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On coupled nonlinear Schrödinger systems

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Abstract A class of coupled Schrödinger equations is investigated. First, in the stationary case, the existence of ground states is obtained and a sharp Gagliardo–Nirenberg inequality is discussed. Second, in the energy critical radial case, global well-posedness and scattering for small data are proved.

Mathematics Subject Classification 35Q55

المخلص

في هذا البحث، قد تم التحقيق في صنف من معادلات شرودينجر المتزاوجة (المقترنة). أولاً، في الحالة الساكنة، قد حصلنا على وجود الحد الأدنى من حلول الطاقة، وناقشنا بعض متراجحات جاجلياردو-نيرنبرج الدقيقة. ثانياً، في حالة الطاقة الحرجة نصف القطرية (الشعاعية)، فقد أثبتنا الصياغة الجيدة الشاملة والانتشار (الفض) لمعطيات صغيرة.

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1 Introduction

Consider the Cauchy problem for a fractional Schrödinger system with power-type coupled non-linearities

$$\begin{cases} i\dot{u}_j - (-\Delta)^s u_j = \gamma \left(\sum_{k=1}^m a_{jk} |u_k|^p \right) |u_j|^{p-2} u_j; \\ u_j(0, \cdot) = \psi_j. \end{cases} \quad (1.1)$$

Here and hereafter, $s \in (0, 1)$, $\gamma = \pm 1$, $u_j : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{C}$, for $j \in [1, m]$ and $a_{jk} = a_{kj}$ are positive real numbers. The fractional Laplacian operator stands for

$$(-\Delta)^s u := \mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F}u),$$

where $\mathcal{F}$ denotes the Fourier transform.

This system of fractional partial differential equations arises in quantum mechanics. It describes how the quantum state of some physical system changes with time [15].

A solution $\mathbf{u} := (u_1, \dots, u_m)$ to (1.1) satisfies (formally) conservation of the mass

$$M(u_j(t)) := \frac{1}{2} \int_{\mathbb{R}^N} |u_j(t, x)|^2 dx = M(u_j(0));$$

and the energy is denoted by

$$\begin{aligned} E(\mathbf{u}(t)) &:= \frac{1}{2} \sum_{j=1}^m \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{s}{2}} u_j(t, x) \right|^2 dx + \frac{\gamma}{2p} \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |u_j(t, x) u_k(t, x)|^p dx \\ &= E(\mathbf{u}(0)). \end{aligned}$$

If $\gamma = 1$, the energy is always positive and the problem (1.1) is said to be defocusing, otherwise a control of a solution to (1.1) with the energy is no longer possible and a local solution may blow-up in finite time, we speak about focusing problem.

In the classical case $s = 1$, the m -component coupled nonlinear Schrödinger system with power-type non-linearities arises in many physical problems in which the field has more than one component such as the interactions of M-wave packets, the nonlinear waveguides and the optical pulse propagation in birefringent fibers. In nonlinear optics [2] u_j denotes the j th component of the beam in Kerr-like photo-refractive media. The coupling constant a_{jk} acts as the interaction between the j th and the k th components of the beam. This system arises also in plasma physics, multispecies, spinor Bose–Einstein condensates, biophysics and nonlinear Rossby waves. Readers are referred, for instance, to [5, 7, 16, 28, 29]. For mathematical point of view, well-posedness issues were investigated by many authors. Indeed, global existence of solutions and scattering hold [3, 22, 24–27].

When $m = 1$, the problem (1.1) is a non-local model known as nonlinear fractional Schrödinger equation which has also attracted much attention recently [9–14]. It is a fundamental equation of fractional quantum mechanics, which was derived by Laskin [17, 18] as a result of extending the Feynman path integral, from the Brownian-like to Levy-like quantum mechanical paths. It is proved that the Cauchy problem is well-posed and scatters in the radial energy space [10, 11], see also [27].

The purpose of this paper is twofold. First, the stationary problem associated to (1.1) is investigated, where the existence of ground states is obtained and a sharp Gagliardo–Nirenberg type inequality is discussed. Second, global well-posedness and scattering for small data are proved in the radial case with energy critical non-linearity.

It is the contribution of this work to extend known results in the case of the scalar fractional Schrödinger equation [4, 23] about the potential well theory, global well-posedness and scattering, to the coupled fractional system. The main difficulty is to overcome the presence of a non-local operator and a combined non-linearity. Indeed, to use Strichartz estimate without loss of regularity, the radial energy space is used. Moreover, in well-posedness, we are restricted to space dimensions because of the combined non-linearity.

The rest of this paper is organized as follows. The next section contains the main results and some technical tools needed in the sequel. In Sects. 3 and 4, the stationary problem associated to (1.1) is investigated, precisely the existence of ground state and a sharp Gagliardo–Nirenberg type inequality are obtained. The goal of the

fifth and sixth sections is to prove well-posedness and scattering of (1.1) in the radial energy space. The two last sections deal with global well-posedness via potential-well method.

We end this section with some definitions. We mention that C will denote a constant which may vary from line to line and if A and B are non negative real numbers, $A \lesssim B$ means that $A \leq CB$. For $1 \leq r \leq \infty$, we denote the Lebesgue space $L^r := L^r(\mathbb{R}^N)$ with the usual norm $\|\cdot\|_r := \|\cdot\|_{L^r}$ and $\|\cdot\| := \|\cdot\|_2$. For simplicity, we denote the usual Sobolev Space $W^{s,p} := W^{s,p}(\mathbb{R}^N)$ and $H^s := W^{s,2}$. If X is an abstract space $C_T(X) := C([0, T], X)$ stands for the set of continuous functions valued in X and X_{rd} is the set of radial elements in X , moreover for an eventual solution to (1.1), $T^* > 0$ denotes its lifespan.

2 Main results and background

In what follows, the main results and some estimates needed in the sequel are given.

2.1 Preliminary

Let us denote the mass critical and energy critical exponents

$$p_* := 1 + \frac{2s}{N} \quad \text{and} \quad p^* := \frac{N}{N-2s}.$$

The so-called energy space is

$$\mathcal{H} := H_{rd}^s(\mathbb{R}^N) \times \cdots \times H_{rd}^s(\mathbb{R}^N) = [H_{rd}^s(\mathbb{R}^N)]^m,$$

where $H^s(\mathbb{R}^N)$ is the usual Sobolev space endowed with the complete norm

$$\|\cdot\|_{H^s} := \left(\|\cdot\|^2 + \|(-\Delta)^{\frac{s}{2}} \cdot\|^2 \right)^{\frac{1}{2}}.$$

For $\mathbf{u} := (u_1, \dots, u_m) \in \mathcal{H}$, define the action functional by

$$S(\mathbf{u}) := \frac{1}{2} \sum_{j=1}^m \|u_j\|_{H^s}^2 - \frac{1}{2p} \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |u_j u_k|^p \, dx.$$

For $\lambda, \alpha, \beta \in \mathbb{R}$, we introduce the scaling

$$(u_j^\lambda)^{\alpha,\beta} := e^{\alpha\lambda} u_j(e^{-\beta\lambda} \cdot), \quad (\mathbf{u}^\lambda)^{\alpha,\beta} := ((u_1^\lambda)^{\alpha,\beta}, \dots, (u_m^\lambda)^{\alpha,\beta})$$

and the differential operator

$$\mathfrak{L}_{\alpha,\beta} : \mathcal{H} \rightarrow \mathcal{H}, \quad \mathbf{u} \mapsto \partial_\lambda ((\mathbf{u}^\lambda)^{\alpha,\beta})|_{\lambda=0}.$$

We extend the previous operator as follows: if $A : \mathcal{H} \rightarrow \mathbb{R}$, then

$$\mathfrak{L}_{\alpha,\beta} A(\mathbf{u}) := \partial_\lambda (A((\mathbf{u}^\lambda)^{\alpha,\beta}))|_{\lambda=0}.$$

Denote also the constraint, when equal to zero, by

$$\begin{aligned} K_{\alpha,\beta}(\mathbf{u}) &:= \mathfrak{L}_{\alpha,\beta} S(\mathbf{u}) \\ &= \frac{1}{2} \sum_{j=1}^m \left((2\alpha + (N-2s)\beta) \|(-\Delta)^{\frac{s}{2}} u_j\|^2 + (2\alpha + N\beta) \|u_j\|^2 \right) \\ &\quad - \frac{(2p\alpha + N\beta)}{2p} \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |u_j u_k|^p \, dx \end{aligned}$$

$$:= \frac{1}{2} \sum_{j=1}^m K_{\alpha,\beta}^Q(u_j) - \frac{(2p\alpha + N\beta)}{2p} \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |u_j u_k|^p dx.$$

Define the real numbers

$$\bar{\mu} := \max(2\alpha + (N - 2s)\beta, 2\alpha + N\beta) \quad \text{and} \quad \underline{\mu} := \min(2\alpha + (N - 2s)\beta, 2\alpha + N\beta).$$

If $\underline{\mu} \neq 0$, write

$$H_{\alpha,\beta} := \left(1 - \frac{\underline{\mu}}{\bar{\mu}}\right) S = \left(S - \frac{1}{\bar{\mu}} K_{\alpha,\beta}\right).$$

Definition 2.1 $\Psi := (\psi_1, \dots, \psi_m)$ is said to be a ground state of (1.1) if

$$-(-\Delta)^s \psi_j - \psi_j + \sum_{k=1}^m a_{jk} |\psi_k|^p |\psi_j|^{p-2} \psi_j = 0, \quad 0 \neq \Psi \in \mathcal{H} \quad (2.2)$$

and it minimizes the problem

$$m_{\alpha,\beta} := \inf_{0 \neq \mathbf{u} \in (H^s)^m} \left\{ S(\mathbf{u}) \text{ s.t. } K_{\alpha,\beta}(\mathbf{u}) = 0 \right\}. \quad (2.3)$$

Moreover, in such a case Ψ is called vector ground state if at least, two components are nonzero.

Remark 2.2 If $\Psi \in \mathcal{H}$ is a solution to (2.2), then $e^{it} \Psi$ is a global solution of the focusing problem (1.1), called standing wave.

For $\alpha, \beta \in \mathbb{R}$, define the sets

$$\begin{aligned} A_{\alpha,\beta}^+ &:= \{\mathbf{u} \in \mathcal{H} \text{ s.t. } S(\mathbf{u}) < m_{\alpha,\beta} \text{ and } K_{\alpha,\beta}(\mathbf{u}) \geq 0\}; \\ A_{\alpha,\beta}^- &:= \{\mathbf{u} \in \mathcal{H} \text{ s.t. } S(\mathbf{u}) < m_{\alpha,\beta} \text{ and } K_{\alpha,\beta}(\mathbf{u}) < 0\}; \\ \mathcal{A} &:= \left\{ (\alpha, \beta) \in \mathbb{R}_+ \times \mathbb{R} \text{ s.t. } \bar{\mu} > 0 \text{ and } \underline{\mu} \geq 0 \right\}. \end{aligned}$$

Remark 2.3 $\mathcal{A} \neq \emptyset$ because $(1, 0) \in \mathcal{A}$. Moreover, if $(\alpha, \beta) \in \mathcal{A}$, then $2\alpha + (N - 2s)\beta > 0$.

Define for $(\psi_1, \dots, \psi_m) \in (H^s)^m$, the functional

$$\begin{aligned} P(\psi) &:= \frac{1}{2p} \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |\psi_j|^p |\psi_k|^p dx \\ &\leq C_{N,p,s} \left(\sum_{j=1}^m \|(-\Delta)^{\frac{s}{2}} \psi_j\|^2 \right)^{\frac{(p-1)N}{2s}} \left(\sum_{j=1}^m \|\psi_j\|^2 \right)^{\frac{N-p(N-2s)}{2s}}. \end{aligned} \quad (2.4)$$

This minimal constant in the previous inequality is determined by the equation

$$\alpha_{N,p,s} := \frac{1}{C_{N,p,s}} = \inf_{\psi \in (H^s)^m} J(\psi), \quad (2.5)$$

where

$$J(\psi) := \frac{\left(\sum_{j=1}^m \|(-\Delta)^{\frac{s}{2}} \psi_j\|^2 \right)^{\frac{(p-1)N}{2s}} \left(\sum_{j=1}^m \|\psi_j\|^2 \right)^{\frac{N-p(N-2s)}{2s}}}{P(\psi)}.$$

Finally, let us give some properties of the free fractional Schrödinger kernel.

Proposition 2.4 Denoting the free operator associated to the fractional Schrödinger system by

$$T(t)\Psi := T_s(t)\Psi := \left(\mathcal{F}^{-1}(e^{-it|y|^{2s}}) * \psi_1, \dots, \mathcal{F}^{-1}(e^{-it|y|^{2s}}) * \psi_m \right),$$

we have

1. $T(t)\Psi - i\gamma \sum_{k=1}^m \int_0^t T_s(t-x)(|u_k|^p |u_1|^{p-2} u_1, \dots, |u_k|^p |u_m|^{p-2} u_m) dx$ is the solution to the problem (1.1);
2. $T^*(t) = T_s(-t)$;
3. $T_s T_t = T_{s+t}$;
4. $T(t)$ is an isometry of L^2 .

In the next sub-section, the main contribution of this note is given.

2.2 Main results

First, we deal with the stationary problem associated to (1.1). The existence of a ground states of (1.1) is claimed.

Theorem 2.5 Take $N \geq 2$, $s \in (0, 1)$, $p_* < p < p^*$ and two real numbers $(\alpha, \beta) \in \mathcal{A}$. Then,

1. $m := m_{\alpha, \beta}$ is nonzero and independent of (α, β) ;
2. there is a minimizer of (2.3), which is some nontrivial solution to (2.2);
3. under the following assumptions

$$a_{jj} = \mu_j \text{ and } a_{jk} = \mu \text{ for } j \neq k \in [1, m],$$

at least two components of the minimizer are non zero if $\mu > 0$ is large enough.

Next, a sharp vector-valued Gagliardo–Nirenberg inequality is studied.

Theorem 2.6 Let $N \geq 2$, $s \in (0, 1)$ and $p_* < p < p^*$. The minimum value for (2.5) is achieved in some minimizer $(\psi_1^*, \dots, \psi_m^*) \in \mathcal{H}$ satisfying

$$\sum_{j=1}^m \|(-\Delta)^{\frac{s}{2}} \psi_j^*\|^2 = 1 = \sum_{j=1}^m \|\psi_j^*\|^2 \text{ and } C_{N,p,s} = P(\psi_1^*, \dots, \psi_m^*).$$

Moreover,

$$\frac{(p-1)N}{s} (-\Delta)^s \psi_j^* + \frac{N-p(N-2s)}{s} \psi_j^* = \alpha_{N,p,s} \sum_{k=1}^m a_{jk} |\psi_k^*|^p |\psi_j^*|^{p-2} \psi_j^*.$$

Now, we are interested on the evolution problem (1.1). First, local well-posedness is claimed.

Theorem 2.7 Let $N = 2$, $s \in (\frac{2}{3}, 1)$ and $\Psi := (\psi_1, \dots, \psi_m) \in \mathcal{H}$. Assume that $2 \leq p < p^*$. Then, there exist $T^* > 0$ and a unique maximal solution to (1.1),

$$\mathbf{u} \in C([0, T^*), \mathcal{H}).$$

Moreover,

1. $\mathbf{u} \in (L^{\frac{4sp}{N(p-1)}}([0, T^*], W^{s,2p}))^{(m)}$;
2. $\mathbf{u}$ satisfies conservation of the energy and the mass;
3. $T^* = \infty$ in the defocusing case ($\gamma = 1$).

In the energy critical case, global well-posedness and scattering of (1.1) hold for small data.

Theorem 2.8 Let $N = 2$, $s \in (\frac{2}{3}, 1)$ and $p = p^*$. Then, there exists $\epsilon_0 > 0$ such that if $(\psi_1, \dots, \psi_m) \in \mathcal{H}$ satisfies $\sum_{j=1}^m \int_{\mathbb{R}^2} |(-\Delta)^{\frac{s}{2}} \psi_j|^2 dx \leq \epsilon_0$, the system (1.1) possesses a unique global solution $\mathbf{u} \in C(\mathbb{R}, \mathcal{H})$ which scatters.

Remark 2.9 Some technical difficulty imposes the condition $p \geq 2$ which requires the restriction $N = 2$ in the two previous results.

Finally, using the potential well method [23], a global well-posedness result about the focusing problem (1.1), is obtained.

Theorem 2.10 Take $N = 2$, $s \in (\frac{2}{3}, 1)$, $\gamma = -1$, $2 \leq p < p^*$, $\Psi \in \mathcal{H}$ and $\mathbf{u} \in C_{T^*}(\mathcal{H})$ be the maximal solution to (1.1). If there exist $(\alpha, \beta) \in \mathcal{A}$ and $t_0 \in [0, T^*)$ such that $\mathbf{u}(t_0) \in A_{\alpha, \beta}^+$, then $\mathbf{u}$ is global.

In what follows, some intermediate estimates are collected.

2.3 Tools

A standard tool to study the Schrödinger problem is the so-called Strichartz estimate [14].

Definition 2.11 A couple of real numbers (q, r) such that $q, r \geq 2$ is said to be admissible if

$$\frac{4N+2}{2N-1} \leq q \leq \infty, \quad \frac{2}{q} + \frac{2N-1}{r} \leq N - \frac{1}{2};$$

or

$$2 \leq q \leq \frac{4N+2}{2N-1}, \quad \frac{2}{q} + \frac{2N-1}{r} < N - \frac{1}{2}.$$

Proposition 2.12 Let $N \geq 2$, $\mu \in \mathbb{R}$, $\frac{N}{2N-1} < s < 1$ and $u_0 \in H_{rd}^\mu$. Then,

$$\|u\|_{L_t^q(L^r) \cap L_t^\infty(\dot{H}^\mu)} \lesssim \|u_0\|_{\dot{H}^\mu} + \|iu - (-\Delta)^s u\|_{L_t^{\tilde{q}'}(L^{\tilde{r}})}, \quad (2.6)$$

whenever (q, r) and $(\tilde{q}, \tilde{r})$ are admissible pairs such that $(q, r, N) \neq (2, \infty, 2)$ and satisfies

$$\frac{2s}{q} + \frac{N}{r} = \frac{N}{2} - \mu, \quad \frac{2s}{\tilde{q}} + \frac{N}{\tilde{r}} = \frac{N}{2} + \mu.$$

Remark 2.13 Taking $\mu = 0$ in the previous result, one obtains the classical Strichartz estimate.

Recall the so-called generalized Pohozaev identity [19].

Proposition 2.14 $\Psi \in \mathcal{H}$ is a solution to (2.2) if and only if $S'(\Psi) = 0$. Moreover, in such a case

$$K_{\alpha, \beta}(\Psi) = 0, \quad \text{for any } (\alpha, \beta) \in \mathbb{R}^2.$$

The following fractional chain rule [6] will be useful.

Lemma 2.15 Let $G \in C^1(\mathbb{C})$, $s \in (0, 1]$ and $1 < p, p_1, p_2 < \infty$ satisfying $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} < \infty$. Then,

$$\| |\nabla|^s G(u) \|_p \lesssim \| G'(u) \|_{p_1} \| |\nabla|^s u \|_{p_2}.$$

Now, we give a Gagliardo–Nirenberg inequality [21].

Lemma 2.16 Let $N \geq 2$, $s \in (0, 1)$ and $1 < p \leq \frac{N}{N-2s}$. Then,

$$\sum_{j,k=1}^m \int_{\mathbb{R}^N} |u_j u_k|^p dx \leq C \left(\sum_{j=1}^m \|(-\Delta)^{\frac{s}{2}} u_j\|^2 \right)^{\frac{(p-1)N}{2s}} \left(\sum_{j=1}^m \|u_j\|^2 \right)^{\frac{N-p(N-2s)}{2s}}.$$

The following Sobolev injections [1, 20] give a meaning to the energy and several computations done in this note.

Lemma 2.17 *Let $N \geq 2$, $p \in (1, \infty)$ and $s \in (0, 1)$. Then,*

1. $W^{\alpha,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ whenever $1 < p < q < \infty$, $\alpha > 0$ and $\frac{1}{p} \leq \frac{1}{q} + \frac{\alpha}{N}$;
2. $H^s(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ for any $q \in [2, \frac{2N}{N-2s}]$;
3. $H^s_{rd}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ for any $q \in (2, \frac{2N}{N-2s})$.

Finally, an absorption result is given.

Lemma 2.18 *Let $T > 0$ and $X \in C([0, T], \mathbb{R}_+)$ such that*

$$X \leq a + bX^\theta \text{ on } [0, T],$$

where $a, b > 0$, $\theta > 1$, $a < (1 - \frac{1}{\theta})(\theta b)^{\frac{1}{\theta-1}}$ and $X(0) \leq (\theta b)^{\frac{1}{\theta-1}}$. Then,

$$X \leq \frac{\theta}{\theta-1} a \text{ on } [0, T].$$

Proof The function $f(x) := bx^\theta - x + a$ is decreasing on $[0, (b\theta)^{\frac{1}{1-\theta}}]$ and increasing on $[(b\theta)^{\frac{1}{1-\theta}}, \infty)$. The assumptions imply that $f((b\theta)^{\frac{1}{1-\theta}}) < 0$ and $f(\frac{\theta}{\theta-1}a) \leq 0$. As $f(X(t)) \geq 0$, $f(0) > 0$ and $X(0) \leq (b\theta)^{\frac{1}{1-\theta}}$, we conclude the proof by a continuity argument. $\square$

3 The stationary problem

The goal of this section is to prove that the elliptic problem (2.2) has a ground state solution which is a vector one in some cases.

3.1 Existence of ground states

Now, we prove Theorem 2.5 about existence of a ground state solution to the stationary problem (2.2).

Remark 3.1 (i) The proof of Theorem 2.5 is based on several lemmas;

(ii) write, for easy notation, $\mathbf{u}^\lambda := (\mathbf{u}^\lambda)^{\alpha,\beta}$, $K := K_{\alpha,\beta}$, $K^Q := K^Q_{\alpha,\beta}$, $\mathbb{E} := \mathbb{E}_{\alpha,\beta}$ and $H := H_{\alpha,\beta}$.

Lemma 3.2 *Let $(\alpha, \beta) \in \mathcal{A}$. Then*

1. $\min(\mathbb{E}H(\mathbf{u}), H(\mathbf{u})) > 0$ for all $0 \neq \mathbf{u} \in \mathcal{H}$;
2. $\lambda \mapsto H(\mathbf{u}^\lambda)$ is increasing.

Proof With the definition,

$$\begin{aligned} H_{\alpha,\beta}(\mathbf{u}) &:= \left(1 - \frac{\mathbb{E}}{\bar{\mu}}\right) S(\mathbf{u}) \\ &= \frac{1}{\bar{\mu}} (\bar{\mu} S(\mathbf{u}) - K(\mathbf{u})) \\ &= \frac{1}{\bar{\mu}} \left[\frac{1}{2} (\bar{\mu} - (2\alpha + (N-2s)\beta)) \sum_{j=1}^m \|(-\Delta)^{\frac{s}{2}} u_j\|^2 + \frac{1}{2} (\bar{\mu} - (2\alpha + N\beta)) \sum_{j=1}^m \|u_j\|^2 \right. \\ &\quad \left. + \frac{1}{2p} (2p\alpha + N\beta - \bar{\mu}) \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |u_j u_k|^p \right]. \end{aligned}$$

Since $\bar{\mu} \geq 0$ and $p > p_*$, we obtain, if $\beta < 0$,

$$\begin{aligned}
2p\alpha + N\beta - \bar{\mu} &= 2\alpha(p-1) + 2s\beta \\
&\geq 2\alpha(p-1) - \frac{2s\alpha}{N} \\
&\geq 2\alpha(p-p_*) \\
&\geq 0.
\end{aligned}$$

Hence, $H_{\alpha,\beta}(\mathbf{u}) > 0$. Moreover, by a direct computation we find

$$\begin{aligned}
\mathfrak{L}H_{\alpha,\beta}(\mathbf{u}) &= \mathfrak{L}\left(1 - \frac{\mathfrak{L}}{\bar{\mu}}\right)S(\mathbf{u}) \\
&= \frac{-1}{\bar{\mu}}(\mathfrak{L} - \bar{\mu})(\mathfrak{L} - \underline{\mu})S(\mathbf{u}) + \underline{\mu}\left(1 - \frac{\mathfrak{L}}{\bar{\mu}}\right)S(\mathbf{u}) \\
&= \frac{-1}{\bar{\mu}}(\mathfrak{L} - \bar{\mu})(\mathfrak{L} - \underline{\mu})S(\mathbf{u}) + \underline{\mu}H_{\alpha,\beta}(\mathbf{u}).
\end{aligned}$$

Since $(\mathfrak{L} - (2\alpha + (N - 2s)\beta))\|(-\Delta)^{\frac{s}{2}}u_j\|^2 = (\mathfrak{L} - (2\alpha + N\beta))\|u_j\|^2 = 0$, we have $(\mathfrak{L} - (2\alpha + (N - 2)\beta))(\mathfrak{L} - (2\alpha + N\beta))\|u_j\|_{H^s}^2 = 0$ and

$$\begin{aligned}
\mathfrak{L}H_{\alpha,\beta}(\mathbf{u}) &\geq \frac{1}{\bar{\mu}}(\mathfrak{L} - \bar{\mu})(\mathfrak{L} - \underline{\mu})\left(\frac{1}{2p} \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |u_j u_k|^p dx\right) \\
&\geq \frac{1}{2p\bar{\mu}}(2p\alpha + N\beta - \bar{\mu})(2p\alpha + N\beta - \underline{\mu}) \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |u_j u_k|^p dx.
\end{aligned}$$

Arguing as previously, it follows that $\mathfrak{L}H_{\alpha,\beta}(\mathbf{u}) > 0$.

The last point is a consequence of the equality $\partial_\lambda H_{\alpha,\beta}(\mathbf{u}^\lambda) = \mathfrak{L}H_{\alpha,\beta}(\mathbf{u}^\lambda)$. □

The next intermediate result is the following.

Lemma 3.3 *Let $(\alpha, \beta) \in \mathcal{A}$ and $0 \neq (u_1^n, \dots, u_m^n)$ be a bounded sequence of $\mathcal{H}$ such that*

$$\lim_n \left(\sum_{j=1}^m K^Q(u_j^n) \right) = 0.$$

Then, there exists $n_0 \in \mathbb{N}$ such that $K(u_1^n, \dots, u_m^n) > 0$ for all $n \geq n_0$.

Proof We have

$$K(u_1^n, \dots, u_m^n) = \frac{1}{2} \sum_{j=1}^m K^Q(u_j^n) - \frac{(2p\alpha + N\beta)}{2p} \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |u_j^n u_k^n|^p dx.$$

Using Lemma 2.16, since $p_* < p < p^*$, $2\alpha + (N - 2s)\beta > 0$, $2\alpha + N\beta \geq 0$ and

$$K^Q(u_j^n) = \left((2\alpha + (N - 2s)\beta) \|(-\Delta)^{\frac{s}{2}} u_j^n\|^2 + (2\alpha + N\beta) \|u_j^n\|^2 \right) \rightarrow 0,$$

we get

$$\sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |u_j^n u_k^n|^p = o \left(\sum_{j=1}^m \|(-\Delta)^{\frac{s}{2}} u_j^n\|^2 \right) = o \left(\sum_{j=1}^m K^Q(u_j^n) \right).$$

Thus,

$$K(u_1^n, \dots, u_m^n) \simeq \frac{1}{2} \sum_{j=1}^m K^Q(u_j^n) > 0.$$

□

Next, we present an auxiliary result.

Lemma 3.4 *Let $(\alpha, \beta) \in \mathcal{A}$. Then,*

$$m_{\alpha, \beta} = \inf_{0 \neq \mathbf{u} \in \mathcal{H}} \{H(\mathbf{u}) \text{ s. t. } K(\mathbf{u}) \leq 0\}.$$

Proof Denoting by a the right hand side of the previous equality, it is sufficient to prove that $m_{\alpha, \beta} \leq a$. Take $\mathbf{u} \in \mathcal{H}$ such that $K(\mathbf{u}) < 0$. Because $\lim_{\lambda \rightarrow -\infty} K^Q(\mathbf{u}^\lambda) = 0$, by the previous Lemma, there exists some $\lambda < 0$ such that $K(\mathbf{u}^\lambda) > 0$. With a continuity argument there exists $\lambda_0 \leq 0$ such that $K(\mathbf{u}^{\lambda_0}) = 0$. Then, since $\lambda \mapsto H(\mathbf{u}^\lambda)$ is increasing, we get

$$m_{\alpha, \beta} \leq H(\mathbf{u}^{\lambda_0}) \leq H(\mathbf{u}).$$

This finishes the proof. $\square$

Proof of Theorem 2.5 Let $(\phi_n) := (\phi_1^n, \dots, \phi_m^n)$ be a minimizing sequence, namely

$$0 \neq (\phi_n) \in \mathcal{H}, \quad K(\phi_n) = 0 \quad \text{and} \quad \lim_n H(\phi_n) = \lim_n S(\phi_n) = m. \quad (3.7)$$

• First step: (ϕ_n) is bounded in $\mathcal{H}$.

First case $\alpha > 0$ and $\beta > 0$. Denoting $\lambda := \frac{\beta}{2\alpha}$, yields

$$\begin{aligned} & \sum_{j=1}^m \|\phi_j^n\|_{H^s}^2 - \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |\phi_j^n \phi_k^n|^p \, dx \\ &= \lambda \left(2s \sum_{j=1}^m \|(-\Delta)^{\frac{s}{2}} \phi_j^n\|^2 - N \sum_{j=1}^m \|\phi_j^n\|_{H^s}^2 + \frac{N}{p} \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |\phi_j^n \phi_k^n|^p \, dx \right) \end{aligned}$$

and

$$\sum_{j=1}^m \|\phi_j^n\|_{H^s}^2 - \frac{1}{p} \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |\phi_j^n \phi_k^n|^p \, dx \rightarrow 2m.$$

Therefore, the following sequences are bounded

$$\begin{aligned} & -2s\lambda \sum_{j=1}^m \|(-\Delta)^{\frac{s}{2}} \phi_j^n\|^2 + \sum_{j=1}^m \|\phi_j^n\|_{H^s}^2 - \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |\phi_j^n \phi_k^n|^p \, dx; \\ & \sum_{j=1}^m \|\phi_j^n\|_{H^s}^2 - \frac{1}{p} \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |\phi_j^n \phi_k^n|^p \, dx. \end{aligned}$$

Thus, for any real number a , the following sequence is also bounded

$$2s\lambda \sum_{j=1}^m \|(-\Delta)^{\frac{s}{2}} \phi_j^n\|^2 + (a-1) \sum_{j=1}^m \|\phi_j^n\|_{H^s}^2 + \left(1 - \frac{a}{p}\right) \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |\phi_j^n \phi_k^n|^p \, dx.$$

Choosing $a \in (1, p)$, it follows that (ϕ_n) is bounded in $\mathcal{H}$. $\square$

Second case $\alpha > 0$ and $\frac{-2\alpha}{N} < \beta \leq 0$. We have

$$\begin{aligned} (\bar{\mu} - \varepsilon)S(\phi_n) &= -s\beta \sum_{j=1}^m \|\phi_j^n\|^2 + (2\alpha(p-1) + 2s\beta) \frac{1}{2p} \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |x|^b |\phi_j^n \phi_k^n|^p \, dx \\ &\geq (2\alpha(p-1) + 2s\beta) \frac{1}{2p} \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |x|^b |\phi_j^n \phi_k^n|^p \, dx. \end{aligned}$$

Moreover, if $\beta < 0$, $\bar{\mu} = 2\alpha + (N - 2s)\beta$. Then, since $\underline{\mu} \geq 0$ and $p > p_c$, we obtain $2\alpha(p - 1) + 2s\beta > 0$. Because $K(\phi_n) = 0$, this implies that

$$\begin{aligned} (\bar{\mu} + (2\alpha(p - 1) + 2s\beta))S(\phi_n) &= (\bar{\mu} - \mathfrak{L})S(\phi_n) + (2\alpha(p - 1) + 2s\beta)S(\phi_n) + \mathfrak{L}S(\phi_n) \\ &\geq (2\alpha(p - 1) + 2s\beta)\frac{1}{2}\sum_{j=1}^m \|\phi_j^n\|_{H^s}^2. \end{aligned}$$

Hence, ϕ_n is bounded in $\mathcal{H}$.

Third case $\alpha = 0$. Since $(\alpha, \beta) \in \mathcal{A}$, it follows that $\beta > 0$. Thus,

$$\sum_{j=1}^m \|(-\Delta)^{\frac{s}{2}} \phi_j^n\|^2 \lesssim H_{0,\beta}(\phi^n) \rightarrow m.$$

Assume that $\lim_n \sum_{j=1}^m \|\phi_j^n\| = \infty$. Then, taking into account Lemma 2.16, we get

$$\sum_{j=1}^m \|\phi_j^n\|^2 \lesssim K^Q(\phi_n) = \frac{N}{2p} \sum_{j,k=1}^m \int_{\mathbb{R}^N} |u_j u_k|^p dx \lesssim \left(\sum_{j=1}^m \|\phi_j^n\|^2 \right)^{\frac{N-p(N-2s)}{2s}}.$$

This is a contradiction because $\frac{N-p(N-2s)}{2s} < 1$.

• Second step: the limit of (ϕ_n) is nonzero and $m > 0$.

Taking into account the compact injection in Lemma 2.17, take

$$(\phi_1^n, \dots, \phi_m^n) \rightharpoonup \phi = (\phi_1, \dots, \phi_m) \quad \text{in } \mathcal{H}$$

and

$$(\phi_1^n, \dots, \phi_m^n) \rightarrow (\phi_1, \dots, \phi_m) \quad \text{in } (L^{2p})^{(m)}.$$

The equality $K(\phi_n) = 0$ implies that

$$(2\alpha + (N - 2s)\beta) \sum_{j=1}^m \|(-\Delta)^{\frac{s}{2}} \phi_j^n\|^2 + (2\alpha + N\beta) \sum_{j=1}^m \|\phi_j^n\|^2 = \frac{2\alpha p + N\beta}{p} \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |\phi_j^n \phi_k^n|^p dx.$$

Assume that $\phi = 0$. Using Hölder inequality we obtain

$$\|\phi_j^n \phi_k^n\|_p^p \leq \|\phi_j^n\|_{2p}^p \|\phi_k^n\|_{2p}^p \rightarrow \|\phi_j\|_{2p}^p \|\phi_k\|_{2p}^p = 0.$$

Now, by Lemma 3.3 yields $K(\phi_n) > 0$ for large n . This contradiction implies that

$$\phi \neq 0.$$

By the lower semi continuity of the H^s norm, we have

$$\begin{aligned} 0 &= \liminf_n K(\phi_n) \\ &\geq \frac{2\alpha + (N - 2s)\beta}{2} \liminf_n \sum_{j=1}^m \|(-\Delta)^{\frac{s}{2}} \phi_j^n\|^2 + \frac{2\alpha + N\beta}{2} \liminf_n \sum_{j=1}^m \|\phi_j^n\|^2 \\ &\quad - \frac{2\alpha p + N\beta}{2p} \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |\phi_j \phi_k|^p dx \\ &\geq K(\phi). \end{aligned}$$

Similarly, we have $H(\phi) \leq m$. Moreover, thanks to Lemma 3.4, we assume that $K(\phi) = 0$ and $S(\phi) = H(\phi) \leq m$. Therefore, ϕ is a minimizer satisfying (3.7) and using previous computation

$$m = H(\phi) > 0.$$

- Third step: the limit ϕ is a solution to (2.2).

There is a Lagrange multiplier $\eta \in \mathbb{R}$ such that $S'(\phi) = \eta K'(\phi)$. Thus

$$0 = K(\phi) = \mathbb{E}S(\phi) = \langle S'(\phi), \mathbb{E}(\phi) \rangle = \eta \langle K'(\phi), \mathbb{E}(\phi) \rangle = \eta \mathbb{E}K(\phi) = \eta \mathbb{E}^2 S(\phi).$$

By a previous computation, we have

$$\begin{aligned} (I) &:= -\mathbb{E}^2 S(\phi) - (2\alpha + (N - 2s)\beta)(2\alpha + N\beta)S(\phi) \\ &= -(\mathbb{E} - (2\alpha + (N - 2s)\beta))(\mathbb{E} - (2\alpha + N\beta))S(\phi) \\ &= \frac{1}{2p} 2\alpha(p - 1)(2\alpha(p - 1) + 2s\beta) \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |\phi_j \phi_k|^p dx \\ &> 0. \end{aligned}$$

Therefore, $\mathbb{E}^2 S(\phi) < 0$. Thus, $\eta = 0$ and $S'(\phi) = 0$. So, ϕ is a ground state and m is independent of (α, β) .

3.2 Existence of vector ground states

Now, we present a proof of the last part of Theorem 2.5, which deals with the existence of a more than one non zero component ground state for large μ . Take $\phi := (\phi_1, \dots, \phi_m)$ such that $(0, \dots, \phi_j, \dots, 0)$ is a ground state solution to (2.2). So, ϕ_j satisfies

$$-(-\Delta)^s \phi_j - \phi_j + \mu_j \phi_j |\phi_j|^{2p-2} = 0 \quad \text{and} \quad \sum_{j=1}^m \|\phi_j\|_{H^s}^2 = \sum_{j=1}^m \mu_j \|\phi_j\|_{2p}^{2p}.$$

Moreover, by Pohozaev identity it follows that

$$(N - 2s) \sum_{j=1}^m \|(-\Delta)^{\frac{s}{2}} \phi_j\|^2 + N \sum_{j=1}^m \|\phi_j\|^2 = \frac{N}{p} \sum_{j=1}^m \mu_j \|\phi_j\|_{2p}^{2p}.$$

Collecting the previous identities, we may write

$$\sum_{j=1}^m \|\phi_j\|^2 = \left(1 - \frac{N}{2s} + \frac{N}{2sp}\right) \sum_{j=1}^m \mu_j \|\phi_j\|_{2p}^{2p}. \quad (3.8)$$

Setting, for $t > 0$, the real variable function $\gamma(t) := (\phi_1(\cdot), \dots, \phi_m(\cdot))$, we compute

$$\begin{aligned} K_{0,1}(\gamma(t)) &= \frac{N - 2s}{2} t^{N-2s} \sum_{j=1}^m \|(-\Delta)^{\frac{s}{2}} \phi_j\|^2 + \frac{N}{2} t^N \sum_{j=1}^m \|\phi_j\|^2 - \frac{N}{2p} t^N \sum_{j=1}^m \mu_j \|\phi_j\|_{2p}^{2p} \\ &\quad - \frac{N}{2p} \mu t^N \sum_{1 \leq k \neq j \leq m} \int_{\mathbb{R}^N} |\phi_j \phi_k|^p dx \end{aligned}$$

and

$$\begin{aligned} g(t) &:= S(\gamma(t)) \\ &= \frac{1}{2} t^{N-2s} \sum_{j=1}^m \|(-\Delta)^{\frac{s}{2}} \phi_j\|^2 + \frac{1}{2} t^N \sum_{j=1}^m \|\phi_j\|^2 - \frac{1}{2p} t^N \sum_{j=1}^m \mu_j \|\phi_j\|_{2p}^{2p} \\ &\quad - \frac{1}{2p} \mu t^N \sum_{1 \leq k \neq j \leq m} \int_{\mathbb{R}^N} |\phi_j \phi_k|^p dx. \end{aligned}$$

Thanks to (3.8), $g(t) < 0$ for large t . Then, since $g(0) \geq 0$, the maximum of $g(t)$ for $t \geq 0$ is achieved at $\bar{t} > 0$. Precisely $g(\bar{t}) = \max_{t \geq 0} g(t)$. Moreover,

$$g'(\bar{t}) = 0 = \bar{t}^{N-1} \left(\frac{N-2s}{2} \bar{t}^{-2s} \sum_{j=1}^m \|(-\Delta)^{\frac{s}{2}} \phi_j\|^2 + \frac{N}{2} \sum_{j=1}^m \|\phi_j\|^2 - \frac{N}{2p} \sum_{j=1}^m \mu_j \|\phi_j\|_{2p}^{2p} - \frac{N}{2p} \mu \sum_{1 \leq j \neq k \leq m} \int_{\mathbb{R}^N} |\phi_j \phi_k|^p dx \right).$$

Then,

$$\bar{t} = \left(\frac{N-2s}{N} \right)^{\frac{1}{2s}} \frac{\left(\sum_{j=1}^m \|(-\Delta)^{\frac{s}{2}} \phi_j\|^2 \right)^{\frac{1}{2s}}}{\left(\frac{1}{p} \sum_{j=1}^m \mu_j \|\phi_j\|_{2p}^{2p} + \frac{1}{p} \mu \sum_{1 \leq j \neq k \leq m} \int_{\mathbb{R}^N} |\phi_j \phi_k|^p dx - \sum_{j=1}^m \|\phi_j\|^2 \right)^{\frac{1}{2s}}}.$$

Thus, the maximum value of g is

$$\begin{aligned} g(\bar{t}) &= \max_{t \geq 0} g(t) \\ &= \frac{2s(N-2s)^{\frac{N-2s}{2s}}}{N^{\frac{N}{2s}}} \frac{\left(\sum_{j=1}^m \|(-\Delta)^{\frac{s}{2}} \phi_j\|^2 \right)^{\frac{N}{2s}}}{\left(\frac{1}{p} \sum_{j=1}^m \mu_j \|\phi_j\|_{2p}^{2p} + \frac{1}{p} \mu \sum_{1 \leq j \neq k \leq m} \int_{\mathbb{R}^N} |\phi_j \phi_k|^p dx - \sum_{j=1}^m \|\phi_j\|^2 \right)^{\frac{N-2s}{2s}}}. \end{aligned}$$

Now, from the previous equality via the fact that $K_{0,1}(\gamma(t)) < 0$, for large μ it follows that

$$m \leq S(\phi_1(\frac{\cdot}{\bar{t}}), \dots, \phi_m(\frac{\cdot}{\bar{t}})) < \min \left(S(\phi_1, 0, \dots, 0), S(0, \phi_2, \dots, 0), \dots, S(0, 0, \dots, \phi_m) \right) \leq m.$$

This contradiction completes the proof.

3.3 Gagliardo–Nirenberg inequality

In this sub-section, we prove Theorem 2.6 on the best constant $C_{N,p,s}$ in the Gagliardo–Nirenberg inequality (2.4).

For $\psi_j \in H^s$ and $v, \mu > 0$, denote the scaling $\psi_j^{v,\mu} = v\psi_j(\mu \cdot)$ and compute

$$\begin{aligned} \|\psi_j^{v,\mu}\|^2 &= v^2 \mu^{-N} \|\psi_j\|^2, \quad \|(-\Delta)^{\frac{s}{2}} \psi_j^{v,\mu}\|^2 = v^2 \mu^{2s-N} \|(-\Delta)^{\frac{s}{2}} \psi_j\|^2; \\ \|\psi_j^{v,\mu}\|_{2p}^{2p} &= v^{2p} \mu^{-N} \|\psi_j\|_{2p}^{2p}, \quad \|\psi_j^{v,\mu} \psi_k^{v,\mu}\|_p^p = v^{2p} \mu^{-N} \|\psi_j \psi_k\|_p^p. \end{aligned}$$

Therefore, $J(\psi_1^{v,\mu}, \dots, \psi_m^{v,\mu}) = J(\psi_1, \dots, \psi_m)$. Let $(\psi_1^n, \dots, \psi_m^n)$ be a minimizing sequence for (2.5), supposed to be radial decreasing with classical rearrangement argument [8] and

$$v_n = \frac{\left(\sum_{j=1}^m \|\psi_j^n\|^2 \right)^{\frac{N-2s}{4s}}}{\left(\sum_{j=1}^m \|(-\Delta)^{\frac{s}{2}} \psi_j^n\|^2 \right)^{\frac{N}{4s}}}, \quad \mu_n = \frac{\left(\sum_{j=1}^m \|\psi_j^n\|^2 \right)^{\frac{1}{2s}}}{\left(\sum_{j=1}^m \|(-\Delta)^{\frac{s}{2}} \psi_j^n\|^2 \right)^{\frac{1}{2s}}}.$$

By the above scaling invariance, $\{((\psi_1^n)^{v_n, \mu_n}, \dots, (\psi_m^n)^{v_n, \mu_n})\}$ is also a minimizing sequence. Moreover, for each $n \in \mathbb{N}$,

$$\sum_{j=1}^m \|(-\Delta)^{\frac{s}{2}} (\psi_j^n)^{v_n, \mu_n}\|^2 = \sum_{j=1}^m \|(\psi_j^n)^{v_n, \mu_n}\|^2 = 1.$$

$(\psi_j^n)^{v_n, \mu_n}$ is also a minimizing sequence which is bounded in $\mathcal{H}$. Therefore, there exist $(\psi_1^*, \dots, \psi_m^*) \in \mathcal{H}$ and a sub-sequence, denoted $((\psi_1^n)^{v_n, \mu_n}, \dots, (\psi_m^n)^{v_n, \mu_n})$, such that the weak convergence holds

$$((\psi_1^n)^{v_n, \mu_n}, \dots, (\psi_m^n)^{v_n, \mu_n}) \rightharpoonup (\psi_1^*, \dots, \psi_m^*) \text{ in } \mathcal{H}.$$

Since $1 < p < \frac{N}{N-2s}$, taking account of the compact Sobolev injection in Lemma 2.17,

$$((\psi_1^n)^{v_n, \mu_n}, \dots, (\psi_m^n)^{v_n, \mu_n}) \rightarrow (\psi_1^*, \dots, \psi_m^*) \text{ in } (L^{2p})^{(m)}.$$

Thanks to the L^2 norm weak lower semi-continuity,

$$\sum_{j=1}^m \|(-\Delta)^{\frac{s}{2}} \psi_j^*\|^2 \leq 1 \quad \text{and} \quad \sum_{j=1}^m \|\psi_j^*\|^2 \leq 1.$$

The strong convergence in L^{2p} implies that

$$P((\psi_1^n)^{v_n, \mu_n}, \dots, (\psi_m^n)^{v_n, \mu_n}) \rightarrow P(\psi_1^*, \dots, \psi_m^*).$$

Hence,

$$\alpha \leq J(\psi_1^*, \dots, \psi_m^*) \leq \frac{1}{P(\psi_1^*, \dots, \psi_m^*)} = \lim_{n \rightarrow \infty} J((\psi_1^n)^{v_n, \mu_n}, \dots, (\psi_m^n)^{v_n, \mu_n}) = \alpha.$$

Therefore,

$$\left(\sum_{j=1}^m \|(-\Delta)^{\frac{s}{2}} \psi_j^*\|^2 \right)^{\frac{(p-1)N}{2s}} \left(\sum_{j=1}^m \|\psi_j^*\|^2 \right)^{\frac{N-p(N-2s)}{2s}} = 1$$

and consequently

$$\sum_{j=1}^m \|(-\Delta)^{\frac{s}{2}} \psi_j^*\|^2 = \sum_{j=1}^m \|\psi_j^*\|^2 = 1.$$

Combined with weak convergence, one concludes that

$$((\psi_1^n)^{v_n, \mu_n}, \dots, (\psi_m^n)^{v_n, \mu_n}) \rightarrow (\psi_1^*, \dots, \psi_m^*) \text{ in } \mathcal{H}.$$

Thus, $\alpha = J(\psi_1^*, \dots, \psi_m^*)$. It follows from the previous equality that $(\psi_1^*, \dots, \psi_m^*)$ is a minimizer of J in $\mathcal{H}$ and satisfies the Euler–Lagrange equation

$$\frac{d}{d\epsilon} J(\psi_1^* + \epsilon v_1, \dots, \psi_m^* + \epsilon v_m)|_{\epsilon=0} = 0 \quad \text{for all } (v_1, \dots, v_m) \in (C_0^\infty(\mathbb{R}^N))^{(m)}.$$

Taking account of the equalities $\sum_{j=1}^m \|\psi_j^*\|^2 = \sum_{j=1}^m \|(-\Delta)^{\frac{s}{2}} \psi_j^*\|^2 = 1$, we see that

$$\frac{(p-1)N}{s} (-\Delta)^s \psi_j^* + \frac{N-p(N-2s)}{s} \psi_j^* = \alpha \sum_{k=1}^m a_{jk} |\psi_k^*|^p |\psi_j^*|^{p-2} \psi_j^*.$$

4 Well-posedness

In what follows, we prove Theorem 2.7, therefore, in all this section we take $N = 2$. The proof contains two steps. First, we prove the existence of a unique local solution to (1.1), second, we establish the global existence. Since the sign of the non-linearity has no local effect, we take $\gamma = 1$.

4.1 Local existence and uniqueness

We use a standard fixed point argument. For $T, \rho > 0$, denote the space $\mathcal{E}_{T,\rho} := \{u \in (C([0, T], H^s) \cap L^{\frac{4sp}{N(p-1)}}([0, T], W^{s,2p}))^{(m)}, \text{ with } \|\mathbf{u}\|_T := \sum_{j=1}^m (\|u_j\|_{L_T^\infty(H^s)} + \|u_j\|_{L_T^{\frac{4sp}{N(p-1)}}(W^{s,2p})}) \leq \rho\}$ and with the complete distance

$$d\left((h_1, \dots, h_m), (g_1, \dots, g_m)\right) := \sum_{i=1}^m \|h_i - g_i\|_{L_T^\infty(L^2) \cap L_T^{\frac{4sp}{N(p-1)}}(L^{2p})}.$$

Define the function

$$\phi(\mathbf{u}) := T(\cdot)\Psi - i \sum_{k=1}^m \int_0^{\cdot} T(\cdot - s)(|u_k|^p |u_1|^{p-2} u_1, \dots, |u_k|^p |u_m|^{p-2} u_m) ds.$$

We prove the existence of some small $T, \rho > 0$ such that ϕ is a contraction of $\mathcal{E}_{T,\rho}$. Taking $\mathbf{u}, \mathbf{v} \in E_T$, applying the Strichartz estimate (2.6), we get

$$d(\phi(\mathbf{u}), \phi(\mathbf{v})) \lesssim \sum_{j,k=1}^m \| |u_k|^p |u_j|^{p-2} u_j - |v_k|^p |v_j|^{p-2} v_j \|_{L_T^{\frac{4sp}{p(4s-N)+N}}(L^{\frac{2p}{2p-1}})}.$$

To derive the contraction, consider the function

$$f_{j,k} : \mathbb{C}^m \rightarrow \mathbb{C}, \quad (u_1, \dots, u_m) \mapsto |u_k|^p |u_j|^{p-2} u_j.$$

Since $p \geq 2$, by the mean value Theorem we see that

$$|f_{j,k}(\mathbf{u}) - f_{j,k}(\mathbf{v})| \lesssim \max \{ |u_k|^{p-1} |u_j|^{p-1} + |u_k|^p |u_j|^{p-2}, |v_k|^p |v_j|^{p-2} + |v_k|^{p-1} |v_j|^{p-1} \} |\mathbf{u} - \mathbf{v}|.$$

Using Hölder inequality, Sobolev embedding and denoting the quantity

$$(\mathcal{I}) := \|f_{j,k}(\mathbf{u}) - f_{j,k}(\mathbf{v})\|_{L_T^{\frac{4sp}{p(4s-N)+N}}(L^{\frac{2p}{2p-1}})},$$

we compute via a symmetry argument

$$\begin{aligned} (\mathcal{I}) &\lesssim \|(|u_k|^{p-1} |u_j|^{p-1} + |u_k|^p |u_j|^{p-2})|\mathbf{u} - \mathbf{v}|\|_{L_T^{\frac{4sp}{p(4s-N)+N}}(L^{\frac{2p}{2p-1}})} \\ &\lesssim \|\mathbf{u} - \mathbf{v}\|_{L_T^{\frac{4sp}{N(p-1)}}(L^{2p})} \| |u_k|^{p-1} |u_j|^{p-1} + |u_k|^p |u_j|^{p-2} \|_{L_T^{\frac{2sp}{2sp-N(p-1)}}(L^{\frac{p}{p-1}})} \\ &\lesssim T^{\frac{2sp-N(p-1)}{2sp}} \|\mathbf{u} - \mathbf{v}\|_{L_T^{\frac{4sp}{N(p-1)}}(L^{2p})} \| |u_k|^{p-1} |u_j|^{p-1} + |u_k|^p |u_j|^{p-2} \|_{L_T^\infty(L^{\frac{p}{p-1}})} \\ &\lesssim T^{\frac{2sp-N(p-1)}{2sp}} \|\mathbf{u} - \mathbf{v}\|_{L_T^{\frac{4sp}{N(p-1)}}(L^{2p})} \left(\|u_k\|_{L_T^\infty(L^{2p})}^{p-1} \|u_j\|_{L_T^\infty(L^{2p})}^{p-1} + \|u_k\|_{L_T^\infty(L^{2p})}^p \|u_j\|_{L_T^\infty(L^{2p})}^{p-2} \right) \\ &\lesssim T^{\frac{2sp-N(p-1)}{2sp}} \|\mathbf{u} - \mathbf{v}\|_{L_T^{\frac{4sp}{N(p-1)}}(L^{2p})} \left(\|u_k\|_{L_T^\infty(H^s)}^{p-1} \|u_j\|_{L_T^\infty(H^s)}^{p-1} + \|u_k\|_{L_T^\infty(H^s)}^p \|u_j\|_{L_T^\infty(H^s)}^{p-2} \right). \end{aligned}$$

Then,

$$\sum_{k,j=1}^m \|f_{j,k}(\mathbf{u}) - f_{j,k}(\mathbf{v})\|_{L_T^{\frac{4sp}{p(4s-N)+N}}(L^{\frac{2p}{2p-1}})} \lesssim T^{\frac{2sp-N(p-1)}{2sp}} \rho^{2(p-1)} d(\mathbf{u}, \mathbf{v}).$$

Thus, for $T > 0$ small enough, ϕ is a contraction satisfying

$$d(\phi(\mathbf{u}), \phi(\mathbf{v})) \lesssim T^{\frac{2sp-N(p-1)}{2sp}} \rho^{2(p-1)} d(\mathbf{u}, \mathbf{v}). \quad (4.9)$$

Taking in the last inequality $\mathbf{v} = 0$, yields

$$\|\phi(\mathbf{u})\|_{L_T^{\frac{4p}{p(4s-N)+N}}(L^{\frac{2p}{2p-1}})} \lesssim T^{\frac{2sp-N(p-1)}{2sp}} \rho^{2p-1} + \|\Psi\|.$$

It remains to estimate

$$(II) := \|(-\Delta)^{\frac{s}{2}} \mathbf{u}\|_{L_T^\infty(L^2) \cap L_T^{\frac{4sp}{N(p-1)}}(L^{2p})}.$$

Using the fractional chain rule via Strichartz estimate and Hölder inequality, we get for $\theta := \frac{4sp(p-1)}{2sp-N(p-1)}$ and $2C\|\Psi\|_{H^s} < \rho$,

$$\begin{aligned} (II) &\leq C\|\Psi\|_{H^s} + C \sum_{k,j=1}^m \left\| (-\Delta)^{\frac{s}{2}} (|u_k|^p |u_j|^{p-2} u_j) \right\|_{L_T^{\frac{4sp}{p(4s-N)+N}}(L^{\frac{2p}{2p-1}})} \\ &\leq \frac{\rho}{2} + C \sum_{k,j=1}^m \|(-\Delta)^{\frac{s}{2}} u_j\|_{L_T^{\frac{4sp}{N(p-1)}}(L^{2p})} \|u_k\|_{L_T^\theta(L^{2p})}^{p-2} \|u_j\|_{L_T^\theta(L^{2p})}^p \\ &\leq \frac{\rho}{2} + C \|\mathbf{u}\|_{L_T^{\frac{4sp}{N(p-1)}}(W^{s,2p})} \|\mathbf{u}\|_{L_T^\theta(L^{2p})}^{2(p-1)} \\ &\leq \frac{\rho}{2} + C \|\mathbf{u}\|_{L_T^{\frac{4sp}{N(p-1)}}(W^{s,2p})} \|\mathbf{u}\|_{L_T^\infty(H^s)}^{2(p-1)} T^{\frac{2(p-1)}{\theta}} \\ &\leq \frac{\rho}{2} + C \rho^{2p-1} T^{\frac{2(p-1)}{\theta}}. \end{aligned}$$

Since $1 < p < p^*$ if $N > 2$, ϕ is a contraction of $\mathcal{E}_{T,\rho}$ for some $T > 0$ small enough. Using (4.9), uniqueness follows for small time and then for all time with a translation argument.

4.2 Global existence

In the sub-critical defocusing case, the global existence is a consequence of energy conservation and previous calculations. Let $\mathbf{u} \in C([0, T^*), \mathcal{H})$ be the unique maximal solution of (1.1). We prove that $\mathbf{u}$ is global. By contradiction, suppose that $T^* < \infty$. Consider for $0 < s < T^*$, the problem

$$(\mathcal{P}_s) \begin{cases} i\dot{v}_j - (-\Delta)^s v_j = \sum_{k,j=1}^m |v_k|^p |v_j|^{p-2} v_j; \\ v_j(s, \cdot) = u_j(s, \cdot). \end{cases}$$

By the same arguments used in the local existence, we can prove the existence of a real $\tau > 0$ and a solution $\mathbf{v} = (v_1, \dots, v_m)$ to $(\mathcal{P}_s)$ on $C([s, s + \tau], \mathcal{H})$. Using the conservation of energy, we see that τ does not depend on s . Thus, if we let s be close to T^* such that $T^* < s + \tau$, this fact contradicts the maximality of T^* .

5 Global existence and scattering in the critical case

In this section, we establish the global existence of a solution to (1.1) in the critical case $p = p^*$ for small data as claimed in Theorem 2.8, therefore, in all this section we take $N = 2$.

Several norms are considered in the analysis of the critical case. Letting $I \subset \mathbb{R}$ be a time slab, we define

$$\begin{aligned} W(I) &:= L^{\frac{2(N+2s)}{N-2s}} \left(I, L_{rd}^{\frac{2N(N+2s)}{N^2+4s^2}} \right) \cap C(I, L_{rd}^2); \\ M(I) &:= L^{\frac{2(N+2s)}{N-2s}} \left(I, \dot{W}_{rd}^{s, \frac{2N(N+2s)}{N^2+4s^2}} \right) \cap C(I, \dot{H}_{rd}^s); \end{aligned}$$

$$S(I) := L^{\frac{2(N+2s)}{N-2s}} \left(I, L_{rd}^{\frac{2(N+2s)}{N-2s}} \right).$$

Remark 5.1 Thanks to Sobolev embedding, clearly $M(I) \hookrightarrow S(I)$.

Let us give an auxiliary result.

Proposition 5.2 Let $p = p^*$, $\Psi := (\psi_1, \dots, \psi_m) \in \mathcal{H} := (\dot{H}_{rd}^s)^{(m)}$. Then, there exists $\delta > 0$ such that for any interval $I = [0, T)$, if

$$\sum_{j=1}^m \|e^{i \cdot (-\Delta)^s} \psi_j\|_{S(I)} < \delta,$$

there exists a unique solution $\mathbf{u} \in C(I, \mathcal{H})$ of (1.1) which satisfies $\mathbf{u} \in M(I)^{(m)}$. Moreover,

$$\sum_{j=1}^m \|u_j\|_{S(I)} \leq 2\delta.$$

Proof The proposition follows from a contraction mapping argument. Let us introduce the function

$$\phi(\mathbf{u}) := T(\cdot)\Psi - i \sum_{k=1}^m \int_0^\cdot T(\cdot - s) \left(|u_k|^{\frac{N}{N-2s}} |u_1|^{\frac{4s-N}{N-2s}} u_1, \dots, |u_k|^{\frac{N}{N-2s}} |u_m|^{\frac{4s-N}{N-2s}} u_m \right) ds.$$

Define the set

$$X_a := \left\{ \mathbf{u} \in (M(I) \cap W(I))^{(m)}; \|\mathbf{u}\|_{M(I)^{(m)}} \leq a \right\}$$

where $a > 0$ is sufficiently small to fix later. Using Strichartz estimate, we get

$$\|\phi(\mathbf{u}) - \phi(\mathbf{v})\|_{W(I)} \lesssim \sum_{j,k=1}^m \|f_{j,k}(\mathbf{u}) - f_{j,k}(\mathbf{v})\|_{L_T^2(L^{\frac{2N}{N+2s}})} := (\mathcal{I}_1).$$

The Hölder inequality and Sobolev embedding yield

$$\begin{aligned} (\mathcal{I}_1) &\lesssim \| |\mathbf{u} - \mathbf{v}| \left(|u_k|^{\frac{2s}{N-2s}} |u_j|^{\frac{2s}{N-2s}} + |u_k|^{\frac{N}{N-2s}} |u_j|^{\frac{4s-N}{N-2s}} \right) \|_{L_T^2(L^{\frac{2N}{N+2s}})} \\ &\lesssim \|\mathbf{u} - \mathbf{v}\|_{L_T^{\frac{2(N+2s)}{N-2s}}(L^{\frac{2N(N+2s)}{N^2+4s^2}})} \left(\|u_k\|_{L_T^{\frac{2(N+2s)}{N-2s}}(L^{\frac{2(N+2s)}{N-2s}})}^{\frac{2s}{N-2s}} \|u_j\|_{L_T^{\frac{2(N+2s)}{N-2s}}(L^{\frac{2(N+2s)}{N-2s}})}^{\frac{2s}{N-2s}} \right. \\ &\quad \left. + \|u_k\|_{L_T^{\frac{2(N+2s)}{N-2s}}(L^{\frac{2(N+2s)}{N-2s}})}^{\frac{N}{N-2s}} \|u_j\|_{L_T^{\frac{2(N+2s)}{N-2s}}(L^{\frac{2(N+2s)}{N-2s}})}^{\frac{4s-N}{N-2s}} \right) \\ &\lesssim \|\mathbf{u} - \mathbf{v}\|_{(W(I))^m} \|\mathbf{u}\|_{(S(I))^m}^{\frac{4s}{N-2s}}. \end{aligned}$$

Then,

$$\|\phi(\mathbf{u}) - \phi(\mathbf{v})\|_{(W(I))^m} \lesssim a^{\frac{4s}{N-2s}} \|\mathbf{u} - \mathbf{v}\|_{(W(I))^m}.$$

Using the fractional chain rule via Strichartz estimate and Hölder inequality, we find

$$\begin{aligned} \|\phi(\mathbf{u})\|_{(M(I))^m} &\leq \delta + C \sum_{k,j=1}^m \left\| (-\Delta)^{\frac{s}{2}} \left(|u_k|^{\frac{N}{N-2s}} |u_j|^{\frac{4s-N}{N-2s}} u_j \right) \right\|_{L_T^2(L^{\frac{2N}{N+2s}})} \\ &\leq \delta + C \|(-\Delta)^{\frac{s}{2}} \mathbf{u}\|_{L_T^{\frac{2(N+2s)}{N-2s}}(L^{\frac{2N(N+2s)}{N^2+4s^2}})} \left(\|u_k\|_{L_T^{\frac{2(N+2s)}{N-2s}}(L^{\frac{2(N+2s)}{N-2s}})}^{\frac{2s}{N-2s}} \|u_j\|_{L_T^{\frac{2(N+2s)}{N-2s}}(L^{\frac{2(N+2s)}{N-2s}})}^{\frac{2s}{N-2s}} \right. \\ &\quad \left. + \|u_k\|_{L_T^{\frac{2(N+2s)}{N-2s}}(L^{\frac{2(N+2s)}{N-2s}})}^{\frac{N}{N-2s}} \|u_j\|_{L_T^{\frac{2(N+2s)}{N-2s}}(L^{\frac{2(N+2s)}{N-2s}})}^{\frac{4s-N}{N-2s}} \right) \end{aligned}$$

$$\begin{aligned}
 & + \|u_k\|_{L_T^{\frac{N}{N-2s}}(L^{\frac{2(N+2s)}{N-2s}})}^{\frac{4s-N}{N-2s}} \|u_j\|_{L_T^{\frac{4s-N}{N-2s}}(L^{\frac{2(N+2s)}{N-2s}})}^{\frac{2(N+2s)}{N-2s}} \\
 & \leq \delta + C \|\mathbf{u}\|_{(M(I))^m} \|\mathbf{u}\|_{(S(I))^m}^{\frac{4s}{N-2s}} \\
 & \leq \delta + Ca^{1+\frac{4s}{N-2s}}.
 \end{aligned}$$

By a classical Picard argument, for small $a > 0$, there exists $\mathbf{u} \in X_a$, a solution to (1.1) satisfying

$$\|\mathbf{u}\|_{(S(I))^m} \leq 2\delta.$$

□

We are ready to prove Theorem 2.8.

Proof of Theorem 2.8 Let us start by proving global well-posedness. Using the previous proposition via the fact that

$$\|e^{i \cdot (-\Delta)^s} \Psi\|_{(S(I))^m} \lesssim \|e^{i \cdot (-\Delta)^s} \Psi\|_{(M(I))^m} \lesssim \|\Psi\|_{\dot{\mathcal{H}}},$$

it suffices to prove that $\|\mathbf{u}\|_{\dot{\mathcal{H}}}$ remains small on the whole interval of existence of $\mathbf{u}$. Letting the functional ξ be defined for $\mathbf{u} \in \mathcal{H}$ by

$$\xi(\mathbf{u}) = \sum_{j=1}^m \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_j|^2 dx,$$

we write using the conservation identities and Lemma 2.16,

$$\begin{aligned}
 \|\mathbf{u}\|_{\dot{\mathcal{H}}}^2 & \leq 2E(\Psi) + \frac{N-2s}{N} \sum_{j,k=1}^m \int_{\mathbb{R}^N} |u_j(x, t)|^{\frac{N}{N-2s}} |u_k(x, t)|^{\frac{N}{N-2s}} dx \\
 & \leq C(\xi(\Psi) + \xi(\Psi)^{\frac{N}{N-2s}}) + C \left(\sum_{j=1}^m \|(-\Delta)^{\frac{s}{2}} u_j\|_2^2 \right)^{\frac{N}{N-2s}} \\
 & \leq C(\xi(\Psi) + \xi(\Psi)^{\frac{N}{N-2s}}) + C \|\mathbf{u}\|_{\dot{\mathcal{H}}}^{\frac{2N}{N-2s}}.
 \end{aligned}$$

Therefore, by Lemma 2.18, if $\xi(\Psi)$ is sufficiently small, then $\mathbf{u}$ stays small in the $\dot{\mathcal{H}}$ norm. We finish this section by proving scattering. Using the previous proposition, it follows that

$$u \in M(\mathbb{R}).$$

Taking account of previous computation and denoting $\mathbf{v}(t) := T(-t)\mathbf{u}(t)$, we get for $t, t' \rightarrow \infty$,

$$\begin{aligned}
 \|\mathbf{v}(t) - \mathbf{v}(t')\|_{\mathcal{H}} & \lesssim \left\| \int_t^{t'} T(-s) \left(|u_k|^{\frac{N}{N-2s}} |u_1|^{\frac{4s-N}{N-2s}} u_1, \dots, |u_k|^{\frac{N}{N-2s}} |u_m|^{\frac{4s-N}{N-2s}} u_m \right) ds \right\|_{H^s} \\
 & \lesssim (1 + \|\mathbf{u}\|_{M(t, t')}) \|\mathbf{u}\|_{M(t, t')}^{\frac{4s}{N-2s}} \rightarrow 0.
 \end{aligned}$$

Finally, taking $\Phi := \lim_{t \rightarrow \infty} \mathbf{v}(t)$ in $\mathcal{H}$, we have

$$\|\mathbf{u} - T(t)\Phi\|_{\mathcal{H}} \rightarrow 0, \quad \text{as } t \rightarrow \infty.$$

Scattering is proved. □

6 Invariant sets and applications

This section is devoted to obtaining global existence of solutions to the focusing system (1.1). Precisely, we prove Theorem 2.10. Let us start with a classical result about stable sets under the flow of (1.1).

Lemma 6.1 *The sets $A_{\alpha,\beta}^+$ and $A_{\alpha,\beta}^-$ are invariant under the flow of (1.1).*

Proof Let $\Psi \in A_{\alpha,\beta}^+$ and $\mathbf{u} \in C_{T^*}(\mathcal{H})$ be the maximal solution to (1.1). Assume that $\mathbf{u}(t_0) \notin A_{\alpha,\beta}^+$ for some $t_0 \in (0, T^*)$. Since $S(\mathbf{u})$ is conserved, we have $K_{\alpha,\beta}(\mathbf{u}(t_0)) < 0$. So, with a continuity argument, there exists a positive time $t_1 \in (0, t_0)$ such that $K_{\alpha,\beta}(\mathbf{u}(t_1)) = 0$ and $S(\mathbf{u}(t_1)) < m$. This contradicts the definition of m . The proof is similar in the case of $A_{\alpha,\beta}^-$. $\square$

The previous stable sets are independent of the parameter (α, β) .

Lemma 6.2 *The sets $A_{\alpha,\beta}^+$ and $A_{\alpha,\beta}^-$ are independent of (α, β) .*

Proof Let (α, β) and (α', β') in $\mathcal{A}$. By Theorem 2.5, the union $A_{\alpha,\beta}^+ \cup A_{\alpha,\beta}^-$ is independent of (α, β) . Therefore, it is sufficient to prove that $A_{\alpha,\beta}^+$ is independent of (α, β) . The rescaling $\mathbf{u}^\lambda := e^{\alpha\lambda}\mathbf{u}(e^{-\beta\lambda}\cdot)$ implies that a neighborhood of zero is in $A_{\alpha,\beta}^+$. If $S(\mathbf{u}) < m$ and $K_{\alpha,\beta}(\mathbf{u}) = 0$, then $\mathbf{u} = 0$. So, $A_{\alpha,\beta}^+$ is open. Moreover, this rescaling with $\lambda \rightarrow -\infty$ gives that $A_{\alpha,\beta}^+$ is contracted to zero and so it is connected. Now, write

$$A_{\alpha,\beta}^+ = A_{\alpha,\beta}^+ \cap (A_{\alpha',\beta'}^+ \cup A_{\alpha',\beta'}^-) = (A_{\alpha,\beta}^+ \cap A_{\alpha',\beta'}^+) \cup (A_{\alpha,\beta}^+ \cap A_{\alpha',\beta'}^-).$$

Since by the definition, $A_{\alpha,\beta}^-$ is open and $0 \in A_{\alpha,\beta}^+ \cap A_{\alpha',\beta'}^+$, using a connectivity argument, we have $A_{\alpha,\beta}^+ = A_{\alpha',\beta'}^+$. $\square$

Now, we are ready to prove Theorem 2.10. By a translation argument, assume that $t_0 = 0$. Thus, $S(\Psi) < m$ and thanks to the two previous lemmas, $\mathbf{u}(t) \in A_{1,1}^+$ for any $t \in [0, T^*)$. Then

$$\begin{aligned} m &\geq \left(S - \frac{1}{2+N} K_{1,1} \right)(\mathbf{u}) \\ &= \frac{1}{2+N} \left(s \sum_{j=1}^m \|(-\Delta)^{\frac{s}{2}} u_j\|^2 + \left(1 - \frac{1}{p}\right) \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |u_j u_k|^p dx \right) \\ &\geq \frac{s}{2+N} \sum_{j=1}^m \|(-\Delta)^{\frac{s}{2}} u_j\|^2. \end{aligned}$$

Thus, $\mathbf{u}$ is bounded in $(\dot{H}^1)^{(m)}$. Precisely

$$\sup_{0 \leq t \leq T^*} \sum_{j=1}^m \|(-\Delta)^{\frac{s}{2}} u_j(t)\|^2 \leq \frac{(2+N)m}{s}.$$

Moreover, since the L^2 norm is conserved, we have

$$\sup_{0 \leq t \leq T^*} \sum_{j=1}^m \|u_j(t)\|_{H^s}^2 < \infty.$$

Finally, $T^* = \infty$.

Remark 6.3 A paper treating scattering and finite time blow-up of solutions to (1.1) is in progress.

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References

1. Adams, R.: Sobolev Spaces. Academic, New York (1975)
2. Akhmediev, N.; Ankiewicz, A.: Partially coherent solitons on a finite background. *Phys. Rev. Lett.* **82**, 2661 (1999)
3. Cassano, B.; Tarulli, M.: H^s -scattering for systems of N-defocusing weakly coupled NLS equations in low space dimensions. *J. Math. Anal. Appl.* **430**, 528–548 (2015)
4. Cazenave, T.: An Introduction to Nonlinear Schrödinger Equations, Textos de Métodos Matemáticos, vol. 26, Instituto de Matemática UFRJ (1996)
5. Chakravarty, S.; Ablowitz, M.J.; Sauer, J.R.; Jenkins, R.B.: Multisoliton interactions and wavelength-division multiplexing. *Opt. Lett.* **20**, 136–138 (1995)
6. Christ, M.; Weinstein, M.: Dispersion of small amplitude solutions of the generalized Korteweg-de Vries equation. *J. Funct. Anal.* **100**, 87–109 (1991)
7. Dalfovo, F.; Giorgini, S.; Pitaevskii, L.P.; Stringari, S.: Theory of Bose-Einstein condensation in trapped gases. *Rev. Mod. Phys.* **71**, 463–512 (1999)
8. Frank, R.L.; Seiringer, R.: Non-linear ground state representations and sharp Hardy inequalities. *J. Funct. Anal.* **255**, 3407–3430 (2008)
9. Guo, B.; Han, Y.; Xin, J.: Existence of the global smooth solution to the period boundary value problem of fractional nonlinear Schrödinger equation. *Appl. Math. Comput.* **204**(1), 468–477 (2008)
10. Guo, B.; Huang, D.: Existence and stability of standing waves for nonlinear fractional Schrödinger equations. *J. Math. Phys.* **53**, 083702 (2012)
11. Guo, B.; Huo, Z.: Global well-posedness for the fractional nonlinear Schrödinger equation. *Commun. Partial Differ. Equ.* **36**(2), 247–255 (2010)
12. Guo, B.; Huo, Z.: Well-posedness for the nonlinear fractional Schrödinger equation and inviscid limit behavior of solution for the fractional Ginzburg-Landau equation. *Fract. Calc. Appl. Anal.* **16**(1), 226–242 (2013)
13. Guo, Z.; Sire, Y.; Wang, Y.; Zhao, L.: On the energy-critical fractional Schrödinger equation in the radial case (2013). [arXiv:1310.6816v1](https://arxiv.org/abs/1310.6816v1) [math.AP]
14. Guo, Z.; Wang, Y.: Improved Strichartz estimates for a class of dispersive equations in the radial case and their applications to nonlinear Schrödinger and wave equations. *J. Anal. Math.* **124**(1), 1–38 (2014)
15. Ibrahim, R.W.; Jalab, H.A.: Analytic and numerical solutions for systems of fractional Schrödinger equation. *J. Inequal. Appl.* **2015**, 23 (2015). <https://doi.org/10.1186/s13660-015-0552-4>
16. Kawaguchi, Y.; Ueda, M.: Spinor Bose-Einstein condensates. *Phys. Rep.* **520**, 253–381 (2012)
17. Laskin, N.: Fractional quantum mechanics and Lévy path integrals. *Phys. Lett.* **268 A**, 298–304 (2000)
18. Laskin, N.: Fractional Schrödinger equation. *Phys. Rev.* **E66**, 056108 (2002)
19. Le Coz, S.: Standing waves in nonlinear Schrödinger equations. In: *Analytical and Numerical Aspects of Partial Differential Equations*, pp. 151–192 (2008)
20. Lions, P.L.: Symétrie et compacité dans les espaces de Sobolev. *J. Funct. Anal.* **49**(3), 315–334 (1982)
21. Morosi, C.; Pizzocchero, L.: On the constants for some fractional Gagliardo–Nirenberg and Sobolev inequalities. *Expositiones Mathematicae* **36**(1), 32–77 (2018)
22. Nguyen, N.V.; Tian, R.; Deconinck, B.; Sheils, N.: Global existence for a coupled system of Schrödinger equations with power-type non-linearities. *J. Math. Phys.* **54**, 011503 (2013)
23. Payne, L.E.; Sattinger, D.H.: Saddle points and instability of nonlinear hyperbolic equations. *Isr. J. Math.* **22**, 273–303 (1975)
24. Saanouni, T.: A note on coupled focusing nonlinear Schrödinger equations. *Appl. Anal.* **95**(9), 2063–2080 (2016)
25. Saanouni, T.: Defocusing coupled nonlinear Schrödinger equations. *J. A. D. E. A.* **7**(1), 78–96 (2016)
26. Saanouni, T.: A note on coupled nonlinear Schrödinger equations. *Adv. Nonlinear Anal.* **3**(4), 247–269 (2014)
27. Saanouni, T.: Remarks on damped fractional Schrödinger equation with pure power non-linearity. *J. Math. Phys.* **56**, 061502 (2015)
28. Som, B.K.; Gupta, M.R.; Dasgupta, B.: Coupled nonlinear Schrödinger equation for Langmuir and dispersive ion acoustic waves. *Phys. Lett. A* **72**, 111–114 (1979)
29. Sun, J.Q.; Ma, Z.Q.; Qin, M.Z.: Simulation of envelope Rossby solitons in a pair of cubic Schrödinger equations. *Appl. Math. Comput.* **183**, 946–952 (2006)

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