B. Mohebbi Najmabadi · T. L. Shateri

On the Cauchy–Schwarz inequality and its reverse in 2- $*$ -semi inner product spaces

Received: 6 October 2017 / Accepted: 9 November 2018 / Published online: 24 November 2018
© The Author(s) 2018

Abstract In this paper, we investigate the $\mathcal{A}$ -valued Cauchy–Schwarz inequality and its reverse in a 2-semi inner product $\mathcal{A}$ -module over a locally C^* -algebra $\mathcal{A}$.

Mathematics Subject Classification 46C50 · 46L08

1 Introduction and preliminaries

The concepts of 2-metric space and linear 2-normed space were first introduced by Gähler in 1963 and 1964 [9, 10]. Since then, many authors, Freese et al. Gähler, Cho et al. Diminnie et al., and Gunawan et al. have developed extensively topological and geometric structures of 2-inner product spaces, 2-normed spaces, 2-metric spaces, semi-2-normed spaces, semi-2-metric spaces (see [1, 3–5, 7–11, 16]). The term locally C^* -algebra was introduced by Inoue in 1971. Locally C^* -algebras are generalizations of C^* -algebras. Instead of being given by a single C^* -norm, the topology of a locally C^* -algebra is defined by a directed family of C^* -seminorms. The basic information about locally C^* -algebras can be found in [6, 12]. The finitely generated modules equipped with inner products over some topological $*$ -algebra and the standard Hilbert module H_A over a locally C^* -algebra $\mathcal{A}$ were first considered by Mallios [14]. Most of the basic properties of Hilbert C^* -modules are still valid for Hilbert modules over locally C^* -algebras. Joita [13] explained the basic notions and theorems of the theory of Hilbert modules over a locally C^* -algebra. The Cauchy–Schwarz inequality plays an important role in the theory of inner product spaces (see, for instance, [20, 21]), which is one of the classical inequalities. It is well known that in a 2-semi inner product space $(X, \langle \cdot, \cdot \rangle)$, the Cauchy–Schwarz inequality will be formed as follows:

$$|\langle x, y|z \rangle|^2 \leq \langle x, x|z \rangle \langle y, y|z \rangle = \|x, z\|^2 \|y, z\|^2 \quad (x, y, z \in X),$$

see [3]. Furthermore, the following additive reverse of the Cauchy–Schwarz inequality holds:

Theorem 1.1 [2, Theorem 1]. *Let $(X, \langle \cdot, \cdot \rangle)$ be a 2-inner product space on $\mathbb{C}$, $x, y, z \in X$ and $a, b \in \mathbb{C}$. If*

$$\operatorname{Re} \langle by - x, x - ay|z \rangle \geq 0$$

B. Mohebbi Najmabadi · T. L. Shateri (✉)
Department of Mathematics and Computer Sciences, Hakim Sabzevari University, P.O. Box 397, Sabzevar, Iran
E-mail: t.shateri@hsu.ac.ir; t.shateri@gmail.com

B. Mohebbi Najmabadi
E-mail: behrozmohebbi1351@gmail.com; behruz_mohebbi@yahoo.com

or equivalently

$$\left\| x - \frac{a+b}{2} y | z \right\| \leq \frac{1}{2} |b-a| \|y, z\|$$

holds, then we obtain

$$0 \leq \|x, z\|^2 \|y, z\|^2 - |\langle x, y | z \rangle|^2 \quad (1.1)$$

$$\leq \frac{1}{4} |b-a|^2 \|y, z\|^4 - \left| \frac{a+b}{2} \|y, z\|^2 - \langle x, y | z \rangle \right|^2 \quad (1.2)$$

$$\leq \frac{1}{4} |b-a|^2 \|y, z\|^4. \quad (1.3)$$

The constant $\frac{1}{4}$ is sharp in (1.3) in the sense that it cannot be replaced by a smaller constant.

The aim of the paper is to extend some inequalities from 2-semi inner product spaces to 2-inner product $\mathcal{A}$ -module spaces. In fact we generalize the above inequalities to a 2-semi inner product C^* -module or a 2- $*$ -semi inner product such that the 2-inner product map takes values in a locally C^* -algebra. First, we recall some definitions and basic properties of 2-inner product spaces over locally C^* -algebras from [15, 17].

Definition 1.2 A C^* -seminorm on a topological $*$ -algebra $\mathcal{A}$ is a seminorm p such that $p(ab) \leq p(a)p(b)$ and $p(aa^*) = p(a)^2$ for all $a, b \in \mathcal{A}$. We denote by $S(\mathcal{A})$ the set of all continuous C^* -seminorms on $\mathcal{A}$.

Definition 1.3 A locally C^* -algebra is a Hausdorff complete complex topological $*$ -algebra $\mathcal{A}$ whose topology is determined by all its continuous C^* -seminorms in the sense that a net $\{a_i\}_{i \in I}$ converges to 0 if and only if the net $\{p(a_i)\}_{i \in I}$ converges to 0 for all continuous C^* -seminorms p on $\mathcal{A}$.

The following example is set by Philips [19]:

Example 1.4 [6] Consider the algebra $C[0, 1]$ of all complex continuous functions on $[0, 1]$. Let $\mathcal{S}$ be the family of all countable compact subsets of $[0, 1]$. Endow $C[0, 1]$ with the topology of uniform convergence on the elements of $\mathcal{S}$. In fact, the topology is defined by the family $\{p_B : B \in \mathcal{S}\}$ of the C^* -seminorms $p_B(f) := \|f|_B\|_\infty$, for all $f \in C[0, 1]$, $B \in \mathcal{S}$. The algebra $C[0, 1]$, equipped with this topology, is a locally C^* -algebra. Note that this topology is different from the supnorm topology of C^* -algebra $C[0, 1]$. In fact, the topology cannot be normable.

Example 1.5 Every C^* -algebra is a locally C^* -algebra.

Definition 1.6 Let $\mathcal{A}$ be a locally C^* -algebra with unit. An element $a \in \mathcal{A}$ is *positive* and we write $a \geq 0$, if $a = a^*$ and $Sp(a) = \{\lambda | a - \lambda 1 \text{ is not invertible}\} \subseteq \mathbb{R}^+$. The set of all positive elements of $\mathcal{A}$ is denoted by $\mathcal{A}^+$. If $a, b \in \mathcal{A}$, then $a \leq b$ means that $b - a \in \mathcal{A}^+$.

Lemma 1.7 [13, Theorem 1.1.16]. Let $\mathcal{A}$ be a locally C^* -algebra. An element a in $\mathcal{A}$ is *bounded* if $\sup\{p(x) | p \in S(\mathcal{A})\} < \infty$. If the set of all bounded elements in $\mathcal{A}$ is denoted by $b(\mathcal{A})$, then the map $\|\cdot\|_\infty$ on $b(\mathcal{A})$ defined by $\|x\|_\infty = \sup\{p(x) | p \in S(\mathcal{A})\}$ is a C^* -norm on $\mathcal{A}$.

Remark 1.8 A C^* -algebra $\mathcal{A}$ can be embedded into a unital C^* -algebra $\tilde{\mathcal{A}}$ containing $\mathcal{A}$ as an ideal. The elements of $\tilde{\mathcal{A}}$ are pairs (a, λ) with $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$; then we may assume that $\mathcal{A}$ is unital with identity 1. The center of $\mathcal{A}$ is denoted by $Z(\mathcal{A})$.

In the following, we recall the concept of a pre-Hilbert $\mathcal{A}$ -module, where $\mathcal{A}$ is a locally C^* -algebra:

Definition 1.9 A *pre-Hilbert $\mathcal{A}$ -module* is a complex vector space E which is also a right $\mathcal{A}$ -module, compatible with the complex algebra structure, equipped with an $\mathcal{A}$ -valued inner product $\langle \cdot, \cdot \rangle : E \times E \rightarrow \mathcal{A}$ which is $\mathbb{C}$ -linear and $\mathcal{A}$ -linear in its second variable and satisfies the following relations:

- (I₁) $\langle x, x \rangle \geq 0$ for every $x \in E$;
- (I₂) $\langle x, y \rangle = \langle y, x \rangle^*$ for every $x, y \in E$;
- (I₃) $\langle x, x \rangle = 0$ if and only if $x = 0$;
- (I₄) $\langle xa, yb \rangle = a^* \langle x, y \rangle b$ for every $x, y \in E$ and $a, b \in \mathcal{A}$;
- (I₅) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ for every $x, y, z \in E$ and $\alpha, \beta \in \mathbb{C}$.

Example 1.10 Let $l^2(\mathcal{A})$ be the set of all sequences $\{a_n\}_{n \in \mathbb{N}}$ of elements of a locally C^* -algebra $\mathcal{A}$ such that the series $\sum_{n \in \mathbb{N}} a_n a_n^*$ is convergent in $\mathcal{A}$. Then $l^2(\mathcal{A})$ is a pre-Hilbert $\mathcal{A}$ -module with respect to the pointwise operations and inner product is defined by

$$\langle \{a_n\}, \{b_n\} \rangle = \sum_{n \in \mathbb{N}} a_n b_n^*.$$

Definition 1.11 Let E be a right $\mathcal{A}$ -module where $\mathcal{A}$ is a locally C^* -algebra. An $\mathcal{A}$ -combination of $x_1, x_2, \dots, x_n$ in E is written as follows:

$$\sum_{i=1}^n x_i a_i = x_1 a_1 + x_2 a_2 + \dots + x_n a_n \quad (a_i \in \mathcal{A}),$$

and $x_1, x_2, \dots, x_n$ are called $\mathcal{A}$ -independent if the equation $x_1 a_1 + x_2 a_2 + \dots + x_n a_n = 0$ has exactly one solution, namely $a_1 = a_2 = \dots = a_n = 0$; otherwise, we say that $x_1, x_2, \dots, x_n$ are $\mathcal{A}$ -dependent. The maximum number of elements in E , that are $\mathcal{A}$ -independent, is called $\mathcal{A}$ -rank of E .

Definition 1.12 Let $\mathcal{A}$ be a locally C^* -algebra and E be a linear space by $\mathcal{A}$ -rank greater than 1, which is also a right $\mathcal{A}$ -module. We define a function $\langle \cdot, \cdot | \cdot \rangle : E \times E \times E \rightarrow \mathcal{A}$ which satisfies the following properties:

- (T_1) $\langle x, x | y \rangle = 0$ if and only if $x = ya$ for $a \in \mathcal{A}$;
- (T_2) $\langle x, x | y \rangle \geq 0$ for all $x, y \in E$;
- (T_3) $\langle x, x | y \rangle = \langle y, y | x \rangle$ for all $x, y \in E$;
- (T_4) $\langle x, y | z \rangle = \langle y, x | z \rangle^*$ for all $x, y, z \in E$;
- (T_5) $\langle xa, yb | z \rangle = a^* \langle x, y | z \rangle b$ for all $x, y, z \in E$ and $a, b \in \mathcal{A}$;
- (T_6) $\langle \alpha x, y | z \rangle = \bar{\alpha} \langle x, y | z \rangle$ for all $x, y \in E$ and $\alpha \in \mathbb{C}$;
- (T_7) $\langle x + y, z | w \rangle = \langle x, z | w \rangle + \langle y, z | w \rangle$ for all $x, y, z, w \in E$.

Then the function $\langle \cdot, \cdot | \cdot \rangle$ is called 2- $*$ - inner product and $(E, \langle \cdot, \cdot | \cdot \rangle)$ is called 2- $*$ - inner product space. If E satisfies all conditions for a 2- $*$ -inner product except the second part of condition (T_1), then we call E is 2- $*$ - semi inner product space.

Example 1.13 Let $\mathcal{A}$ be an unital commutative locally C^* -algebra and E be a pre-Hilbert $\mathcal{A}$ -module with inner product $\langle \cdot, \cdot \rangle$. Define $\langle \cdot, \cdot | \cdot \rangle : E \times E \times E \rightarrow \mathcal{A}$ by $(x, y, z) \mapsto \langle x, y | z \rangle = \langle x, y \rangle \langle z, z \rangle - \langle x, z \rangle \langle z, y \rangle$. Then $(E, \langle \cdot, \cdot | \cdot \rangle)$ is a 2- $*$ -semi inner product space.

Example 1.14 Let $\mathcal{A}$ be a commutative unital C^* -algebra. By the commutative Gelfand–Naimark theorem, we can identify $\mathcal{A}$ with $C(X)$, the algebra of continuous complex-valued functions on a compact Hausdorff space X . If X was an Euclidean manifold, then one would analyse it by geometric techniques, among the most important of which is the study of vector bundles over X . A vector bundle E can be described as follows: Take a fixed Euclidean space H (note that, for example, if $(H, \langle \cdot, \cdot \rangle)$ is a Hilbert space a continuous 2-inner product which is defined on H by $\langle \langle a, b | c \rangle = \langle a, b \rangle \|c\|^2 - \langle a, c \rangle \langle b, c \rangle, a, b, c \in H$) and for each t in X let H_t be a subspace of H . Let E be the space of all continuous functions ξ from x to H such that, for all t in X , $\xi(t) \in H_t$. Then E is naturally endowed with a $C(X)$ -valued 2-semi inner product. Namely, if $\xi, \eta, \gamma \in E$, then we define $\langle \xi, \eta | \gamma \rangle$ to be the function $t \mapsto \langle \xi(t), \eta(t) | \gamma(t) \rangle_H$. Also, E has the structure of a $C(X)$ -module: given ξ in E and f in $C(X)$, we define ξf to be the pointwise product $t \mapsto \xi(t) f(t)$, which is an element of E .

The following theorems are proved in [17]:

Theorem 1.15 [17, Lemma 2.3, Theorem 2.4]. For every $x, y, z \in E$ and $\alpha \in \mathbb{C}$,

- (i) $\langle x, y | z \rangle = \frac{1}{4} [\langle x + y, x + y | z \rangle - \langle x - y, x - y | z \rangle] + \frac{-i}{4} [\langle x + iy, x + iy | z \rangle - \langle x - iy, x - iy | z \rangle]$.
- (ii) $\langle x, y | \alpha z \rangle = |\alpha|^2 \langle x, y | z \rangle$.
- (iii) $\langle x, y | za \rangle = a^* \langle x, y | z \rangle a \quad (a \in \mathcal{A})$.

Recall that

$$\operatorname{Re}\langle x, y | z \rangle = \frac{1}{4} \left[\langle x + y, x + y | z \rangle - \langle x - y, x - y | z \rangle \right]$$

and

$$\operatorname{Im}\langle x, y | z \rangle = \frac{-1}{4} \left[\langle x + iy, x + iy | z \rangle - \langle x - iy, x - iy | z \rangle \right].$$

Theorem 1.16 [17, Theorem 2.6]. *Let $(E, \langle \cdot, \cdot | \cdot \rangle)$ be a 2- $*$ -inner product space. For every $x, y, z \in E$ and $p \in S(\mathcal{A})$, the following inequality is satisfied:*

$$p(\langle x, y | z \rangle) \leq \sqrt{p(\langle x, x | z \rangle)} \sqrt{p(\langle y, y | z \rangle)}.$$

Proposition 1.17 [15, Proposition 1]. *Let $(E, \langle \cdot, \cdot | \cdot \rangle)$ be a $\mathcal{A}$ -2- semi inner product space on a C^* -algebra $\mathcal{A}$. Then the following inequalities hold:*

$$\begin{aligned} (i) & |\langle x, y | z \rangle|^2 = \langle x, y | z \rangle^* \langle x, y | z \rangle \leq \|\langle x, x | z \rangle\| \|\langle y, y | z \rangle\|. \\ (ii) & \|\langle x, y | z \rangle\|^2 \leq \|\langle x, x | z \rangle\| \|\langle y, y | z \rangle\|, \end{aligned}$$

for all $x, y, z \in E$.

Since $\langle x, x | z \rangle$ is a positive element in $\mathcal{A}$, there is a positive square root of $\langle x, x | z \rangle$ denoted by $|x, z|$ and for every $a \in \mathcal{A}$, $|a| = (a^*a)^{1/2}$ and $\|x, z\| = \|\langle x, x | z \rangle\|^{1/2}$.

2 The Cauchy–Schwarz inequality

In this section we state and prove some results about the Cauchy–Schwarz inequality and its reverse in a 2- $*$ -semi inner product space.

Theorem 2.1 *Let $\mathcal{A}$ be a locally C^* -algebra and $(X, \langle \cdot, \cdot | \cdot \rangle)$ be a 2- $*$ - semi inner product space. If $x, y, z \in X$ and $|x, z| \in Z(\mathcal{A})$, then*

$$|\langle x, y | z \rangle|^2 = \langle x, y | z \rangle^* \langle x, y | z \rangle \leq |x, z|^2 |y, z|^2.$$

Proof For all $x, y, z \in X$ and $a \in \mathcal{A}$, we have

$$0 \leq \langle xa - y, xa - y | z \rangle = a^* \langle x, x | z \rangle a - a^* \langle x, y | z \rangle - \langle y, x | z \rangle a + \langle y, y | z \rangle,$$

which implies

$$a^* \langle x, y | z \rangle + \langle y, x | z \rangle a \leq a^* \langle x, x | z \rangle a + \langle y, y | z \rangle.$$

We know the positive number ϵ , $|x, z| + \epsilon e$ is invertible [6]; take $a = \langle x, y | z \rangle (|x, z| + \epsilon e)^{-2}$, then we get

$$2(|x, z| + \epsilon e)^{-2} |\langle x, y | z \rangle|^2 \leq (|x, z| + \epsilon e)^{-4} |x, z|^2 |\langle x, y | z \rangle|^2 + |y, z|^2.$$

Multiplying this by $(|x, z| + \epsilon e)^2$, using the fact that $|x, z| + \epsilon e \in Z(\mathcal{A})$ and applying [18, Theorem 2.2.5], we get

$$2|\langle x, y | z \rangle|^2 \leq (|x, z| + \epsilon e)^{-2} |x, z|^2 |\langle x, y | z \rangle|^2 + (|x, z| + \epsilon e)^2 |y, z|^2,$$

that is

$$|\langle x, y | z \rangle|^2 - (|x, z| + \epsilon e)^2 |y, z|^2 \leq (|x, z| + \epsilon e)^{-2} |x, z|^2 |\langle x, y | z \rangle|^2 - |\langle x, y | z \rangle|^2. \quad (2.1)$$

Multiplying $|x, z|^2 \leq (|x, z| + \epsilon e)^2$ by $(|x, z| + \epsilon e)^{-1} \langle y, x | z \rangle$ from the left and by $(|x, z| + \epsilon e)^{-1} \langle x, y | z \rangle$ from the right, we get

$$(|x, z| + \epsilon e)^{-2} |\langle x, y | z \rangle|^2 |y, z|^2 \leq |\langle x, y | z \rangle|^2.$$

Taking this into account in (2.1), we have

$$|\langle x, y|z \rangle|^2 - (|x, z| + \epsilon e)^2 |y, z|^2 \leq 0,$$

or

$$|\langle x, y|z \rangle|^2 - (|x, z|^2 + 2\epsilon e|x, z| + \epsilon^2 e^2) |y, z|^2 \leq 0.$$

Thus, we conclude

$$|\langle x, y|z \rangle|^2 - |x, z|^2 |y, z|^2 \leq \epsilon(2|x, z| + \epsilon e) |y, z|^2.$$

Letting $\epsilon \rightarrow 0$, we finally obtain $|\langle x, y|z \rangle|^2 \leq |x, z|^2 |y, z|^2$. $\square$

We get the following corollaries from Theorem 2.1.

Corollary 2.2 *Let $\mathcal{A}$ be a commutative locally C^* -algebra, $(X, \langle \cdot, \cdot | \cdot \rangle)$ be a 2- $*$ - semi inner product space, and $|x, z| \in b(\mathcal{A})$. Then we have*

$$p^2(\langle x, y | z \rangle) \leq p(\langle x, x | z \rangle) p(\langle y, y | z \rangle) \leq \|x, x|z\|_\infty p(\langle y, y | z \rangle).$$

Corollary 2.3 *Let $\mathcal{A}$ be a C^* -algebra, $(X, \langle \cdot, \cdot | \cdot \rangle)$ be a 2- $*$ - semi inner product space, and $|x, z| \in Z(\mathcal{A})$. Then we have*

$$|\langle x, y|z \rangle|^2 \leq |x, z|^2 |y, z|^2 \leq \|x, z\|^2 |y, z|^2.$$

Proof It is well known that $a^*b^*ba \leq \|b\|^2 a^*a$ for all $a, b \in \mathcal{A}$ [18, Theorem 2.2.5]. In particular, if $|x, z| \in Z(\mathcal{A})$, then $|x, z|^2 |y, z|^2 = |y, z| |x, z|^2 |y, z| \leq \|x, z\|^2 |y, z|^2$. $\square$

Corollary 2.4 *Let $\mathcal{A}$ be a C^* -algebra and $(X, \langle \cdot, \cdot | \cdot \rangle)$ be a 2- $*$ -inner product space. Then for $x, y, z \in X$ and $a \in \mathcal{A}$ we have*

$$|\langle x, y|z \rangle|^2 = \|x, z\|^2 |y, z|^2 \text{ if and only if } y = \frac{1}{\|x, z\|^2} x \langle x, y | z \rangle + za.$$

Proof We may assume that $\|x, z\| = 1$. First let us $y = x \langle x, y | z \rangle + za$. Then suppose that $\langle y, x | z \rangle \langle x, y | z \rangle = \langle y, x | z \rangle \langle x, x | z \rangle \langle x, y | z \rangle + za | z \rangle = \langle y, x | z \rangle \langle x, x | z \rangle \langle x, y | z \rangle$, which implies that

$$\begin{aligned} 0 &= \langle x \langle x, y | z \rangle + za - y, x \langle x, y | z \rangle + za - y | z \rangle \\ &= \langle y, x | z \rangle \langle x, x | z \rangle \langle x, y | z \rangle - \langle y, x | z \rangle \langle x, y | z \rangle + \langle y, y | z \rangle. \end{aligned}$$

Hence,

$$\langle y, x | z \rangle \langle x, y | z \rangle = \langle y, y | z \rangle.$$

Conversely, suppose that $\langle y, x | z \rangle \langle x, y | z \rangle = \langle y, y | z \rangle$. Then, we prove there exists $a \in \mathcal{A}$ such that $y = x \langle x, y | z \rangle + za$. Since $\langle y, x | z \rangle \langle x, x | z \rangle \langle x, y | z \rangle \leq \|x, x|z\| \langle y, x | z \rangle \langle x, y | z \rangle$, we have

$$\begin{aligned} 0 &\leq \langle x \langle x, y | z \rangle - y, x \langle x, y | z \rangle - y | z \rangle \\ &= \langle y, x | z \rangle \langle x, x | z \rangle \langle x, y | z \rangle - \langle y, x | z \rangle \langle x, y | z \rangle - \langle y, x | z \rangle \langle x, y | z \rangle + \langle y, y | z \rangle \\ &\leq \langle y, x | z \rangle \langle x, y | z \rangle - \langle y, x | z \rangle \langle x, y | z \rangle = 0. \end{aligned}$$

Thus, there exists $a \in \mathcal{A}$ such that $y = x \langle x, y | z \rangle + za$. $\square$

Now, we give the reverse of the Cauchy–Schwarz inequality.

Theorem 2.5 Let $\mathcal{A}$ be a locally C^* -algebra and $(X, \langle \cdot, \cdot \rangle, |\cdot|)$ be a 2- $*$ -semi inner product space. If $a \in \mathcal{A}$ and $x, y, z \in X$ are such that $|x, z| \in Z(\mathcal{A})$ and

$$\operatorname{Re}\langle y - xa, xb - y|z \rangle \geq 0$$

holds, then we have the inequality

$$\begin{aligned} 0 \leq |x, z|^2 |y, z|^2 - |\langle x, y|z \rangle|^2 &\leq \frac{1}{4} |x, z|^4 |b - a|^2 - \left| |x, z|^2 (a + b)/2 - \langle x, y|z \rangle \right|^2 \\ &\leq \frac{1}{4} |x, z|^4 |b - a|^2. \end{aligned}$$

Proof From Theorem 1.15 we have

$$\Re\langle y - xa, xb - y|z \rangle = \frac{1}{4} \left(\langle z, z|x(b - a) \rangle - \langle z, z|2y - x(a + b) \rangle \right) \geq 0.$$

Then

$$\begin{aligned} \langle y - x(a + b)/2, y - x(a + b)/2|z \rangle &\leq \frac{1}{4} (b - a)^* \langle z, z|x \rangle (b - a) \\ &= \frac{1}{4} |b - a|^2 |x, z|^2. \end{aligned}$$

This is equivalent to

$$|y, z|^2 \leq \frac{1}{4} |b - a|^2 |x, z|^2 - \langle y, x|z \rangle \frac{a + b}{2} - \frac{a^* + b^*}{2} \langle x, y|z \rangle - \frac{|a + b|^2}{4} |x, z|^2.$$

Using $|x, z| \in Z(\mathcal{A})$ and [18, Theorem 2.2.5], we get

$$\begin{aligned} |x, z|^2 |y, z|^2 - |\langle x, y|z \rangle|^2 &\leq \frac{1}{4} |b - a|^2 |x, z|^4 + |x, z|^2 \langle y, x|z \rangle \frac{a + b}{2} + \frac{a^* + b^*}{2} |x, z|^2 \langle x, y|z \rangle \\ &\quad - \frac{|a + b|^2}{4} |x, z|^4 - |\langle x, y|z \rangle|^2 \\ &= \frac{1}{4} |b - a|^2 |x, z|^4 - \left(|x, z|^4 \frac{|a + b|^2}{4} - \frac{a^* + b^*}{2} |x, z|^2 \langle x, y|z \rangle \right. \\ &\quad \left. - \frac{a + b}{2} |x, z|^2 \langle y, x|z \rangle + |\langle x, y|z \rangle|^2 \right) \\ &= \frac{1}{4} |b - a|^2 |x, z|^4 - \left(\frac{1}{2} |x, z|^2 (a + b) - \langle x, y|z \rangle \right)^* \left(\frac{1}{2} |x, z|^2 (a + b) - \langle x, y|z \rangle \right) \\ &= \frac{1}{4} |b - a|^2 |x, z|^4 - \left| \frac{1}{2} |x, z|^2 (a + b) - \langle x, y|z \rangle \right|^2 \\ &\leq \frac{1}{4} |b - a|^2 |x, z|^4. \end{aligned}$$

□

Remark 2.6 Let $x, z \in X$ such that $|x, z| \in Z(\mathcal{A})$ and let $a, b \in \mathcal{A}$ such that $a \neq b$. Assume that there exists $w \in X$ satisfying $|w, z| = |x, z|$ and $\langle x, w|z \rangle = 0$. Let

$$y = x \left(\frac{a + b}{2} \right) + w \left(\frac{b - a}{2} \right).$$

Then

$$\begin{aligned} |y, z|^2 &= \left(\frac{a + b}{2} \right)^* |x, z|^2 \left(\frac{a + b}{2} \right) + \left(\frac{b - a}{2} \right)^* |w, z|^2 \left(\frac{b - a}{2} \right) \\ &= \frac{1}{4} |x, z|^2 |a + b|^2 + \frac{1}{4} |x, z|^2 |b - a|^2 \end{aligned}$$

and

$$|\langle x, y|z \rangle|^2 = \frac{1}{4}|x, z|^4|a + b|^2.$$

Thus

$$|x, z|^2|y, z|^2 - |\langle x, y|z \rangle|^2 = \frac{1}{4}|x, z|^4|b - a|^2.$$

Hence, the constant $\frac{1}{4}$ in Theorem 2.11 is the best choice.

Remark 2.7 Let $(\mathcal{H}, \langle \cdot, \cdot | \cdot \rangle)$ be a 2- semi inner product space on $\mathbb{C}$. Then, $|x, z| = \|x, z\|$ for all $x, z \in \mathcal{H}$ and if $\dim \mathcal{H} > 2$, then Theorem 2.11 yields Theorem 2.11. The proof of Theorem 2.11 is simpler than the given proof of Theorem 2.10 in [2].

We come into the following conclusion that provides the property for the constant in the Cachy–Schwarz inequality:

Theorem 2.8 Let $\mathcal{A}$ be a locally C^* -algebra and $(X, \langle \cdot, \cdot | \cdot \rangle)$ be a 2- $*$ - semi inner product space. If $\lambda \in \mathcal{A}$ and $x, y, z \in X$ are such that $\lambda, |x, z| \in Z(\mathcal{A})$, then

$$|x, z|^2|y, z|^2 - |\langle x, y | z \rangle|^2 = |x - y\lambda, z|^2|y, z|^2 - |\langle x - y\lambda, y | z \rangle|^2.$$

Proof By properties of 2- $*$ -inner product, it follows that, for any $\lambda, |x, z| \in Z(\mathcal{A})$,

$$\begin{aligned} |x - y\lambda, z|^2|y, z|^2 - |\langle x - y\lambda, y | z \rangle|^2 &= \langle x - y\lambda, x - y\lambda | z \rangle \langle y, y | z \rangle \\ &\quad - (\langle y, x | z \rangle - \langle y, y | z \rangle \lambda)(\langle x, y | z \rangle - \lambda^* \langle y, y | z \rangle) \\ &= |x, z|^2|y, z|^2 - 2|y, z|^2 \operatorname{Re}(\langle x, y | z \rangle \lambda) - |\lambda|^2|y, z|^4 \\ &\quad - |\langle x, y | z \rangle|^2 + 2|y, z|^2 \operatorname{Re}(\langle x, y | z \rangle \lambda) + |\lambda|^2|y, z|^4 \\ &= |x, z|^2|y, z|^2 - |\langle x, y | z \rangle|^2. \end{aligned}$$

This completes the proof. $\square$

Corollary 2.9 Let $\mathcal{A}$ be a locally C^* -algebra and $(X, \langle \cdot, \cdot | \cdot \rangle)$ be a 2- $*$ - semi inner product space. If $\lambda \in \mathcal{A}$ and $x, y, z \in X$ are such that $\lambda, |x, z| \in Z(\mathcal{A})$, then

$$|x, z|^2|y, z|^2 - |\langle x, y | z \rangle|^2 \leq |x - y\lambda, z|^2|y, z|^2. \quad (2.2)$$

The equalities in 2.2 hold if and only if $\langle x, y | z \rangle = \lambda^* \langle y, y | z \rangle$.

For two parameters, we can have the following:

Theorem 2.10 Let $\mathcal{A}$ be a locally C^* -algebra and $(X, \langle \cdot, \cdot | \cdot \rangle)$ be a 2- $*$ - semi inner product space. If $\lambda, \mu \in \mathcal{A}$ and $x, y, z \in X$ are such that $\lambda, \mu, |x - y\lambda, z| \in Z(\mathcal{A})$, then

$$\left(|x, z|^2|y, z|^2 - |\langle x, y | z \rangle|^2 \right) |\mu - \lambda|^2 = |x - y\lambda, z|^2|x - y\mu, z|^2 - |\langle x - y\lambda, x - y\mu | z \rangle|^2. \quad (2.3)$$

Proof Take $w := x - y\lambda$. Using some properties of a 2- $*$ -inner product, we have

$$\begin{aligned} |\langle x - y\lambda, x - y\mu | z \rangle|^2 &= \langle w + y(\lambda - \mu), w | z \rangle \langle w, w + y(\lambda - \mu) | z \rangle \\ &= (\langle w, w | z \rangle + (\lambda - \mu)^* \langle y, w | z \rangle)(\langle w, w | z \rangle + \langle y, w | z \rangle (\lambda - \mu)) \\ &= |w, z|^4 + |w, z|^2 \operatorname{Re}(\langle y, w | z \rangle (\lambda - \mu)) + |\lambda - \mu|^2 |w, z|^2 |y, z|^2 \\ &\quad - |\lambda - \mu|^2 \left(|w, z|^2 |y, z|^2 - |\langle w, y | z \rangle|^2 \right). \end{aligned} \quad (2.4)$$

Also, we notice that

$$\begin{aligned}
 & |w, z|^4 + |w, z|^2 \operatorname{Re}(\langle y, w | z \rangle (\lambda - \mu)) + |\lambda - \mu|^2 |w, z|^2 |y, z|^2 \\
 &= |w, z|^2 \left(|w, z|^2 + 2 \operatorname{Re}(\langle y, w | z \rangle (\lambda - \mu)) + |\lambda - \mu|^2 |y, z|^2 \right) \\
 &= |w, z|^2 \left(|w + y(\lambda - \mu), z|^2 \right) \\
 &= |x - y\lambda, z|^2 |x - \mu y, z|^2.
 \end{aligned} \tag{2.5}$$

Therefore, from (2.5) and (2.4), we obtain the desired result. This completes the proof. $\square$

Corollary 2.11 *Let $\mathcal{A}$ be a locally C^* -algebra and $(X, \langle \cdot, \cdot | \cdot \rangle)$ be a 2- $*$ -semi inner product space. If $\lambda, \mu \in \mathbb{C}e$ and $x, y, z \in X$ are such that $|x - y\lambda, z| \in Z(\mathcal{A})$ and $\lambda \neq \mu$, then*

$$|x, z|^2 |y, z|^2 - |\langle x, y | z \rangle|^2 \leq \frac{1}{|\mu - \lambda|^2} |x - y\lambda, z|^2 |x - y\mu, z|^2.$$

As an application of Theorem 2.10, we have the following:

Proposition 2.12 *Let $\mathcal{A}$ be a commutative C^* -algebra and $(X, \langle \cdot, \cdot | \cdot \rangle)$ be a 2- $*$ -semi inner product space. Then, for all $x, y, z, e \in X$ with $\langle e, e | z \rangle = 1$ and $\lambda, \mu, \gamma, \eta \in \mathbb{C}e$ with $\lambda \neq \mu$ and $\gamma \neq \eta$, we have*

$$\|\langle x, y | z \rangle - \langle x, e | z \rangle \langle e, y | z \rangle\| \leq \frac{1}{|\lambda - \mu| |\gamma - \eta|} \left(\|x - e\lambda, z\| \|x - e\mu, z\| \|x - e\gamma, z\| \|x - e\eta, z\| \right).$$

Proof We have $\langle x - \langle e, x | z \rangle e, y - \langle e, y | z \rangle e \rangle = \langle x, y | z \rangle - \langle x, e | z \rangle \langle e, y | z \rangle$,

$$|\langle x - \langle e, x | z \rangle e, z \rangle|^2 = |x, z|^2 - |\langle e, x | z \rangle|^2$$

and

$$|\langle y - \langle e, y | z \rangle e, z \rangle|^2 = |y, z|^2 - |\langle e, y | z \rangle|^2.$$

Applying the Cauchy–Schwarz inequality and Corollary 2.11, we have

$$\begin{aligned}
 \|\langle x, y | z \rangle - \langle x, e | z \rangle \langle e, y | z \rangle\| &\leq \|\langle x - \langle e, x | z \rangle e, z \rangle\| \|\langle y - \langle e, y | z \rangle e, z \rangle\| \\
 &= \| |x, z|^2 - |\langle e, x | z \rangle|^2 \|^{1/2} \| |y, z|^2 - |\langle e, y | z \rangle|^2 \|^{1/2} \\
 &\leq \frac{1}{|\lambda - \mu| |\gamma - \eta|} \left(\|x - e\lambda, z\| \|x - e\mu, z\| \|x - e\gamma, z\| \|x - e\eta, z\| \right).
 \end{aligned}$$

$\square$

Open Access This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

References

1. Cho, Y.J.; Diminnie, C.R.; Freese, R.W.; Andalafte, E.Z.: Isosceles orthogonal triples in linear 2-normed spaces. *Math. Nachr.* **157**, 225–234 (1992)
2. Cerone, P.; Cho, Y.J.; Dragomir, S.S.; Kim, S.S.: Refinements of some reverses of Schwarz inequality in 2-inner product spaces and applications for integrals. *J. Indones. Math. Soc.* **12**(2), 185–199 (2006)
3. Cho, Y.J.; Lin, P.C.S.; Kim, S.S.; Misiak, A.: *Theory of 2-Inner Product Spaces*. Nova Science Publishers Inc, New York (2001)
4. Diminnie, C.; Ghler, S.; White, A.: 2-Inner product spaces II. *Demonstr. Math.* **1**, 169–188 (1977)
5. Dragomir, S.S.; Cho, Y.J.; Kim, S.S.; Sofo, A.: Some Boas–Bellman type inequalities in 2-inner product spaces. *J. Inequal. Pure Appl. Math.* **6**(2), 1–13 (2003)
6. Fragouloupoulou, M.: *Topological Algebras with Involution*. North Holland, Amsterdam (2005)
7. Freese, R.W.; Cho, Y.J.: *Geometry of Linear 2- Normed Spaces*. Nova Science Publishers Inc, New York (2001)

8. Freese, R.W.; Gähler, S.: Remarks on semi-2-normed spaces. *Math. Nachr.* **105**, 151–161 (1982)
9. Gähler, S.: 2-metrische Räume und ihre topologische Struktur. *Math. Nachr.* **26**(1–4), 115–148 (1963)
10. Gähler, S.: Lineare 2-normierte Räume. *Math. Nachr.* **28**, 1–43 (1965)
11. Gozali, S.M.; Gunawan, H.: On b-orthogonality in 2-normed spaces. *J. Indones. Math. Soc.* **16**, 127–132 (2009)
12. Inoue, A.: Locally C^* -algebras. *Mem. Fac. Sci. Kyushu Univ. Ser. A* **25**, 197–235 (1971)
13. Joita, M.: Hilbert modules over locally C^* -algebras. Editura Universitatii din Bucuresti, Bucharest (2006)
14. Lance, C.: Hilbert C^* -Modules, London Mathematical Society. Lecture Note Series 210. Cambridge University Press, Cambridge (1995)
15. Mehdiabad Mahchari, T.; Nazari, A.: 2-Hilbert C^* -modules and some Grüss type inequalities in $\mathcal{A}$ -2-inner product spaces. *Math. Inequal. Appl.* **18**(2), 721–754 (2015)
16. Misiak, A.: n -Inner product spaces. *Math. Nachr.* **140**(1), 299–319 (1989)
17. Mohebbi Najmabadi, B.; Shateri, T.L.: 2-Inner product which takes values on a locally C^* -algebra. *Indian J. Math. Soc.* **85**(1–2), 218–226 (2018)
18. Murphy, G.J.: C^* -Algebras and Operator Theory. Academic Press, Boston (1990)
19. Phillips, N.C.: Inverse limits of C^* -algebras. *J. Oper. Theory* **19**(1), 159–195 (1988)
20. Rassias, T.M.: Survey on Classical Inequalities. Kluwer Academic Publishers, Dordrecht (2000)
21. Rassias, T.M.: Functional Equations, Inequalities and Applications. Kluwer Academic Publishers, Dordrecht (2003)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.