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Type I and Type II codes over the ring $\mathbb{F}_2 + s\mathbb{F}_2 + s^2\mathbb{F}_2$

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Abstract We take a ring $R = \mathbb{F}_2 + s\mathbb{F}_2 + s^2\mathbb{F}_2$. We consider a Gray map on this ring, discuss self-dual codes, define various weight enumerators over the ring, and discuss equivalence class of codes over the ring. We construct self-dual codes of Type I and Type II over the given ring for different lengths.

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المخلص

نأخذ حلقة $R = \mathbb{F}_2 + s\mathbb{F}_2 + s^2\mathbb{F}_2$. نعتبر خريطة جري على هذه الحلقة، نناقش الشفرات ذاتية الثنائية، نعرف عدادات لأوزان مختلفة على الحلقة، ونناقش أصناف تكافؤ من الشفرات على الحلقة. ننشئ شفرات ذاتية الثنائية من النوعين الأول والثاني على الحلقة المعطاة لأطوال مختلفة.

1 Introduction

Several researchers are studying codes over the ring from long period of time. At the beginning, people started working on codes over the field and latter on they extended the scope of codes over the ring. Mallows and Sloane [13] discuss a self-dual code over GF(2) and GF(3) with their weight distribution and also discuss the bounds to compute self-dual codes. FJ MacWilliams et al. [12] discuss the self-dual codes over GF(4); they also discuss the concept of generator matrix to find the self-dual codes. In [4], Conway and Sloane discuss the self-dual codes for the modulo 4 integers and find various codes for the given ring for different lengths. For detailed description, we refer to the following series of papers [1, 2, 6–8, 10, 11, 14–16, 18, 20, 21].

Dougherty et al. [6] discuss the Type II code over the ring $\mathbb{F}_2 + s\mathbb{F}_2$. They prove various results to find the self-dual code over the ring by computing generator matrix. Huffman [9] discusses the decomposition of codes over the ring $\mathbb{F}_2 + s\mathbb{F}_2$, $s^2 = 0$. He finds the bound to compute the extremal self-dual codes for the given ring and computes the generator matrix to find the self-dual codes of various lengths over the given ring. Qian et al. [17] discuss the Type II code over the ring $\mathbb{F}_2 + s\mathbb{F}_2 + s^2\mathbb{F}_2$; they consider a Gray map to construct self-dual codes over the ring. We use the motivation from Dougherty's paper and some results from Qian paper to discuss some properties of the codes over the ring $\mathbb{F}_2 + s\mathbb{F}_2 + s^2\mathbb{F}_2$. We define different types of weight enumerators for codes over the ring. In [3] and [19], we found the various relationship among the weight enumerators.

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This paper is organized as follows: In Sect. 2, we discuss codes over the ring $R = \mathbb{F}_2 + s\mathbb{F}_2 + s^2\mathbb{F}_2$, $s^3 = 0$ and some results for the self-dual codes and also we define different types of weight enumerators over the ring R and talk about the equivalence class. In Sect. 3, we discuss the Mac Williams identity for the weight enumerator, compute the matrix form for the equivalence class, and see some results on the weight enumerator. In Sect. 4, we construct some codes over the ring $\mathbb{F}_2 + s\mathbb{F}_2 + s^2\mathbb{F}_2$.

2 Preliminaries

2.1 Codes over the ring

We consider the finite ring $R := \mathbb{F}_2 + s\mathbb{F}_2 + s^2\mathbb{F}_2$ with order of nilpotency for s being 3, i.e., $s^3 = 0$. A code over R is a subset of R^n , and if the code is a submodule of R^n , then it is a linear code. We take all the codes to be linear unless otherwise specified.

We define the inner product on the ring R as :

$$[v, u] = \sum_i v_i u_i$$

as the standard inner product. We define $C^\perp$ as:

$$C^\perp = \{v : [v, u] = 0 \text{ for all } u \in C\}.$$

It is easy to see that $C^\perp$ of a linear code C is linear and that $|C||C^\perp| = |R^n|$. If $C \subseteq C^\perp$, then the code is said to be orthogonal and if $C = C^\perp$ then it is self-dual. The Gray map is defined by $\phi : R \rightarrow \mathbb{F}_2^3$

$$\phi(a + sb + s^2c) = (a + b + c, b + c, b).$$

The map ϕ is linear.

The Gray map ϕ can be extended to R^n in the same manner as extended in [17]. This extended map ϕ is a bijection from R^n to $\mathbb{F}_2^{3n}$.

Note that ϕ is a distance preserving map or isometry from $(R^n, \text{Lee distance})$ to $\mathbb{F}_2^{3n}$, (Hamming distance).

Lemma 2.1 *Let C be a code of length n over R which is self-dual. Then, $\phi(C)$ is a code of length $3n$ over $\mathbb{F}_2$, which is also self-dual.*

Proof For vectors v and v' in C , let us assume that the code C is self-dual. Then, we have $[v, v'] = \sum v_j v'_j = 0$.

$$\text{Set } v_j = a_j + sb_j + s^2c_j, v'_j = a'_j + sb'_j + s^2c'_j$$

$$\sum v_j v'_j = \sum (a_j + sb_j + s^2c_j)(a'_j + sb'_j + s^2c'_j) = \sum (a_j + sb_j + s^2c_j)(a'_j + sb'_j + s^2c'_j) = \sum a_j a'_j + s(a_j b'_j + b_j a'_j) + s^2(a_j c'_j + c_j a'_j + b_j b'_j).$$

$$\text{Hence, } \sum a_j a'_j \equiv 0 \pmod{2} \quad \sum (a_j b'_j + b_j a'_j) \equiv 0 \pmod{2} \quad \sum (a_j c'_j + c_j a'_j + b_j b'_j) \equiv 0 \pmod{2}.$$

To show $\phi(C)$ is self-dual, we shall show $[\phi(v), \phi(v')] = 0$. We have $\phi(v_j) = \phi(a_j + sb_j + s^2c_j) = (a_j + b_j + c_j, b_j + c_j, b_j)$ and $\phi(v'_j) = \phi(a'_j + sb'_j + s^2c'_j) = (a'_j + b'_j + c'_j, b'_j + c'_j, b'_j)$.

$$[\phi(v), \phi(v')] = \sum (a_j + b_j + c_j, b_j + c_j, b_j)(a'_j + b'_j + c'_j, b'_j + c'_j, b'_j).$$

$$\sum a_j a'_j + a_j b'_j + b_j a'_j + a_j c'_j + c_j a'_j + b_j b'_j = 0.$$

Since we know that

$$\sum a_j a'_j \equiv 0 \pmod{2} \quad \sum (a_j b'_j + b_j a'_j) \equiv 0 \pmod{2} \quad \sum (a_j c'_j + c_j a'_j + b_j b'_j) \equiv 0 \pmod{2}.$$

Hence, the code C is self-orthogonal.

Since $\phi(C) = |C|$, as the code C is self-dual, i.e., $|C| = |C^\perp|$. Therefore, $|\phi(C)| = |\phi(C^\perp)|$. Hence, $\phi(C)$ is a self-dual code of length $3n$ over $\mathbb{F}_2$. $\square$

The elements of the ring R are indexed as $\{0, 1, s, s^2, 1 + s, 1 + s^2, s + s^2, 1 + s + s^2\}$. We define the Lee weight of a codeword as the sum of its coordinates Lee weights. Lee weights of the elements 0 and s are 0 and 3, respectively, whereas Lee weights of $\{1, 1 + s^2, s + s^2\}$ and $\{1 + s, s^2, 1 + s + s^2\}$ are 1 and 2, respectively.

Definition A self-dual code is said to be of Type II over R if the Lee weight of all the codewords are divisible by 4. A self-dual code in which the Lee weight of each codewords is not divisible by 4 is said to be of Type I.

Theorem 2.2 *Let C be a Type II (respectively, Type I) code of length n over R . Then, $\phi(C)$ is a Type II (respectively, Type I) code of length $3n$ over $\mathbb{F}_2$.*

Proof From Lemma 2.1, if C is self-orthogonal, so is $\phi(C)$. Since the Gray map ϕ is a linear isometry. Therefore, from [17], Theorem 5.1, $\phi(C)$ is a Type II (respectively, Type I) code over $\mathbb{F}_2$. $\square$

The elements of the ring R are $\{0, 1, s, s^2, 1+s, 1+s^2, s+s^2, 1+s+s^2\}$. The Euclidean weight of 0 is 0. Weight of $1, 1+s^2, s+s^2$ is 1. Weight of the elements $s^2, 1+s$ and $1+s+s^2$ is 2 and s is of weight 3.

2.2 Weight enumerators

We define the Complete weight enumerator of a code C over the ring R by

$$cwe_C(x_0, x_1, x_s, x_{s^2}, x_{1+s}, x_{1+s^2}, x_{s+s^2}, x_{1+s+s^2}) = \sum_{v \in C} \prod_{a \in R} x_a^{n_a(v)},$$

where $n_a(v) = |\{j : v_j = a\}|$. We define relation $\sim$ on R by $a \sim b$ iff $a = bq$, where q is a unit in R . Here $\sim$ stands for the equivalence relation and we take $\bar{R} = R/\sim$.

Symmetrized weight enumerator for a code $\mathfrak{C}$ over R is

$$swe_{\mathfrak{C}}(x_{[a]} : a \in \bar{R}) = \sum_{v \in \mathfrak{C}} \prod_{[a] \in \bar{R}} x_{[a]}^{\text{sn}_{[a]}(v)},$$

where $\text{sn}_{[a]}(v) = |\{j : v_j = [a]\}|$.

Complete joint weight enumerator for codes $\mathfrak{C}$ and $\mathfrak{D}$ over R is defined by

$$\mathfrak{J}_{\mathfrak{C}, \mathfrak{D}}(X) = \sum_v \sum_{v'} \prod_{(a,b) \in R^2} x_{(a,b)}^{n_{(a,b)}(v,v')},$$

where the exponent $n_{(a,b)}(v, v')$ is the cardinality of the set $\{j : v_j = a, v'_j = b\}$.

We consider $R^2/\sim$, where $(s, \beta) \sim (s', \beta')$ iff $s' = sq, \beta' = \beta q$, for a unit q in the ring R , we define symmetrized joint weight enumerator by

$$\mathfrak{S}\mathfrak{J}_{\mathfrak{C}, \mathfrak{D}}(X) = \sum_v \sum_{v'} \prod_{[(a,b)] \in (R^2/\sim)} x_{[(a,b)]}^{n_{[(a,b)]}(v,v')},$$

where the exponent $n_{[(a,b)]}(v, v')$ is the cardinality of the set $\{j : (v_j, v'_j) \in [(a, b)]\}$.

The units of the ring R are $\{1, 1+s, 1+s^2, 1+s+s^2\}$ and non-units are $\{0, s, s^2, s+s^2\}$. Now, we shall show the structure of $R^2/\sim$. Each element of R^2 is equivalent to (e, f) , where e is in $\{0, 1, s, s^2\}$. We need to consider all the different classes of R and corresponding to these classes we have to find (e, f) and a unit $q = a + sb + s^2c$ such that $e' = eq$ and $f' = fq$.

Suppose we take $e = 1$ at the first place. Then the second coordinate f will have 8 possibilities and each element will represent different equivalence class.

For $e = 0$ at the first coordinate, there will be 4 possibilities for the f corresponding to the classes $[0], [1], [s], [s^2]$.

If we take $e = s$ at the first coordinate, then there will be 3 possibilities for f corresponding to the classes $\{1, 1+s^2\}, \{0\}, \{s^2\}$ and will represent different equivalence class.

For $e = s^2$ at the first place and corresponding to the classes $\{1, 1+s, 1+s^2, 1+s+s^2\}, \{0\}, \{s\}, \{s^2\}$ and $\{s+s^2\}$, we will have 5 possibilities for the second coordinate

Hence $|R^2/\sim| = 20$.

3 MacWilliams identity

We now define MacWilliams identities for different weight enumerators described earlier. We compute some matrices with the weight enumerators and the equivalence classes of the self-dual codes. These matrices help us give some results on the different weight enumerators.

We define a character from the additive group of R as $\chi : R \rightarrow \mathbb{C}^\times$ as $\chi(a + sb + s^2c) = (-1)^{w_H(a,b,c)}$, where w_H is the Hamming weight on binary codes. Action of the matrix T on the weight enumerator gives us the MacWilliams relations, where T is indexed by the elements of the ring and is defined as $T_{i,j} = \chi(a_i a_j)$. The matrix is thus given by

$$T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \end{bmatrix}$$

Let $X = (x_0, x_1, x_u, \dots, x_{1+u+u^2})$. We consider $R' = \{\pi \mid \pi \text{ is a character of additive group of } R\}$. We note that the additive groups of R and R' are isomorphic.

We define orthogonal of C as $C^\perp = \{(a_1, a_2, \dots, a_n) \mid \prod_{i=1}^n \chi_{a_i}(c_i) = 1, \forall (c_1, c_2, \dots, c_n) \in C\}$. The code $C^\perp$ is associated with the set $\{\chi \in R^{n'} \mid \chi(c) = 1, \forall c \in C\}$. X^t denotes the transpose of X .

MacWilliams identity has been proved for the general Finite Frobenius ring for any code of length n by J A Wood [19] and latter on Dougherty explained it in simpler form [5]. Thus, we are able to give MacWilliams identity for the ring $\mathbb{F}_2 + s\mathbb{F}_2 + s^2\mathbb{F}_2$, with $s^3 = 0$.

Theorem 3.1 *If C is a code over R , then*

$$cwe_{C^\perp}(X) = \frac{1}{|C|} cwe_C(T \cdot X^t).$$

Proof The matrix form is given by

$$\chi(a_1 + sb_1 + s^2c_1)(a_2 + sb_2 + s^2c_2) = \chi_1((a_1 + sb_1 + s^2c_1)(a_2 + sb_2 + s^2c_2))$$

The character corresponding to 1 in the character group is given by $\chi_1(a_1 + ub_1 + u^2c_1) = (-1)^{w_H(a_1, b_1, c_1)}$. Hence, MacWilliams relation follows from Theorem 3.1 of [5]. $\square$

We now give Corollary 3.3 from [3] and Corollary 3.1 from [5] for complete joint weight enumerator of two codes, i.e., genus 2 codes for the ring R . This can be proved using the above Theorem 2 and definition of complete joint weight enumerator.

Corollary 3.2 *Let $\mathcal{C}$ and $\mathcal{D}$ be codes over R . Then*

$$\begin{aligned} \mathfrak{J}_{\mathcal{C}^\perp, \mathcal{D}^\perp}(X) &= \frac{1}{|\mathcal{C}|} \frac{1}{|\mathcal{D}|} \mathfrak{J}_{\mathcal{C}, \mathcal{D}}((T \otimes T) \cdot X^t). \\ \mathfrak{J}_{\mathcal{C}^\perp, \mathcal{D}}(X) &= \frac{1}{|\mathcal{C}|} \mathfrak{J}_{\mathcal{C}, \mathcal{D}}((T \otimes I) \cdot X^t). \\ \mathfrak{J}_{\mathcal{C}, \mathcal{D}^\perp}(X) &= \frac{1}{|\mathcal{D}|} \mathfrak{J}_{\mathcal{C}, \mathcal{D}}((I \otimes T) \cdot X^t). \end{aligned}$$

Let T' be a $|\bar{R}|$ by $|\bar{R}|$ matrix formed from the elements of $\bar{R}$. Set $(T')_{c,d} = \sum_{e \sim f} (T)_{c,e}$. Then, the following results hold.

Corollary 3.3 *If $\mathcal{C}$ is a code over R , then*

$$swe_{\mathcal{C}^\perp}(X) = \frac{1}{|\mathcal{C}|} swe_{\mathcal{C}}(T' \cdot X^t).$$

Proof Proof of the corollary follows from the above Theorem 2 and from the definition of symmetrized weight enumerator given in Theorem 3.1 of [5]. $\square$

The matrix T' is indexed by the classes $[0], [1], [s], [s^2]$ of the ring and is given by

$$T' = \begin{bmatrix} 1 & 4 & 2 & 0 \\ 1 & 0 & 2 & 1 \\ 1 & 0 & -2 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix}.$$

Let T'_1, T'_2, T'_3 be $|R^2/\sim|$ by $|R^2/\sim|$ matrix indexed with elements from $R^2/\sim$. Set

$$T'_1{}_{(a,b)(c,d)} = \sum_{(e,f) \sim (c,d)} (T \otimes T)_{(a,b),(e,f)}$$

$$T'_2{}_{(a,b)(c,d)} = \sum_{(e,f) \sim (c,d)} (T \otimes I)_{(a,b),(e,f)}$$

and

$$T'_3{}_{(a,b)(c,d)} = \sum_{(e,f) \sim (c,d)} (I \otimes T)_{(a,b),(e,f)}.$$

Note that the symmetrized joint weight enumerator of codes C and D can be given as:
 $\mathfrak{S}\mathfrak{J}_{\mathfrak{C},\mathfrak{D}}(X \mid X = (x_{[a]}), [a] \in R^2/\sim) = \mathfrak{J}_{\mathfrak{C},\mathfrak{D}}(X_a \mid a \in R^2, x_a \text{ is identified with } x_b \text{ if and only if } a \sim b).$

Corollary 3.4 *The weight enumerator $\mathfrak{S}\mathfrak{J}_{\mathfrak{C},\mathfrak{D}}$ of codes $\mathfrak{C}$ and $\mathfrak{D}$ over R satisfies*

$$\mathfrak{S}\mathfrak{J}_{\mathfrak{C}^\perp, \mathfrak{D}^\perp}(X) = \frac{1}{|\mathfrak{C}|} \frac{1}{|\mathfrak{D}|} \mathfrak{S}\mathfrak{J}_{\mathfrak{C}, \mathfrak{D}}(T'_1 \cdot X')$$

$$\mathfrak{S}\mathfrak{J}_{\mathfrak{C}^\perp, \mathfrak{D}}(X) = \frac{1}{|\mathfrak{C}|} \mathfrak{S}\mathfrak{J}_{\mathfrak{C}, \mathfrak{D}}(T'_2 \cdot X')$$

$$\mathfrak{S}\mathfrak{J}_{\mathfrak{C}, \mathfrak{D}^\perp}(X) = \frac{1}{|\mathfrak{D}|} \mathfrak{S}\mathfrak{J}_{\mathfrak{C}, \mathfrak{D}}(T'_3 \cdot X').$$

4 Type I and Type II codes

The generator matrix G represents the permutation equivalent matrix of a code C over the ring R and is of the form:

$$G = \begin{pmatrix} I_{k_0} & B_{01} & B_{02} & B_{03} \\ 0 & sI_{k_1} & sB_{12} & sB_{13} \\ 0 & 0 & s^2I_{k_2} & s^2B_{23} \end{pmatrix}$$

where B_{ij} are matrices over $\mathbb{F}_2$. Any code which has generator matrix like G is of type $\{k_0, k_1, k_2\}$ and has $2^{3k_0+2k_1+k_2}$ elements. We define torsion code $Tor_i(C)$ over $\mathbb{F}_2$ for a code C as:

$$Tor_i(C) = \{v(\bmod s) \mid v \in C\}$$

where $0 \leq i \leq 2$. Also, we have

$$Res(C) = Tor_0(C) = \{v(\bmod s) \mid v \in C\}.$$

We now find Type I and Type II codes over the ring $\mathbb{F}_2 + s\mathbb{F}_2 + s^2\mathbb{F}_2$ using some results already discussed in [17]. These results will help us using residue and torsion codes to construct self-dual codes over the ring $\mathbb{F}_2 + s\mathbb{F}_2 + s^2\mathbb{F}_2$.

4.1 Example

Self-dual codes up to length 4 have already been discussed in [17]. We find self-dual codes of length up to 10.

Codes of length 6

$$C_{6,1} = \{(1, 1 + s, s^2, 0, s, 0), (1, 1 + s + s^2, 0, s, 0, s^2), (s^2, s^2, 1 + s, 0, s, 0)\}$$

$C_{6,1}$ is a Type I code, as the w_L of each codeword is divisible by 4. It is not a Type II code as the Lee weight of the binary code is not divisible by 8.

$$C_{6,2} = \{(1, s, 0, 1 + s, 0, s^2), (0, 0, s, s^2, 1 + s, 1), (1, s + s^2, 0, 1 + s, 0, s^2)\}$$

$C_{6,2}$ is a Type I code, as the w_L of few codewords is not divisible by 4.

Codes of length 8

$$C_{8,1} = \{(s + s^2, 0, 0, 1 + s, 1 + s + s^2, 0, s, 0), (0, 0, s, 1 + s + s^2, 1 + s, s + s^2, 0, 0), \\ (0, 1 + s, s + s^2, 0, 0, s, 0, 1 + s + s^2), (0, 1 + s + s^2, s, 0, 0, s + s^2, 0, 1 + s)\}.$$

$C_{8,1}$ is a Type II code over the ring R , as the Lee weight of binary code is divisible by 8.

$$C_{8,2} = \{(1 + s, 0, 0, s, 1 + s + s^2, 0, s + s^2, 0), (1 + s + s^2, 0, s^2, 0, 1 + s, 0, 0, 0), \\ (0, 1 + s, s + s^2, 0, 0, 1 + s + s^2, 0, s), (0, 1 + s + s^2, s, 0, 0, 1 + s, 0, s + s^2)\}.$$

$C_{8,2}$ is a Type I code over the ring R .

Codes of length 10

$$C_{10,1} = \{(1 + s, 1, s, 0, s^2, 1 + s, 0, 0, 1 + s + s^2, 0), (1, 1 + s, s^2, 0, s, 1 + s + s^2, 0, 0, 1 + s, 0), \\ (0, 0, 1 + s, 0, 1 + s + s^2, s, 0, 0, s^2, s), (0, 0, 1 + s + s^2, s, 1 + s, s^2, 0, 0, s, 0), \\ (1 + s + s^2, 1 + s, 0, s, 0, s^2, 0, 0, s, 0)\}.$$

$C_{10,1}$ is a Type I code over the ring R .

$$C_{10,2} = \{(1, 0, s, 1 + s, 1 + s, s^2, 0, 1 + s + s^2, 1, 1 + s^2), (1 + s, 0, s^2, 1, 1 + s + s^2, s, 0, 1 + s, 1 + s^2, 1), \\ (0, 0, 1 + s, 0, s, 1 + s + s^2, s, s^2, s^2, s^2), (0, s, 1 + s + s^2, 0, s^2, 1 + s, 0, s, s, s), \\ (1 + s, s, 0, 1 + s + s^2, s^2, 0, 0, s, 0, 0)\}.$$

$C_{10,2}$ is also a Type I code over the ring R .

5 Conclusion

The construction of self-dual codes has been carried out for different lengths over the ring $\mathbb{F}_2 + s\mathbb{F}_2 + s^2\mathbb{F}_2$ with $s^3 = 0$. Self-dual codes have been computed by Quin in his paper up to length 4 using generator matrix and by defining a Gray map. In many papers, people have computed self-dual codes using residue and torsion codes over the given ring and also used permutation equivalent matrix G for computing Torsion and Residue codes. We defined various weight enumerators over the ring for a self-dual code and discussed the MacWilliams identity and also defined equivalence relation to talk about the matrices T and T' . On this ring if we can construct lattices over totally real field and find unimodular lattices, then we can consider the modular forms and Clifford Weil group to investigate the algebra homomorphism and weight and index of the modular form. Also, weight enumerator can be used to find the relation with the theta series.

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