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Generalized Weyl's theorem and property (gw) for upper triangular operator matrices

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Abstract It is known that if $A \in \mathscr{L}(\mathscr{X})$ and $B \in \mathscr{L}(\mathscr{Y})$ are Banach operators with the single-valued extension property, SVEP, then the matrix operator $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ has SVEP for every operator $C \in \mathscr{L}(\mathscr{Y}, \mathscr{X})$, and hence obeys generalized Browder's theorem. This paper considers conditions on operators A, B, and M_0 ensuring generalized Weyl's theorem and property (Bw) for operators M_C. Moreover, certain conditions are explored on Banach space operators T and S so that $T \oplus S$ obeys property (gw).

Mathematics Subject Classification 47A10 · 47A53

الملخص

من المعروف أنه إذا كان $A \in L(X)$ و $B \in L(Y)$ مؤثري بناخ بخاصية الامتداد ذات القيمة المفردة، فإن لمصفوفة المؤثرات $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ خاصية الامتداد ذات القيمة المفردة وذلك لكل مؤثر $C \in L(Y, X)$، وبالتالي يخضع لنظرية براودر المعممة. هذا البحث يناقش الشروط على المؤثرات A، B، وM_0 التي تضمن مبرهنة ويل وخاصية $(B\omega)$ للمؤثرات M_C. بالإضافة، نستكشف بعض الشروط المعينة على فضاء بناخ للموثرات T و S وذلك حتى تخضع $T \oplus S$ للخاصية $(g\omega)$.

1 Introduction

Throughout this paper, $\mathscr{X}$ and $\mathscr{Y}$ are Banach spaces and $\mathscr{L}(\mathscr{X}, \mathscr{Y})$ denotes the space of all bounded linear operators from $\mathscr{X}$ to $\mathscr{Y}$. For $\mathscr{X} = \mathscr{Y}$, we write $\mathscr{L}(\mathscr{X}, \mathscr{Y}) = \mathscr{L}(\mathscr{X})$. For $T \in \mathscr{L}(\mathscr{X})$, let T^*, $\ker(T)$, $\Re(T)$, $\sigma(T)$, $\sigma_d(T)$, $\sigma_p(T)$ and $\sigma_a(T)$ denote the *adjoint*, the null space, the range, the spectrum, the surjective spectrum, the point spectrum and the approximate point spectrum of T, respectively. Let $\alpha(T)$ and $\beta(T)$ be the nullity and the deficiency of T defined by $\alpha(T) = \dim \ker(T)$ and $\beta(T) = co \dim \Re(T)$. Let $a := a(T)$ be the ascent of an operator T ; i.e., the smallest nonnegative integer p such that $\ker(T^p) = \ker(T^{p+1})$. If such an integer does not exist, we put $a(T) = \infty$. Analogously, let $d := d(T)$ be the descent of an operator T , i.e., the smallest nonnegative integer s such that $\Re(T^s) = \Re(T^{s+1})$, and if such an integer does not exist we put $d(T) = \infty$. It is well known that if $a(T)$ and $d(T)$ are both finite, then $a(T) = d(T)$ [1].

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For $A \in \mathscr{L}(\mathscr{X})$, $B \in \mathscr{L}(\mathscr{Y})$ and $C \in \mathscr{L}(\mathscr{X}, \mathscr{Y})$, let M_{C} denote the upper triangular operator matrix $M_{\mathrm{C}} = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ and $M_0 = A \oplus B$. The spectrum of the operators M_{C} and M_0 has been studied by a number of authors in the recent past. Of particular interest here is the relationship between the spectral, the Fredholm, the Browder, and the Weyl properties.

In this paper, we introduce most of our notations and terminologies in Sect. 2. In Sect. 3, we prove a sufficient condition for the implication M_0 satisfies property (Bw) $\Rightarrow M_{\mathrm{C}}$ satisfies property (Bw). Section 4 is devoted to explore certain conditions on Banach operators T and S so that $T \oplus S$ obeys property (gw). We consider generalized Weyl's theorem and generalized a-Weyl's theorem for the operators M_0 and M_{C} in Sect. 5. Here, we prove a necessary and sufficient condition for the equivalence $M_0 \in \mathrm{g}\mathcal{W} \Leftrightarrow M_{\mathrm{C}} \in \mathrm{g}\mathcal{W}$. For operators M_0 and M_{C} such that $\sigma_{\mathrm{SBF}_+^-}(M_0) = \sigma_{\mathrm{SBF}_+^-}(M_{\mathrm{C}})$, we prove a sufficient condition for the implications $M_0 \in \mathrm{ga}\mathcal{W} \Rightarrow M_{\mathrm{C}} \in \mathrm{ga}\mathcal{W}$ and $M_{\mathrm{C}} \in \mathrm{ga}\mathcal{W} \Rightarrow M_0 \in \mathrm{ga}\mathcal{W}$.

2 Notation and terminology

Given two Banach spaces $\mathscr{X}$ and $\mathscr{Y}$, the set of all *upper semi-Fredholm* operators is defined by

$$\Phi_+(\mathscr{X}, \mathscr{Y}) := \{T \in \mathscr{L}(\mathscr{X}, \mathscr{Y}) : \alpha(T) < \infty \quad \text{and } \Re(T) \text{ is closed}\},$$

while the set of all *lower semi-Fredholm* operators is defined by

$$\Phi_-(\mathscr{X}, \mathscr{Y}) := \{T \in \mathscr{L}(\mathscr{X}, \mathscr{Y}) : \beta(T) < \infty\}.$$

The set of all *semi-Fredholm* operators is defined by

$$\Phi_\pm(\mathscr{X}, \mathscr{Y}) := \Phi_+(\mathscr{X}, \mathscr{Y}) \cup \Phi_-(\mathscr{X}, \mathscr{Y}).$$

We shall set

$$\Phi_+(\mathscr{X}) := \Phi_+(\mathscr{X}, \mathscr{X}) \quad \text{and} \quad \Phi_-(\mathscr{X}) := \Phi_-(\mathscr{X}, \mathscr{X}),$$

while

$$\Phi(\mathscr{X}) := \Phi(\mathscr{X}, \mathscr{X}) \quad \text{and} \quad \Phi_\pm(\mathscr{X}) = \Phi_\pm(\mathscr{X}, \mathscr{X}).$$

Note that $T \in \Phi(\mathscr{X}, \mathscr{Y})$ if and only if $\alpha(T)$ and $\beta(T)$ are finite. The index of a semi-Fredholm operator $T \in \Phi_\pm(\mathscr{X}, \mathscr{Y})$ is defined by

$$\mathrm{ind}(T) := \alpha(T) - \beta(T).$$

Clearly, $\mathrm{ind}(T)$ is an integer or $\pm\infty$. Recall that a bounded operator T is said *bounded below* if it is injective and has closed range. Evidently, if T is bounded below, then $T \in \Phi_+(\mathscr{X})$ and $\mathrm{ind}(T) \leq 0$. Define

$$W_+(\mathscr{X}) := \{T \in \Phi_+(\mathscr{X}) : \mathrm{ind}(T) \leq 0\},$$

and

$$W_-(\mathscr{X}) := \{T \in \Phi_-(\mathscr{X}) : \mathrm{ind}(T) \geq 0\}.$$

The set of *Weyl* operators is defined by

$$W(\mathscr{X}) := W_+(\mathscr{X}) \cap W_-(\mathscr{X}) = \{T \in \Phi(\mathscr{X}) : \mathrm{ind}(T) = 0\}.$$

In the following, let

$$\begin{aligned}
\Phi_+(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \quad \text{is upper semi-Fredholm}\}, \\
\Phi_-(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \quad \text{is lower semi-Fredholm}\}, \\
\Phi_+^-(T) &= \{\lambda \in \Phi_+(T) : \mathrm{ind}(T - \lambda I) \leq 0\}, \\
\Phi_-^+(T) &= \{\lambda \in \Phi_+(T) : \mathrm{ind}(T - \lambda I) \geq 0\}, \\
\Phi(T) &= \Phi_+(T) \cap \Phi_-(T), \quad \text{and}
\end{aligned}$$

$$\Phi^0(T) = \{\lambda \in \mathbb{C} : \mathrm{ind}(T - \lambda I) = 0\}.$$

The classes of operators defined above generate the following spectra. Denote by

$$\sigma_a(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not bounded below}\},$$

the *approximate point spectrum,* and by

$$\sigma_d(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not surjective}\},$$

the *surjectivity spectrum* of $T \in \mathscr{L}(\mathscr{X})$. The *Weyl spectrum* is defined by

$$\sigma_w(T) := \{\lambda \in \mathbb{C} : T - \lambda I \notin W(\mathscr{X})\} \text{ and}$$

the *Weyl essential approximate point spectrum* is defined by

$$\sigma_{aw}(T) := \{\lambda \in \mathbb{C} : T - \lambda I \notin W_+(\mathscr{X})\},$$

while the *Weyl essential surjectivity spectrum* is defined by

$$\sigma_{lw}(T) := \{\lambda \in \mathbb{C} : T - \lambda I \notin W_-(\mathscr{X})\}.$$

Obviously, $\sigma_w(T) = \sigma_{aw}(T) \cup \sigma_{lw}(T)$ and from basic Fredholm theory we have

$$\sigma_{aw}(T) = \sigma_{ws}(T^*) \qquad \sigma_{ws}(T) = \sigma_{aw}(T^*).$$

Note that $\sigma_{aw}(T)$ is the intersection of all approximate point spectra $\sigma_a(T + K)$ of compact perturbations K of T , while $\sigma_{lw}(T)$ is the intersection of all surjectivity spectra $\sigma_s(T + K)$ of compact perturbations K of T; see, for instance, [1, Theorem 3.65].

The class of all *upper semi-Browder* operators on a Banach space $\mathscr{X}$ is defined by

$$\mathcal{B}_+(\mathscr{X}) := \{T \in \Phi_+(\mathscr{X}) : a(T) < \infty\},$$

and The class of all *lower semi-Browder* operators on a Banach space $\mathscr{X}$ is defined by

$$\mathcal{B}_-(\mathscr{X}) := \{T \in \Phi_-(\mathscr{X}) : d(T) < \infty\}.$$

The class of all *Browder* operators is defined by

$$\mathcal{B}(\mathscr{X}) := \mathcal{B}_+(\mathscr{X}) \cap \mathcal{B}_-(\mathscr{X}) = \{T \in \Phi(\mathscr{X}); a(T), d(T) < \infty\}.$$

The *Browder spectrum* of $T \in \mathscr{L}$ is defined by

$$\sigma_b(T) := \{\lambda \in \mathbb{C} : T - \lambda I \notin B(\mathscr{X})\},$$

the *upper Browder spectrum* is defined by

$$\sigma_{ab}(T) := \{\lambda \in \mathbb{C} : T - \lambda I \notin B_+(\mathscr{X})\},$$

and analogously the *lower Browder spectrum* is defined by

$$\sigma_{lb}(T) := \{\lambda \in \mathbb{C} : T - \lambda I \notin B_-(\mathscr{X})\}.$$

Clearly, $\sigma_b(T) = \sigma_{ab}(T) \cup \sigma_{lb}(T)$ and $\sigma_w(T) \subseteq \sigma_b(T)$.
Let $\Delta(T) = \sigma(T) \setminus \sigma_w(T)$ and $\Delta_a(T) = \sigma_a(T) \setminus \sigma_{aw}(T)$. We say that Weyl's theorem holds for $T \in \mathscr{L}(\mathscr{X})$ (in symbols, $T \in \mathcal{W}$) if $\Delta(T) = E^0(T)$, where $E^0(T) = \left\{\lambda \in \sigma(T)^{iso} : 0 < \alpha(T - \lambda I) < \infty\right\}$ and that Browder's theorem holds for T (in symbols, $T \in \mathcal{B}$) if $\sigma_b(T) = \sigma_w(T)$ and that a-Browder's theorem holds for T (in symbols, $T \in a\mathcal{B}$) if $\sigma_{ab}(T) = \sigma_{aw}(T)$.

Here and elsewhere in this paper, for $K \subset \mathbb{C}$, K^{iso} is the set of isolated points of K and K^{acc} is the set of accumulation points of K. According to Rakočević [21], an operator $T \in \mathscr{L}(\mathscr{X})$ is said to satisfy a-Weyl's theorem (in symbols, $T \in a\mathcal{W}$) if $\Delta_a(T) = E_a^0(T)$, where

$$E_a^0(T) = \left\{\lambda \in \sigma_a(T)^{iso} : 0 < \alpha(T - \lambda I) < \infty\right\}.$$

It is known [21] that an operator satisfying a-Weyl's theorem satisfies Weyl's theorem, but the converse does not hold in general.

For $T \in \mathscr{L}(\mathscr{X})$ and a nonnegative integer n, define $T_{[n]}$ to be the restriction of T to $\Re(T^n)$ viewed as a map from $\Re(T^n)$ into $\Re(T^n)$ (in particular $T_{[0]} = T$). If for some integer n the range space $\Re(T^n)$ is closed and $T_{[n]}$ is an upper (resp., a lower) semi-Fredholm operator, then T is called an *upper* (resp., a *lower*) *semi- B-Fredholm* operator. In this case, the index of T is defined as the index of the semi-B-Fredholm operator $T_{[n]}$. Moreover, if $T_{[n]}$ is a Fredholm operator, then T is called a *B-Fredholm* operator (see [6,7]). A *semi-B-Fredholm* operator is an upper or a lower semi-B-Fredholm operator. The *upper semi-B-Fredholm spectrum* $\sigma_{\mathrm{UBF}}(T)$, the *lower semi-B-Fredholm spectrum* $\sigma_{\mathrm{LBF}}(T)$ and the *B-Fredholm spectrum* $\sigma_{\mathrm{BF}}(T)$ of T are defined by

$$\sigma_{\mathrm{UBF}}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not an upper semi-}B\text{-Fredholm operator}\},$$
$$\sigma_{\mathrm{LBF}}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a lower semi-}B\text{-Fredholm operator}\},$$
$$\sigma_{\mathrm{BF}}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a } B\text{-Fredholm operator}\}.$$

We have

$$\sigma_{\mathrm{BF}}(T) = \sigma_{\mathrm{UBF}}(T) \cup \sigma_{\mathrm{LBF}}(T).$$

An operator $T \in \mathscr{L}(\mathscr{X})$ is said to be a B-Weyl operator if it is a B-Fredholm operator of index zero. The B-Weyl spectrum $\sigma_{\mathrm{BW}}(T)$ of T is defined by

$$\sigma_{\mathrm{BW}}(T) := \{\lambda \in \mathbb{C} : T - \lambda I \quad \text{is not a B-Weyl operator}\}.$$

We shall denote by $\mathrm{SBF}_+^-(\mathscr{X})$ ($\mathrm{SBF}_-^+(\mathscr{X})$) the class of all T upper semi-B- Fredholm operators (T lower semi-B-Fredholm operators) such that $\mathrm{ind}(T) \leq 0$ ($\mathrm{ind}(T) \geq 0$). The spectrum associated with $\mathrm{SBF}_+^-(\mathscr{X})$ is called the *semi-essential approximate point spectrum* and is denoted by

$$\sigma_{\mathrm{SBF}_+^-}(T) = \left\{\lambda \in \mathbb{C} : T - \lambda I \notin \mathrm{SBF}_+^-(\mathscr{X})\right\},$$

while the spectrum associated with $\mathrm{SBF}_-^+(\mathscr{X})$ is denoted by

$$\sigma_{\mathrm{SBF}_-^+}(T) = \left\{\lambda \in \mathbb{C} : T - \lambda I \notin \mathrm{SBF}_-^+(\mathscr{X})\right\}.$$

Given $T \in \mathscr{L}(\mathscr{X})$, let $\Delta^g(T) = \sigma(T) \setminus \sigma_{\mathrm{BW}}(T)$. We say that the generalized Weyl's theorem holds for T (and we write $T \in \mathrm{g}\mathcal{W}$) if $\Delta^g(T) = E(T)$, where $E(T)$ is the set of all isolated eigenvalues of T, and that the generalized Browder's theorem holds for T (in symbols, $T \in \mathrm{g}\mathcal{B}$) if $\Delta^g(T) = \pi(T)$, where $\pi(T)$ is the set of all poles of T; see [8, Definition 2.13]. It is known [8] that

$$\mathrm{g}\mathcal{W} \subseteq \mathrm{g}\mathcal{B} \cap \mathcal{W} \quad \text{and that} \quad \mathrm{g}\mathcal{B} \cup \mathcal{W} \subseteq \mathcal{B}.$$

Moreover, given $T \in \mathrm{g}\mathcal{B}$, it is clear that $T \in \mathrm{g}\mathcal{W}$ if and only if $E(T) = \pi(T)$.

For $T \in \mathscr{L}(\mathscr{X})$. Let $\Delta_a^g(T) = \sigma_a(T) \setminus \sigma_{\mathrm{SBF}_+^-}(T)$. We say that T obeys generalized a-Weyl's theorem (in symbols, $T \in \mathrm{ga}\mathcal{W}$), if $\Delta_a^g(T) = E_a(T)$, where $E_a(T)$ is the set of all eigenvalues of T which are isolated in $\sigma_a(T)$ ([8, Definition 2.13]).

Define the set $D(\mathscr{X})$ by $D(\mathscr{X}) = \{T \in \mathscr{L}(\mathscr{X}) : a(T), d(T) < \infty\}$. An operator T is *Drazin invertible* if $T \in D(\mathscr{X})$. The Drazin spectrum

$$\sigma_D(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Drazin invertible}\}.$$

We observe that $\sigma_D(T) = \sigma(T) \setminus \pi(T)$. Define the set $LD(\mathscr{X})$ by

$$LD(\mathscr{X}) = \left\{T \in \mathscr{L}(\mathscr{X}) : a(T) < \infty \text{ and } \Re(T^{a(T)+1}) \text{ is closed}\right\}.$$

Recall that an operator $T \in \mathscr{L}(\mathscr{X})$ is called left Drazin invertible if $T \in LD(\mathscr{X})$. The left Drazin spectrum is defined by

$$\sigma_{LD}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin LD(\mathscr{X})\}.$$

We will say that $\lambda \in \sigma_a(T)$ is a left pole of T if $T - \lambda I$ is left Drazin invertible and that $\lambda \in \sigma_a(T)$ is a left pole of T of finite rank if λ is a left pole of T and $\alpha(T - \lambda I) < \infty$. We will denote by $\pi^a(T)$ the set of all left poles of T, and by $\pi_0^a(T)$ the set of all left poles of T of finite rank. It follows from the preceding description that $\sigma_{LD} = \sigma_a(T) \setminus \pi^a(T)$. Following [2], we say that T obeys generalized a-Browder's theorem (in symbol, $T \in \mathrm{g}a\mathcal{B}$) if $\sigma_a(T) \setminus \sigma_{\mathrm{SBF}_+^-}(T) = \pi^a(T)$ or, equivalently, $\sigma_{LD}(T) = \sigma_{\mathrm{SBF}_+^-}(T)$.

An operator $T \in \mathscr{L}(\mathscr{X})$ has the *single-valued extension property* at $\lambda_0 \in \mathbb{C}$, SVEP at λ_0, if for every open disc U_{λ_0} centered at λ_0 the only analytic function $f : U_{\lambda_0} \longrightarrow \mathscr{X}$ which satisfies $(T - \lambda I) f(\lambda) = 0$ for all $\lambda \in U_{\lambda_0}$ is the function $f \equiv 0$. Trivially, every operator T has SVEP on the resolvent set $\rho(T) = \mathbb{C} \setminus \sigma(T)$; also, T has SVEP at points $\lambda \in \sigma^{iso}(T)$. Let $S(T)$ denote the set of $\lambda \in \mathbb{C}$ where T does not have SVEP: we say that T has SVEP if $S(T) = \emptyset$. SVEP plays an important role in determining the relationship between the Browder and Weyl spectra, and the Browder and Weyl theorems. Thus $\sigma_b(T) = \sigma_w(T) \cup S(T) = \sigma_w(T) \cup S(T^*)$, and if T^* has SVEP, then $\sigma_b(T) = \sigma_w(T) = \sigma_{ab}(T) = \sigma_{aw}(T)$ [1, Page 141- 142]; T satisfies Browder's theorem (resp., a-Browder's theorem) if and only if T has SVEP at $\lambda \notin \sigma_w(T)$ (resp., $\lambda \notin \sigma_{aw}(T)$) [14, Lemma 2.18]; and if T^* has SVEP, then $T \in \mathcal{W}$ if and only if $T \in a\mathcal{W}$ (see [20]).

A study of the spectrum, the Browder and Weyl spectra, and the Browder and Weyl theorems for the operator M_{C}, and the related diagonal operator $M_0 = A \oplus B$, has been carried out by a number of authors in the recent past (see [12,19]). Thus, if either $S(A^*) = \emptyset$ or $S(B) = \emptyset$, then $\sigma(M_{\mathrm{C}}) = \sigma(M_0) = \sigma(A) \cup \sigma(B)$; if $S(A) \cup S(B) = \emptyset$, then M_{C} has SVEP, $\sigma_b(M_c) = \sigma_w(M_{\mathrm{C}}) = \sigma_w(M_0) = \sigma_b(M_0)$, and $M_{\mathrm{C}} \in \mathrm{g}\mathcal{B}$. Browder's theorem, much less Weyl's theorem, does not transfer from individual operators to direct sums: for example, the forward unilateral shift and the backward unilateral shift on a Hilbert space satisfy Browder's theorem, but their direct sum does not. However, if $(S(A) \cap S(B^*)) \cup S(A^*) = \emptyset$, then : M_0 satisfies Browder's theorem (resp., a-Browder's theorem) implying M_{C} satisfies Browder's theorem (resp., a-Browder's theorem); if points $\lambda \in \sigma^{iso}(A)$ are eigenvalues of $A \in \mathcal{W}$, then $M_0 \in \mathcal{W}$ implies $M_{\mathrm{C}} \in \mathcal{W}$ [12, Proposition 4.1 and Theorem 4.2].

It is known from [5,11,12] that

(i) $\sigma_x(M_0) = \sigma_x(A) \cup \sigma_x(B) = \sigma_x(M_{\mathrm{C}}) \cup \{\sigma_x(A) \cap \sigma_x(B)\}$, where $\sigma_x = \sigma, \sigma_b$ or σ_e;
(ii) $\sigma_w(M_0) \subseteq \sigma_w(A) \cup \sigma_w(B) = \sigma_w(M_{\mathrm{C}}) \cup \{\sigma_w(A) \cap \sigma_w(B)\}$;
(iii) if $\sigma_w(M_{\mathrm{C}}) = \sigma_w(A) \cup \sigma_w(B)$, then $\sigma(M_{\mathrm{C}}) = \sigma(M_0)$ and
(iv) $\sigma_{aw}(M_0) \subseteq \sigma_{aw}(A) \cup \sigma_{aw}(B) = \sigma_{aw}(M_{\mathrm{C}}) \cup \{S(A) \cup S(A^*)\}$.

Remark 2.1 The spectral picture of T (notation: $SP(T)$) is the structure consisting of the set $\sigma_e(T)$, the collection of holes and pseudoholes in $\sigma_e(T)$ and the Fredholm indices associated with those holes and pseudoholes. The concept of the spectral picture of an operator has been useful in operator theory (see [9]). It is known that: if either $SP(A)$ or $SP(B)$ has no pseudoholes, then $\sigma^{acc}(M_0) \subseteq \sigma_w(M_0) \Rightarrow \sigma^{acc}(M_{\mathrm{C}}) \subseteq \sigma_w(M_{\mathrm{C}})$ [19, Theorem 2.3]; if additionally A is an isoloid (the isolated points of $\sigma(A)$ are eigenvalues of A) and A satisfies Weyl's theorem, then $M_0 \in \mathcal{W} \Rightarrow M_{\mathrm{C}} \in \mathcal{W}$ [19, Theorem 2.4]. If $\{S(A) \cap S(B^*)\} \cup S(A^*) = \emptyset$, then $\sigma^{acc}(M_0) \subseteq \sigma_w(M_0) \Rightarrow \sigma^{acc}(M_{\mathrm{C}}) \subseteq \sigma_w(M_{\mathrm{C}})$ [12, Proposition 4.1]. Again, if $\sigma_a(A^*)$ has empty interior, A is an a-isoloid (isolated points of $\sigma_a(A)$ are eigenvalues of A) and $A \in a\mathcal{W}$, then $M_0 \in a\mathcal{W} \Rightarrow M_{\mathrm{C}} \in a\mathcal{W}$ [11, Theorem 3.3].

3 Property (Bw)

Following [17], an operator $T \in \mathscr{L}(\mathscr{X})$ is said to satisfy property (Bw) if $\Delta^g(T) = E^0(T)$. The authors proved that T satisfied property (Bw) if and only if generalized Browder's theorem holds for T and $\pi(T) = E^0(T)$.

In general, the fact that property (Bw) holds for A and B does not imply that property (Bw) holds for $M_0 = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. Indeed, let I_1 and I_2 be the identities on $\mathbb{C}$ and ℓ^2, respectively. Let S_1 and S_2 be defined on ℓ^2 by

$$S_1(x_1, x_2, \ldots) = \left(0, \frac{1}{3}x_1, \frac{1}{3}x_2, \ldots\right), \qquad S_2(x_1, x_2, \ldots) = \left(0, \frac{1}{2}x_1, \frac{1}{3}x_2, \ldots\right).$$

Let $T - 1 = I_1 \oplus S_1$, $T_2 = S_2 - I_2$, $A = T_1^2$ and $B = T_2^2$. It follows from [23, Example 1] that

$$\sigma(T_2) = \sigma_{\mathrm{BW}}(T_2) = \{-1\},$$

and

$$\sigma(T_1) = \left\{\lambda \in \mathbb{C} : |\lambda| \leq \frac{1}{3}\right\} \cup \{1\}.$$

Then,

$$\sigma(B) = \sigma_{\mathrm{BW}}(B) = \{1\}, E^0(B) = \emptyset.$$

Hence, B obeys property (Bw). Since

$$\sigma(A) = \left\{\lambda \in \mathbb{C} : |\lambda| \leq \frac{1}{9}\right\} \cup \{1\}, \sigma_{\mathrm{BW}}(A) = \left\{\lambda \in \mathbb{C} : |\lambda| \leq \frac{1}{9}\right\}, E^0(A) = \{1\},$$

A satisfies property (Bw). Now since

$$\sigma(M_0) = \left\{\lambda \in \mathbb{C} : |\lambda| \leq \frac{1}{9}\right\} \cup \{1\} = \sigma_{\mathrm{BW}}(M_0), E^0(M_0) = \{1\},$$

M_0 does not obey property (Bw).

It also may happen that M_{C} obeys property (Bw), while M_0 does not obey it. Let A be the unilateral unweighted shift operator. For $B = A^*$ and $C = I - AA^*$, we have that M_{C} is unitary without eigenvalues. Hence M_{C} satisfies property (Bw), but $\sigma_{\mathrm{BW}}(M_0) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and $\sigma(M_0) \setminus E^0(M_0) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$. Hence, M_0 does not satisfy property (Bw).

Proposition 3.1 *Let A and B be isoloids. Assume that $\sigma_{\mathrm{BW}}(M_0) = \sigma_{\mathrm{BW}}(A) \cup \sigma_{\mathrm{BW}}(B)$. If A and B obey property* (Bw)*, then M_0 obey property* (Bw).

Proof Since A and B are isoloids, we have

$$E^0(M_0) = [E^0(A) \cap \rho(A)] \cup [E^0(B) \cap \rho(B)] \cup [E^0(A) \cap E^0(B)].$$

Now if A and B obey property (Bw), then

$$\begin{aligned} E^0(M_0) &= [\sigma(A) \cup \sigma(B)] \setminus [\sigma_{\mathrm{BW}}(A) \cup \sigma_{\mathrm{BW}}(B)] \\ &= \sigma(M_0) \setminus \sigma_{\mathrm{BW}}(M_0). \end{aligned}$$

Thus, M_0 obeys property (Bw). □

Theorem 3.2 *Let A and B be isoloids with SVEP. If A and B obey property* (Bw)*, then M_{C} obeys property* (Bw) *for every $C \in \mathscr{L}(\mathscr{Y}, \mathscr{X})$.*

Proof Since A and B have SVEP, then it follows from Proposition 3.1 of [18] that M_{C} also has SVEP. Hence, $\sigma_D(M_{\mathrm{C}}) = \sigma_{BW}(M_{\mathrm{C}})$ [4, Corollar 2.4]. Also since A and B have SVEP, it follows from [24, Corollary 2.1] that $\sigma_D(M_{\mathrm{C}}) = \sigma_D(A) \cup \sigma_D(B)$. Therefore, $\sigma_{\mathrm{BW}}(M_{\mathrm{C}}) = \sigma_{\mathrm{BW}}(A) \cup \sigma_{\mathrm{BW}}(B)$ by [4, Corollary 2.4]. It follows from Proposition 3.1 that

$$E^0(M_0) = \sigma(M_0) \setminus \sigma_{\mathrm{BW}}(M_0) = \sigma(M_{\mathrm{C}}) \setminus \sigma_{\mathrm{BW}}(M_{\mathrm{C}}).$$

Hence, it is enough to show that $E^0(M_0) = E^0(M_{\mathrm{C}})$. Let $\lambda \in E^0(M_{\mathrm{C}})$. Then, $\lambda \in \sigma_p(M_{\mathrm{C}}) \subseteq \sigma_p(A) \cup \sigma_p(B)$. Hence, $\lambda \in \sigma_p(M_0)$. Since $\lambda \in \sigma^{iso}(M_{\mathrm{C}}) = \sigma^{iso}(M_0)$, we have $\lambda \in E^0(M_0)$. Now, let $\lambda \in E^0(M_0)$. If $\lambda \in \sigma(A)$ then $\lambda \in \sigma^{iso}(A)$. Since A is an isoloid, we have $\lambda \in \sigma_p(A) \subseteq \sigma_p(M_{\mathrm{C}})$. Hence $\lambda \in E^0(M_{\mathrm{C}})$. If $\lambda \in \sigma(B) \setminus \sigma(A)$, then $\lambda \in \sigma_p(B)$. Since A is invertible, we conclude that $\lambda \in \sigma_p(M_{\mathrm{C}})$. Thus, $\lambda \in E^0(M_{\mathrm{C}})$. Therefore, $E^0(M_0) = E^0(M_{\mathrm{C}})$. □

Recall that an operator $T \in \mathscr{L}(\mathscr{X})$ is a polaroid (finite polaroid) if $\sigma^{iso}(T) \subseteq \pi(T)$ ($\sigma^{iso}(T) \subseteq \pi_0(T)$). Since $\pi_0(T) \subseteq E(T)$ with no restriction on T, then if T is finite polaroid then $E(T) = \pi_0(T)$.

Corollary 3.3 *Let A and B be finite polaroids with SVEP. Then M_{C} obeys property* (Bw) *for every $C \in \mathscr{L}(\mathscr{Y}, \mathscr{X})$.*

Proof A and B are finite polaroids, hence $E(A) = \pi_0(A)$ and $E(B) = \pi_0(B)$. Since A and B have the SVEP, we have by [2] that A and B satisfy the generalized Browder's theorem and $E^0(A) = \pi(A)$ and $E^0(B) = \pi(B)$. Hence, A and B obey property (Bw). Therefore, we complete the proof by Theorem 3.2. □

Recall that a bounded linear operator $T \in \mathscr{L}(\mathscr{X})$ is said to be a finite isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of T of finite multiplicity. Note that if T is finite isoloid then T is an isoloid, but the converse is not true.

Theorem 3.4 *Let A be a finite isoloid. Assume that A and B (or A^* and B^*) have SVEP. If A and M_0 satisfy property* (Bw), *then M_C satisfies property* (Bw) *for every $C \in \mathscr{L}(\mathscr{Y}, \mathscr{X})$.*

Proof Let $\lambda \in \sigma(M_C) \setminus \sigma_{BW}(M_C)$. From [18, Theorem 2.1], we have $\sigma(M_0) = \sigma(M_C)$. Then by [24, Corollary 2.7], $\Delta^g(M_C) = \Delta^g(M_0)$, which equal to $E^0(M_0)$ since M_0 satisfies property (Bw). Thus, $\lambda \in \sigma^{iso}(M_0) = \sigma^{iso}(M_C)$. If $\lambda \in \sigma^{iso}(A)$, since A is finite isoloid then $\lambda \in \sigma_p(A)$. hence $\lambda \in \sigma_p(M_C)$. Then $\lambda \in E^0(M_C)$. Now, assume that $\lambda \in \sigma^{iso}(B) \setminus \sigma^{iso}(A)$. If $\lambda \notin \sigma(A)$, then it is easy to see that $\lambda \in \sigma_p(M_C)$. If $\lambda \in \sigma_p(A)$ then $\lambda \in \sigma_p(M_C)$, then assume that $\lambda \in \sigma_p(B) \setminus \sigma_p(A)$. Then $\lambda \notin E^0(A)$. Since A satisfies property (Bw), then $\lambda \in \sigma_{BW}(A)$. This is impossible (since $\lambda \notin \sigma(A)$). Therefore, $\lambda \in E^0(M_C)$. Conversely, assume that $\lambda \in E^0(M_C)$. Then, $\lambda \in \sigma^{iso}(M_C) = \sigma^{iso}(M_0)$. On the other hand, $\lambda \in \sigma_p(M_C) \subseteq \sigma_p(A) \cup \sigma_p(B)$. Hence, $\lambda \in \sigma_p(M_0)$. Thus, $\lambda \in E^0(M_0) = \sigma(M_0) \setminus \sigma_{BW}(M_0)$ which equal to $\sigma(M_C) \setminus \sigma_{BW}(M_C)$. Therefore, $\lambda \in \Delta^g(M_C)$. □

4 Property (gw) for direct sum

According to [3], we say that $T \in \mathscr{L}(\mathscr{X})$ possesses property (gw) if $\Delta_a^g(T) = E(T)$. Property (gw) has been introduced and studied in [3].

Theorem 4.1 *Suppose that property (gw) holds for $T \in \mathscr{L}(\mathscr{X})$ and $S \in \mathscr{L}(\mathscr{Y})$. If T and S are a-isoloid and $\sigma_{SBF_+^-}(T \oplus S) = \sigma_{SBF_+^-}(T) \cup \sigma_{SBF_+^-}(S)$, then property (gw) holds for $T \oplus S$.*

Proof We know $\sigma_a(T \oplus S) = \sigma_a(T) \cup \sigma_a(S)$ for any pair of operators. If T and S are a-isoloid, then

$$E(T \oplus S) = [E(T) \cap \rho_a(S)] \cup [\rho_a(T) \cap E(S)] \cup [E(T) \cap E(S)],$$

where $\rho_a(.) = \mathbb{C} \setminus \sigma_a(.)$.
If property (gw) holds for T and S, then

$$\begin{aligned}[]
[\sigma_a(T) \cup \sigma_a(S)] \backslash [\sigma_{SBF_+^-}(T) \cup \sigma_{SBF_+^-}(S)] \\
= [E(T) \cap \rho_a(S)] \cup [\rho_a(T) \cap E(S)] \cup [E(T) \cap E(S)].
\end{aligned}$$

Thus, $E(T \oplus S) = \sigma_a(T) \cup \sigma_a(S)] \setminus \sigma_{SBF_+^-}(T) \cup \sigma_{SBF_+^-}(S)$. Then,

$$E(T \oplus S) = \sigma_a(T \oplus S) \setminus \sigma_{SBF_+^-}(T \oplus S).$$

That is, property (gw) holds for $T \oplus S$. □

Theorem 4.2 *Suppose that $T \in \mathscr{L}(\mathscr{X})$ such that $\sigma_a^{iso}(T) = \emptyset$, $\sigma(T) = \sigma_a(T)$ and $S \in \mathscr{L}(\mathscr{Y})$ satisfies property (gw). If $\sigma_{SBF_+^-}(T \oplus S) = \sigma_a(T) \cup \sigma_{SBF_+^-}(S)$, then property (gw) holds for $T \oplus S$.*

Proof We know that $\sigma_a(T \oplus S) = \sigma_a(T) \cup \sigma_a(S)$ for any pair of operators. Then,

$$\begin{aligned}
\sigma_a(T \oplus S) \setminus \sigma_{SBF_+^-}(T \oplus S) &= [\sigma_a(T) \cup \sigma_a(S)] \setminus \left[\sigma_a(T) \cup \sigma_{SBF_+^-}(S)\right] \\
&= \sigma_a(S) \setminus \left[\sigma_a(T) \cup \sigma_{SBF_+^-}(S)\right] \\
&= \left[\sigma_a(S) \setminus \sigma_{SBF_+^-}(S)\right] \setminus \sigma_a(T) \\
&= E(S) \cap \rho_a(T)
\end{aligned}$$

If $\sigma_a^{iso}(T) = \emptyset$, it implies that $\sigma^{iso}(T) = \emptyset$ and $\sigma(T) = \sigma^{acc}(T)$, where $\sigma^{acc}(T) = \sigma(T) \setminus \sigma^{iso}(T)$ is the set of all accumulation points of $\sigma(T)$. Thus, we have

$$\begin{aligned}\sigma^{iso}(T \oplus S) &= \left[\sigma^{iso}(T) \cup \sigma^{iso}(S)\right] \setminus \left[\left(\sigma^{iso}(T) \cap \sigma^{acc}(S)\right) \cup \left(\sigma^{acc}(T) \cap \sigma^{iso}(S)\right)\right] \\ &= \left[\sigma^{iso}(T) \setminus \sigma^{acc}(S)\right] \cup \left[\sigma^{iso}(S) \setminus \sigma^{acc}(T)\right] \\ &= \sigma^{iso}(S) \setminus \sigma_a(T) \\ &= \sigma^{iso}(S) \cap \rho_a(T).\end{aligned}$$

We know that $\sigma_p(T \oplus S) = \sigma_p(T) \cup \sigma_p(S)$. Therefore,

$$\begin{aligned}E(T \oplus S) &= \sigma^{iso}(T \oplus S) \cap \sigma_p(T \oplus S) \\ &= \sigma^{iso}(S) \cap \rho_a(T) \cap \sigma_P(S) \\ &= E(S) \cap \rho_a(T).\end{aligned}$$

Thus, $\sigma_a(T \oplus S) \setminus \sigma_{\mathrm{SBF}_+^-}(T \oplus S) = E(T \oplus S)$. Hence, $T \oplus S$ satisfies property (gw). □

Corollary 4.3 *Suppose that $T \in \mathscr{L}(\mathscr{X})$ such that $\sigma_a^{iso}(T) = \emptyset$, $\sigma(T) = \sigma_a(T)$ and $S \in \mathscr{L}(\mathscr{Y})$ satisfies property (gw) with $\sigma_a^{iso}(S) \cap \sigma_p(S) = \emptyset$, and $\Delta_a^g(T \oplus S) = \emptyset$. Then $T \oplus S$ satisfies property (gw).*

Proof Since S satisfies property (gw), therefore given condition $\sigma_a^{iso}(S) \cap \sigma_p(S) = \emptyset$ implies that $\sigma_a(S) = \sigma_{\mathrm{SBF}_+^-}(S)$. Now, $\Delta_a^g(T \oplus S) = \emptyset$ gives that $\sigma_{\mathrm{SBF}_+^-}(T \oplus S) = \sigma_a(T \oplus S) = \sigma_a(T) \cup \sigma_{\mathrm{SBF}_+^-}(S)$. Thus from Theorem 4.2, we have that $T \oplus S$ satisfies property (gw). □

Corollary 4.4 *Suppose that $T \in \mathscr{L}(\mathscr{X})$ such that $\sigma_a^{iso}(T) \cup \Delta_a^g(T) = \emptyset$ and $S \in \mathscr{L}(\mathscr{Y})$ satisfies property (gw). If $\sigma_{\mathrm{SBF}_+^-}(T \oplus S) = \sigma_{\mathrm{SBF}_+^-}(T) \cup \sigma_{\mathrm{SBF}_+^-}(S)$, then $T \oplus S$ satisfies property (gw).*

Theorem 4.5 *Suppose that generalized a-Browder's theorem holds for $T \in \mathscr{L}(\mathscr{X})$ and $S \in \mathscr{L}(\mathscr{Y})$. Suppose T and S are a-polaroid and $\sigma_{\mathrm{SBF}_+^-}(T \oplus S) = \sigma_{\mathrm{SBF}_+^-}(T) \cup \sigma_{\mathrm{SBF}_+^-}(S)$. Then property (gw) holds for $T \oplus S$.*

Proof If T and S are a-polaroid, then

$$\begin{aligned}\pi^a(T \oplus S) &= [\pi^a(T) \cap \rho_a(S)] \cup [\pi^a(S) \cap \rho_a(T)] \cup [\pi^a(T) \cap \pi^a(S)] \\ &= [E(T) \cap \rho_a(S)] \cup [E(S) \cap \rho_a(T)] \cup [E(T) \cap E(S)] \\ &= E(T \oplus S),\end{aligned}$$

where $\rho_a(.) = \mathbb{C} \setminus \sigma_a(.)$.

Since generalized a-Browder's theorem holds for T and S, then

$$\begin{aligned}&[\sigma_a(T) \cup \sigma_a(S)] \setminus [\sigma_{\mathrm{SBF}_+^-}(T) \cup \sigma_{\mathrm{SBF}_+^-}(S)] \\ &\qquad = [E(T) \cap \rho_a(S)] \cup [E(S) \cap \rho_a(T)] \cup [E(T) \cap E(S)].\end{aligned}$$

Thus, $[\sigma_a(T) \cup \sigma_a(S)] \setminus [\sigma_{\mathrm{SBF}_+^-}(T) \cup \sigma_{\mathrm{SBF}_+^-}(S)] = E(T \oplus S)$. Hence,

$$E(T \oplus S) = \sigma_a(T \oplus S) \setminus \sigma_{\mathrm{SBF}_+^-}(T \oplus S).$$

Therefore, property (gw) holds for $T \oplus S$. □

5 Generalized Weyl's theorem for M_C

In the following, let

$$\begin{aligned}
\Psi_+(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \quad \text{is upper semi-B-Fredholm}\},\\
\Psi_+^-(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \in \mathrm{SBF}_+(\mathscr{X}) \quad \text{and} \quad \mathrm{ind}(T - \lambda I) \le 0\},\\
\Psi_-(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \quad \text{is lower semi-B-Fredholm}\},\\
\Psi_-^+(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \in \mathrm{SBF}_-(\mathscr{X}) \quad \text{and} \quad \mathrm{ind}(T - \lambda I) \ge 0\},\\
\Psi(T) &= \Psi_+(T) \cap \Psi_-(T),\\
\Psi^0(T) &= \{\lambda \in \Psi(T) : \mathrm{ind}(T - \lambda I) = 0\}, \quad \text{and}\\
D(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \in D(\mathscr{X})\}.
\end{aligned}$$

Then the upper semiFredholm spectrum $\sigma_{\mathrm{SBF}_+}(T)$and the lower semiFredholm spectrum $\sigma_{\mathrm{SBF}_+}(T)$ of T are the sets

$$\begin{aligned}
\sigma_{\mathrm{SBF}_+}(T) &= \{\lambda \in \sigma(T) : \lambda \notin \Psi_+(T)\},\\
\sigma_{\mathrm{SBF}_-}(T) &= \{\lambda \in \sigma(T) : \lambda \notin \Psi_-(T)\}.
\end{aligned}$$

It is easily verified, see [26, Exercise 7, Page 293], that

$$\begin{aligned}
a(A - \lambda I) &\le a(M_C - \lambda I) \le a(A - \lambda I) + a(B - \lambda I);\\
d(A - \lambda I) &\le d(M_C - \lambda I) \le d(A - \lambda I) + d(B - \lambda I)
\end{aligned}$$

for every $\lambda \in \mathbb{C}$.

Lemma 5.1 ([3]) *Let $T \in \mathscr{L}(\mathscr{X})$ be an upper semi-B-Fredholm operator. If $\alpha(T) < \infty$, then T is an upper semi-Fredholm operator.*

Remark 5.2 The following implications hold [1, Theorem 3.4]: $a(T-\lambda I) < \infty \Rightarrow \alpha(T-\lambda I) \le \beta(T-\lambda I)$); $d(T - \lambda I)) < \infty \Rightarrow \beta(T - \lambda I) \le \alpha(T - \lambda I)$; if $\alpha(T - \lambda I) = \beta(T - \lambda I)$, then either $a(T - \lambda I) < \infty$ and $d(T - \lambda I) < \infty \Rightarrow a(T - \lambda I) = d(T - \lambda I) < \infty$. Furthermore, if both T and T^* have SVEP at λ, then $a(T - \lambda I) = d(T - \lambda I) < \infty$, $\lambda \in \sigma^{iso}(T)$ and λ is a pole of (the resolvent of) T [1, Corollary 3.21].

For an operator $S \in \mathscr{L}(\mathscr{X})$ and $\sigma_x(T)$ a subset of $\sigma(T)$, let

$$S_{\sigma_x(T)}(S) = \{\lambda \in \sigma(T) \setminus \sigma_x(T) : S \quad \text{does not have SVEP at } \lambda\}.$$

Remark 5.3 From [15,16], the Following relations hold:

$$\begin{aligned}
(i) \qquad \sigma(M_0) &= \sigma(A) \cup \sigma(B) = \sigma(M_C) \cup \{\sigma(A) \cap \sigma(B)\}\\
&= \sigma(M_C) \cup \left\{S_{\sigma_a(A)}(A^*) \cap S_{\sigma_a(B)}(B)\right\},\\
(ii) \qquad \sigma_b(M_0) &= \sigma_b(A) \cup \sigma_b(B) = \sigma_b(M_C) \cup \{\sigma_b(A) \cap \sigma_b(B)\}\\
&= \sigma_b(M_C) \cup \left\{S_{\sigma_b(M_C)}(A^*) \cap S_{\sigma_b(M_C)}(B)\right\}.\\
(iii) \qquad \sigma_w(A) \cup \sigma_w(B) &\subseteq \sigma_w(M_C) \cup \left\{S_{\sigma_w(M_C)}(P) \cup S_{\sigma_w(M_C)}(Q)\right\},\\
&\text{where} \quad (P, Q) = (A, A^*), (B, B^*), (A, B), \quad \text{or } (A^*, B^*).
\end{aligned}$$

Proposition 5.4 *If $\sigma_{\mathrm{BW}}(M_C) = \sigma_{\mathrm{BW}}(A) \cup \sigma_{\mathrm{BW}}(B)$, or $\sigma_{\mathrm{SBF}_+^-}(M_C) = \sigma_{\mathrm{SBF}_+^-}(A) \cup \sigma_{\mathrm{SBF}_+^-}(B)$, then $\sigma(M_C) = \sigma(A) \cup \sigma(B)$.*

Proof The first result ($\sigma_{\mathrm{BW}}(M_C) = \sigma_{\mathrm{BW}}(A) \cup \sigma_{\mathrm{BW}}(B)$ implies $\sigma(M_C) = \sigma(A) \cup \sigma(B)$) is in immediate consequence of Proposition 3.7 and Proposition 4.2 of [25].

Assume now that $\sigma_{\mathrm{SBF}_+^-}(M_C) = \sigma_{\mathrm{SBF}_+^-}(A) \cup \sigma_{\mathrm{SBF}_+^-}(B)$. If $\beta(A-\lambda I) = \alpha(B-\lambda I) \neq 0$, then $\mathrm{ind}(A-\lambda I) < 0$ and $\mathrm{ind}(B - \lambda I) > 0$. This, since already $\lambda \in \Psi_+(A) \cap \Psi_-(B)$, implies that $\lambda \in \Psi_+^-(A) \cap \Psi_-^+(B)$. Observe that if (also) $\lambda \notin \sigma_{\mathrm{SBF}_+^-}(B)$, then $\lambda \in \Psi^0(A) \cap \Psi^0(B)$ implies $\beta(A - \lambda I) = \alpha(B - \lambda I) = 0$. Consequently, $\lambda \in \sigma_{\mathrm{SBF}_+^-}(B)$. But then $\lambda \in \sigma(M_C)$—once again a contradiction. Hence, $\beta(A - \lambda I) = \alpha(B - \lambda I) = 0$ and so $\lambda \notin \sigma(A) \cup \sigma(B)$. □

Remark 5.5 If $\lambda \in \pi(A)\cup\pi(B)$, then the inequality $a(M_C-\lambda I) \leq a(A-\lambda I)+a(B-\lambda I)$ and $d(M_C-\lambda I) \leq d(A-\lambda I)+d(B-\lambda I)$ imply that $\lambda \in \pi(M_C)$. Observe that if $\lambda \in \pi(A)\cup\pi(B)$, then A, A^*, B and B^* have SVEP at λ.

Proposition 5.6 *(a) If either A^* or B has SVEP on $\pi(M_C)$, then $\lambda \in \pi(M_C)$ if and only if $\lambda \in \pi(A)\cup\pi(B)$. (b) If $\sigma_b(M_C) = \sigma_b(A)\cup\sigma_b(B)$ or $\sigma(M_C) = \sigma(A)\cup\sigma(B)$ or $\sigma_w(M_C) = \sigma_w(A)\cup\sigma_w(B)$, then $\lambda \in \pi(M_C)$ if and only if $\lambda \in \pi(A)\cup\pi(B)$.*

Proof (a). We have proven that $\lambda \in \pi(M_C)$ implies $\lambda \in \pi(A)\cup\pi(B)$. Let $\lambda \in \pi(M_C)$. Without loss of generality, we may assume that $\lambda = 0$. Then, M_C is of finite ascent and descent. Hence from [12, Lemma 2.1], we have A is of finite ascent and B is of finite descent. Also by duality, A^* is of finite descent and $B^*$is of finite ascent. For the sake of contradiction, assume that $0 \notin S(A^*)\cap S(B)$.

Case 1. $0 \notin S(A^*)$: Since M_C is Drazin invertible, then there exists $\epsilon > 0$ such that for every λ, $0 < |\lambda| < \epsilon$, $M_C-\lambda I$ is invertible. Hence, $A-\lambda I$ is left invertible. Thus, $0 \notin \sigma_a^{acc}(A) = \sigma_d^{acc}(A^*)$. If $0 \notin \sigma(A^*)$, then A^* is Drazin invertible and so A is. Now if $0 \in \sigma(A^*)$, since $\sigma(A^*) = S(A^*)\cup\sigma_d(A^*)$, then 0 is an isolated point of $\sigma(A^*)$. Now, A^* is of finite descent and $0 \in \sigma^{iso}(A^*)$ hence it follows from [26, Theorem 10.5] that A^* is Drazin invertible. Thus, A is Drazin invertible. Since M_C is Drazin invertible, it follows from [22, Lemma 2.7] that B is also Drazin invertible which contradicts our assumption. Therefore, $0 \in \pi(A)\cup\pi(B)$.

Case 2. If $0 \notin S(B)$, by similar argument in case 1, we have the result.
(b) Suppose that $\sigma_b(M_C) = \sigma_b(A)\cup\sigma_b(B)$, we have proved that $\lambda \in \pi(M_C)$ implies $\lambda \in \pi(A)\cup\pi(B)$. Without loss of generality,we suppose that $0 \in \pi(M_C)$, which implies $0 \in \rho(M_C)\cup\sigma^{iso}(M_C)$. Thus there exists $\epsilon > 0$ such that for every λ, $0 < |\lambda| < \epsilon$, $M_C-\lambda I$ is invertible and it is easy to prove that $B-\lambda I$ is right invertible. Therefore we have $\beta(B-\lambda I) = 0$. Moreover, since $M_C-\lambda I$ is invertible for every λ, $0 < |\lambda| < \epsilon$, then $\lambda \notin \sigma_b(M_C) = \sigma_b(A)\cup\sigma_b(B)$. Thus, $B-\lambda I$ is Browder. Therefore, $\alpha(B-\lambda I) = \beta(B-\lambda I) = 0$, that is, $B-\lambda I$ is invertible for every λ, $0 < |\lambda| < \epsilon$. Since $0 \in \pi(M_C)$, we have $d(B) < \infty$ from Lemma 2.6 of [22]. By Corollary 2.3 of [22] we know that B is Drzain invertible. Hence, A is also Drazin invertible from lemma 2.7 of [22]. So $0 \in \pi(A)\cup\pi(B)$. It shows that $\pi(M_C) \subseteq \pi(A)\cup\pi(B)$. From Lemma 2.7 of [22], we have $\sigma_D(M_C) \subseteq \sigma_D(A)\cup\sigma_D(B)$ for every $C \in \mathscr{L}(\mathscr{Y}.\mathscr{X})$. Hence if $\lambda \in \pi(A)\cup\pi(B)$, then $\lambda \notin \sigma_D(A)\cup\sigma_D(B)$ and so $\lambda \notin \sigma_D(M_C)$. Therefore, $\lambda \in \pi(M_C)$. If $\sigma(M_C) = \sigma(A)\cup\sigma(B)$ or $\sigma_w(M_C) = \sigma_w(A)\cup\sigma_w(B)$, similarly we can prove the result. □

The problem we consider in this section is that of finding necessary and/or sufficient conditions for the equivalence M_0 satisfies generalized Weyl's theorem $\Leftrightarrow$ M_C satisfies generalized Weyl's theorem to hold.

Theorem 5.7 *Assume A and B have SVEP and $\dim\chi_B(\{\lambda\}) < \infty$ for all $\lambda \in \sigma^{iso}(B)$. If generalized Weyl's theorem holds for M_0, then generalized Weyl's theorem holds for M_C for every $C \in \mathscr{L}(\mathscr{Y}, \mathscr{X})$.*

Proof Since A and B have SVEP, M_C has SVEP [18, Proposition 3.1], and so it follows from [13, Lemma 2.1] and [2, Theorem 2.1] that M_C obeys generalized Browder's theorem. Hence,

$$\sigma(M_C) \setminus \sigma_{BW}(M_C) = \pi(M_C) \subseteq E(M_C).$$

Let $\lambda \in E(M_C)$. Then, $\lambda \in \sigma^{iso}(M_C)$. By Lemma 2.3 of [14], $\lambda \in \sigma^{iso}(A)\cup\sigma^{iso}(B)$. Hence, $\lambda \in \sigma^{iso}(M_0)$. Since $\ker(A-\lambda I)\oplus\{0\} \subset \ker(M_C-\lambda I)$, $\dim\ker(A-\lambda I) < \infty$ (because $\lambda \in E(M_C)$) in the case in which $\lambda \in \sigma^{iso}(A)\cup\rho(A)$. Again, if $\lambda \in \sigma^{iso}(B)$, or $\lambda \in \rho(B)$, then the assumption that $\dim\chi_B(\{\lambda\}) < \infty$ implies that $\dim\ker(B-\lambda I) < \infty$, and hence that

$$\dim(\ker(A-\lambda I)\oplus\ker(B-\lambda I)) < \infty$$

Evidently, the non-triviality of $\ker(M_C-\lambda I)$ implies that $\ker(A-\lambda I)\cup\ker(B-\lambda I) \neq \{0\}$, i.e., $0 < \dim(\ker(A-\lambda I)\oplus\ker(B-\lambda I))$. Hence, $\lambda \in \sigma^{iso}(M_0)$ and

$$0 < \dim(\ker(A-\lambda I)\oplus\ker(B-\lambda I)) < \infty,$$

i.e., $\lambda \in E(M_0) = \sigma(M_0)\setminus\sigma_{BW}(M_0)$. □

Theorem 5.8 *If $\sigma_{BW}(M_C) = \sigma_{BW}(A)\cup\sigma_{BW}(B)$, then the equivalence*

$$M_0 \text{ satisfies generalized Weyl's theorem} \Leftrightarrow M_C \text{ satisfies generalized Weyl's theorem}$$

holds if and only if $E(M_0) = E(M_C)$.

Proof The hypothesis $\sigma_{\mathrm{BW}}(M_{\mathrm{C}}) = \sigma_{\mathrm{BW}}(A) \cup \sigma_{\mathrm{BW}}(B)$ implies that $\sigma_{\mathrm{BW}}(M_0) = \sigma_{\mathrm{BW}}(M_{\mathrm{C}})$ and $\sigma(M_0) = \sigma(M_{\mathrm{C}})$ [25, Proposition 3.7, Proposition 4.2]. Suppose that M_0 satisfies generalized Weyl's theorem; then, M_0 satisfies generalized Browder's theorem and so

$$\sigma(M_{\mathrm{C}}) \setminus \sigma_{\mathrm{BW}}(M_{\mathrm{C}}) = \sigma(M_0) \setminus \sigma_{\mathrm{BW}}(M_0) = E(M_0) = \pi(M_0) = \pi(M_{\mathrm{C}}) \subseteq E(M_{\mathrm{C}}).$$

Again, if M_{C} satisfies generalized Weyl's theorem, then M_{C} satisfies generalized Browder's theorem and so

$$\sigma(M_0) \setminus \sigma_{\mathrm{BW}}(M_0) = \sigma(M_{\mathrm{C}}) \setminus \sigma_{\mathrm{BW}}(M_{\mathrm{C}}) = E(M_{\mathrm{C}}) = \pi(M_{\mathrm{C}}) = \pi(M_0) \subseteq E(M_0).$$

Thus, the statements of the theorem are equivalent if and only if $E(M_0) = E(M_{\mathrm{C}})$. □

Corollary 5.9 *If $\sigma_{\mathrm{BW}}(M_0) = \sigma_{\mathrm{BW}}(M_{\mathrm{C}})$, A is isoloid and satisfies generalized Weyl's theorem, then M_0 satisfies generalized Weyl's theorem implying M_{C} satisfies generalized Weyl's theorem.*

Proof We prove that if the hypotheses of the corollary are satisfied, then $\sigma_{\mathrm{BW}}(M_{\mathrm{C}}) = \sigma_{\mathrm{BW}}(A) \cup \sigma_{\mathrm{BW}}$ and $E(M_0) = E(M_{\mathrm{C}})$; the proof of the corollary would them follow from Theorem 5.8 that $\pi_0(M_0) = \pi_0(M_{\mathrm{C}})$. The hypothesis A satisfies generalized Weyl's theorem implying that $\sigma(A) \setminus \sigma_{\mathrm{BW}}(A) = \pi(A) = E(A)$ (so that both A and A^* have SVEP on $\Delta^g(A)$. If $\lambda \in \Delta^g(A)$, then M_0 satisfies generalized Weyl's theorem, which implies that $\lambda \in E(M_0)$. Hence, $\lambda \in \sigma^{iso}(A) \cup \rho(A)$ and $\alpha(A - \lambda I) < \infty$. By hypothesis, A is isoloid; hence $\lambda \in E(A)$, which implies that both A and A^* have SVEP on $\Delta^g(M_0)$. Since $\lambda \in \Delta^g(M_0)$, and A and A^* have SVEP at λ, it implies that $A - \lambda I$ and $B - \lambda I$ are B-Weyl, and it follows that $\sigma_{\mathrm{BW}}(M_0) = \sigma_{\mathrm{BW}}(M_{\mathrm{C}}) = \sigma_{\mathrm{BW}}(A) \cup \sigma_{\mathrm{BW}}(B)$, which is implied by [25, Proposition 3.7, Proposition 4.2] that $\sigma(M_{\mathrm{C}}) = \sigma(A) \cup \sigma(B)$. Again, since A and A^* have SVEP on $\Delta^g(M_0) = \Delta^g(M_))$, $\pi(M_0) = \pi(M_{\mathrm{C}})$. Hence, $E(M_0) = \pi(M_0) = \pi(M_{\mathrm{C}}) \subseteq E(M_{\mathrm{C}})$. Finally, since $\sigma^{iso}(M_{\mathrm{C}}) = \sigma^{iso}(A) \cup \sigma^{iso}(B)$, $\lambda \in E(M_{\mathrm{C}})$ implies that $\lambda \in E(A) \cup E(B) = E(M_0)$. Hence, $E(M_0) = E(M_{\mathrm{C}})$. □

Theorem 5.10 *Suppose that $A \in \mathscr{L}(\mathscr{X})$ and $B \in \mathscr{L}(\mathscr{Y})$ are polaroid and satisfy generalized Browder's theorem. If $\sigma_{\mathrm{BW}}(M_{\mathrm{C}}) = \sigma_{\mathrm{BW}}(A) \cup \sigma_{\mathrm{BW}}(B)$, then M_{C} satisfies generalized Weyl's theorem.*

Proof Evidently, $\sigma_{\mathrm{BW}}(M_{\mathrm{C}}) = \sigma_{\mathrm{BW}}(M_0) = \sigma_{\mathrm{BW}}(A) \cup \sigma_{\mathrm{BW}}(B)$ implies that $\sigma(M_{\mathrm{C}}) = \sigma(A) \cup \sigma(B)$. The hypothesis A is polaroid and implies that $E(A) = \pi(A)$, and the hypotheses B is polaroid and implies that $E(B) = \pi(B)$, which imply that $E(M_0) \subseteq \pi(M_0)$; hence, since A and B satisfy generalized Browder's theorem, it implies that M_0 (has SVEP on $\Delta^g(M_0) = \Delta^g(A) \cap \Delta^g(B) = \pi(A) \cap \pi(B)$ implies M_0) satisfies generalized Browder's theorem, and A and M_0 satisfy generalized Weyl's theorem. Since A is evidently isoloid, the proof follows from Corollary 5.9. □

Proposition 5.11 *Let $\Delta_x(T) = \sigma(T) \setminus \sigma_x(T)$. Suppose that $A \in \mathscr{L}(\mathscr{X})$ and $B \in \mathscr{L}(\mathscr{Y})$. Then,*

$$\sigma_D(A) \cup \sigma_D(B) = \sigma_D(M_{\mathrm{C}}) \cup \left\{S_{\sigma_D(M_{\mathrm{C}})}(A^*) \cap S_{\sigma_D(M_{\mathrm{C}})}(B)\right\}$$

for every $C \in \mathscr{L}(\mathscr{Y}, \mathscr{X})$.

Proof The implications

$$\begin{aligned}\lambda \notin \sigma_D(A) \cup \sigma_D(B) &\Leftrightarrow \lambda \in \Delta_D(A) \cap \Delta_D(B)\\ &\Leftrightarrow \lambda \in D(A) \cap D(B), a(A - \lambda I) = d(A - \lambda I) < \infty,\\ &a(B - \lambda I) = d(B - \lambda I) < \infty\\ &\Leftrightarrow \lambda \in D(M_{\mathrm{C}}), a(M_{\mathrm{C}} - \lambda I) = d(M_{\mathrm{C}} - \lambda I) < \infty,\\ &A^* \text{has SVEP at} \lambda \text{or} B \text{has SVEP at} \lambda\end{aligned}$$

show that

$$\sigma_D(A) \cup \sigma_D(B) = \sigma_D(M_{\mathrm{C}}) \cup \left\{S_{\sigma_D(M_{\mathrm{C}})}(A^*) \cap S_{\sigma_D(M_{\mathrm{C}})}(B)\right\}.$$

□

Theorem 5.12 *(i) If $\sigma_{\mathrm{SBF}_+^-}(M_0) = \sigma_{\mathrm{SBF}_+^-}(M_{\mathrm{C}})$, then $M_0 \in \mathrm{ga}\mathcal{W}$ implies $M_{\mathrm{C}} \in \mathrm{ga}\mathcal{W}$ if and only if $E_a(M_{\mathrm{C}}) \subseteq E_a(M_0)$.*
(ii) If $\sigma_{\mathrm{SBF}_+^-}(M_0) = \sigma_{\mathrm{SBF}_+^-}(M_{\mathrm{C}})$ and A^* has SVEP on $\Delta_a^g(M_{\mathrm{C}})$, then $M_{\mathrm{C}} \in \mathrm{ga}\mathcal{W}$ implies $M_0 \in \mathrm{ga}\mathcal{W}$ if and only if $E_a(M_0) \subseteq E_a(M_{\mathrm{C}})$.

Proof (i). Since $M_0 \in \text{ga}\mathcal{W}$ implies $M_0 \in \text{ga}\mathcal{B}$, A and B have SVEP on $\Delta_a^g(M_C) = \Delta_a^g(M_0) = \Delta_a^g(A) \cup \Delta_a^g(B)$. Hence, it follows from Theorem 4.12 of [16] and Theorem 2.2 of [2] that $M_C \in \text{ga}\mathcal{B}$. Thus, $\lambda \in \pi^a(M_C) \Leftrightarrow \lambda \in \Delta_a^g(M_C) = \Delta_a^g(M_0) = \pi^a(M_0)$. Since $\pi^a(M_C) \subseteq E_a(M_C)$, it follows that

$$\sigma_a(M_C) \setminus \sigma_{\text{SBF}_+^-}(M_C) = \pi^a(M_C) = \pi^a(M_0) = E_a(M_0) \subseteq E_a(M_C),$$

which proves that $M_C \in \text{ga}\mathcal{W}$ if and only if $E_a(M_C) \subseteq E_a(M_0)$.
(ii). Since $M_C \in \text{ga}\mathcal{B}$, it implies that A has SVEP on $\Delta_a^g(M_C)$. Assume that A^* has SVEP on $\Delta_a^g(M_C)$. We prove that $\sigma_a(M_C) = \sigma_a(A) \cup \sigma_a(B)$. If $\mu \notin \sigma_a(M_C)$, then $M_C - \mu I$ and $A - \mu I$ are left invertible, $\mu \in \Delta_a^g(M_C)$. The left invertibility of $A - \mu I$ implies the right invertibility of $A^* - \mu I^*$; hence, since A^* has SVEP on $\Delta_a^g(M_C)$, $A^* - \mu I^*$. But then the invertibility of $A - \mu I$, taken along with the left invertibility of $M_C - \mu I$, implies that $B - \mu I$ is left invertible. Hence, $\mu \notin \sigma_a(A) \cup \sigma_a(B)$. Assume now that $M_C \in \text{ga}\mathcal{B}$. Then, $\lambda \in \Delta_a^g(M_C)$ implies that $\lambda \in \sigma_a^{iso}(M_C) = \sigma_a^{iso}(A) \cup \sigma_a^{iso}(B)$; hence A and B have SVEP on $\Delta_a^g(M_0) = \Delta_a^g(M_C)$. So, M_0 has SVEP on $\Delta_a^g(M_0)$, and hence $M_0 \in \text{ga}\mathcal{B}$. Therefore,

$$\sigma(M_0) \setminus \sigma_{\text{BW}}(M_0) = \sigma(M_C) \setminus \sigma_{\text{BW}}(M_C) = \pi^a(M_C) = \pi^a(M_0) = E_a(M_C) \subseteq E_a(M_0),$$

where the equality $\pi^a(M_0) = \pi^a(M_C)$ follows from the implications $\lambda \in \pi^a(M_C) \Leftrightarrow \lambda \in \Delta_a^g(M_C) = \Delta_a^g(M_0) \Leftrightarrow \lambda \in \pi^a(M_0)$. Hence, $M_0 \in \text{ga}\mathcal{W}$ if and only if $E_a(M_0) \subseteq E_a(M_C)$. □

Corollary 5.13 *If $\sigma_{\text{SBF}_+^-}(A) = \sigma_{\text{SBF}_+}(B)$, A is a-isoloid and $A \in \text{ga}\mathcal{W}$, then $M_0 \in \text{ga}\mathcal{W}$ implies $M_C \in \text{ga}\mathcal{W}$.*

Proof Start by observing that if $\lambda \in \Psi_+^-(M_C)$ and $\text{ind}(A - \lambda I) > 0$, then $\lambda \in \Psi(A) \cap \Psi_+(B)$ and $\text{ind}(A - \lambda I) + \text{ind}(B - \lambda I) \le 0$; if, instead, $\text{ind}(A - \lambda I) \le 0$, then $\sigma_{\text{SBF}_+^-}(A) = \sigma_{\text{SBF}_+}(B)$ and $\lambda \in \Psi_+^-(M_C)$ imply that $\lambda \in \Psi_+^-(A) \cap \psi_+(B)$ and $\text{ind}(A - \lambda I) + \text{ind}(B - \lambda I) \le 0$. In either case, $\lambda \in \psi_+^-(M_C)$ implies $\lambda \in \Psi_+^-(M_0)$; hence $\sigma_{\text{SBF}_+^-}(M_C) = \sigma_{\text{SBF}_+^-}(M_0)$. In view of Theorem 5.12, we are thus left to prove that $E_a(M_C) \subseteq E_a(M_0)$. If $\lambda \in E_a(M_C)$, then $\lambda \in \sigma_a^{iso}(A) \cup \sigma_a^{iso}(B)$, and so $\lambda \in E_a(A) = \Delta_a^g(A) = \sigma_a(B) \setminus \sigma_{\text{SBF}_+}(B)$ (since A is a-isoloid, $A \in \text{ga}\mathcal{W}$ and $\sigma_{\text{SBF}_+^-}(A) = \sigma_{\text{SBF}_+}(B)$. But then, since $M_0 \in \text{ga}\mathcal{B}$ implies B has SVEP at λ, $\lambda \in \pi^a(B)$. Hence $\lambda \in \pi^a(M_0) = E_a(M_0)$. □

Remark 5.14 If A^* has SVEP, then $\lambda \in \Delta_a^g(M_C)$ implies $\lambda \in \Psi(A) \cap \Psi_+^-(B)$, $\text{ind}(A - \lambda I) \ge 0$ and $\text{ind}(A - \lambda I) + \text{ind}(B - \lambda I) \le 0$; this in turn implies that $\lambda \notin \sigma_{\text{SBF}_+^-}(A) \cup \sigma_{\text{SBF}_+^-}(B)$. Thus, if A^* has SVEP and $M_0 \in \text{ga}\mathcal{B}$, then

$$\sigma_{\text{SBF}_+^-}(M_0) = \sigma_{\text{SBF}_+^-}(A) \cup \sigma_{\text{SBF}_+^-}(B) = \sigma_{\text{SBF}_+^-}(M_C).$$

Corollary 5.15 *If $\sigma_a(A^*)$ has empty interior, A is a-isoloid and $A \in \text{ga}\mathcal{W}$, then $M_0 \in \text{ga}\mathcal{W}$ implies $M_C \in \text{ga}\mathcal{W}$.*

Proof Evidently, A^* has SVEP, $M_0 \in \text{ga}\mathcal{B}$ and $\sigma_{\text{SBF}_+^-}(M_0) = \sigma_{\text{SBF}_+^-}(M_C)$. In view of Theorem 5.12, we are thus left to prove that $E_a(M_C) \subseteq E_a(M_0)$. If $\lambda \in E_a(M_C)$, then $\lambda \in \sigma_a^{iso}(A) \cup \sigma_a^{iso}(B)$, and so $\lambda \in E_a(A) = \Delta_a^g(A) = \sigma_a(B) \setminus \sigma_{\text{SBF}_+}(B)$ (since A is a-isoloid, $A \in \text{ga}\mathcal{W}$ and $\sigma_{\text{SBF}_+^-}(A) = \sigma_{\text{SBF}_+}(B)$. But then, since $M_0 \in \text{ga}\mathcal{B}$ implies B has SVEP at λ, $\lambda \in \pi^a(B)$. Hence, $\lambda \in \pi^a(M_0) = E_a(M_0)$. □

For an operator $T \in \mathscr{L}(\mathscr{X})$ such that T^* has SVEP, T satisfies generalized Weyl's theorem if and only if T satisfies generalized a-Weyl's theorem [3, Theorem 2.7]. Thus, if A^* and B^* have SVEP, then M_X^* has SVEP, and the (two way) implication M_X satisfies generalized Weyl's theorem if and only if M_X satisfies generalized a-Weyl's theorem, where $M_X = M_0$ or $M_X = M_C$. The following theorem proves more.

Theorem 5.16 *If $S_{\sigma_{\text{SBF}_+}(A)}(A^*) \cup S_{\sigma_{\text{SBF}_+}(B)}(B^*) = \emptyset$, then M_C satisfies generalized Weyl's theorem if and only M_C satisfies generalized a-Weyl's theorem if and only if M_C satisfies property (gw).*

Proof The implication M_C satisfies generalized a-Weyl's theorem or M_C satisfies property (gw) implies M_C satisfies generalized Weyl's theorem being clear, we prove the reverse implication. For this, it would suffice to prove that $\sigma(M_C) = \sigma_a(M_C)$ (which would then imply $E(M_C) = E_a(M_C)$ and $\sigma_{\text{BW}}(M_C) = \sigma_{\text{SBF}_+^-}(M_C)$).

Evidently, $\sigma_a(M_C) \subseteq \sigma(M_C)$. Let $\lambda \notin \sigma_a(M_C)$. Then, $M_C - \lambda I$ and $A - \lambda I$ are bounded below. The boundedness below of $A - \lambda I$ implies $\lambda \in \Psi_+(A)$. Since A^* has SVEP at points $\lambda \in \Psi_+(A)$, it follows that $A - \lambda I)$ is invertible. But then $B - \lambda I$ is bounded below, which (because B^* has SVEP at points $\lambda \in \Psi_+(B)$

implies that $B - \lambda I$ is invertible. Thus, $\lambda \notin \sigma(A) \cup \sigma(B)$, i.e., $\sigma(M_C) \subseteq \sigma(A) \cup \sigma(B) \subseteq \sigma_a(M_C)$. Next, we prove that $\sigma_{BW}(M_C) \subseteq \sigma_{SBF_+^-}(M_C)$: this would then imply the equality $\sigma_{BW}(M_C) = \sigma_{SBF_+^-}(M_C)$. Let $\lambda \notin \sigma_{SBF_+^-}(M_C)$; then, $\lambda \in \Psi_+(A)$ (and $\text{ind}(A - \lambda I) + \text{ind}(B - \lambda I) \leq 0$). Since A^* has SVEP at points $\lambda \in \Psi_+(A)$, it follows that $\text{ind}(A - \lambda I) \geq 0)$ implies $\lambda \in \Psi(A)$ (with $\text{ind}(A - \lambda I) \geq 0$). Since this forces $\lambda \in \Psi_+(B)$, it follows (from the hypothesis B^* has SVEP on the set of $\lambda \in \Psi_+(B)$) that $\lambda \in \Psi(B)$ and $\text{ind}(B - \lambda I) \geq 0$. Since $\text{ind}(A - \lambda I) + \text{ind}(B - \lambda I) \leq 0$, we conclude that $\lambda \in \Psi^0(A) \cap \Psi^0(B)$. Hence $\sigma_{BW}(M_C) \subseteq \sigma_{BW}(A) \cup \sigma_{BW}(B) \subseteq \sigma_{SBF_+^-}(M_C)$, and the proof is achieved. □

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References

1. Aiena, P.: Fredholm and Local Spectral Theory with Applications to Multipliers. Kluwer, Boston (2004)
2. Amouch, M.; Zguitti, H.: On the equivalence of Browder's and generalized Browder's theorem. Glasgow Math. J. **48**, 179–185 (2006)
3. Amouch, M.; Berkani, M.: On the property (gw). Mediterr. J. Math. **5**, 371–378 (2008)
4. Amouch, M.; Zguitti, H.: B-Fredholm and Drazin invertible operators through localized SVEP. Math. Bohemica **136**, 39–49 (2011)
5. Barnes, B.A.: Riesz points of upper triangular operator matrices. Proc. Am. Math. Soc. **133**, 1343–1347 (2005)
6. Berkani, M.: On a class of quasi-Fredholm operators. Integral Equations Oper. Theory **34**, 244–249 (1999)
7. Berkani, M.: Index of Fredholm operators and generalization of a Weyl theorem. Proc. Am. Math. Soc. **130**, 1717–1723 (2002)
8. Berkani, M.; Koliha, J.: Weyl type theorems for bounded linear operators. Acta Sci. Math. (Szeged) **69**, 359–376 (2003)
9. Bosch, C.; Hernandez, C.; De Oteyza, E.; Pearcy, C.: Spectral pictures of functions of operators. J. Oper. Theory **8**, 391–400 (1982)
10. Cao, X.H.: a-Browder's theorem and generalized a-Weyl's theorem. Acta Math. Sinica **23**, 951–960 (2007)
11. Cao, X.; Guo, M.; Meng, B.: Weyl's theorem for upper triangular operator matrices. Linear Algebra Appl. **402**, 61–73 (2005)
12. Djordjević, S.V.; Zguitti, H.: Essential point spectra of operator matrices through local spectral theory. J. Math. Anal. Appl. **338**, 285–291 (2008)
13. Duggal, B.P.; Djordjević, S.V.: Dunford's property (C) and Weyl's theorems. Integral Equations Oper. Theory **43**, 290–297 (2002)
14. Duggal, B.P.: Hereditarily normaloid operators. Extracta Math. **20**, 203–217 (2005)
15. Duggal, B.P.: Upper triangular operator matrices, SVEP and Browder, Weyl theorems. Integral Equations Oper. Theory **63**, 17–28 (2009)
16. Duggal, B.P.: Browder and Weyl spectra of upper triangular operator matrices. Filomat **24**, 111–130 (2010)
17. Gupta, A.; Kashyap, N.: Property (Bw) and Weyl type theorems. Bull. Math. Anal. Appl. **3**, 1–7 (2011)
18. Houimdi, M.; Zguitti, H.: Propriétés spectrales locales d'une matrice carrée des opérateurs. Acta Math. Vietnam. **25**, 137–144 (2000)
19. Lee, W.Y.: Weyl's theorem for operator matrices. Integral Equation Oper. Theory **32**, 319–331 (1998)
20. Oudghiri, M.: a-Weyl's theorem and the single valued extension property. Exctracta Math. **21**(1), 41–50 (2006)
21. Rakočevic, V.: Operatorsobeying a-Weyl's theorem. Rev. Roumaine Math. Pures Appl. **10**, 915–919 (1986)
22. Zhang, S.; Zhong, H.; Jiang, Q.: Drazin spectrum of operator matrices on the Banach space. Linear Alg. Appl. **429**, 2067–2075 (2008)
23. Zguitti, H.: A note on generlized Weyl's theorem. J. Math. Anal. Appl. **316**, 373–381 (2006)
24. Zguitti, H.: On the Drazin inverse for upper triangular operator matrices. Bull. Math. Anal. Appl. **2**, 27–33 (2010)
25. Zguitti, H.: A note on Drazin invertibility for upper triangular block operators. Mediterr. J. Math. **10**, 1497–1507 (2013)
26. Taylor, A.E.; Lay, D.C.: Introduction to Functional Analysis. Wiley, New York (1980)