)$ are right α -skew RNP for any endomorphism α .*

Proof For a reduced ring R , we have

$$N(U_2(R)) = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} : a \in R \right\} = N(D_2(R)).$$

□

For an algebra R over a commutative ring S , the Dorroh extension [5] of R by S is the abelian group $D = R \oplus S$ with multiplication given by

$$(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2).$$

For an S -endomorphism α of R and the Dorroh extension D of R by S , the mapping $\bar{\alpha} : D \rightarrow D$ defined by $\bar{\alpha}(r, s) = (\alpha(r), s)$, is an S -algebra endomorphism.

Proposition 2.9 *Let R be an algebra with unity over a commutative reduced ring S and α an S -endomorphism of R . Then, R is right α -skew RNP if and only if the Dorroh extension D of R by S is right $\bar{\alpha}$ -skew RNP.*

Proof Note that any $s \in S$ can be written as $s = s1 \in R$; therefore,

$$R = \{r + s : (r, s) \in D\}.$$

Also $N(D) = (N(R), 0)$ as S is a reduced ring. Let R be right α -skew RNP. Suppose that $(a, 0)D(b, 0) = 0$ for $a, b \in N(R)$. Then for any $(r, s) \in D$, we have $(a, 0)(r, s)(b, 0) = 0$. This gives $arb + sab = 0$ and so $a(r + s)b = 0$, entailing $aRb = 0$. Since R is right α -skew RNP, therefore $bR\alpha(a) = 0$ and so $(b, 0)D\bar{\alpha}(a, 0) = 0$. Thus, D is right $\bar{\alpha}$ -skew RNP.

Conversely, assume that D is right $\bar{\alpha}$ -skew RNP. Let $a, b \in N(R)$ such that $aRb = 0$. Then $a(r + s)b = 0$ for any $(r, s) \in D$ and so $(a, 0)D(b, 0) = 0$. Since D is right $\bar{\alpha}$ -skew RNP, therefore $(b, 0)D\bar{\alpha}(a, 0) = 0$ and so $bR\alpha(a) = 0$. Hence R is right α -skew RNP. □

For a ring R with an endomorphism α , the mapping $\bar{\alpha} : R[x] \rightarrow R[x]$ defined by

$$\bar{\alpha} \left(\sum_{i=0}^n a_i x^i \right) = \sum_{i=0}^n \alpha(a_i) x^i$$

induces an endomorphism of $R[x]$.

Proposition 2.10 *For a ring R with an endomorphism α , if $R[x]$ is right $\bar{\alpha}$ -skew RNP, then R is right α -skew RNP.*

Proof Let R be a ring with an endomorphism α such that $R[x]$ is right $\bar{\alpha}$ -skew RNP. Suppose that $a, b \in N(R)$ such that $aRb = 0$. By [7, Lemma 2.1], $aR[x]b = 0$ and so $bR[x]\bar{\alpha}(a) = 0$, entailing $bR\alpha(a) = 0$. Therefore, R is right α -skew RNP. □

Next, we show that for a right α -skew RNP ring R , $R[x]$ need not be right $\bar{\alpha}$ -skew RNP.

Example 2.11 We refer to the ring in [11, Example 2.8]. Let $A = \mathbb{Z}_2 \{a_0, a_1, a_2, b_0, b_1, b_2, c\}$ be the free algebra with zero constant terms in noncommuting indeterminates $a_0, a_1, a_2, b_0, b_1, b_2, c$ over $\mathbb{Z}_2$. Note that A is a ring without unity and consider an ideal of the ring $\mathbb{Z}_2 + A$, say I , generated by

$$\begin{aligned} & a_0b_0, a_0b_1 + a_1b_0, a_0b_2 + a_1b_1 + a_2b_0, a_1b_2 + a_2b_1, a_2b_2, a_0rb_0, a_2rb_2, \\ & b_0a_0, b_0a_1 + b_1a_0, b_0a_2 + b_1a_1 + b_2a_0, b_1a_2 + b_2a_1, b_2a_2, b_0ra_0, b_2ra_2, \\ & (a_0 + a_1 + a_2)r(b_0 + b_1 + b_2), (b_0 + b_1 + b_2)r(a_0 + a_1 + a_2), \\ & a_0a_0, a_2a_2, a_0ra_0, a_2ra_2, b_0b_0, b_2b_2, b_0rb_0, b_2rb_2, r_1r_2r_3r_4, \\ & a_0a_1 + a_1a_0, a_0a_2 + a_1a_1 + a_2a_0, a_1a_2 + a_2a_1, \\ & b_0b_1 + b_1b_0, b_0b_2 + b_1b_1 + b_2b_0, b_1b_2 + b_2b_1, \\ & (a_0 + a_1 + a_2)r(a_0 + a_1 + a_2), (b_0 + b_1 + b_2)r(b_0 + b_1 + b_2), \end{aligned}$$

where $r, r_1, r_2, r_3, r_4 \in A$. Then, clearly $A^4 \subseteq I$. Let $R = (\mathbb{Z}_2 + A)/I$. Let σ be an automorphism of $\mathbb{Z}_2 + A$ defined by

$$a_0, a_1, a_2, b_0, b_1, b_2, c \mapsto b_0, b_1, b_2, a_0, a_1, a_2, c.$$

Since $\sigma(I) \subseteq I$, we obtain an endomorphism α of R defined by $\alpha(s + I) = \sigma(s) + I$ for $s \in \mathbb{Z}_2 + A$. By [4, Example 2.8], R is right α -skew reflexive and so R is right α -skew RNP by Remark 1.2(2). Consider $R[x] \cong (\mathbb{Z}_2 + A)[x]/I[x]$. For simplicity, we identify the elements of $\mathbb{Z}_2 + A$ with their images in R .

For $f(x) = ca_0 + ca_1x + ca_2x^2$, $g(x) = b_0 + b_1x + b_2x^2 \in N(R[x])$ (since $A^4 \subseteq I$) and any $r = k + h \in R$, where $k \in \mathbb{Z}_2$ and $h \in A$, we have

$$\begin{aligned} f(x)rg(x) &= (ca_0 + ca_1x + ca_2x^2)r(b_0 + b_1x + b_2x^2) \\ &= c(a_0b_1 + a_1b_0)kx + c(a_0b_2 + a_1b_1 + a_2b_0)kx^2 + c(a_1b_2 + a_2b_1)kx^3 \\ &\quad + c(a_0hb_1 + a_1hb_0)x + c(a_0hb_2 + a_1hb_1 + a_2hb_0)x^2 + c(a_1hb_2 + a_2hb_1)x^3 = 0, \end{aligned}$$

by the construction of I . Therefore $f(x)Rg(x) = 0$. By [7, Lemma 2.1], $f(x)R[x]g(x) = 0$. But $g(x)\bar{\alpha}(f(x)) \neq 0$ as $b_0cb_1 + b_1cb_0 \neq 0$, entailing $g(x)R[x]\bar{\alpha}(f(x)) \neq 0$. Hence $R[x]$ is not right $\bar{\alpha}$ -skew RNP.

Following [18], a ring R is called Armendariz if for $f(x) = \sum_{i=0}^m a_i x^i$, $g(x) = \sum_{j=0}^n b_j x^j$ in $R[x]$, $f(x)g(x) = 0$ implies $a_i b_j = 0$ for all i, j . Hirano generalized the notion of Armendariz rings to quasi-Armendariz rings. A ring R is called quasi-Armendariz [7] if for $f(x) = \sum_{i=0}^m a_i x^i$, $g(x) = \sum_{j=0}^n b_j x^j$ in $R[x]$, $f(x)R[x]g(x) = 0$ implies $a_i R b_j = 0$ for all i, j . It is well known that reduced rings are Armendariz and Armendariz rings are obviously quasi-Armendariz, but these implications are strict. Following [13], a ring R is called CN, if $N(R[x]) \subseteq N(R)[x]$. Both NI rings and Armendariz rings are CN but not conversely [13]. The classes of quasi-Armendariz and CN rings are independent of each other by means of [10, Examples 3.15(1) and (2)].

Proposition 2.12 *Let R be a ring with an endomorphism α . Suppose that R is quasi-Armendariz and CN. Then, R is right α -skew RNP if and only if $R[x]$ is right $\bar{\alpha}$ -skew RNP.*

Proof Let R be right α -skew RNP. Let $f(x) = \sum_{i=0}^m a_i x^i$, $g(x) = \sum_{j=0}^n b_j x^j \in N(R[x])$ such that $f(x)R[x]g(x) = 0$. Since R is quasi-Armendariz, therefore $a_i R b_j = 0$ for all i, j . Since R is CN, therefore $a_i, b_j \in N(R)$ for all i, j . By hypothesis, $b_j R \alpha(a_i) = 0$ for all i, j and so $g(x)R[x]\bar{\alpha}(f(x)) = 0$. Hence, $R[x]$ is right $\bar{\alpha}$ -skew RNP. Converse is straightforward. $\square$

Corollary 2.13 *Let R be a ring with an endomorphism α . Suppose that R is Armendariz. Then, R is right α -skew RNP if and only if $R[x]$ is right $\bar{\alpha}$ -skew RNP.*

For a ring R with an endomorphism α , the correspondence $(a_{ij}) \xrightarrow{\bar{\alpha}} (\alpha(a_{ij}))$ induce endomorphisms of $M_n(R)$, $U_n(R)$, $D_n(R)$ and $V_n(R)$.

Proposition 2.14 *Let R be a ring with an endomorphism α such that $\alpha(1) = 1$. Then, R is right α -skew RNP, if any of the following conditions hold:*

- (1) $M_n(R)$ is right $\bar{\alpha}$ -skew RNP for $n \geq 2$.
- (2) $U_2(R)$ is right $\bar{\alpha}$ -skew RNP.
- (3) $D_2(R)$ is right $\bar{\alpha}$ -skew RNP.
- (4) $V_n(R)$ is right $\bar{\alpha}$ -skew RNP for $n \geq 2$.

Proof (1) Let $n \geq 2$. Assume that $M_n(R)$ is right $\bar{\alpha}$ -skew RNP. Let $a, b \in N(R)$ such that $aRb = 0$. Then, $AM_n(R)B = 0$ for $A = aI_n$, $B = bI_n \in N(M_n(R))$, where I_n denotes the identity matrix of order n . By assumption, $BM_n(R)\bar{\alpha}(A) = 0$, entailing $bR\alpha(a) = 0$. Therefore, R is right α -skew RNP.

(2), (3) and (4) can be proved similarly. $\square$

For a right α -skew RNP ring R , $M_n(R)$ need not be right $\bar{\alpha}$ -skew RNP for $n \geq 2$, by the following example.

Example 2.15 Consider $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with usual addition and multiplication. Let $\alpha : R \rightarrow R$ be an endomorphism defined by $\alpha((a, b)) = (b, a)$. Then, R is right α -skew RNP by Remark 1.2(1). Let $a = (0, 1), b = (1, 0) \in R$. For $A = aE_{21}, B = bE_{12} \in N(M_n(R))$, we have $AM_n(R)B = 0$ but $B\bar{\alpha}(A) = bE_{11} \neq 0$, entailing $BM_n(R)\bar{\alpha}(A) \neq 0$. Therefore, $M_n(R)$ is not right $\bar{\alpha}$ -skew RNP.

Next, we show that for any ring R with an endomorphism α such that $\alpha(1) = 1$, neither $U_n(R)$ nor $D_n(R)$ is right $\bar{\alpha}$ -skew RNP for $n \geq 3$.

Example 2.16 Let $n \geq 3$. Suppose that R is a ring with an endomorphism α such that $\alpha(1) = 1$. For $A = E_{23}, B = E_{12} \in N(D_n(R))$, we have $AD_n(R)B = 0$ but $B\bar{\alpha}(A) = E_{13} \neq 0$, entailing $BD_n(R)\bar{\alpha}(A) \neq 0$. Therefore, $D_n(R)$ is not right $\bar{\alpha}$ -skew RNP. By the same argument, $U_n(R)$ is not right $\bar{\alpha}$ -skew RNP.

By the help of Example 2.16, we show that the class of right α -skew RNP rings is not closed under subrings.

Example 2.17 Let A be a ring with an endomorphism α such that $\alpha(1) = 1$. Assume that A is right α -skew reflexive. Then by [14, Theorem 3.3], $R = M_3(A)$ is right $\bar{\alpha}$ -skew reflexive and so R is right $\bar{\alpha}$ -skew RNP by Remark 1.2(2). However, the subring $S = U_3(A)$ of R is not right $\bar{\alpha}$ -skew RNP by Example 2.16.

For a right α -skew RNP ring R , both $U_2(R)$ and $D_2(R)$ need not be right $\bar{\alpha}$ -skew RNP by the next example.

Example 2.18 Consider a ring $R = D_2(\mathbb{Z}_4)$. Let $\alpha : R \rightarrow R$ be an endomorphism defined by

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}.$$

By [14, Proposition 2.11], R is right α -skew reflexive and so R is right α -skew RNP by Remark 1.2(2). For

$$A = \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}\right), B = \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) \in N(D_2(R)),$$

we have $AD_2(R)B = 0$, but

$$B\bar{\alpha}(A) = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}\right) \neq 0,$$

entailing $BD_2(R)\bar{\alpha}(A) \neq 0$ and so $D_2(R)$ is not right $\bar{\alpha}$ -skew RNP. By the same argument, $U_2(R)$ is not right $\bar{\alpha}$ -skew RNP.

However, we have an affirmative answer, if we consider a stronger condition as in the following.

Proposition 2.19 Let R be an α -rigid ring. Then, both $U_2(R)$ and $D_2(R)$ are right $\bar{\alpha}$ -skew RNP.

Proof Since α -rigid rings are reduced (see [8, pp. 218]), therefore the result follows directly from Corollary 2.8. $\square$

Next, we show that for a right α -skew RNP ring R , $V_n(R)$ need not be right $\bar{\alpha}$ -skew RNP for $n \geq 2$.

Example 2.20 We consider the ring R with the endomorphism α of Example 2.18. Then, R is right α -skew RNP. Assume that $n \geq 2$. We use $(a_1, a_2, \dots, a_n) \in V_n(R)$ to denote

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ 0 & a_1 & \cdots & a_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_1 \end{pmatrix}.$$

For

$$A = \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}\right), B = \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) \in N(V_n(R)),$$

we have $AV_n(R)B = 0$, but

$$B\bar{\alpha}(A) = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}\right) \neq 0,$$

entailing $BV_n(R)\bar{\alpha}(A) \neq 0$. Therefore $V_n(R)$ is not right $\bar{\alpha}$ -skew RNP.

However, we have the following.

Proposition 2.21 *For a ring R with an endomorphism α , if R is a semiprime right α -skew reflexive ring, then $V_n(R)$ is right $\bar{\alpha}$ -skew RNP for $n \geq 2$.*

Proof For a semiprime right α -skew reflexive ring R , $V_n(R)$ is right $\bar{\alpha}$ -skew reflexive by [14, Theorem 3.8(1)] and so the result follows from Remark 1.2(2). $\square$

The condition “ R is a semiprime right α -skew reflexive ring” in Proposition 2.21 is not superfluous by means of Example 2.20 and [14, Example 3.6].

Next, we give an example of a right α -skew RNP ring which is not nil-IFP.

Example 2.22 Let A be a ring with an endomorphism α such that A is right α -skew reflexive. By [14, Theorem 3.3], $R = M_2(A)$ is right $\bar{\alpha}$ -skew reflexive and so by Remark 1.2(2), R is right $\bar{\alpha}$ -skew RNP. However, R is not nil-IFP by [3, Example 1.1].

Recall that an element u of a ring R is called right regular if $ur = 0$ implies $r = 0$ for all $r \in R$. Left regular elements are defined analogously. An element is called regular if it is both right and left regular (i.e., not a zero divisor).

Let S be a multiplicatively closed subset of a ring R consisting of central regular elements. Let α be an automorphism of R such that $\alpha(S) \subseteq S$. Then, the mapping $\bar{\alpha} : S^{-1}R \rightarrow S^{-1}R$ defined by $\bar{\alpha}(u^{-1}a) = \alpha(u)^{-1}\alpha(a)$ for $u \in S$ and $a \in R$, induces an endomorphism of $S^{-1}R$.

Proposition 2.23 *Let R be a ring with an automorphism α and S a multiplicatively closed subset of R consisting of central regular elements such that $\alpha(S) \subseteq S$. Then, R is right α -skew RNP if and only if $S^{-1}R$ is right $\bar{\alpha}$ -skew RNP.*

Proof It follows from the fact that $N(S^{-1}R) = S^{-1}N(R)$. $\square$

The ring of Laurent polynomials in x over a ring R , consisting of all formal sums $\sum_{i=k}^n r_i x^i$ with usual addition and multiplication, where $r_i \in R$ and k, n are integers, is denoted by $R[x; x^{-1}]$.

For a ring R with an endomorphism α , the mapping $\bar{\alpha} : R[x; x^{-1}] \rightarrow R[x; x^{-1}]$ defined by

$$\bar{\alpha} \left(\sum_{i=k}^n r_i x^i \right) = \sum_{i=k}^n \alpha(r_i) x^i$$

induces an endomorphism of $R[x; x^{-1}]$.

Corollary 2.24 *For a ring R with an automorphism α , $R[x]$ is right $\bar{\alpha}$ -skew RNP if and only if $R[x; x^{-1}]$ is right $\bar{\alpha}$ -skew RNP.*

Proof It follows directly as $S = \{1, x, x^2, \dots\}$ is a multiplicatively closed subset of central regular elements in $R[x]$ and $R[x; x^{-1}] = S^{-1}R[x]$. $\square$

Recall that a multiplicatively closed subset S of a ring R is said to satisfy the right Ore condition if for each $a \in R$ and $b \in S$, there exist $a_1 \in R$ and $b_1 \in S$ such that $ab_1 = ba_1$. By [17, Theorem 2.1.12], a multiplicatively closed subset S of a ring R satisfies the right Ore condition and S consists of regular elements if and only if the right quotient ring Q of R with respect to S exists. For an automorphism α of R with $\alpha(S) \subseteq S$, the mapping $\bar{\alpha} : Q \rightarrow Q$ defined by $\bar{\alpha}(au^{-1}) = \alpha(a)\alpha(u)^{-1}$ for $a \in R$ and $u \in S$, induces an automorphism of Q .

Proposition 2.25 *Let S be a multiplicatively closed subset of a ring R and α an automorphism of R with $\alpha(S) \subseteq S$. Suppose that S satisfies the right Ore condition and S consists of regular elements. If R is right α -skew RNP and Q is NI, then the right quotient ring Q of R with respect to S is right $\bar{\alpha}$ -skew RNP.*

Proof Let R be right α -skew RNP and Q be NI. Then, R is NI by [9, Lemma 2.1]. We will freely use the assumption Q is NI, and the fact that R is NI without reference. Let $A = au^{-1}$, $B = bv^{-1} \in N(Q)$ such that $AQB = 0$, where $a, b \in R$, $u, v \in S$. Since $u^{-1}Q = Q$, therefore $0 = AQB = aQ(bv^{-1}) = a(rs^{-1})(bv^{-1})$ for any $rs^{-1} \in Q$. Let I and J be the ideals of Q generated by A and B , respectively. Then, I and J are nil such that $a = Au \in I$ and $b = Bv \in J$, entailing $a, b \in N(R)$. Since $b \in R$ and $s \in S$, therefore there exist $b_1 \in R$ and $s_1 \in S$ such that $bs_1 = sb_1$ and $s^{-1}b = b_1s_1^{-1}$ as S satisfies the right Ore condition. Then

$$0 = AQB = ars^{-1}bv^{-1} = arb_1s_1^{-1}v^{-1}$$

for any $r \in R$ and so $aRb_1 = 0$. From $bs_1 = sb_1$ and $aRb_1 = 0$, we have $0 = arsb_1 = arbs_1$ for any $r \in R$ and so $aRb = 0$. Since $v^{-1}Q = Q$, therefore

$$BQ\bar{\alpha}(A) = bQ\alpha(a)\alpha(u)^{-1} = b(rs^{-1})(\alpha(a)\alpha(u)^{-1})$$

for any $rs^{-1} \in Q$. Since $\alpha(a) \in R$ and $s \in S$, therefore there exist $a_2 \in R$ and $s_2 \in S$ such that $\alpha(a)s_2 = sa_2$ and $s^{-1}\alpha(a) = a_2s_2^{-1}$. Since α is bijective and $\alpha(S) \subseteq S$, therefore there exist $a'_2 \in R$, $s', s'_2 \in S$ such that $\alpha(a'_2) = a_2$, $\alpha(s') = s$, $\alpha(s'_2) = s_2$ and $as'_2 = s'a'_2$. Then $a'_2 = s'^{-1}(as'_2) \in I$ and so $a'_2 \in N(R)$. From $aRb = 0$ and $as'_2 = s'a'_2$, we have $0 = as'_2rb = s'a'_2rb$ for any $r \in R$ and so $a'_2Rb = 0$. Since R is right α -skew RNP, therefore $bRa_2 = bR\alpha(a'_2) = 0$. Thus, we have

$$BQ\bar{\alpha}(A) = brs^{-1}\alpha(a)\alpha(u)^{-1} = bra_2s_2^{-1}\alpha(u)^{-1} = 0$$

and hence Q is right $\bar{\alpha}$ -skew RNP. $\square$

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