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Local fractional integrals involving generalized strongly m-convex mappings

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Abstract In this paper, we first obtain a generalized integral identity for twice local fractional differentiable mappings on fractal sets $\mathbb{R}^{\alpha}$ $(0 < \alpha \leq 1)$ of real line numbers. Then, using twice local fractional differentiable mappings that are in absolute value at certain powers generalized strongly m-convex, we obtain some new estimates on generalization of trapezium-like inequalities. We also discuss some new special cases which can be deduced from our main results.

Mathematics Subject Classification Primary 26A51; Secondary 26A33 · 26D07 · 26D10 · 26D15

الملخص

في هذا البحث، نحصل أولا على متطابقة تكاملية معمّمة لراسمات (تطبيقات) قابلة مرتين للاشتقاق الكسري المحلي على مجموعات كسورية $\mathbb{R}^{\alpha}$ $(0 < \alpha \leq 1)$ لخط الأعداد الحقيقية. بعد ذلك، باستعمال راسمات قابلة مرتين للاشتقاق الكسري المحلي والتي هي في القيمة المطلقة لبعض القوى م− محدبة معممة بقوة، نحصل على بعض التقديرات الجديدة لتعميم متراجحات أشباه المنحرف. ونناقش أيضا بعض الحالات الخاصة الجديدة والتي يمكن استنتاجها من نتائجنا الرئيسية.

1 Introduction

Throughout this paper, let $\mathbb{R}$, $\mathbb{R}^{+}$, $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$ be the sets of real numbers, positive real numbers, rational numbers, integers, and positive integers, respectively, and

$$\mathbb{I} := \mathbb{R}\backslash\mathbb{Q} \quad \text{and} \quad \mathbb{N}_0 := \mathbb{N} \cup \{0\}.$$

To describe the definition of the local fractional derivative and local fractional integral, recently, one has introduced the following sets (see [7,21,24,27]). In this paper, we are also motivated by (see [3–5]). Recently,

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the theory of Yang's fractional sets, see [24] and references therein, was introduced as follows. For $0 < \alpha \leq 1$, we have the following α-type set of element sets:

(1) The α-type set of integers $\mathbb{Z}^\alpha$ is defined by
$$\mathbb{Z}^\alpha := \left\{0^\alpha\right\} \cup \left\{\pm n^\alpha : \ n \in \mathbb{N}\right\}.$$

(2) The α-type set of rational numbers $\mathbb{Q}^\alpha$ is defined by
$$\mathbb{Q}^\alpha := \left\{q^\alpha : \ q \in \mathbb{Q}\right\} = \left\{q^\alpha = \left(\frac{r}{s}\right)^\alpha : \ r \in \mathbb{Z}, \ s \in \mathbb{N}\right\}.$$

(3) The α-type set of irrational numbers $\mathbb{I}^\alpha$ is defined by
$$\mathbb{I}^\alpha := \left\{i^\alpha : \ i \in \mathbb{I}\right\} = \left\{i^\alpha \neq \left(\frac{r}{s}\right)^\alpha : \ r \in \mathbb{Z}, \ s \in \mathbb{N}\right\}.$$

(4) The α-type set of real line numbers $\mathbb{R}^\alpha$ is defined by $\mathbb{R}^\alpha := \mathbb{Q}^\alpha \cup \mathbb{I}^\alpha$.

Throughout this paper, whenever the α-type set $\mathbb{R}^\alpha$ of real line numbers is involved, the α is assumed to be tacitly $0 < \alpha \leq 1$. One has also defined by two binary operations the addition $+$ and the multiplication $\cdot$ (which is conventionally omitted) on the α-type set $\mathbb{R}^\alpha$ of real line numbers as follows. For $a^\alpha, b^\alpha \in \mathbb{R}^\alpha$, two binary operations the addition $+$ and the multiplication $\cdot$ are defined as
$$a^\alpha + b^\alpha := (a+b)^\alpha \quad \text{and} \quad a^\alpha \cdot b^\alpha = a^\alpha b^\alpha := (ab)^\alpha.$$

Then, one finds that

- $(\mathbb{R}^\alpha, +)$ is a commutative group. For $a^\alpha, b^\alpha, c^\alpha \in \mathbb{R}^\alpha$ the following holds:

(1) $a^\alpha + b^\alpha \in \mathbb{R}^\alpha$.
(2) $a^\alpha + b^\alpha = b^\alpha + a^\alpha$.
(3) $a^\alpha + (b^\alpha + c^\alpha) = (a^\alpha + b^\alpha) + c^\alpha$.
(4) 0^α is the identity for $(\mathbb{R}^\alpha, +)$. For any $a^\alpha \in \mathbb{R}^\alpha$, $a^\alpha + 0^\alpha = 0^\alpha + a^\alpha = a^\alpha$.
(5) For each $a^\alpha \in \mathbb{R}^\alpha$, $(-a)^\alpha$ is the inverse element of a^α for $(\mathbb{R}^\alpha, +)$, so we have
$$a^\alpha + (-a)^\alpha = (a + (-a))^\alpha = 0^\alpha.$$

- $(\mathbb{R}^\alpha \backslash \{0^\alpha\}, \cdot)$ is a commutative group. For $a^\alpha, b^\alpha, c^\alpha \in \mathbb{R}^\alpha$, the following holds:

(1) $a^\alpha \cdot b^\alpha \in \mathbb{R}^\alpha$.
(2) $a^\alpha \cdot b^\alpha = b^\alpha \cdot a^\alpha$.
(3) $a^\alpha \cdot (b^\alpha \cdot c^\alpha) = (a^\alpha \cdot b^\alpha) \cdot c^\alpha$.
(4) 1^α is the identity for $(\mathbb{R}^\alpha, \cdot)$. For any $a^\alpha \in \mathbb{R}^\alpha$, $a^\alpha \cdot 1^\alpha = 1^\alpha \cdot a^\alpha = a^\alpha$.
(5) For each $a^\alpha \in \mathbb{R}^\alpha \backslash \{0^\alpha\}$, $\left(\frac{1}{a}\right)^\alpha$ is the inverse element of a^α for $(\mathbb{R}^\alpha, \cdot)$, so we have
$$a^\alpha \cdot \left(\frac{1}{a}\right)^\alpha = \left(a \cdot \left(\frac{1}{a}\right)\right)^\alpha = 1^\alpha.$$

- Distributive law holds: $a^\alpha \cdot (b^\alpha + c^\alpha) = a^\alpha \cdot b^\alpha + a^\alpha \cdot c^\alpha$.

Furthermore, we observe some additional properties for $(\mathbb{R}^\alpha, +, \cdot)$ which are stated in the following proposition.

Proposition 1.1 [7] *Each of the following statements holds true:*

(a) *Like the usual real number system* $(\mathbb{R}, +, \cdot)$, $(\mathbb{R}^\alpha, +, \cdot)$ *is a field.*
(b) *The additive identity* 0^α *and the multiplicative identity* 1^α *are unique.*
(c) *The additive inverse element* $(-a)^\alpha$ *and the multiplicative inverse element* $\left(\frac{1}{a}\right)^\alpha$ *of element* a^α *are unique.*
(d) *For each* $a^\alpha \in \mathbb{R}^\alpha$, *its inverse element* $(-a)^\alpha$ *may be written as* $-a^\alpha$. *For each* $b^\alpha \in \mathbb{R}^\alpha \backslash \{0^\alpha\}$, *its inverse element* $\left(\frac{1}{b}\right)^\alpha$ *may be written as* $\frac{1^\alpha}{b^\alpha}$ *but not as* $\frac{1}{b^\alpha}$.
(e) *If the order* $<$ *is defined on* $(\mathbb{R}^\alpha, +, \cdot)$ *as follows:* $a^\alpha < b^\alpha$ *in* $\mathbb{R}^\alpha$ *if and only if* $a < b$ *in* $\mathbb{R}$, *then* $(\mathbb{R}^\alpha, +, \cdot, <)$ *is an ordered field like* $(\mathbb{R}, +, \cdot, <)$.

To introduce the local fractional calculus on $\mathbb{R}^\alpha$, we begin with the concept of the local fractional continuity as in Definition 1.2.

Definition 1.2 A non-differentiable function $f: \mathbb{R} \longrightarrow \mathbb{R}^\alpha$, $x \longrightarrow f(x)$, is called to be local fractional continuous at x_0 if, for any $\epsilon \in \mathbb{R}^+$, there exists $\delta \in \mathbb{R}^+$, such that

$$\left|f(x) - f(x_0)\right| < \epsilon^\alpha$$

holds for $|x - x_0| < \delta$. If a function f is local continuous on the interval (a, b), we denote that $f \in C_\alpha(a, b)$.

Among several attempts to have defined local fractional derivative and local fractional integral (see [23], Section 2.1), we choose to recall the following definitions of local fractional calculus (see [8,23,24]).

Definition 1.3 The local fractional derivative of $f(x)$ of order α at $x = x_0$ is defined by

$$f^{(\alpha)}(x_0) = {}_{x_0}D_x^\alpha f(x) = \left.\frac{d^\alpha f(x)}{\mathrm{d}x^\alpha}\right|_{x=x_0} = \lim_{x \longrightarrow x_0} \frac{\Delta^\alpha(f(x) - f(x_0))}{(x - x_0)^\alpha},$$

where $\Delta^\alpha(f(x) - f(x_0)) \cong \Gamma(1+\alpha)(f(x) - f(x_0))$ and Γ is the familiar gamma function (see [20], Section 1.1).

Let $f^{(\alpha)}(x) = D_x^\alpha f(x)$. If there exists $f^{(k+1)\alpha}(x) = \overbrace{D_x^\alpha \dots D_x^\alpha}^{k+1 \text{ times}} f(x)$ for any $x \in I \subseteq \mathbb{R}$, then we denote $f \in D_{(k+1)\alpha}(I)$, where $k \in \mathbb{N}_0$.

Remark 1.4 It is found that, see [24] and references therein, in this expression, α is precisely the Hölder exponent of function defined Cantor's set. That is to say, $\left[d(x - x_0)\right]^\alpha$, which is a fractal span, is a fractal geometrical meaning.

Definition 1.5 Let $f \in C_\alpha[a, b]$. In addition, let $P = \{t_0, \dots, t_N\}$, $(N \in \mathbb{N})$ be a partition of the interval $[a, b]$ which satisfies $a = t_0 < t_1 < \cdots < t_{N-1} < t_N = b$. Furthermore, for this partition P, let $\Delta t := \max_{0 \le j \le N-1} \Delta t_j$, where $\Delta t_j := t_{j+1} - t_j$ and $j = 0, \dots, N-1$. Then, the local fractional integral of f on the interval $[a, b]$ of order α (denoted by ${}_aI_b^{(\alpha)} f$) is defined by

$${}_aI_b^{(\alpha)} f(t) = \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)(\mathrm{d}t)^\alpha := \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \longrightarrow 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^\alpha,$$

provided that the limit exists (in fact, this limit exists if $f \in C_\alpha[a, b]$).

Here, it follows that ${}_aI_b^{(\alpha)} f = 0$ if $a = b$ and ${}_aI_b^{(\alpha)} f = -{}_bI_a^{(\alpha)} f$ if $a < b$. If ${}_aI_x^{(\alpha)} g$ exists for any $x \in [a, b]$ and a function $g : [a, b] \longrightarrow \mathbb{R}^\alpha$, then we denote $g \in I_x^{(\alpha)}[a, b]$.

We give some of the features related to the local fractional calculus that will be required for our main results.

Lemma 1.6 [24] *The following identities hold true:*

(1) *(Local fractional derivative of $x^{k\alpha}$):*

$$\frac{d^\alpha x^{k\alpha}}{dx^\alpha} = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\alpha)} x^{(k-1)\alpha}.$$

(2) *(Local fractional integration is anti-differentiation). Suppose that $f(x) = g^{(\alpha)}(x) \in C_\alpha[a, b]$. Then, we have*

$${}_aI_b^{(\alpha)} f(x) = g(b) - g(a).$$

(3) *(Local fractional integration by parts). Suppose that $f(x), g(x) \in D_\alpha[a, b]$ and $f^{(\alpha)}(x)$, $g^{(\alpha)}(x) \in C_\alpha[a, b]$. Then, we have*

$${}_aI_b^{(\alpha)} f(x) g^{(\alpha)}(x) = f(x)g(x)\big|_a^b - {}_aI_b^{(\alpha)} f^{(\alpha)}(x) g(x).$$

(4) *(Local fractional definite integrals of $x^{k\alpha}$):*

$$\frac{1}{\Gamma(1+\alpha)}\int_a^b x^{k\alpha}(dx)^\alpha = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)}\left(b^{(k+1)\alpha}-a^{(k+1)\alpha}\right), \quad k \in \mathbb{R}.$$

For further details on local fractional calculus, one may refer to (see [22–26]).

Definition 1.7 [21] A function $f: I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be convex on I, if

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)$$

holds for every $x, y \in I$ and $t \in [0, 1]$.

Definition 1.8 [10] A function $f: [0, b] \longrightarrow \mathbb{R}$ is called m-convex with $m \in [0, 1]$, if, for any $x, y \in [0, b]$ and $t \in [0, 1]$, we have

$$f(tx+m(1-t)y) \leq tf(x)+m(1-t)f(y).$$

Definition 1.9 [10] A function $f: I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is called strongly convex with modulus $c \in \mathbb{R}^+$, if

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)-ct(1-t)(x-y)^2$$

holds for every $x, y \in I$ and $t \in [0, 1]$.

Strongly convex functions have been introduced by Polyak (see [10] and references therein). Since strong convexity is a strengthening of the notion of convexity, some properties of strongly convex functions are just stronger versions of known properties of convex functions. Strongly convex functions have been used for proving the convergence of a gradient-type algorithm for minimizing a function. They play an important role in optimization theory and mathematical economics.

Definition 1.10 (see [10,11]) A function $f: I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is called strongly m-convex with $m \in [0, 1]$ and modulus $c \in \mathbb{R}^+$, if

$$f(tx+m(1-t)y) \leq tf(x)+m(1-t)f(y)-cmt(1-t)(x-y)^2$$

holds for every $x, y \in I$ and $t \in [0, 1]$.

Remark 1.11 Any strongly m-convex function is, in particular, m-convex. However, there are m-convex functions, which are not strongly m-convex with modulus c, for some $c \in \mathbb{R}^+$ (see [10], Example 1.8).

The following inequality, named Hermite–Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1.12 *Let $f: I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on I and $a, b \in I$ with $a < b$. Then, the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Mo et al. (see [15]) introduced the following generalized convex function.

Definition 1.13 Let $f: I \subseteq \mathbb{R} \longrightarrow \mathbb{R}^\alpha$ be a function. For any $x_1, x_2 \in I$ and $\lambda \in [0, 1]$, if the following inequality

$$f(\lambda x_1+(1-\lambda)x_2) \leq \lambda^\alpha f(x_1)+(1-\lambda)^\alpha f(x_2)$$

holds, then f is called a generalized convex function on I.

Here are two basic examples of generalized convex functions:

(1) $f(x) = x^{\alpha p}$, where $x \geq 0$ and $p > 1$.

(2) $g(x) = E_\alpha(x^\alpha)$, $x \in \mathbb{R}$, where $E_\alpha(x^\alpha) := \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{\Gamma(1+k\alpha)}$ is the Mittag–Leffler function.

Recently, the fractal theory has received a significant attention (see [1–19,22–27]). Mo et al. (see [15]) proved the following analog of the Hermite–Hadamard inequality (1.1) for generalized convex functions.

Theorem 1.14 *Let $f : [a, b] \longrightarrow \mathbb{R}^\alpha$ be a generalized convex function with $a < b$. Then, for all $x \in [a, b]$, the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(1+\alpha)}{(b-a)^\alpha}\, {}_aI_b^{(\alpha)} f(x) \leq \frac{f(a)+f(b)}{2^\alpha}. \tag{1.2}$$

Remark 1.15 The double inequality (1.2) is known in the literature as generalized Hermite–Hadamard integral inequality for generalized convex functions. Some of the classical inequalities for means can be derived from (1.2) with appropriate selections of the mapping f. Both inequalities in (1.1) and (1.2) hold in the reverse direction if f is concave and generalized concave, respectively. For some more results which generalize, improve, and extend the inequality (1.2), one may refer to the recent papers (see [6,12,14,16–18] and references therein).

An analog in the fractal set $\mathbb{R}^\alpha$ of the classical Hölder's inequality has been established by Yang (see [24]), which is asserted by the following lemma.

Lemma 1.16 *Let $f, g \in C_\alpha[a, b]$ with $p^{-1} + q^{-1} = 1$, where $p, q > 1$. Then, we have*

$$\frac{1}{\Gamma(1+\alpha)} \int_a^b \left|f(x)g(x)\right|(dx)^\alpha \leq \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b \left|f(x)\right|^p (dx)^\alpha\right)^{\frac{1}{p}} \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b \left|g(x)\right|^q (dx)^\alpha\right)^{\frac{1}{q}}.$$

Theorem 1.17 *(Generalized Ostrowski inequality) Let $I \subseteq \mathbb{R}$ be an interval, $f : I^0 \subseteq \mathbb{R} \longrightarrow \mathbb{R}^\alpha$ (I^0 is the interior of I), such that $f \in D_\alpha\left(I^0\right)$, and $f^{(\alpha)} \in C_\alpha[a, b]$ for $a, b \in I^0$ with $a < b$. Then, for all $x \in [a, b]$, the following inequality holds:*

$$\left|f(x) - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha}\, {}_aI_b^{(\alpha)} f(t)\right| \leq 2^\alpha \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \left[\frac{1}{4^\alpha} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^{2\alpha}\right](b-a)^\alpha \|f^{(\alpha)}\|_\infty. \tag{1.3}$$

Motivated by the above literatures, in the next section, we first introduce the notion of generalized strongly m-convex mappings and after that we will obtain a generalized integral identity for twice local differentiable mappings on fractal sets $\mathbb{R}^\alpha$ $(0 < \alpha \leq 1)$ of real line numbers. In addition, we use this identity to obtain some new estimates on generalization of trapezium-like inequalities for twice local fractional differentiable mappings that are in absolute value at certain powers generalized strongly m-convex. We will discuss some new special cases which can be deduced from our main results.

2 Main results

The following definitions will be used in the sequel.

Definition 2.1 [9] For $0 < \alpha \leq 1$ and $x \in \mathbb{R}^+$, the local gamma function is defined by

$$\Gamma_\alpha(x) := \frac{1}{\alpha!} \int_0^\infty E_\alpha(-t^\alpha) t^{(x-1)\alpha} (dt)^\alpha. \tag{2.1}$$

For $\alpha = 1$, (2.1) gives integral representation of classical Euler gamma function $\Gamma(x)$. Therefore, in this case, $\Gamma_\alpha(x) = \Gamma(x)$.
In addition, the following relations holds for local gamma function:

(1) $\Gamma_\alpha(x+1) = (\alpha!)x\Gamma_\alpha(x)$ for $x \in \mathbb{R}^+$.
(2) $\Gamma_\alpha(n+1) = (\alpha!)^n n!$ for $n \in \mathbb{N}$.

Definition 2.2 [9] For $0 < \alpha \leq 1$ and $x, y \in \mathbb{R}^+$, the local beta function with two parameters x and y is defined as

$$B_\alpha(x, y) := \int_0^1 t^{(x-1)\alpha} (1-t)^{(y-1)\alpha} (dt)^\alpha. \tag{2.2}$$

For $\alpha = 1$, (2.2) gives integral representation of classical Euler beta function $\beta(x, y)$. Therefore, in this case, $B_\alpha(x, y) = \beta(x, y)$.

Remark 2.3 In (see [9]), Jumarie considered the formulations via fractional calculus, which is not called the local fractional derivative. Thus, author has defined the local gamma and local beta function via local fractional calculus.

Theorem 2.4 *[9] Let $x, y \in \mathbb{R}^+$. Then for local gamma and local beta function, the following equality holds:*

$$B_\alpha(x, y) = \frac{\Gamma_\alpha(x)\Gamma_\alpha(y)}{\Gamma_\alpha(x+y)}. \tag{2.3}$$

We are now in position to introduce a new class called generalized strongly m-convex.

Definition 2.5 A function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}^\alpha$ is called generalized strongly m-convex with $m \in [0, 1]$ and modulus $c \in \mathbb{R}^+$, if

$$f(\lambda x_1 + m(1-\lambda)x_2) \leq \lambda^\alpha f(x_1) + m^\alpha(1-\lambda)^\alpha f(x_2) - (cm)^\alpha \lambda^\alpha (1-\lambda)^\alpha (x_1 - x_2)^{2\alpha} \tag{2.4}$$

holds for any $x_1, x_2 \in I$ and $\lambda \in [0, 1]$.

Remark 2.6 In Definition 2.5, if we choose $\alpha = 1$, then we get Definition 1.10. These mean that any generalized strongly m-convex mapping is, in particular, strongly m-convex. Moreover, if f is generalized strongly m-convex with modulus c, then f is generalized strongly m-convex with modulus k, for any constant $0 < k < c$.

For establishing our main results regarding some new estimates on generalization of trapezium-like integral inequalities on fractal sets $\mathbb{R}^\alpha$ $(0 < \alpha \leq 1)$, we need the following lemma.

Lemma 2.7 *Let $I \subseteq \mathbb{R}$ be an interval, $m \in [0, 1]$, $f : I^0 \subseteq \mathbb{R} \longrightarrow \mathbb{R}^\alpha$ (I^0 is the interior of I), such that $f^{(\alpha)} \in D_\alpha(I^0)$ and $f^{(2\alpha)} \in C_\alpha[ma, mb]$ for $ma, mb \in I^0$ with $a < b$. Then, for all $x \in [ma, mb]$, the following identity holds:*

$$\begin{aligned}
&\frac{1}{(b-a)^\alpha}\, {}_{ma}I_{mb}^{(\alpha)} f(t) - \frac{1}{\Gamma^2(1+\alpha)(b-a)^\alpha}\Big[(x-ma)^\alpha f(x) + (mb-x)^\alpha f(mb)\Big] \\
&+ \frac{1}{\Gamma(1+\alpha)\Gamma(1+2\alpha)(b-a)^\alpha}\Big[(x-ma)^{2\alpha} f^{(\alpha)}(x) + (mb-x)^{2\alpha} f^{(\alpha)}(mb)\Big] \\
&= \frac{(x-ma)^{3\alpha}}{\Gamma(1+\alpha)\Gamma(1+2\alpha)(b-a)^\alpha} \int_0^1 t^{2\alpha} f^{(2\alpha)}(tx + m(1-t)a)(dt)^\alpha \\
&+ \frac{(mb-x)^{3\alpha}}{\Gamma(1+\alpha)\Gamma(1+2\alpha)(b-a)^\alpha} \int_0^1 t^{2\alpha} f^{(2\alpha)}(mtb + (1-t)x)(dt)^\alpha.
\end{aligned} \tag{2.5}$$

We denote

$$\begin{aligned}
T_f^{(\alpha)}(x; m, a, b) &:= \frac{(x-ma)^{3\alpha}}{\Gamma(1+\alpha)\Gamma(1+2\alpha)(b-a)^\alpha} \int_0^1 t^{2\alpha} f^{(2\alpha)}(tx + m(1-t)a)(dt)^\alpha \\
&+ \frac{(mb-x)^{3\alpha}}{\Gamma(1+\alpha)\Gamma(1+2\alpha)(b-a)^\alpha} \int_0^1 t^{2\alpha} f^{(2\alpha)}(mtb + (1-t)x)(dt)^\alpha.
\end{aligned} \tag{2.6}$$

Proof Using twice the local fractional integration by parts and changing the variables, respectively, $u = tx + m(1-t)a$ and $v = mtb + (1-t)x$ for all $t \in [0, 1]$, we have

$$\begin{aligned}
T_f^{(\alpha)}(x; m, a, b) &= \frac{(x-ma)^{3\alpha}}{\Gamma(1+\alpha)\Gamma(1+2\alpha)(b-a)^\alpha} \\
&\times \left[\frac{t^{2\alpha}}{(x-ma)^\alpha} f^{(\alpha)}(tx + m(1-t)a)\Big|_0^1 - \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)(x-ma)^\alpha} \int_0^1 t^\alpha f^{(\alpha)}(tx + m(1-t)a)(dt)^\alpha\right] \\
&+ \frac{(mb-x)^{3\alpha}}{\Gamma(1+\alpha)\Gamma(1+2\alpha)(b-a)^\alpha}
\end{aligned}$$

$$\times\left[\frac{t^{2\alpha}}{(mb-x)^{\alpha}}f^{(\alpha)}(mtb+(1-t)x)\Big|_0^1-\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)(mb-x)^{\alpha}}\int_0^1 t^{\alpha}f^{(\alpha)}(mtb+(1-t)x)(dt)^{\alpha}\right]$$
$$=\frac{(x-ma)^{3\alpha}}{\Gamma(1+\alpha)\Gamma(1+2\alpha)(b-a)^{\alpha}}\times\left\{\frac{f^{(\alpha)}(x)}{(x-ma)^{\alpha}}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)(x-ma)^{\alpha}}\right.$$
$$\left.\times\left[\frac{t^{\alpha}}{(x-ma)^{\alpha}}f(tx+m(1-t)a)\Big|_0^1-\frac{\Gamma(1+\alpha)}{(x-ma)^{\alpha}}\int_0^1 f(tx+m(1-t)a)(dt)^{\alpha}\right]\right\}$$
$$+\frac{(mb-x)^{3\alpha}}{\Gamma(1+\alpha)\Gamma(1+2\alpha)(b-a)^{\alpha}}\times\left\{\frac{f^{(\alpha)}(mb)}{(mb-x)^{\alpha}}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)(mb-x)^{\alpha}}\right.$$
$$\left.\times\left[\frac{t^{\alpha}}{(mb-x)^{\alpha}}f(mtb+(1-t)x)\Big|_0^1-\frac{\Gamma(1+\alpha)}{(mb-x)^{\alpha}}\int_0^1 f(mtb+(1-t)x)(dt)^{\alpha}\right]\right\}$$
$$=\frac{(x-ma)^{3\alpha}}{\Gamma(1+\alpha)\Gamma(1+2\alpha)(b-a)^{\alpha}}\times\left\{\frac{f^{(\alpha)}(x)}{(x-ma)^{\alpha}}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)(x-ma)^{\alpha}}\right.$$
$$\left.\times\left[\frac{f(x)}{(x-ma)^{\alpha}}-\frac{\Gamma^2(1+\alpha)}{(x-ma)^{2\alpha}}\,{}_{ma}I_x^{(\alpha)}f(t)\right]\right\}$$
$$+\frac{(mb-x)^{3\alpha}}{\Gamma(1+\alpha)\Gamma(1+2\alpha)(b-a)^{\alpha}}\times\left\{\frac{f^{(\alpha)}(mb)}{(mb-x)^{\alpha}}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)(mb-x)^{\alpha}}\right.$$
$$\left.\times\left[\frac{f(mb)}{(mb-x)^{\alpha}}-\frac{\Gamma^2(1+\alpha)}{(mb-x)^{2\alpha}}\,{}_{x}I_{mb}^{(\alpha)}f(t)\right]\right\}$$
$$=\frac{1}{(b-a)^{\alpha}}\,{}_{ma}I_{mb}^{(\alpha)}f(t)-\frac{1}{\Gamma^2(1+\alpha)(b-a)^{\alpha}}\left[(x-ma)^{\alpha}f(x)+(mb-x)^{\alpha}f(mb)\right]$$
$$+\frac{1}{\Gamma(1+\alpha)\Gamma(1+2\alpha)(b-a)^{\alpha}}\left[(x-ma)^{2\alpha}f^{(\alpha)}(x)+(mb-x)^{2\alpha}f^{(\alpha)}(mb)\right].$$

This completes the proof of the lemma. □

Using Lemma 2.7, we now state the following theorems for twice local fractional differentiable mappings that are in absolute value at certain powers generalized strongly m-convex.

Theorem 2.8 *Suppose that the assumptions of Lemma 2.7 are satisfied. If $m \in (0,1]$ and $\left|f^{(2\alpha)}\right|^q$ is generalized strongly m-convex, then, for $p, q > 1$, where $p^{-1}+q^{-1}=1$ and $c \in \mathbb{R}^+$, the following inequality holds:*

$$\left|T_f^{(\alpha)}(x;m,a,b)\right| \le \frac{(x-ma)^{3\alpha}}{\Gamma(1+2\alpha)(b-a)^{\alpha}}\left(\frac{\Gamma(1+2p\alpha)}{\Gamma(1+(2p+1)\alpha)}\right)^{\frac{1}{p}}$$
$$\times\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\left|f^{(2\alpha)}(x)\right|^q+m^{\alpha}\left|f^{(2\alpha)}(a)\right|^q\right)-(cm)^{\alpha}(x-a)^{2\alpha}\frac{B_{\alpha}(2,2)}{\Gamma(1+\alpha)}\right]^{\frac{1}{q}}$$
$$+\frac{(mb-x)^{3\alpha}}{\Gamma(1+2\alpha)(b-a)^{\alpha}}\left(\frac{\Gamma(1+2p\alpha)}{\Gamma(1+(2p+1)\alpha)}\right)^{\frac{1}{p}}$$
$$\times\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\left|f^{(2\alpha)}(mb)\right|^q+m^{\alpha}\left|f^{(2\alpha)}\left(\frac{x}{m}\right)\right|^q\right)-(cm)^{\alpha}\left(mb-\frac{x}{m}\right)^{2\alpha}\frac{B_{\alpha}(2,2)}{\Gamma(1+\alpha)}\right]^{\frac{1}{q}}. \quad (2.7)$$

Proof From Lemmas 1.6, 1.16, and 2.7, Definition 2.1, generalized strongly m-convexity of $\left|f^{(2\alpha)}\right|^q$, and properties of the modulus, we have

$$\left|T_f^{(\alpha)}(x;m,a,b)\right| \le \frac{(x-ma)^{3\alpha}}{\Gamma(1+\alpha)\Gamma(1+2\alpha)(b-a)^{\alpha}}\int_0^1 t^{2\alpha}\left|f^{(2\alpha)}(tx+m(1-t)a)\right|(dt)^{\alpha}$$

$$+\frac{(mb-x)^{3\alpha}}{\Gamma(1+\alpha)\Gamma(1+2\alpha)(b-a)^{\alpha}}\int_0^1 t^{2\alpha}\left|f^{(2\alpha)}(mtb+(1-t)x)\right|(dt)^{\alpha}$$

$$\leq\frac{(x-ma)^{3\alpha}}{\Gamma(1+2\alpha)(b-a)^{\alpha}}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1 t^{2p\alpha}(dt)^{\alpha}\right)^{\frac{1}{p}}$$

$$\times\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|f^{(2\alpha)}(tx+m(1-t)a)\right|^q(dt)^{\alpha}\right)^{\frac{1}{q}}$$

$$+\frac{(mb-x)^{3\alpha}}{\Gamma(1+2\alpha)(b-a)^{\alpha}}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1 t^{2p\alpha}(dt)^{\alpha}\right)^{\frac{1}{p}}$$

$$\times\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|f^{(2\alpha)}(mtb+(1-t)x)\right|^q(dt)^{\alpha}\right)^{\frac{1}{q}}$$

$$\leq\frac{(x-ma)^{3\alpha}}{\Gamma(1+2\alpha)(b-a)^{\alpha}}\left(\frac{\Gamma(1+2p\alpha)}{\Gamma(1+(2p+1)\alpha)}\right)^{\frac{1}{p}}$$

$$\times\left[\frac{1}{\Gamma(1+\alpha)}\int_0^1\left(t^{\alpha}\left|f^{(2\alpha)}(x)\right|^q+m^{\alpha}(1-t)^{\alpha}\left|f^{(2\alpha)}(a)\right|^q-(cm)^{\alpha}t^{\alpha}(1-t)^{\alpha}(x-a)^{2\alpha}\right)(dt)^{\alpha}\right]^{\frac{1}{q}}$$

$$+\frac{(mb-x)^{3\alpha}}{\Gamma(1+2\alpha)(b-a)^{\alpha}}\left(\frac{\Gamma(1+2p\alpha)}{\Gamma(1+(2p+1)\alpha)}\right)^{\frac{1}{p}}$$

$$\times\left[\frac{1}{\Gamma(1+\alpha)}\int_0^1\left(t^{\alpha}\left|f^{(2\alpha)}(mb)\right|^q+m^{\alpha}(1-t)^{\alpha}\left|f^{(2\alpha)}\left(\frac{x}{m}\right)\right|^q\right.\right.$$

$$\left.\left.-(cm)^{\alpha}t^{\alpha}(1-t)^{\alpha}\left(mb-\frac{x}{m}\right)^{2\alpha}\right)(dt)^{\alpha}\right]^{\frac{1}{q}}$$

$$=\frac{(x-ma)^{3\alpha}}{\Gamma(1+2\alpha)(b-a)^{\alpha}}\left(\frac{\Gamma(1+2p\alpha)}{\Gamma(1+(2p+1)\alpha)}\right)^{\frac{1}{p}}$$

$$\times\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\left|f^{(2\alpha)}(x)\right|^q+m^{\alpha}\left|f^{(2\alpha)}(a)\right|^q\right)-(cm)^{\alpha}(x-a)^{2\alpha}\frac{B_{\alpha}(2,2)}{\Gamma(1+\alpha)}\right]^{\frac{1}{q}}$$

$$+\frac{(mb-x)^{3\alpha}}{\Gamma(1+2\alpha)(b-a)^{\alpha}}\left(\frac{\Gamma(1+2p\alpha)}{\Gamma(1+(2p+1)\alpha)}\right)^{\frac{1}{p}}$$

$$\times\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\left|f^{(2\alpha)}(mb)\right|^q+m^{\alpha}\left|f^{(2\alpha)}\left(\frac{x}{m}\right)\right|^q\right)-(cm)^{\alpha}\left(mb-\frac{x}{m}\right)^{2\alpha}\frac{B_{\alpha}(2,2)}{\Gamma(1+\alpha)}\right]^{\frac{1}{q}}.$$

Therefore, the proof of this theorem is completed. □

We point out some special cases of Theorem 2.8.

Corollary 2.9 *Under assumptions of Theorem 2.8, if we choose $p=q=2$, we get the following generalized trapezium-like inequality:*

$$\left|T_f^{(\alpha)}(x;m,a,b)\right|\leq\frac{(x-ma)^{3\alpha}}{\Gamma(1+2\alpha)(b-a)^{\alpha}}\sqrt{\frac{\Gamma(1+4\alpha)}{\Gamma(1+5\alpha)}}$$

$$\times\sqrt{\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\left|f^{(2\alpha)}(x)\right|^2+m^{\alpha}\left|f^{(2\alpha)}(a)\right|^2\right)-(cm)^{\alpha}(x-a)^{2\alpha}\frac{B_{\alpha}(2,2)}{\Gamma(1+\alpha)}}$$

$$+\frac{(mb-x)^{3\alpha}}{\Gamma(1+2\alpha)(b-a)^{\alpha}}\sqrt{\frac{\Gamma(1+4\alpha)}{\Gamma(1+5\alpha)}}$$

$$\times\sqrt{\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\left|f^{(2\alpha)}(mb)\right|^2+m^\alpha\left|f^{(2\alpha)}\left(\frac{x}{m}\right)\right|^2\right)-(cm)^\alpha\left(mb-\frac{x}{m}\right)^{2\alpha}\frac{B_\alpha(2,2)}{\Gamma(1+\alpha)}}. \tag{2.8}$$

Corollary 2.10 *Under assumptions of Theorem 2.8, if we choose* $x=\dfrac{a+b}{2}$ *and* $m=1$*, we get the following generalized trapezium-like inequality:*

$$\begin{aligned}&\left|\frac{(b-a)^\alpha}{4^\alpha\Gamma(1+\alpha)\Gamma(1+2\alpha)}\left(f^{(\alpha)}\left(\frac{a+b}{2}\right)+f^{(\alpha)}(b)\right)\right.\\&\quad\left.-\frac{1}{2^\alpha\Gamma^2(1+\alpha)}\left(f\left(\frac{a+b}{2}\right)+f(b)\right)+\frac{1}{(b-a)^\alpha}\,{}_aI_b^{(\alpha)}f(t)\right|\\&\le\frac{(b-a)^{2\alpha}}{8^\alpha\Gamma(1+2\alpha)}\left(\frac{\Gamma(1+2p\alpha)}{\Gamma(1+(2p+1)\alpha)}\right)^{\frac{1}{p}}\\&\quad\times\left[\left\{\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\left|f^{(2\alpha)}(a)\right|^q+\left|f^{(2\alpha)}\left(\frac{a+b}{2}\right)\right|^q\right)-c^\alpha\left(\frac{b-a}{2}\right)^{2\alpha}\frac{B_\alpha(2,2)}{\Gamma(1+\alpha)}\right\}^{\frac{1}{q}}\right.\\&\quad\left.+\left\{\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\left|f^{(2\alpha)}\left(\frac{a+b}{2}\right)\right|^q+\left|f^{(2\alpha)}(b)\right|^q\right)-c^\alpha\left(\frac{b-a}{2}\right)^{2\alpha}\frac{B_\alpha(2,2)}{\Gamma(1+\alpha)}\right\}^{\frac{1}{q}}\right].\end{aligned} \tag{2.9}$$

Corollary 2.11 *Under assumptions of Theorem 2.8, by taking* $K:=\left\|f^{(2\alpha)}(x)\right\|_\infty$*, we get the following generalized trapezium-like inequality:*

$$\begin{aligned}\left|T_f^{(\alpha)}(x;m,a,b)\right|&\le\frac{(x-ma)^{3\alpha}}{\Gamma(1+2\alpha)(b-a)^\alpha}\left(\frac{\Gamma(1+2p\alpha)}{\Gamma(1+(2p+1)\alpha)}\right)^{\frac{1}{p}}\\&\times\left[K^q\left(m^\alpha+1\right)\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-(cm)^\alpha(x-a)^{2\alpha}\frac{B_\alpha(2,2)}{\Gamma(1+\alpha)}\right]^{\frac{1}{q}}\\&+\frac{(mb-x)^{3\alpha}}{\Gamma(1+2\alpha)(b-a)^\alpha}\left(\frac{\Gamma(1+2p\alpha)}{\Gamma(1+(2p+1)\alpha)}\right)^{\frac{1}{p}}\\&\times\left[K^q\left(m^\alpha+1\right)\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-(cm)^\alpha\left(mb-\frac{x}{m}\right)^{2\alpha}\frac{B_\alpha(2,2)}{\Gamma(1+\alpha)}\right]^{\frac{1}{q}}.\end{aligned} \tag{2.10}$$

Corollary 2.12 *Under assumptions of Corollary 2.11, if we choose* $x=\dfrac{a+b}{2}$ *and* $m=1$*, we get the following generalized trapezium-like inequality:*

$$\begin{aligned}&\left|\frac{(b-a)^\alpha}{4^\alpha\Gamma(1+\alpha)\Gamma(1+2\alpha)}\left(f^{(\alpha)}\left(\frac{a+b}{2}\right)+f^{(\alpha)}(b)\right)\right.\\&\quad\left.-\frac{1}{2^\alpha\Gamma^2(1+\alpha)}\left(f\left(\frac{a+b}{2}\right)+f(b)\right)+\frac{1}{(b-a)^\alpha}\,{}_aI_b^{(\alpha)}f(t)\right|\\&\le\frac{2(b-a)^{2\alpha}}{8^\alpha\Gamma(1+2\alpha)}\left(\frac{\Gamma(1+2p\alpha)}{\Gamma(1+(2p+1)\alpha)}\right)^{\frac{1}{p}}\left[2K^q\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-c^\alpha\left(\frac{b-a}{2}\right)^{2\alpha}\frac{B_\alpha(2,2)}{\Gamma(1+\alpha)}\right]^{\frac{1}{q}}.\end{aligned} \tag{2.11}$$

Theorem 2.13 *Suppose that the assumptions of Lemma 2.7 are satisfied. If* $m\in(0,1]$ *and* $\left|f^{(2\alpha)}\right|^q$ *is generalized strongly m-convex, then, for* $q\ge1$ *and* $c\in\mathbb{R}^+$*, the following inequality holds:*

$$\left|T_f^{(\alpha)}(x;m,a,b)\right|\le\frac{(x-ma)^{3\alpha}}{\Gamma(1+2\alpha)(b-a)^\alpha}\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)^{1-\frac{1}{q}}$$

$$\times\left[\left|f^{(2\alpha)}(x)\right|^{q}C(\alpha)+m^{\alpha}\left|f^{(2\alpha)}(a)\right|^{q}D(\alpha)-(cm)^{\alpha}(x-a)^{2\alpha}\frac{B_{\alpha}(4,2)}{\Gamma(1+\alpha)}\right]^{\frac{1}{q}}$$

$$+\frac{(mb-x)^{3\alpha}}{\Gamma(1+2\alpha)(b-a)^{\alpha}}\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)^{1-\frac{1}{q}}$$

$$\times\left[\left|f^{(2\alpha)}(mb)\right|^{q}C(\alpha)+m^{\alpha}\left|f^{(2\alpha)}\left(\frac{x}{m}\right)\right|^{q}D(\alpha)-(cm)^{\alpha}\left(mb-\frac{x}{m}\right)^{2\alpha}\frac{B_{\alpha}(4,2)}{\Gamma(1+\alpha)}\right]^{\frac{1}{q}}, \tag{2.12}$$

where

$$C(\alpha)=\frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)},\quad D(\alpha)=\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-2^{\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}+C(\alpha).$$

Proof From Lemmas 1.6 and 2.7 , Definition 2.1, generalized power mean inequality, generalized strongly m-convexity of $\left|f^{(2\alpha)}\right|^{q}$, and properties of the modulus, we have

$$\left|T_{f}^{(\alpha)}(x;m,a,b)\right|\leq\frac{(x-ma)^{3\alpha}}{\Gamma(1+\alpha)\Gamma(1+2\alpha)(b-a)^{\alpha}}\int_{0}^{1}t^{2\alpha}\left|f^{(2\alpha)}(tx+m(1-t)a)\right|(dt)^{\alpha}$$

$$+\frac{(mb-x)^{3\alpha}}{\Gamma(1+\alpha)\Gamma(1+2\alpha)(b-a)^{\alpha}}\int_{0}^{1}t^{2\alpha}\left|f^{(2\alpha)}(mtb+(1-t)x)\right|(dt)^{\alpha}$$

$$\leq\frac{(x-ma)^{3\alpha}}{\Gamma(1+2\alpha)(b-a)^{\alpha}}\left(\frac{1}{\Gamma(1+\alpha)}\int_{0}^{1}t^{2\alpha}(dt)^{\alpha}\right)^{1-\frac{1}{q}}$$

$$\times\left(\frac{1}{\Gamma(1+\alpha)}\int_{0}^{1}t^{2\alpha}\left|f^{(2\alpha)}(tx+m(1-t)a)\right|^{q}(dt)^{\alpha}\right)^{\frac{1}{q}}$$

$$+\frac{(mb-x)^{3\alpha}}{\Gamma(1+2\alpha)(b-a)^{\alpha}}\left(\frac{1}{\Gamma(1+\alpha)}\int_{0}^{1}t^{2\alpha}(dt)^{\alpha}\right)^{1-\frac{1}{q}}$$

$$\times\left(\frac{1}{\Gamma(1+\alpha)}\int_{0}^{1}t^{2\alpha}\left|f^{(2\alpha)}(mtb+(1-t)x)\right|^{q}(dt)^{\alpha}\right)^{\frac{1}{q}}$$

$$\leq\frac{(x-ma)^{3\alpha}}{\Gamma(1+2\alpha)(b-a)^{\alpha}}\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)^{1-\frac{1}{q}}$$

$$\times\left[\frac{1}{\Gamma(1+\alpha)}\int_{0}^{1}t^{2\alpha}\left(t^{\alpha}\left|f^{(2\alpha)}(x)\right|^{q}+m^{\alpha}(1-t)^{\alpha}\left|f^{(2\alpha)}(a)\right|^{q}-(cm)^{\alpha}t^{\alpha}(1-t)^{\alpha}(x-a)^{2\alpha}\right)(dt)^{\alpha}\right]^{\frac{1}{q}}$$

$$+\frac{(mb-x)^{3\alpha}}{\Gamma(1+2\alpha)(b-a)^{\alpha}}\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)^{1-\frac{1}{q}}$$

$$\times\left[\frac{1}{\Gamma(1+\alpha)}\int_{0}^{1}t^{2\alpha}\left(t^{\alpha}\left|f^{(2\alpha)}(mb)\right|^{q}+m^{\alpha}(1-t)^{\alpha}\left|f^{(2\alpha)}\left(\frac{x}{m}\right)\right|^{q}\right.\right.$$

$$\left.\left.-(cm)^{\alpha}t^{\alpha}(1-t)^{\alpha}\left(mb-\frac{x}{m}\right)^{2\alpha}\right)(dt)^{\alpha}\right]^{\frac{1}{q}}$$

$$=\frac{(x-ma)^{3\alpha}}{\Gamma(1+2\alpha)(b-a)^{\alpha}}\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)^{1-\frac{1}{q}}$$

$$\times\left[\left|f^{(2\alpha)}(x)\right|^{q}C(\alpha)+m^{\alpha}\left|f^{(2\alpha)}(a)\right|^{q}D(\alpha)-(cm)^{\alpha}(x-a)^{2\alpha}\frac{B_{\alpha}(4,2)}{\Gamma(1+\alpha)}\right]^{\frac{1}{q}}$$

$$+\frac{(mb-x)^{3\alpha}}{\Gamma(1+2\alpha)(b-a)^{\alpha}}\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)^{1-\frac{1}{q}}$$

$$\times\left[\left|f^{(2\alpha)}(mb)\right|^{q}C(\alpha)+m^{\alpha}\left|f^{(2\alpha)}\left(\frac{x}{m}\right)\right|^{q}D(\alpha)-(cm)^{\alpha}\left(mb-\frac{x}{m}\right)^{2\alpha}\frac{B_{\alpha}(4,2)}{\Gamma(1+\alpha)}\right]^{\frac{1}{q}}.$$

Therefore, the proof of this theorem is completed. □

We point out some special cases of Theorem 2.13.

Corollary 2.14 *Under assumptions of Theorem 2.13, if we choose $q=1$, we get the following generalized trapezium-like inequality:*

$$\begin{aligned}\left|T_{f}^{(\alpha)}(x;m,a,b)\right| &\leq \frac{(x-ma)^{3\alpha}}{\Gamma(1+2\alpha)(b-a)^{\alpha}}\\ &\times\left[\left|f^{(2\alpha)}(x)\right|C(\alpha)+m^{\alpha}\left|f^{(2\alpha)}(a)\right|D(\alpha)-(cm)^{\alpha}(x-a)^{2\alpha}\frac{B_{\alpha}(4,2)}{\Gamma(1+\alpha)}\right]\\ &+\frac{(mb-x)^{3\alpha}}{\Gamma(1+2\alpha)(b-a)^{\alpha}}\\ &\times\left[\left|f^{(2\alpha)}(mb)\right|C(\alpha)+m^{\alpha}\left|f^{(2\alpha)}\left(\frac{x}{m}\right)\right|D(\alpha)-(cm)^{\alpha}\left(mb-\frac{x}{m}\right)^{2\alpha}\frac{B_{\alpha}(4,2)}{\Gamma(1+\alpha)}\right]. \end{aligned}\tag{2.13}$$

Corollary 2.15 *Under assumptions of Theorem 2.13, if we choose $x=\dfrac{a+b}{2}$ and $m=1$, we get the following generalized trapezium-like inequality:*

$$\begin{aligned}&\left|\frac{(b-a)^{\alpha}}{4^{\alpha}\Gamma(1+\alpha)\Gamma(1+2\alpha)}\left(f^{(\alpha)}\left(\frac{a+b}{2}\right)+f^{(\alpha)}(b)\right)\right.\\ &\left.\quad-\frac{1}{2^{\alpha}\Gamma^{2}(1+\alpha)}\left(f\left(\frac{a+b}{2}\right)+f(b)\right)+\frac{1}{(b-a)^{\alpha}}\,{}_{a}I_{b}^{(\alpha)}f(t)\right|\\ &\leq\frac{(b-a)^{2\alpha}}{8^{\alpha}\Gamma(1+2\alpha)}\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)^{1-\frac{1}{q}}\\ &\quad\times\left[\left\{\left|f^{(2\alpha)}(a)\right|^{q}D(\alpha)+\left|f^{(2\alpha)}\left(\frac{a+b}{2}\right)\right|^{q}C(\alpha)-c^{\alpha}\left(\frac{b-a}{2}\right)^{2\alpha}\frac{B_{\alpha}(4,2)}{\Gamma(1+\alpha)}\right\}^{\frac{1}{q}}\right.\\ &\quad\left.+\left\{\left|f^{(2\alpha)}\left(\frac{a+b}{2}\right)\right|^{q}D(\alpha)+\left|f^{(2\alpha)}(b)\right|^{q}C(\alpha)-c^{\alpha}\left(\frac{b-a}{2}\right)^{2\alpha}\frac{B_{\alpha}(4,2)}{\Gamma(1+\alpha)}\right\}^{\frac{1}{q}}\right].\end{aligned}\tag{2.14}$$

Corollary 2.16 *Under assumptions of Theorem 2.13, by taking $K:=\left\|f^{(2\alpha)}(x)\right\|_{\infty}$, we get the following generalized trapezium-like inequality:*

$$\begin{aligned}\left|T_{f}^{(\alpha)}(x;m,a,b)\right| &\leq\frac{(x-ma)^{3\alpha}}{\Gamma(1+2\alpha)(b-a)^{\alpha}}\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)^{1-\frac{1}{q}}\\ &\times\left[K^{q}\left(C(\alpha)+m^{\alpha}D(\alpha)\right)-(cm)^{\alpha}(x-a)^{2\alpha}\frac{B_{\alpha}(4,2)}{\Gamma(1+\alpha)}\right]^{\frac{1}{q}}\\ &+\frac{(mb-x)^{3\alpha}}{\Gamma(1+2\alpha)(b-a)^{\alpha}}\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)^{1-\frac{1}{q}}\\ &\times\left[K^{q}\left(C(\alpha)+m^{\alpha}D(\alpha)\right)-(cm)^{\alpha}\left(mb-\frac{x}{m}\right)^{2\alpha}\frac{B_{\alpha}(4,2)}{\Gamma(1+\alpha)}\right]^{\frac{1}{q}}.\end{aligned}\tag{2.15}$$

Corollary 2.17 *Under assumptions of Corollary 2.16, if we choose* $x = \frac{a+b}{2}$ *and* $m = 1$*, we get the following generalized trapezium-like inequality:*

$$\begin{aligned}&\Bigg|\frac{(b-a)^{\alpha}}{4^{\alpha}\Gamma(1+\alpha)\Gamma(1+2\alpha)}\left(f^{(\alpha)}\left(\frac{a+b}{2}\right)+f^{(\alpha)}(b)\right)\\&-\frac{1}{2^{\alpha}\Gamma^{2}(1+\alpha)}\left(f\left(\frac{a+b}{2}\right)+f(b)\right)+\frac{1}{(b-a)^{\alpha}}\,{}_{a}I_{b}^{(\alpha)}f(t)\Bigg|\\&\le\frac{2(b-a)^{2\alpha}}{8^{\alpha}\Gamma(1+2\alpha)}\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)^{1-\frac{1}{q}}\left[K^{q}(C(\alpha)+D(\alpha))-c^{\alpha}\left(\frac{b-a}{2}\right)^{2\alpha}\frac{B_{\alpha}(4,2)}{\Gamma(1+\alpha)}\right]^{\frac{1}{q}}. \qquad (2.16)\end{aligned}$$

Remark 2.18 For $\alpha = 1$, by our Theorems 2.8 and 2.13, we can obtain some new estimates on generalization of trapezium-like inequalities for twice differentiable mappings that are in absolute value at certain powers strongly m-convex.

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