Dilip Kumar · Harish Chandra

Antinormal weighted composition operators on $L^2(\mu)$ –space of an atomic measure space

Received: 31 January 2018 / Accepted: 19 November 2018 / Published online: 27 November 2018
© The Author(s) 2018

Abstract Let $L^2(\mu)$ denote the separable Hilbert space associated with a σ -finite atomic measure μ . In this paper, we determine necessary and sufficient conditions for boundedness of weighted composition transformation on $L^2(\mu)$ and give a characterization of antinormal weighted composition operators on $L^2(\mu)$.

Mathematics Subject Classification 47B33 · 47B37

1 Introduction

The problem of finding the proximal set in the class of all bounded linear operators is a frequently appearing problem in operator theory. This problem has been resolved for many important classes of operators such as Hermitian, positive, compact and unitary operators. That is, every bounded linear operator has a best approximation or a nearest point from within each of these classes. See, for example [3, 4, 6] and [8] and the references therein. But the class of normal operators fails to be a proximal set [13] in the class of all bounded linear operators. In 1974, Holmes [7] discovered that there are operators for which the largest possible distance from the set of normal operators can be achieved. He named such operators as antinormal operators and showed that no compact operator is antinormal. Subsequently, this class has been extensively studied by several authors in [5, 9, 13] and [10]. Over the years, antinormal operators have been studied in context of composition operators on the Hilbert spaces l^2 and $l^2(\lambda)$ in [15] and [12], respectively. In this paper, we investigate antinormality of weighted composition operators on $L^2(\mu)$.

Notation and Terminology: Let $\mathbb{N}$ and $\mathbb{C}$ denote the set of all positive integers and the set of all complex numbers, respectively. Then for $n \in \mathbb{N}$, $\chi_{A_n} : X \rightarrow \{0, 1\}$ is defined as

$$\chi_{A_n}(x) = \begin{cases} 1, & \text{if } x \in A_n \\ 0, & \text{otherwise.} \end{cases}$$

Suppose H is a separable complex Hilbert space and $B(H)$ denotes the algebra of all bounded linear operators on H . Further, for $T \in B(H)$, let $N(T)$ and $R(T)$, respectively, denote the null space and the range space of T .

Now we record certain definitions and results which are useful in our context.

D. Kumar
Department of Mathematics, Hindu College, Moradabad 244001, India
E-mail: dilipmathsbhu@gmail.com

H. Chandra (✉)
Department of Mathematics, Institute of Science, Banaras Hindu University, Varanasi 221005, India
E-mail: harishc@bhu.ac.in

Definition 1.1 Let $(X, \mathcal{S}, \mu)$ be a measure space. A measurable set E is called an atom if $\mu(E) \neq 0$ and for each measurable subset F of E either $\mu(F) = 0$ or $\mu(F) = \mu(E)$. A measure space $(X, \mathcal{S}, \mu)$ is called atomic measure space if each measurable subset of non-zero measure contains an atom.

A typical example of an atomic measure space is $(X, \mathcal{S}, \mu)$, where X is any non-empty set, $\mathcal{S}$ is a σ -algebra and μ is the counting measure.

Definition 1.2 An atomic measure space $(X, \mathcal{S}, \mu)$ is called σ -finite atomic measure space if X is expressible as countable union of its atoms of finite measure.

Now onwards we assume that $(X, \mathcal{S}, \mu)$ is a σ -finite atomic measure space. Then $X = \bigcup_{n \in \mathbb{N}} A_n$, where A_n 's are disjoint atoms of finite measure. These atoms are unique in the sense that if $X = \bigcup_{n \in \mathbb{N}} B_n$, where B_n 's are disjoint atoms of finite measure, then $A_n = B_n$ upto a null set for each $n \in \mathbb{N}$. A measurable transformation $\varphi : X \rightarrow X$ is called non-singular if $\mu\varphi^{-1}$ is absolutely continuous with respect to the measure μ . If a non-singular transformation on X takes some part of an atom A_n to a subset of an atom A_m and the remaining part of A_n to a subset of another atom A_r , then both parts must be null sets as the image of an atom under a non-singular transformation cannot be a null set. Therefore a non-singular measurable transformation takes atoms into atoms. A non-singular transformation φ of X into X is called injective a.e. (almost everywhere) if the inverse image of every atom under φ contains at most one atom. It is called surjective a.e. if the inverse image of every atom under φ contains at least one atom. If φ is both injective and surjective a.e., then it is called bijective a.e.. Henceforth, we consider $L^2(\mu)$ of a σ -finite atomic measure space $(X, \mathcal{S}, \mu)$, where $X = \bigcup_{n \in \mathbb{N}} A_n$ and A_n 's are disjoint atoms of finite measure. Further, each $f \in L^2(\mu)$ is constant a.e. on an atom. Hence, the span of the characteristic functions $\{\chi_{A_n} : n \in \mathbb{N}\}$ forms a dense subset of $L^2(\mu)$. Thus, $\{K_{A_n} = \frac{1}{(\mu(A_n))^{1/2}} \chi_{A_n} : n \in \mathbb{N}\}$ forms an orthonormal basis for $L^2(\mu)$.

Definition 1.3 ([14]) Let X be a non-empty set and $V(X)$ be a linear space of complex valued functions on X under pointwise addition and scalar multiplication. If ϕ is a selfmap on X into itself such that composition $f \circ \phi$ belongs to $V(X)$ for each $f \in V(X)$, then ϕ induces a linear transformation on $V(X)$ into itself given by $C_\phi f = f \circ \phi$. The transformation C_ϕ is known as composition transformation. When $V(X)$ is a Banach space or Hilbert space and C_ϕ is a bounded linear operator on $V(X)$, then C_ϕ is called composition operator.

Definition 1.4 A function $u : X \rightarrow \mathbb{C}$ is said to be bounded away from zero if there exists a positive real number r such that $0 < \frac{1}{|u(x)|} \leq r$ for each $x \in X$.

Definition 1.5 ([1]) An operator $T \in B(H)$ is said to be Fredholm operator if dimension of $N(T)$ and the dimension of the quotient space $H/R(T)$ are both finite.

We also note here that, equivalently, T is Fredholm if both $N(T)$ and $N(T^*)$ are finite dimensional.

Definition 1.6 Essential spectrum of an operator $T \in B(H)$ is defined as $\sigma_e(T) = \{\alpha \in \mathbb{C} : T - \alpha I \text{ is not Fredholm}\}$.

Since every invertible operator is Fredholm operator, hence $\sigma_e(T) \subseteq \sigma(T)$ see [2].

Definition 1.7 Minimum modulus of an operator $T \in B(H)$ is defined as $m(T) = \inf\{\|Tx\| : \|x\| = 1\}$.

Definition 1.8 Essential minimum modulus of an operator $T \in B(H)$ is defined as $m_e(T) = \inf\{\alpha \geq 0 : \alpha \in \sigma_e(|T|)\}$, where $|T| = (T^*T)^{\frac{1}{2}}$.

Definition 1.9 An operator $T \in B(H)$ is said to be antinormal if $d(T, \mathcal{N}) = \inf_{N \in \mathcal{N}} \|T - N\| = \|T\|$, where $\mathcal{N}$ the class of all normal operators in $B(H)$.

Remark 1.10 An operator $T \in B(H)$ is antinormal if and only if its adjoint T^* is antinormal.

Definition 1.11 For an operator T in $B(H)$, the index of T is defined as

$$\text{index}(T) = \begin{cases} \dim(N(T)) - \dim(N(T^*)), & \text{if } \dim(N(T)) \text{ or} \\ & \dim(N(T^*)) < \infty. \\ 0, & \text{otherwise.} \end{cases}$$

Remark 1.12 Observe that $\text{index}(T) = -\text{index}(T^*)$.

The following results will be used later in the paper.

Theorem 1.13 ([9]) *Let $T \in B(H)$.*

- (i) *If $\text{index}(T) = 0$, then $d(T, \mathcal{N}) \leq \frac{\|T\| - m(T)}{2}$,*
- (ii) *If $\text{index}(T) < 0$, then $m_e(T) \leq d(T, \mathcal{N}) \leq \frac{\|T\| + m_e(T)}{2}$.*

Remark 1.14 If $\text{index}(T) = 0$, then T can not be antinormal.

Theorem 1.15 ([9]) *Let $T \in B(H)$ with $\text{index}(T) < 0$. Then, the following conditions are equivalent:*

- (i) *T is antinormal*
- (ii) *$m_e(T) = \|T\|$*
- (iii) *$d(T, \mathcal{U}) = 1 + \|T\|$, where $\mathcal{U}$ is class of all unitary operators in $B(H)$*
- (iv) *$T = \alpha W(1 - K)$ for some $\alpha > 0$, isometry W and positive compact contraction K .*

2 Antinormal weighted composition operators on $L^2(\mu)$

Let $\varphi : X \rightarrow X$ be a self-map. For $u \in L^\infty(\mu)$, a weighted composition transformation, induced by u and φ , is defined as

$$(uC_\varphi)(f)(x) = u(x)f(\varphi(x)) \quad \forall f \in L^2(\mu) \text{ and } \forall x \in \mathbb{X}.$$

If uC_φ is bounded then it is called weighted composition operator induced by u and φ . The following result gives a necessary and sufficient condition for the boundedness of uC_φ .

Theorem 2.1 *The transformation uC_φ on $L^2(\mu)$ is bounded if and only if $\sup_{n \in \mathbb{N}} \{ |u(A_n)|^2 \frac{\mu(\varphi^{-1}(A_n))}{\mu(A_n)} \} < \infty$. In this case*

$$\|uC_\varphi\| = \sup_{n \in \mathbb{N}} \left\{ |u(A_n)|^2 \frac{\mu(\varphi^{-1}(A_n))}{\mu(A_n)} \right\}^{\frac{1}{2}}.$$

Proof Let $f = \sum_{n \in \mathbb{N}} f|_{A_n} \chi_{A_n}$ be an element in $L^2(\mu)$. Then

$$\begin{aligned} \|uC_\varphi f\|^2 &= \left\| uC_\varphi \left(\sum_{n \in \mathbb{N}} f|_{A_n} \chi_{A_n} \right) \right\|^2 \\ &= \left\| \sum_{n \in \mathbb{N}} u(A_n) f(A_n) \chi_{\varphi^{-1}(A_n)} \right\|^2 \\ &\leq \sum_{n \in \mathbb{N}} |u(A_n)|^2 |f(A_n)|^2 \mu(\varphi^{-1}(A_n)) \\ &\leq \sup_{n \in \mathbb{N}} \left\{ |u(A_n)|^2 \frac{\mu(\varphi^{-1}(A_n))}{\mu(A_n)} \right\} \sum_{n \in \mathbb{N}} |f(A_n)|^2 \mu(A_n) \\ &= \sup_{n \in \mathbb{N}} \left\{ |u(A_n)|^2 \frac{\mu(\varphi^{-1}(A_n))}{\mu(A_n)} \right\} \|f\|^2. \end{aligned}$$

This implies that $\|uC_\varphi\|^2 \leq \sup_{n \in \mathbb{N}} \left\{ |u(A_n)|^2 \frac{\mu(\varphi^{-1}(A_n))}{\mu(A_n)} \right\}$.

Conversely, we have

$$\left\| uC_\varphi \frac{\chi_{A_n}}{(\mu(A_n))^{1/2}} \right\|^2 = |u(A_n)|^2 \frac{\mu(\varphi^{-1}(A_n))}{\mu(A_n)} \quad \forall n \in \mathbb{N}.$$

Now it is easy to see that $\sup_{n \in \mathbb{N}} \left\{ |u(A_n)|^2 \frac{\mu(\varphi^{-1}(A_n))}{\mu(A_n)} \right\} \leq \|uC_\varphi\|^2$. Hence the proof. $\square$

The following theorem defines the adjoint of a weighted composition operator.

Theorem 2.2 *Let uC_φ be a weighted composition operator and let T be defined as*

$$(Tf)(A_n) = \frac{1}{\mu(A_n)} \int_{\varphi^{-1}(A_n)} f \bar{u} d\mu$$

a.e. for $f \in L^2(\mu)$ and for every atom A_n . Then $T = (uC_\varphi)^$.*

Proof Let $f, g \in L^2(\mu)$. Then

$$\begin{aligned} \langle f, (uC_\varphi)g \rangle &= \int_X f \overline{(uC_\varphi)g} d\mu \\ &= \sum_{n \in \mathbb{N}} \int_{\varphi^{-1}(A_n)} f \bar{u} \overline{g \circ \varphi} d\mu \\ &= \sum_{n \in \mathbb{N}} \bar{g} \int_{\varphi^{-1}(A_n)} f \bar{u} d\mu \\ &= \sum_{n \in \mathbb{N}} \overline{g(A_n)} \mu(A_n) (Tf)(A_n) \\ &= \langle Tf, g \rangle. \end{aligned}$$

Hence, $T = (uC_\varphi)^*$ □

We now give a sufficient condition which excludes certain class of composition operators from the consideration of antinormality

Theorem 2.3 *Let $u \in L^\infty(\mu)$ be bounded away from zero and φ be a bijective map on $\mathbb{N}$. Then uC_φ is not antinormal.*

Proof It is easy to see that $\text{index}(uC_\varphi) = 0$ by showing that both uC_φ and $(uC_\varphi)^*$ are injective. Indeed, suppose that $f, g \in L^2(\mu)$ such that

$$(uC_\varphi)f = (uC_\varphi)g.$$

This implies

$$(uf \circ \varphi)(\varphi^{-1}(A_n)) = (ug \circ \varphi)(\varphi^{-1}(A_n)) \quad \forall n \in \mathbb{N}.$$

Then

$$\begin{aligned} u(\varphi^{-1}(A_n))f(\varphi(\varphi^{-1}(A_n))) &= u(\varphi^{-1}(A_n))g(\varphi(\varphi^{-1}(A_n))) \quad \forall n \in \mathbb{N} \\ \Rightarrow f(A_n) &= g(A_n) \quad \forall n \in \mathbb{N}. \end{aligned}$$

Thus, $f = g$. Therefore uC_φ is injective. To see that $(uC_\varphi)^*$ is also injective, suppose that $(uC_\varphi)^*f = 0$ for some $f \in L^2(\mu)$. Then

$$\begin{aligned} \langle (uC_\varphi)^*f, g \rangle &= 0 \quad \forall g \in L^2(\mu) \\ \Rightarrow \langle (uC_\varphi)^*f, \chi_{A_n} \rangle &= 0 \quad \forall n \in \mathbb{N} \\ \Rightarrow \sum_{m \in \mathbb{N}} f|A_m \langle (uC_\varphi)^* \chi_{A_m}, \chi_{A_n} \rangle &= 0 \quad \forall n \in \mathbb{N} \\ \Rightarrow \sum_{m \in \mathbb{N}} f|A_m \langle \bar{u} \chi_{\varphi(A_m)}, \chi_{A_n} \rangle &= 0 \quad \forall n \in \mathbb{N} \\ \Rightarrow f|A_m \overline{u(A_n)} &= 0 \quad \forall n \in \mathbb{N} \\ \Rightarrow f(A_n) &= 0 \quad \forall n \in \mathbb{N} \\ \Rightarrow f &= 0. \end{aligned}$$

Hence, uC_φ is not antinormal by Remark 1.14. □

The following remark shows that the condition that “ u is bounded away from zero” is not essential for the validity of the above Theorem.

Remark 2.4 Let $u \in L^\infty$ be such that $u(A_n) \rightarrow 0$ as $n \rightarrow \infty$. Then it is easy to see that uC_φ is compact. Hence, it is not antinormal.

The following result gives a necessary and sufficient condition for antinormality of uC_φ when φ is injective but not surjective.

Theorem 2.5 Suppose $u(n) \neq 0$ for each $n \in \mathbb{N}$ and φ is injective but not surjective. Then uC_φ is antinormal if and only if the following conditions hold:

- (a) For each $0 \leq \alpha < \|uC_\varphi\|$, $\overline{u(A_n)u(\varphi(A_n))} \neq \alpha$ except for finitely many $n \in \mathbb{N}$,
- (b) $\|uC_\varphi\|^2 = \overline{u(A_n)u(\varphi(A_n))}$ for infinitely many $n \in \mathbb{N}$.

Proof Since φ is not surjective so there exists $n \in \mathbb{N}$ with $\mu(\varphi^{-1}(A_n)) = 0$. Therefore,

$$(uC_\varphi)\chi_{A_n} = u(A_n)\chi_{\varphi^{-1}(A_n)} = 0.$$

But $\chi_{A_n} \neq 0$. Consequently, uC_φ is not injective. Moreover, $(uC_\varphi)^*$ is injective by Theorem 2.3. Hence, $\text{index}((uC_\varphi)^*) < 0$. For $f = \sum_{n \in \mathbb{N}} f(A_n)\chi_{A_n} \in L^2(\mu)$ and for given complex number α , we have

$$((uC_\varphi)(uC_\varphi)^* - \alpha I)f = \sum_{n \in \mathbb{N}} (\overline{u(A_n)u(\varphi(A_n))} - \alpha)f(A_n)\chi_{A_n}. \quad (1)$$

From the above equation and condition (a) it follows that

$\dimker((uC_\varphi)(uC_\varphi)^* - \alpha I) = \dimker((uC_\varphi)(uC_\varphi)^* - \alpha I)^*$ is finite whenever $0 \leq \alpha < \|uC_\varphi\|^2$. Hence $((uC_\varphi)(uC_\varphi)^* - \alpha I)$ is Fredholm for each $0 \leq \alpha < \|uC_\varphi\|^2$. Thus $\sqrt{\alpha} \notin \sigma_e(|u(C_\varphi)^*|)$ for each $0 \leq \alpha < \|uC_\varphi\|^2$. Again by condition (b) $\|uC_\varphi\|^2 = \overline{u(n)u(\varphi(n))}$ for infinitely many $n \in \mathbb{N}$. Hence $\dimker((uC_\varphi)(uC_\varphi)^* - \|uC_\varphi\|^2 I)$ is infinite. Therefore $\|uC_\varphi\| \in \sigma_e(|u(C_\varphi)^*|)$. Thus $m_e((uC_\varphi)^*) = \|uC_\varphi\|$. Hence, $(uC_\varphi)^*$ is antinormal by Theorem 1.15. Since adjoint of an antinormal operator is antinormal, therefore, uC_φ is antinormal. Conversely, if either of the conditions fail, then we claim that (uC_φ) is not antinormal. For, if condition (a) fails, then there exist an α_0 with $0 \leq \alpha_0 < \|uC_\varphi\|^2$ such that $\sqrt{\alpha_0} \in \sigma_e(|u(C_\varphi)^*|)$. Therefore, $m_e((uC_\varphi)^*) \leq \sqrt{\alpha_0} < \|uC_\varphi\|$. Hence, again by, Theorem ef302 $(uC_\varphi)^*$ is not antinormal. Now suppose condition (b) fails. Then, by equation (1), $\|u(C_\varphi)^*\| \notin \sigma_e(|u(C_\varphi)^*|)$. Therefore, $m_e((uC_\varphi)^*) < \|uC_\varphi\|$. Therefore, uC_φ is not antinormal. $\square$

We now prove the following lemma.

Lemma 2.6 Suppose φ is surjective but not injective and $u(A_n) \neq 0$ for each $n \in \mathbb{N}$. Then uC_φ is injective but $(uC_\varphi)^*$ is not injective.

Proof Let $f \in L^2(\mu)$ be such that $uC_\varphi(f) = 0$. Therefore, $u(A_n)f(\varphi(A_n)) = 0 \forall n \in \mathbb{N}$. This implies $f(\varphi(A_n)) = 0 \forall n \in \mathbb{N}$ as $u(A_n) \neq 0 \forall n \in \mathbb{N}$. Now surjectivity φ implies $f = 0$. Further, Since φ is not injective, there exist $m, n \in \mathbb{N}$ with $m \neq n$ such that $\varphi(A_m) = \varphi(A_n)$. Now define $f = \overline{u(A_m)}\chi_{A_m}$ and $g = \overline{u(A_n)}\chi_{A_n}$. It is easy to see that $f, g \in L^2$ and $f \neq g$ but $(uC_\varphi)^*(f) = (uC_\varphi)^*(g)$. Therefore, $(uC_\varphi)^*$ is not injective. $\square$

Using the above lemma, we prove the following result.

Theorem 2.7 Suppose $u(n) \neq 0$ for each $n \in \mathbb{N}$ and φ is surjective but not injective. Then uC_φ is antinormal if and only if the following conditions hold:

- (a) For each $0 \leq \alpha < \|uC_\varphi\|$, $u(n) \sum_{A_m \in \varphi^{-1}(A_n)} \overline{u(m)} \neq \alpha$ except for finitely many $n \in \mathbb{N}$
- (b) $\|uC_\varphi\|^2 = u(n) \sum_{A_m \in \varphi^{-1}(A_n)} \overline{u(m)}$ for infinitely many $n \in \mathbb{N}$.

Proof Since φ is surjective but not injective and $u(n) \neq 0$ for each $n \in \mathbb{N}$, hence by above lemma $uC_\varphi^* < 0$. For $f \in L^2(\mu)$ and $\alpha \in \mathbb{C}$, we have

$$\begin{aligned} & ((uC_\varphi)^*(uC_\varphi) - \alpha I)f \\ &= \sum_{n \in \mathbb{N}} \left(u(n) \sum_{A_m \in \varphi^{-1}(A_n)} \overline{u(m)} - \alpha \right) f(A_n) \chi_{A_n}. \end{aligned} \quad (2)$$

Therefore, $\dim \ker((uC_\varphi)^*(uC_\varphi) - \alpha I) = \dim \ker((uC_\varphi)^*(uC_\varphi) - \alpha I)^*$ is finite for each $0 \leq \alpha < \|uC_\varphi\|^2$ by condition (a). Hence, $\sqrt{\alpha} \notin \sigma_e(|uC_\varphi|)$ for each $0 \leq \alpha < \|uC_\varphi\|^2$. Also $u(n) \sum_{m \in \varphi^{-1}(n)} \overline{u(m)} = \|uC_\varphi\|^2$ for infinitely many $n \in \mathbb{N}$ so $\|uC_\varphi\| \in \sigma_e(|uC_\varphi|)$. Thus $m_e(uC_\varphi) = \|uC_\varphi\|$. Consequently uC_φ is antinormal. Conversely, if either of the conditions is not true, then we claim that uC_φ is not antinormal. If condition (a) fails, then there exists an α_0 with $0 \leq \sqrt{\alpha_0} < \|uC_\varphi\|^2$, such that $\sqrt{\alpha_0} \in \sigma_e(|uC_\varphi|)$. Therefore, $m_e(uC_\varphi) \leq \sqrt{\alpha_0} < \|uC_\varphi\|$ and so uC_φ is not antinormal. Now suppose condition (b) fails. Then, by equation (2), $\|uC_\varphi\| \notin \sigma_e(|uC_\varphi|)$. Therefore, $m_e(uC_\varphi) < \|uC_\varphi\|$. Thus uC_φ is not antinormal in either case. $\square$

Theorem 2.8 Suppose $u(A_n) \neq 0 \forall n \in \mathbb{N}$ and φ is neither injective nor surjective.

- (i) If $\text{index}(uC_\varphi) = 0$, then uC_φ is not antinormal.
- (ii) If $\text{index}(uC_\varphi) < 0$, then uC_φ is antinormal if and only if the following conditions hold:
 - (a) For each $0 \leq \alpha < \|uC_\varphi\|$, $u(n) \sum_{A_m \in \varphi^{-1}(A_n)} \overline{u(m)} \neq \alpha$ except for finitely many $n \in \mathbb{N}$
 - (b) $\|uC_\varphi\|^2 = u(n) \sum_{A_m \in \varphi^{-1}(A_n)} \overline{u(m)}$ for infinitely many $n \in \mathbb{N}$.

Proof (i) It follows by Remark 1.14.

(ii) Proof is same as in case of Theorem 2.7. $\square$

We now give examples, one in each case, satisfying Theorems 2.5 and 2.7.

Example 2.9 Let l^2 denote the Hilbert space of all square summable sequences of complex numbers. Let $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ be defined as $\varphi(n) = n + 1$ and

$$u(n) = \begin{cases} 1, & n = 1 \\ 2, & \text{otherwise.} \end{cases}$$

Clearly, φ is injective but not surjective and $\overline{u(n)}u(\varphi(n)) = 4 = \|uC_\varphi\|^2$ for $n \geq 2$. Hence by Theorem 2.5, uC_φ is antinormal on l^2 .

Example 2.10 Let $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ be defined as $\varphi(n) = m$ if $n = 3m, 3m - 1, 3m - 2$. and

$$u(n) = \begin{cases} 1, & n = 1 \\ 3, & \text{otherwise.} \end{cases}$$

Clearly, φ is surjective but not injective and

$\|uC_\varphi\|^2 = 9 = u(n) \sum_{m \in \varphi^{-1}(n)} \overline{u(m)}$ for $n \geq 2$. Then, by Theorem 2.7, uC_φ is antinormal on l^2 .

Open Access This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

References

1. Abramovich, Y.A.; Aliprantis, C.D.: An Invitation to Operator Theory, Graduate Studies in Mathematics, American Mathematical Society, Providence (2002)
2. Conway, J.B.: A Course in Functional Analysis, 2nd edn. Springer, Berlin (1990)
3. ter Elst, A.F.M.: Antinormal operators. *Acta. Sci. Math.* **54**, 151–158 (1990)
4. Fan, K.; Hoffman, A.: Some metric inequalities in the space of matrices. *Proc. Am. Math. Soc.* **6**, 111–116 (1955)
5. Fujii, M.; Nakamoto, R.: Antinormal operators and theorems of Izumino. *Math. Jpn.* **24**, 41–44 (1979)
6. Halmos, P.R.: Positive approximants of operators. *Indiana Univ. Math. J.* **21**, 951–960 (1972)
7. Holmes, R.B.: Best approximation by normal operators. *J. Approx. Theory* **12**, 412–417 (1974)
8. Holmes, R.B.; Kripke, B.R.: Best approximation by compact operators. *Ind. Univ. Math. J.* **21**, 255–263 (1971)
9. Izumino, S.: Inequalities on normal and antinormal operators. *Math. Jpn.* **23**, 211–215 (1978)
10. Izumino, S.: Inequalities on nilpotent operators. *Math. Jpn.* **24**, 31–34 (1979)
11. Kumar, D.; Chandra, H.: Composition operators on Poisson weighted sequence spaces. *Funct. Anal. Approx. Comput.* **8**(1), 21–37 (2016)
12. Kumar, D.; Chandra, H.: Antinormal composition operators on $l^2(\lambda)$. *Mat. Vesn.* **68**(4), 259–266 (2016)
13. Rogers, D.D.: On proximal sets of normal operators. *Proc. Am. Math. Soc.* **61**, 44–48 (1976)
14. Singh, R.K.; Manhas, J.S.: Composition Operators on Function Spaces. North-Holland, New York (1993)
15. Tripathi, G.P.; Lal, N.: Antinormal composition operators on l^2 . *Tamkang J. Math.* **39**, 347–352 (2008)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.