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On dynamic coloring of certain cycle-related graphs

Received: 31 January 2018 / Accepted: 7 August 2018 / Published online: 23 August 2018
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Abstract Coloring the vertices of a particular graph has often been motivated by its utility to various applied fields and its mathematical interest. A dynamic coloring of a graph G is a proper coloring of the vertex set $V(G)$ such that for each vertex of degree at least 2, its neighbors receive at least two distinct colors. A dynamic k -coloring of a graph is a dynamic coloring with k colors. A dynamic k -coloring is also called a conditional $(k, 2)$ -coloring. The smallest integer k such that G has a dynamic k -coloring is called the dynamic chromatic number $\chi_d(G)$ of G . In this paper, we investigate the dynamic chromatic number for the line graph of sunlet graph and middle graph, total graph and central graph of sunlet graphs, paths and cycles. Also, we find the dynamic chromatic number for Mycielskian of paths and cycles and the join graph of paths and cycles.

Mathematics Subject Classification 05C15

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المخلص

إن تلوين رؤوس رسم بياني معيّن عادة ما يحفز من خلال فائدته في مختلف الميادين التطبيقية والاهتمامات الرياضية. فالتلوين الديناميكي (الحركي) للرسم البياني G هو تلوين مناسب لمجموعة الرؤوس $V(G)$ ، بحيث أن لكل رأس من الدرجة 2 على الأقل مجاور بلونين مختلفين على الأقل. إن التلوين الديناميكي من الصنف k (k -تلوين ديناميكي) لرسم بياني هو تلوين ديناميكي بـ k لونا ويطلق عليه أيضا إسم $(k;2)$ -تلوين مشروط. ويسمى أصغر عدد صحيح k ، بحيث أن للرسم البياني G k -تلوين ديناميكي، العدد اللوني الديناميكي $\chi_d(G)$ لـ G . في هذا المقال، نقوم بالبحث عن العدد اللوني الديناميكي للرسم البياني سانليت والرسم البياني الوسيط، والرسم البياني الكلي، والرسم البياني المركزي، والمسارات والدوريات. أيضا نجد العدد اللوني الديناميكي للمسارات والدوريات المائيسيسكية والرسم البياني الرابط للمسارات والدوريات.

1 Introduction

Throughout this paper, all graphs are finite and simple. The dynamic chromatic number was first introduced by Montgomery [14]. A dynamic coloring is defined as a proper coloring in which any multiple degree vertex is adjacent to more than one color class. A dynamic coloring is thus a map c from V to the set of colors such that

- If $uv \in E(G)$, then $c(u) \neq c(v)$, and
- For each vertex $v \in V(G)$, $|c(N(v))| \geq \min\{2, d(v)\}$.

The first condition characterizes proper colorings; the adjacency condition and second condition are double-adjacency condition. The dynamic chromatic number $\chi_d = \chi_d(G)$ is the minimum k for which G has a dynamic k -coloring.

The dynamic chromatic number, $\chi_d(G)$, has been investigated in several papers, see, [1–4, 6, 9, 14, 16]. In 2001, Montgomery [14] conjectured that for a regular graph G , $\chi_d(G) - \chi(G) \leq 2$. Akbari et al. [2] proved this conjecture for bipartite regular graphs. Some upper bounds for the dynamic chromatic number of graphs have been studied in recent years. In 2017, Bowler et al. [5] disproved Montgomery's conjecture on dynamic coloring of regular graphs. In [12, 13], Mohanapriya et al. studied the δ -dynamic chromatic number of helm and fan graph families. There are many upper bounds and lower bounds for $\chi_d(G)$ in terms of graph parameters. For example,

Theorem A [14] *Let G be a graph with maximum degree $\Delta(G)$. Then, $\chi_d(G) \leq \Delta(G) + 3$.*

In this regard, for a graph G with $\Delta(G) \geq 3$, it was proved that $\chi_d(G) \leq \Delta(G) + 1$ [9]. Also, for a regular graph G , it was shown by Alishahi:

Theorem B [4] *If G is an r -regular graph, then $\chi_d(G) \leq \chi(G) + 14.06 \log r + 1$.*

Another upper bound on $\chi_d(G)$ is $\chi_d(G) \leq 1 + l(G)$, where $l(G)$ is the length of a longest path in G [14].

Theorem C [8] *If G is a connected planar graph with $G \neq C_5$, then $\chi_d(G) \leq 4$.*

Alishahi [4] proved that for every graph G with $\chi(G) \geq 4$, $\chi_d(G) \leq \chi(G) + \gamma(G)$, where $\gamma(G)$ is the domination number of a graph G .

Another upper bound for the dynamic chromatic number of a d -regular graph G in terms of $\chi(G)$ and the independence number of G , $\alpha(G)$, was introduced in [6]. In fact, it was proved that $\chi_d(G) \leq \chi(G) + 2 \log_2 \alpha(G) + 3$.

In [11], it has been proved that determining $\chi_d(G)$ for a 3-regular graph is an NP-complete problem. Furthermore, in [10] it is shown that it is NP-complete to determine whether there exists a 3-dynamic coloring for a claw free graph with the maximum degree 3.

Theorem D [14]

$$\chi_d(C_n) = \begin{cases} 5; & \text{for } n = 5 \\ 3; & \text{for } n = 3k, \forall k \geq 1 \\ 4; & \text{otherwise.} \end{cases}$$

2 Preliminaries

When it is required for an edge $uv = e \in E(G)$ to be represented by a vertex, such vertex will be denoted by e' .

The line graph [7] of a graph G , denoted by $L(G)$, is the graph in which all edges $e_i \in E(G)$ are represented by $e'_i \in V(L(G))$ and an edge $e'_i e'_j \in E(L(G))$ if and only if the edges e_i, e_j share a vertex (are incident) in G .

The middle graph of G , denoted by $M(G)$, is defined as follows. The vertex set of $M(G)$ is $V(G) \cup E(G)$. Two vertices x, y in the vertex set of $M(G)$ are adjacent in $M(G)$ in case one of the following holds:

- (i) x, y are in $E(G)$ and x, y are adjacent in G .
- (ii) x is in $V(G)$, y is in $E(G)$, and x, y are incident in G .

The total graph [7] of G denoted by $T(G)$ has vertex set $V(G) \cup E(G)$ and edges joining all elements of this vertex set which are adjacent or incident in G .

The central graph [17] of a graph G denoted by $C(G)$ is formed by subdividing each edge of G by a vertex and joining each pair of vertices of the original graph which were previously non-adjacent.

The path graph P_n is a tree with two vertices of degree 1 and the other $n - 2$ vertices of degree 2.

A cycle graph $C_n, n \geq 3$ (also called a cycle for brevity) is a closed path, i.e., $v_0 = v_n$.

An n -sunlet graph (also called a sunlet graph for brevity) $S_n, n \geq 3$ on $2n$ vertices is obtained by attaching a pendant vertex $u_i, 1 \leq i \leq n$ to each vertex v_i of the cycle C_n . The pendant edges are correspondingly labeled $s_i, 1 \leq i \leq n$.

2.1 Mycielskian graph $\mu(G)$ of a graph G

Mycielski [15] introduced an interesting graph transformation in 1955. The transformation can be described as follows:

- (1) Consider any simple connected graph G on $n \geq 2$ vertices labeled $v_1, v_2, v_3, \dots, v_n$.
- (2) Consider the extended vertex set $V(G) \cup \{x_1, x_2, x_3, \dots, x_n\}$ and add the edges $\{v_i x_j : \text{if and only if } v_i v_j \in E(G)\}$.
- (3) Add one more vertex w together with the edges $\{w x_i : \forall i\}$.

Formally stated, the transformed graph (Mycielskian graph of G or Mycielski G) denoted by $\mu(G)$ is the simple connected graph with $V(\mu(G)) = V(G) \cup \{x_1, x_2, x_3, \dots, x_n\} \cup \{w\}$ and $E(\mu(G)) = E(G) \cup \{v_i x_j : \text{if and only if } v_i v_j \in E(G)\} \cup \{w x_i : \forall i\}$.

The join [7] $G = G_1 + G_2$ of graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and disjoint edge sets E_1 and E_2 is the graph union $G_1 \cup G_2$ together with all the edges joining V_1 and V_2 .

The work we present in this paper investigates the dynamic chromatic number for the line graph of sunlet graph and middle graph, total graph and central graph of sunlet graphs, paths and cycles. Also, we find the dynamic chromatic number for Mycielskian of paths and cycles and the join graph of paths and cycles.

3 Dynamic coloring of sunlet graph-related graphs

Theorem 3.1 For a sunlet graph $S_n, \chi_d(L(S_n)) = 3$.

Proof Let $V(S_n) = \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\}$ and $E(S_n) = \{e_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq n\}$. Let vertices e'_i and $s'_i, 1 \leq i \leq n$ be represent in $L(S_n)$ the edges e_i and s_i , respectively. By the definition of the line graph corresponding to S_n , assign a proper 3-coloring to $V(L(S_n))$ as follows:

Case 1: If $n \equiv 0 \pmod{3}$

Assign colors $1, 2, 3, 1, 2, 3, \dots, 1, 2, 3$ to consecutive vertices e'_i , $1 \leq i \leq n$ and colors $2, 3, 1, 2, 3, 1, \dots, 2, 3, 1$ to consecutive vertices of s'_i , $1 \leq i \leq n$. An easy check shows that this is a dynamic 3-coloring.

Case 2: If $n \equiv 1 \pmod{3}$

Assign colors $1, 2, 3, 1, 2, 3, \dots, 1, 2, 3, 2$ to consecutive vertices of e'_i , $1 \leq i \leq n$, and colors $3, 3, 1, 2, 3, 1, \dots, 3, 1, 1$ to consecutive vertices of s'_i , $1 \leq i \leq n$. An easy check shows that this is a dynamic 3-coloring.

Case 3: If $n \equiv 2 \pmod{3}$

Assign colors $1, 2, 3, 1, 2, 3, \dots, 1, 2, 3, 1, 2$ to consecutive vertices of e'_i , $1 \leq i \leq n$, and colors $3, 3, 1, 2, 3, \dots, 1, 2, 3$ to consecutive vertices of s'_i , $1 \leq i \leq n$. An easy check shows that this is a dynamic 3-coloring.

From the cases above, it follows that: $\chi_d(L(S_n)) \leq 3$.

Without loss of generality, note that the three consecutive vertices of the path $e'_1 s'_2 e'_2$ must be colored differently in any dynamic coloring of $L(S_n)$, since the first and third vertices are the only neighbors of the second vertex and must be colored differently (by double-adjacency conditions) and also differently from the second vertex. Therefore, $\chi_d(L(S_n)) \geq 3$. Hence, $\chi_d(L(S_n)) = 3$. $\square$

Theorem 3.2 For a sunlet graph S_n , $\chi_d(M(S_n)) = 4$.

Proof Let $V(S_n) = \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\}$ and $E(S_n) = \{e_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq n\}$.

Let vertices e'_i and s'_i , $1 \leq i \leq n$ represent the edges e_i and s_i , respectively. By the definition of the middle graph

$$V(M(S_n)) = V(S_n) \cup E(S_n) = \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq n\} \cup \{s'_i : 1 \leq i \leq n\},$$

Assign a proper 4-coloring to $V(M(S_n))$ as follows:

Case 1: If $n \equiv 0 \pmod{2}$

For $1 \leq i \leq n$, assign colors $1, 1, 1, 1, 1, 1, \dots, 1, 1$ to consecutive vertices of v_i , colors $2, 3, 2, 3, \dots, 2, 3$ to consecutive vertices of e'_i , colors $3, 3, 3, 3, \dots, 3, 3$ to consecutive vertices of u_i and colors $4, 4, 4, 4, \dots, 4, 4$ to consecutive vertices of s'_i .

Case 2: If $n \equiv 1 \pmod{2}$

For $1 \leq i \leq n$, assign colors $1, 1, 1, 1, \dots, 1, 1, 3, 2$ to consecutive vertices of v_i , colors $2, 3, 2, 3, \dots, 2, 3, 2, 1, 3$ to consecutive vertices of e'_i , colors $3, 3, 3, 3, \dots, 3$ to consecutive vertices of u_i and colors $4, 4, 4, 4, \dots, 4$ to consecutive vertices of s'_i .

It follows that $\chi_d(M(S_n)) \leq 4$. Since $\langle \{v_1, s'_1, e'_1, e'_n\} \rangle$ is complete, we have $\chi_d(M(S_n)) \geq 4$. Hence, $\chi_d(M(S_n)) = 4$. $\square$

Theorem 3.3 For a sunlet graph S_n , $\chi_d(T(S_n)) = 4$.

Proof Let $V(S_n) = \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\}$ and $E(S_n) = \{e_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq n\}$.

Let vertices e'_i and s'_i , $1 \leq i \leq n$ represent the edges e_i and s_i , respectively. By the definition of the total graph

$$V(T(S_n)) = V(S_n) \cup E(S_n) = \{v_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{s'_i : 1 \leq i \leq n\},$$

Assign a proper 4-coloring to $V(T(S_n))$ as follows:

Case 1: If $n \equiv 0 \pmod{3}$

For $1 \leq i \leq n$, assign colors $1, 2, 3, 1, 2, 3, \dots, 1, 2, 3$ to consecutive vertices of v_i , colors $3, 1, 2, 3, 1, 2, \dots, 3, 1, 2$ to consecutive vertices of e'_i , colors $3, 1, 2, 3, 1, 2, \dots, 3, 1, 2$ to consecutive vertices of u_i and colors $4, 4, 4, 4, \dots, 4, 4$ to consecutive vertices of s'_i .

Case 2: If $n \equiv 1 \pmod{3}$

For $1 \leq i \leq n$, assign colors $1, 2, 3, 4, 1, 2, 3, 4, \dots, 1, 2, 3, 4$ to consecutive vertices of v_i , colors $4, 1, 2, 3, \dots, 4, 1, 2, 3$ to consecutive vertices of e'_i , colors $4, 1, 2, 3, \dots, 4, 1, 2, 3$ to consecutive vertices of u_i and colors $2, 3, 4, 1, \dots, 2, 3, 4, 1$ to consecutive vertices of s'_i .

Case 3: If $n \equiv 2 \pmod{3}$

For $1 \leq i \leq n$, assign colors 1, 2, 3, 1, 2, 3, 4, ..., 1, 2 to consecutive vertices of v_i , colors 3, 1, 2, 3, 4, 2, ..., 3, 4, 2, 3, 4 to consecutive vertices of e'_i , colors 3, 1, 2, 3, 4, 2, ..., 3, 4, 2, 3, 4 to consecutive vertices of u_i and colors 2, 4, 4, 4, 1, 1, 4, 1, 1, 4, ..., 1, 1, 4, 1 to consecutive vertices of s'_i .

Hence, $\chi_d(T(S_n)) \leq 4$. Since $\{v_1, s'_1, e'_1, u_1\}$ is complete, we have $\chi_d(T(S_n)) \geq 4$. Hence, $\chi_d(T(S_n)) = 4$. $\square$

Theorem 3.4 For a sunlet graph S_n , $\chi_d(C(S_n)) = 2n$.

Proof For a sunlet graph S_n , $n \geq 3$, let

$$V(S_n) = \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \text{ and} \\ E(S_n) = \{e_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq n\}.$$

Let vertices e'_i and s'_i , $1 \leq i \leq n$ represent the edges e_i and s_i , respectively. By the definition of central graph each edge e_i and s_i are subdivided by the vertices $\{e'_i : 1 \leq i \leq n\}$ and $\{s'_i : 1 \leq i \leq n\}$, respectively.

$$V(C(S_n)) = \{v_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{s'_i : 1 \leq i \leq n\}.$$

Note that in $C(S_n)$, the induced subgraph $\langle u_1, u_2, \dots, u_n \rangle$ is complete.

Thus, $\chi_d(C(S_n)) \geq n$.

Now consider the vertex set $V(C(S_n))$ and assign a proper dynamic coloring to $V(C(S_n))$ as follows:

Since the coloring 1, 2, 3, ..., n of consecutive vertices of $v_1, v_2, \dots, v_n$, the coloring $n, 2, 1, 3, \dots, n-1$ of consecutive vertices of e'_i , $1 \leq i \leq n$, the coloring 4, 4, 1, 4, ..., 4 of consecutive vertices of $u_1, u_2, \dots, u_n$ and the coloring $n+1, n+2, \dots, 2n$ of consecutive vertices of s'_i , $1 \leq i \leq n$ are a proper $2n$ -coloring; an easy check shows that this is minimum dynamic $2n$ -coloring. $\square$

4 Dynamic coloring of path-related graphs

Theorem 4.1 For a non-trivial path P_n , $\chi_d(M(P_n)) = 3$.

Proof Let $V(P_n) = \{v_i : 1 \leq i \leq n\}$ and $E(P_n) = \{e_i : 1 \leq i \leq n-1\}$. Let vertices e'_i , $1 \leq i \leq n-1$ represent the edges e_i . For $n = 2$ color the vertices to be $c(v_1) = 1$, $c(v_2) = 3$ and $c(e'_1) = 2$. For $n \geq 3$ color the vertices to be $c(v_i) = 1$, $\forall i$, $c(e'_i) = 2$ if i is odd and $c(e'_i) = 3$ if i is even.

An easy check shows that both colorings are dynamic, hence $\chi_d(M(P_n)) \leq 3$.

Clearly for $n = 2$, the dynamic coloring is minimum. Also for $n \geq 3$ the set of vertices $\{e'_1, v_2, e'_2\}$ induces a clique of order 3 and we have $\chi_d(M(P_n)) \geq 3$. Hence, the result. $\square$

Theorem 4.2 For a non-trivial path P_n , $\chi_d(T(P_n)) = 3$.

Proof Let

$$V(P_n) = \{v_1, v_2, \dots, v_n\}$$

and let

$$V(T(P_n)) = \{v_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq n-1\},$$

where e'_i is the vertex of $T(P_n)$ corresponding to edge $v_i v_{i+1}$ of P_n .

Note that any three consecutive vertices of a sub-path must be colored differently in any dynamic coloring, since the first and third vertices are the only neighbors of the second vertex and must be colored differently (by the condition of dynamic coloring) and also differently from the second vertex. Therefore, $\chi_d(T(P_n)) \geq 3$.

Consider a dynamic 3-coloring of $T(P_n)$ as follows. Assign colors 1, 2, 3, 1, 2, 3, ..., 1, 2, 3 to consecutive vertices $v_1, v_2, \dots, v_n$ and colors 3, 1, 2, 3, 1, 2, ..., 3, 1, 2 to consecutive vertices $e'_1, e'_2, \dots, e'_n$. Clearly, such coloring is dynamic 3-coloring.

Thus, $\chi_d(T(P_n)) \leq 3$. Hence, $\chi_d(T(P_n)) = 3$. $\square$

Theorem 4.3 For a non-trivial path P_n ,

$$\chi_d(C(P_n)) = \begin{cases} n+1; & \text{if } n = 2 \text{ or } 3 \\ n; & \text{otherwise.} \end{cases}$$

Proof Let $v_1, v_2, \dots, v_n$ be the vertices of P_n . If $n = 2$ or 3 , then it can be easily verified. Assume $n \geq 4$. Let the edge $v_i v_{i+1}$ be subdivided by the vertex e'_i , $1 \leq i \leq n-1$ in $C(P_n)$, and let $V = \{v_1, v_2, \dots, v_n\}$, $V' = \{e'_1, e'_2, \dots, e'_{n-1}\}$.

Consider a dynamic n -coloring $(c_1, c_2, \dots, c_n)$ of $C(P_n)$ as follows. For $1 \leq i \leq n$, assign the color c_i to v_i and assign colors $c_3, c_1, c_2, c_3, c_1, c_2, \dots, c_3, c_1, c_2$ to consecutive vertices $e'_1, e'_2, \dots, e'_{n-1}$ to obtain a dynamic n -coloring. Thus, $\chi_d(C(P_n)) \leq n$.

Since every pair of non-adjacent vertices in P_n are adjacent in $C(P_n)$, they received the different colors in $C(P_n)$. If any pair of adjacent vertices in P_n received same color in $C(P_n)$, then the double-adjacency conditions failed. Hence, every vertices of P_n received distinct colors in $C(P_n)$. Hence, $\chi_d(C(P_n)) \geq n$. Thus, the result follows. $\square$

5 Dynamic coloring of cycle-related graphs

Theorem 5.1 For a cycle C_n , $\chi_d(M(C_n)) = 3$.

Proof Let $C_n = (v_1, v_2, \dots, v_n, v_1)$, $e_i = v_i v_{i+1}$, $1 \leq i \leq n-1$, $e_n = v_n v_1$ and let the vertices e'_i represent the edges e_i in $M(C_n)$, $1 \leq i \leq n$.

Case 1: If n is even

Consider the color function $c : V(M(C_n)) \rightarrow \{c_1, c_2, c_3\}$ defined by $c(v_i) = c_1$, $1 \leq i \leq n$ and

$$c(e'_i) = \begin{cases} c_2 & \text{if } n \text{ is odd} \\ c_3 & \text{if } n \text{ is even} \end{cases}$$

It is clear that c is a dynamic coloring and hence $\chi_d(M(C_n)) \leq 3$.

Case 2: If n is odd

Consider the color function $c : V(M(C_n)) \rightarrow \{c_1, c_2, c_3\}$ defined by

$$c(v_i) = \begin{cases} c_1 & \text{if } 2 \leq i \leq n-1 \\ c_2 & \text{if } i = n \\ c_3 & \text{if } i = 1 \end{cases}$$

and

$$c(e'_i) = \begin{cases} c_2 & \text{if } n \text{ is odd, } i < n \\ c_3 & \text{if } n \text{ is even} \\ c_1 & \text{if } i = n \end{cases}$$

It is clear that c is a dynamic coloring and hence $\chi_d(M(C_n)) \leq 3$.

Since $\{\{e'_n, v_1, e'_1\}\}$ is complete, we have $\chi_d(M(C_n)) \geq 3$ and hence the result. $\square$

Theorem 5.2 For a cycle C_n ,

$$\chi_d(T(C_n)) = \begin{cases} 3; & \text{if } n \equiv 0 \pmod{3} \\ 4; & \text{otherwise.} \end{cases}$$

Proof Let $C_n = (v_1, v_2, \dots, v_n, v_1)$, $e_i = v_i v_{i+1}$, $1 \leq i \leq n-1$, $e_n = v_n v_1$ and let the vertices e'_i represent the edges e_i in $T(C_n)$, $1 \leq i \leq n$.

Case 1: If $n \equiv 0(\text{mod } 3)$

Consider the color function $c : V(T(C_n)) \longrightarrow \{c_1, c_2, c_3\}$ defined by

$$c(v_i) = \begin{cases} c_1 & \text{if } i = 3t + 1, t \geq 0, 1 \leq i \leq n \\ c_2 & \text{if } i = 3t + 2, t \geq 0, 1 \leq i \leq n \\ c_3 & \text{if } i = 3t, t \geq 1, 1 \leq i \leq n \end{cases}$$

and

$$c(e_i) = \begin{cases} c_3 & \text{if } i = 3t + 1, t \geq 0, 1 \leq i \leq n \\ c_1 & \text{if } i = 3t + 2, t \geq 0, 1 \leq i \leq n \\ c_2 & \text{if } i = 3t, t \geq 1, 1 \leq i \leq n \end{cases}$$

It is clear that c is a dynamic coloring and hence $\chi_d(T(C_n)) \leq 3$.

Case 2: If $n \equiv 1(\text{mod } 3)$ or $n \equiv 2(\text{mod } 3)$

Subcase 2.1: If n is even

Consider the color function $c : V(T(C_n)) \longrightarrow \{c_1, c_2, c_3, c_4\}$ defined by

$$c(v_i) = \begin{cases} c_1 & \text{if } n \text{ is odd } 1 \leq i \leq n \\ c_2 & \text{if } n \text{ is even } 1 \leq i \leq n \end{cases}$$

and

$$c(e'_i) = \begin{cases} c_3 & \text{if } n \text{ is odd } 1 \leq i \leq n \\ c_4 & \text{if } n \text{ is even } 1 \leq i \leq n \end{cases}$$

It is clear that c is a dynamic coloring and hence $\chi_d(T(C_n)) \leq 4$.

Subcase 2.2: If n is odd

Consider the color function $c : V(T(C_n)) \longrightarrow \{c_1, c_2, c_3, c_4\}$ defined by

$$c(v_i) = \begin{cases} c_1 & \text{if } i \text{ is odd, } i \leq n \\ c_2 & \text{if } i \text{ is even} \\ c_3 & \text{if } i = n \end{cases}$$

and

$$c(e'_i) = \begin{cases} c_3 & \text{if } i \text{ is odd, } i \leq n \\ c_4 & \text{if } i \text{ is even} \\ c_2 & \text{if } i = n \end{cases}$$

It is clear that c is a dynamic coloring and hence $\chi_d(T(C_n)) \leq 4$.

It is clear that c is a dynamic coloring and hence

$$\chi_d(T(C_n)) = \begin{cases} 3; & \text{if } n \equiv 0(\text{mod } 3) \\ 4; & \text{otherwise.} \end{cases}$$

□

Theorem 5.3 For a cycle C_n , $\chi_d(C(C_n)) = n$.

Proof Let $v_1, v_2, \dots, v_n$ be the vertices of C_n . Let vertices e'_i $1 \leq i \leq n$ represent the edges e_i in $C(C_n)$, and let $V = \{v_1, v_2, \dots, v_n\}$, $V' = \{e'_1, e'_2, \dots, e'_n\}$.

Consider a dynamic n -coloring of $C(C_n)$ as follows. For $1 \leq i \leq n$, assign the color i to v_i and, for $2 \leq i \leq n-1$, assign the color i to e'_{i+1} and assign the color n to e'_1 . Thus, $\chi_d(C(C_n)) \leq n$.

Suppose $\chi_d(C(C_n)) = n-1$. Here, all distinct pairs e'_i and e'_j are non-adjacent vertices. A $(n-1)$ -coloring of $C(C_n)$ in which e'_i and e'_{i+1} receive the same color must satisfy both adjacency and double-adjacency conditions unless the double-adjacency condition is not satisfied for some vertex v_j adjacent to e'_i and e'_{i+1} . This is the contradiction; hence, such dynamic-coloring with $n-1$ colors is impossible. Therefore, $\chi_d(C(C_n)) \geq n$. Hence, $\chi_d(C(C_n)) = n$, $\forall n \geq 3$. $\square$

6 Dynamic coloring of Mycielskian graphs

Theorem 6.1 For a non-trivial path P_n ,

$$\chi_d(\mu(P_n)) = \begin{cases} 5; & \text{if } n = 2 \\ 4; & \text{otherwise.} \end{cases}$$

Proof Refer to the construction of the Mycielski graph (see Sect. 2.1). Now consider the vertex set $V(\mu(P_n))$ and assign a proper dynamic coloring to $V(\mu(P_n))$ as follows:

Case 1: If $n = 2$

If $n = 2$, then $\mu(P_2)$ is isomorphic to C_5 . Therefore, $\chi_d(\mu(P_2)) = 5$. (By Theorem D).

Case 2: If $n \geq 3$

Assign colors 1, 4, 1, 4, ... to consecutive vertices of $\{v_1, v_2, \dots, v_n\}$ and colors 1, 2, 2, 2, ... to consecutive vertices of $\{x_1, x_2, \dots, x_n\}$ and assign color 3 to vertex w . By Theorem D, it is easily verified that this coloring is a minimum dynamic 4-coloring.

Hence, $\chi_d(\mu(P_n)) = 5$ if $n = 2$ else, $\chi_d(\mu(P_n)) = 4$. $\square$

Theorem 6.2 For a cycle C_n , $\chi_d(\mu(C_n)) = 4$.

Proof Refer to the definition of the Mycielski graph. Now consider the vertex set $V(\mu(C_n))$ and assign a proper dynamic coloring to $V(\mu(C_n))$ as follows:

Case 1: If $n \equiv 0(\text{mod } 3)$

Assign colors 1, 2, 3, 1, 2, 3, ... to consecutive vertices of $\{v_1, v_2, \dots, v_n\}$ and colors 1, 2, 3, 1, 2, 3, ... to consecutive vertices of $\{x_1, x_2, \dots, x_n\}$ and assign color 4 to w . An easy check shows that this coloring is a minimum dynamic 4-coloring.

Case 2: If $n \equiv 1(\text{mod } 3)$

Assign colors 1, 2, 3, 4, 1, 2, 3, 4, ... to consecutive vertices of $\{v_1, v_2, \dots, v_n\}$ and colors 1, 2, 3, 2, 1, 2, 3, 2, 1, 2, 3, ... to consecutive vertices of $\{x_1, x_2, \dots, x_n\}$ and assign color 4 to w . An easy check shows that this coloring is a minimum dynamic 4-coloring.

Case 3: If $n \equiv 2(\text{mod } 3)$

Assign colors 1, 2, 3, 1, 2, 3, 4, 2, 1, 2, 3, 4, 1, 2, 3, 4, ... to consecutive vertices of $\{v_1, v_2, \dots, v_n\}$ and colors 1, 2, 3, 1, 2, 3, 1, 3, 1, 2, 3, 1, 2, 3, ... to consecutive vertices of $\{x_1, x_2, \dots, x_n\}$ and assign color 4 to w . An easy check shows that this coloring is a minimum dynamic 4-coloring.

Hence, $\chi_d(\mu(C_n)) = 4$. $\square$

7 Dynamic coloring of join graphs

Theorem 7.1 For a non-trivial path P_n and a cycle C_n ,

$$\chi_d(P_m + C_n) = \begin{cases} 4; & \text{if } n \equiv 0(\text{mod } 2) \\ 5; & \text{otherwise.} \end{cases}$$

Proof Let $V = \{v_1, v_2, \dots, v_m\}$ be the vertices of P_m and let $V' = \{u_1, u_2, \dots, u_n\}$ be the vertices of C_n . Therefore, the vertices of $P_m + C_n$ is $V \cup V'$.

Now consider the vertex set $V(P_m + C_n)$ and assign a proper dynamic coloring to $V(P_m + C_n)$ as follows:

Case 1: If $n \equiv 0 \pmod{2}$

Assign colors 1, 2, 1, 2, $\dots$, 1, 2 to consecutive vertices $v_1, v_2, \dots, v_n$ and colors 3, 4, 3, 4, $\dots$, 3, 4 to consecutive vertices $u_1, u_2, \dots, u_n$. An easy check shows that this coloring is a minimum dynamic 4-coloring.

Case 2: If $n \equiv 1 \pmod{2}$

Assign colors 1, 2, 1, 2, $\dots$, 1, 2 to consecutive vertices $v_1, v_2, \dots, v_n$ and colors 3, 4, 3, 4, $\dots$, 5 to consecutive vertices $u_1, u_2, \dots, u_n$. An easy check shows that this coloring is a minimum dynamic 5-coloring.

Hence, $\chi_d(P_m + C_n) = 4$ if $n \equiv 0 \pmod{2}$ else, $\chi_d(P_m + C_n) = 5$. $\square$

Acknowledgements The authors thank the referees for their excellent comments and valuable suggestions which led to substantial improvement of this paper.

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