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Boundedness and power-type decay of solutions for a class of generalized fractional Langevin equations

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Abstract In this paper we study the long-time behavior of solutions for a general class of Langevin-type fractional integro-differential equations. The involved fractional derivatives are either of Riemann–Liouville or Caputo type. Reasonable sufficient conditions under which the solutions are bounded or decay like power functions are established. For this purpose, we combine and generalize some well-known integral inequalities with some crucial estimates. Our findings are supported by examples and special cases.

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المخلص

في هذا البحث نقوم بدراسة السلوك البعيد المدى لحلول صنف عام من معادلات تفاضلية تكاملية كسرية من نوع لانجوفين. فالمشتقات المستخدمة هي إما من نوع ريمان-ليوفيل أو من نوع كبوتو. فبعض شروط الكفاية التي تكون بها الحلول محدودة أو تضمحل مثل دوال أسية قد أسست. ولهذا الغرض فقد مزجنا بعض المتراجحات التكاملية المعروفة ببعض التقديرات الهامة. فنتائجنا مدعومة بأمثلة وحالات خاصة.

1 Introduction

We consider the following class of nonlinear fractional integro-differential equations:

$$(\mathbf{D}_{0+}^{\alpha}x)(t) = f\left(t, (\mathbf{D}_{0+}^{\beta}x)(t), \int_0^t k(t,s, (\mathbf{D}_{0+}^{\gamma}x)(s))ds\right), \quad t > 0, \quad (1)$$

supplied with appropriate initial data. The derivatives \mathbf{D}_{0+}^{α} , \mathbf{D}_{0+}^{β} and \mathbf{D}_{0+}^{γ} represent either the Caputo or the Riemann–Liouville fractional derivatives of orders α , β and γ , respectively, where $0 \leq \beta, \gamma < \alpha \leq 1$. The definitions of the Caputo and the Riemann–Liouville fractional derivatives are given in the next section.

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Clearly, many treated fractional integro-differential equations in the literature become special cases of this general class.

It is well-known that $x(t) = a_1$ and $x(t) = a_2 t^{\alpha-1}$, $0 < \alpha < 1$, are solutions for $({}^C\mathbf{D}_{0+}^\alpha x)(t) = 0$ and $(\mathbf{D}_{0+}^\alpha x)(t) = 0$, respectively, where a_1 and a_2 are constants depending on the initial values of x . Under some sufficient conditions, we prove that the solutions of (1) are bounded by a constant when the fractional derivatives are of Caputo type, ${}^C\mathbf{D}_{0+}^\alpha$, and decay like $t^{\alpha-1}$ when the derivatives are of Riemann–Liouville type, \mathbf{D}_{0+}^α . We use fractional calculus tools to treat the terms involving fractional integrals and derivatives and to find appropriate underlying spaces for the solutions. An argument is borrowed from Medved [26] to deal with the singularities inside the fractional derivatives. Also, to treat the associated Volterra integral equations corresponding to (1), we use and generalize some known nonlinear integral inequalities.

From both the theoretical point of view and the application point of view, it is of great importance to have an idea about the behavior of solutions for large values of the time variable. It is known from the definitions of fractional derivatives that all the history of the state is taken into account through a convolution with a singular kernel. Moreover, in our case, the nonlinear term may involve additional singularities. Because of all these features, it is difficult to apply the existing approaches and methods in the literature for integer order to the noninteger order case.

Langevin equation is a widely used model to describe the evolution of physical phenomena in fluctuating environments [10]. Several fractional generalizations of Langevin equations have been suggested to describe dynamical systems in fractal media, see e.g. [5, 9, 24, 25, 36, 38, 39] and the references therein. There is a large volume of literature on existence of solutions for various classes of fractional differential and integro-differential equations (see for instance [1–4, 8, 12, 17, 18, 20, 35, 37]). Agarwal et al. surveyed in [1] many of these existence results. They focused on initial and boundary value problems for fractional differential equations with Caputo fractional derivatives of orders between 0 and 1 and between 1 and 2. Furati and Tatar considered in [13], the Cauchy-type fractional differential problem:

$$\begin{cases} (\mathbf{D}_{0+}^\alpha x)(t) = f(t, x(t)) + \int_0^t k(t, s, x(s)) ds, & t > 0, \\ \lim_{t \rightarrow 0+} t^{1-\alpha} x(0^+) = c_0, & c_0 \in \mathbb{R}, \end{cases} \quad (2)$$

where \mathbf{D}_{0+}^α is the Riemann–Liouville fractional derivative of order α , $0 < \alpha < 1$. They used the Schauder fixed point theorem to prove a local existence result in the space $C_{1-\alpha}[0, \infty)$ (see (9), for some classes of the nonlinearities f and k involving some power functions of t , s and x . The case $k \equiv 0$ has been studied by Delbosco and Rodino [11], Kilbas et al. [21], and many others. In 2012, Trujillo et al. [3] established the existence and uniqueness of solutions for the nonlinear fractional integro-differential problem:

$$\begin{cases} ({}^C\mathbf{D}_{a+}^\alpha x)(t) = f\left(t, ({}^C\mathbf{D}_{a+}^\beta x)(t), \int_a^t g(t, s, x(s)) ds\right), & t \in (a, b], \\ x^{(k)}(a) = c_k, & k = 0, 1, \dots, m-1, \end{cases} \quad (3)$$

where ${}^C\mathbf{D}_{a+}^\alpha$ is the Caputo fractional derivative of order α , $m-1 < \alpha < m$, $n-1 < \beta < n$, $\beta < \alpha$, $m, n \in \mathbb{N}$, $f : [a, b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and $g : [a, b] \times [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions. They showed that this problem has a unique solution $x \in C^{m-1}[a, b]$ with ${}^C\mathbf{D}_{a+}^\alpha x \in C[a, b]$. Their main tool is a fixed point theorem for non-self mappings. First, using a suitable substitution, they constructed an equivalent fractional integral equation. Then, they used some fractional integral inequalities and a nonlinear alternative of Leray–Schauder type to achieve their existence result. The uniqueness of the solution was established with the help of the Banach contraction principle.

The long-time behavior of solutions of differential equations has attracted many researchers, see [6, 7, 14, 16, 19, 27, 30, 31]. In many cases, the main idea is to establish sufficient reasonable conditions ensuring comparison or similarity with the long-time behavior of solutions of simpler differential equations.

In 2004, Momani et al. [30] discussed the Lyapunov stability and asymptotic stability for solutions of the fractional integro-differential equation (2) for $t \geq a > 0$. The assumptions:

$$|f(t, x(t))| \leq \gamma(t) |x|, \quad (4)$$

$$\int_s^t k(\sigma, s, x(s)) d\sigma \leq \delta(t) |x|, \quad s \in [a, t], \quad (5)$$



where $\gamma(t)$ and $\delta(t)$ are continuous nonnegative functions and

$$\int_a^t (t-s)^{\alpha-1} [\gamma(s) + \delta(s)] ds = O((t-a)^{\alpha-1}), \tag{6}$$

were imposed. The authors proved that every solution $x(t)$ of (2) satisfies $|x(t)| \leq C_0(t-a)^{\alpha-1}$ where C_0 is a positive constant, and hence the solution of (2) is asymptotically stable.

Furati and Tatar [14] showed that the solutions of (2) decay polynomially for some nonlinear functions f and k . When $k \equiv 0$, they proved in [15] that solutions of the problem exist globally and decay as a power function in the space $C_{1-\alpha}^\alpha [0, \infty)$ defined in (26). The same authors considered in [16] Eq. (2) and found bounds for solutions on infinite time intervals and also provided sufficient conditions assuring decay of power type for the solutions.

In [31], Mustafa and Băleanu proved that the solution of the nonlinear fractional differential problem:

$$\begin{cases} (\mathbf{D}_{0^+}^\alpha (x - x_0))(t) = f(t, x(t)), & 0 < \alpha < 1, \quad t > 0, \\ x(0^+) = x_0, & x_0 \in \mathbb{R}, \end{cases} \tag{7}$$

is asymptotic to $o(t^{\alpha\alpha})$ as $t \rightarrow \infty$, $0 < 1 - a < \alpha$. They assumed that,

$$|f(t, x)| \leq h(t) g\left(\frac{|x|}{(t+1)^\alpha}\right),$$

and

$$t^{(q_3/q_1)[1-q_1(1-\alpha)]} \left\{ \int_0^t [h(s)]^{q_2} ds \right\}^{q_3/q_2} \leq M(t+1)^\alpha, \quad t \geq 0,$$

for some sufficiently large constant M , $q_1, q_2, q_3 > 1$, $a \in (0, 1)$, $g : [0, \infty) \rightarrow [0, \infty)$ is continuous, nondecreasing function and the function $h : [0, \infty) \rightarrow [0, \infty)$ is continuous with

$$t^{(q_3/q_1)[1-q_1(1-\alpha)]} \|h\|_{L^{q_2}(0,t)}^{q_3} = O(t^\alpha) \text{ when } t \rightarrow \infty.$$

Băleanu et al. discussed in [7] the long-time behavior of solutions to the linear fractional differential problem:

$$\begin{cases} (\mathbf{D}_{0^+}^\alpha x)'(t) = h(t)x(t), & 0 < \alpha < 1, \quad t > 0, \\ \lim_{t \rightarrow 0^+} (t^{1-\alpha}x(t)) = c_0, & c_0 \in \mathbb{R}, \end{cases}$$

where $\mathbf{D}_{0^+}^\alpha$ is the Riemann–Liouville fractional derivative of order α . They proved that it has a solution $x \in C((0, \infty), \mathbb{R})$ such that,

$$\lim_{t \rightarrow \infty} (t^{1-\alpha}x(t)) = c_1, \quad c_1 \in \mathbb{R},$$

and it has the asymptotic expansion,

$$x(t) = (c_0 + O(1))t^{\alpha-1} + \left(\frac{c_1}{\Gamma(\alpha+1)} + o(1)\right)t^\alpha \text{ when } t \rightarrow \infty,$$

provided that $h : (0, \infty) \rightarrow \mathbb{R}$ is continuous function such that for some $T > 0$,

$$\int_T^\infty s^{\alpha+1} |h(s)| ds < \infty,$$

and

$$\max\{1, T\} \int_{0^+}^T \frac{|h(s)|}{s^{1-\alpha}} ds + \int_T^\infty s^\alpha |h(s)| ds < \Gamma(\alpha+1).$$

In 2011, the authors in [6] established under the condition:

$$|f(t, x)| \leq \phi\left(t, \frac{|x|}{(1+t)^\alpha}\right), \quad t \geq 0, \quad x \in \mathbb{R},$$

($f : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ and $\phi : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ are continuous functions and ϕ is nondecreasing in the second argument), that the solution of the nonlinear fractional differential equation,

$$(\mathbf{D}_{0+}^{\alpha} x')(t) + f(t, x) = 0, \quad 0 < \alpha < 1, \quad t > 0,$$

can be expressed asymptotically as $c_1 + c_2 t^{\alpha} + O(t^{\alpha-1})$ when $t \rightarrow \infty$, $c_1, c_2 \in \mathbb{R}$.

In 2015, Medved and Pospíšil considered in the paper [27] the case when the right-hand side depends on a Caputo fractional derivative of the solution. They proved that there exists a constant $c \in \mathbb{R}$ such that any global solution of the initial value problem,

$$\begin{cases} ({}^C \mathbf{D}_{a+}^{\alpha} x)(t) = f(t, x(t), ({}^C \mathbf{D}_{a+}^{\beta} x)(t)), & t \geq a > 0, \\ x'(a) = c_0, \end{cases}$$

where $0 < \beta < \alpha < 1$, is asymptotic to ct^{β} . Also, they considered the case when the right hand side depends on ordinary derivatives up to order $n - 1$ and Caputo derivatives of fractional orders $n - 1 < \alpha_j < \alpha < n$, $j = 1, 2, \dots, m$, $m \in \mathbb{N}$. It has been shown that the solution in this case is asymptotic to bt^r with $r = \max\{n - 1, \alpha_m\}$.

In 2016, Kassim et al. studied in [19] the boundedness and asymptotic behavior of solutions for the fractional differential problem:

$$\begin{cases} (\mathbf{D}_{0+}^{\alpha} x)(t) = f(t, x(t), (\mathbf{D}_{0+}^{\beta} x)(t)), & 0 \leq \beta < \alpha < 1, \quad t > 0, \\ (I_{0+}^{1-\alpha} x)(t) \Big|_{t=0} = c_0, & c_0 \in \mathbb{R}, \end{cases}$$

in the space

$$C_{1-\alpha}^{\alpha}[0, \infty) = \{x \in C_{1-\alpha}[0, \infty) : \mathbf{D}_{0+}^{\alpha} x \in C_{1-\alpha}[0, \infty)\}. \quad (8)$$

They showed that there exists a positive constant A such that,

$$|x(t)| \leq At^{\alpha-1}, \quad \left| \mathbf{D}_{0+}^{\beta} x(t) \right| \leq At^{\alpha-\beta-1}, \quad t > 0$$

provided that,

$$|f(t, x, y)| \leq t^{\sigma} e^{-\lambda t} h(t) \psi_1(t^{1-\alpha} |x|) \psi_2(t^{1-(\alpha-\beta)} |y|),$$

and

$$\int_T^{\infty} \frac{ds}{\psi_1^p\left(s^{\frac{1}{p}}\right) \psi_2^p\left(s^{\frac{1}{p}}\right)} = \infty, \quad T > 0.$$

The functions $h, \psi_1, \psi_2 : [0, \infty) \rightarrow [0, \infty)$ are assumed to be continuous with $h \in L^p(0, \infty)$, $p(\alpha - \beta) > 1$, $\sigma > \frac{1}{p} - 1$, $\lambda > 0$ and ψ_1, ψ_2 are nondecreasing.

In this paper, we consider a much wider class of nonlinearities than the ones considered in the previous papers [14, 16, 30]. Also, we improve the results in these papers by weakening the imposed conditions. Namely, we consider the problem:

$$(\mathbf{D}_{0+}^{\alpha} x)(t) = f\left(t, (\mathbf{D}_{0+}^{\beta} x)(t), \int_0^t k(t, s, (\mathbf{D}_{0+}^{\gamma} x)(s)) ds\right), \quad t > 0,$$

with appropriate initial conditions depending on \mathbf{D} being the Caputo or the Riemann–Liouville fractional derivative. Clearly, this nonlinearity is much more general than the one in [14, 30] and even more general than the ones in [6, 7, 15, 31].

The rest of this paper is organized as follows: In the next section we briefly present the used notation, underlying function spaces, background material and some useful lemmas and inequalities. Section 3 is devoted to the main results of the study of the long-time behavior of solutions of (1).



2 Preliminaries

In this section, we briefly introduce some basic definitions, notions and properties from the theory of fractional differential equations.

Definition 2.1 [22] Let $-\infty \leq a < b \leq \infty$. The space $L^p(a, b)$ ($1 \leq p \leq \infty$) consists of all (Lebesgue) real-valued measurable functions f on (a, b) for which $\|f\|_p < \infty$, where

$$\|f\|_p = \left(\int_a^b |f(s)|^p ds \right)^{1/p}, \quad 1 \leq p < \infty,$$

$$\|f\|_\infty = \text{ess sup}_{a \leq t \leq b} |f(t)|,$$

and $\text{ess sup } |f(t)|$ is the essential supremum of the function $|f(t)|$.

Definition 2.2 [22] We denote by $C[a, b]$ and $C^n[a, b]$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the spaces of continuous and n -times continuously differentiable functions on $[a, b]$, with $C[a, b] = C^0[a, b]$.

Definition 2.3 [22] We denote by $C_\gamma[a, b]$, $0 \leq \gamma < 1$, the following weighted space of continuous functions

$$C_\gamma[a, b] = \{f : (a, b) \rightarrow \mathbb{R} : (t - a)^\gamma f(t) \in C[a, b]\}, \tag{9}$$

where $C[a, b]$ is the space of continuous functions on $[a, b]$.

Definition 2.4 [22] For $n \in \mathbb{N}$ and $0 \leq \gamma < 1$, we denote by $C_\gamma^n[a, b]$, the following weighted space of continuously differentiable functions up to order $n - 1$ with n th derivative in $C_\gamma[a, b]$,

$$C_\gamma^n[a, b] = \left\{ f : (a, b) \rightarrow \mathbb{R} \mid f \in C^{n-1}[a, b], f^{(n)} \in C_\gamma[a, b] \right\}.$$

In particular, $C_\gamma[a, b] = C_\gamma^0[a, b]$.

Definition 2.5 The Riemann–Liouville left-sided fractional integral of order $\alpha > 0$ is defined by

$$\left(\mathbf{I}_{a+}^\alpha u\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha-1} u(s) ds, \quad t > a, \tag{10}$$

provided the right-hand side exists. We define $\mathbf{I}_{a+}^0 u = u$. The function Γ is the Euler gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$, $\alpha > 0$.

Definition 2.6 The Riemann–Liouville left-sided fractional derivative of order $\alpha \geq 0$, is defined by:

$$\left(\mathbf{D}_{a+}^\alpha u\right)(t) = \mathbf{D}^n \left(\mathbf{I}_{a+}^{n-\alpha} u\right)(t), \quad t > a, \tag{11}$$

where $\mathbf{D}^n = \frac{d^n}{dt^n}$, $n = [\alpha] + 1$, $[\alpha]$ is the integral part of α . In particular, when $\alpha = m \in \mathbb{N}_0$, it follows from the definition that, $\mathbf{D}_{a+}^m u = \mathbf{D}^m u$.

Definition 2.7 The Caputo left-sided fractional derivative of order $\alpha \geq 0$, is defined by:

$$\left({}^C\mathbf{D}_{a+}^\alpha u\right)(t) = \left(\mathbf{I}_{a+}^{n-\alpha} \mathbf{D}^n u\right)(t), \tag{12}$$

where $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}_0$ and $n = \alpha$ for $\alpha \in \mathbb{N}_0$. In particular, when $\alpha = n \in \mathbb{N}_0$, it follows from the definition that ${}^C\mathbf{D}_{a+}^0 u = u$, ${}^C\mathbf{D}_{a+}^n u = \mathbf{D}^n u$.

The next lemma shows that the Riemann–Liouville fractional integral and derivative of the power functions yield power functions multiplied by certain coefficients and with the order of the fractional derivative added or subtracted from the power.

Lemma 2.8 [22] *If $\alpha \geq 0$, $\beta > 0$, then*

$$(\mathbf{I}_{a^+}^\alpha (s-a)^{\beta-1})(t) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (t-a)^{\beta+\alpha-1}, \quad t > a,$$

$$(\mathbf{D}_{a^+}^\alpha (s-a)^{\beta-1})(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (t-a)^{\beta-\alpha-1}, \quad t > a.$$

Lemma 2.9 [22, p.77 Lemma 2.9 (c)] *Let $0 < \beta < \alpha$ and $0 \leq \gamma < 1$. If $u \in C_\gamma[a, b]$, then*

$$\mathbf{D}_{a^+}^\beta \mathbf{I}_{a^+}^\alpha u = \mathbf{I}_{a^+}^{\alpha-\beta} u$$

at every point in $(a, b]$. When $u \in C[a, b]$, this relation is valid at every point in $[a, b]$. In particular, if $\beta = m \in \mathbb{N}$ and $\alpha > m$, then $\mathbf{D}_{a^+}^m \mathbf{I}_{a^+}^\alpha u = \mathbf{I}_{a^+}^{\alpha-m} u$.

Lemma 2.10 [3] *Let $n-1 < \alpha < n$, $m-1 < \beta < m$ and $\beta < \alpha$.*

1. *If $u \in C[a, b]$, then*

$${}^C \mathbf{D}_{a^+}^\beta \mathbf{I}_{a^+}^\alpha u = \mathbf{I}_{a^+}^{\alpha-\beta} u$$

at every point in $[a, b]$,

2. *If $u \in C^{n-1}[a, b]$ and ${}^C \mathbf{D}_{a^+}^\alpha u \in C[a, b]$, then ${}^C \mathbf{D}_{a^+}^\beta u \in C[a, b]$.*

The following result is about the composition $\mathbf{I}_{a^+}^\alpha \mathbf{D}_{a^+}^\alpha$ of the Riemann–Liouville fractional integration and differentiation operators.

Lemma 2.11 [22] *Let $0 < \alpha \leq 1$. If $u \in C_\gamma[a, b]$, $0 \leq \gamma < 1$ and $\mathbf{I}_{a^+}^{1-\alpha} u \in C_\gamma^1[a, b]$ then*

$$(\mathbf{I}_{a^+}^\alpha \mathbf{D}_{a^+}^\alpha u)(t) = u(t) - \frac{(\mathbf{I}_{a^+}^{1-\alpha} u)(a)}{\Gamma(\alpha)} (t-a)^{\alpha-1}, \quad (13)$$

for all $t \in (a, b]$.

The next lemma is an analog of Lemma 2.11 for the Caputo fractional derivative.

Lemma 2.12 [22] *Let $0 < \alpha \leq 1$. If $u \in C^1[a, b]$, then*

$$(\mathbf{I}_{a^+}^\alpha {}^C \mathbf{D}_{a^+}^\alpha u)(t) = u(t) - u(a), \quad t \in (a, b], \quad (14)$$

for all $t \in (a, b]$.

For more details about fractional integrals and fractional derivatives, the reader is referred to the books [22, 29, 33, 34]. We mention here some useful basic inequalities and some linear and nonlinear integral inequalities to be used in the next sections.

Lemma 2.13 [28] *If $\lambda, \nu, \omega > 0$, then, for any $t > 0$, we have*

$$\int_0^t (t-s)^{\nu-1} s^{\lambda-1} e^{-\omega s} ds \leq Ct^{\nu-1},$$

where C is a positive constant independent of t . In fact,

$$C = \max\{1, 2^{1-\nu}\} \Gamma(\lambda) (1 + \lambda(\lambda+1)/\nu) \omega^{-\lambda}.$$

The inequalities in the next lemma are of Jensen's inequalities type.



Lemma 2.14 [23] *Let $a_i, i = 1, \dots, m, m \in \mathbb{N}$, be nonnegative real numbers. Then,*

$$\left(\sum_{i=1}^m a_i\right)^q \leq m^{q-1} \sum_{i=1}^m a_i^q \text{ for } q \geq 1.$$

Moreover, if $a_i > 0$ for all $i = 1, \dots, m$, then

$$\left(\sum_{i=1}^m a_i\right)^q \geq m^{q-1} \sum_{i=1}^m a_i^q \text{ for } 0 \leq q \leq 1.$$

Notation 2.15 *Let $S \subseteq \mathbb{R}$. For two functions $f, g : S \rightarrow \mathbb{R}/\{0\}$, we say that $f \propto g$ if g/f is nondecreasing on S .*

Lemma 2.16 [32, Theorem 4] *Let $u, \lambda_i, i = 1, 2, 3$ be continuous and nonnegative functions on $I = [a, b]$ and the functions $\omega_i, i = 1, 2, 3$ be continuous nonnegative and nondecreasing on $[0, \infty)$ such that $\omega_1 \propto \omega_2 \propto \omega_3$. Assume further that c is a positive constant. If*

$$u(t) \leq c + \int_a^t \lambda_1(s)\omega_1(u(s))ds + \int_a^t \lambda_2(s)\omega_2\left(\int_a^s \lambda_3(\tau)\omega_3(u(\tau))d\tau\right)ds,$$

then, for $t \in [a, b_1]$,

$$u(t) \leq W_3^{-1}\left(W_3(c_2) + \int_a^t \lambda_3(s)ds\right),$$

where

1. $W_i(v) = \int_{v_i}^v \frac{d\tau}{\omega_i(\tau)}$, $v > 0, v_i > 0, i = 1, 2, 3$ and W_i^{-1} is the inverse function of W_i .
2. The constants c_i are given by $c_0 = c$ and $c_i = W_i^{-1}\left(W_i(c_{i-1}) + \int_a^{b_1} \lambda_i(s)ds\right)$, $i = 1, 2$.
3. The number $b_1 \in [a, b]$ is the largest number such that,

$$\int_a^{b_1} \lambda_i(s)ds \leq \int_{c_{i-1}}^\infty \frac{d\tau}{\omega_i(\tau)}, \quad i = 1, 2, 3.$$

Remark 2.17 The monotonicity and the ordering requirements of Lemma 2.16, can be dropped using the functions:

$$\begin{aligned} \varphi_1(t) &:= \max_{s \in [0,t]} \{\omega_1(s)\}, \\ \varphi_i(t) &:= \max_{s \in [0,t]} \left\{ \frac{\omega_i(s)}{\varphi_{i-1}(s)} \right\} \varphi_{i-1}(t), \quad i = 2, 3. \end{aligned} \tag{15}$$

Note that $\varphi_i, i = 1, 2, 3$ are nonnegative nondecreasing functions on $[0, \infty)$, $\omega_i(t) \leq \varphi_i(t), i = 1, 2, 3$ for all $t \in [0, \infty)$ and $\varphi_1 \propto \varphi_2 \propto \varphi_3$.

3 Main results

In this section, the long-time behavior of solutions is investigated for the initial value problem composed of Eq. (1) with the initial conditions $x(0) = c_0$ and $\mathbf{I}_{0+}^{1-\alpha} x(0^+) = c_1$, where $c_0, c_1 \in \mathbb{R}$, when the fractional derivatives are of Caputo and Riemann–Liouville types, respectively. The first section is dedicated to the study of the long-time behavior of the solutions for (1) with Caputo fractional derivatives. The case of the Riemann–Liouville derivatives is discussed in Sect. 3.2.

Before presenting our results we need to define the following classes of functions:

Definition 3.1 We say that a function $h : [0, \infty) \rightarrow [0, \infty)$ is of type $\mathcal{H}_{\sigma,\delta}$ if $h \in C[0, \infty)$ and $t^\sigma h^\delta(t) \in L^1(1, \infty), \sigma \geq -1, \delta \geq 1$.

Definition 3.2 We say that a function $h : [0, \infty) \rightarrow [0, \infty)$ is of type ${}_q\mathcal{H}_{r,\eta}$ if $h \in C[0, \infty)$ and $t^{rq}e^{q\eta t}h^q \in L^1(0, \infty)$, $0 \leq r < \frac{q-1}{q}$, $\eta > 0$ and $q \geq 1$.

Definition 3.3 We say that a function g is of type \mathcal{G} if it is continuous nondecreasing on $[0, \infty)$ and positive on $(0, \infty)$.

The above classes are not empty. Examples showing this fact are given in the next sections.

3.1 Equations with Caputo fractional derivatives

Equation (1) with the Caputo fractional derivatives of orders $0 \leq \beta, \gamma < \alpha \leq 1$, is considered in this section, that is,

$$\begin{cases} ({}^C\mathbf{D}_{0+}^\alpha x)(t) = f\left(t, ({}^C\mathbf{D}_{0+}^\beta x)(t), \int_0^t k(t,s, ({}^C\mathbf{D}_{0+}^\gamma x)(s)) ds\right), & t \geq 0, \\ x(0) = c_0, & c_0 \in \mathbb{R}. \end{cases} \quad (16)$$

We discuss the boundedness of the continuable solutions of (16) in the space $C^\alpha[0, \infty)$ defined by:

$$C^\alpha[0, \infty) := \left\{ f : [0, \infty) \rightarrow \mathbb{R} \mid f, {}^C\mathbf{D}_{0+}^\alpha f \in C[0, \infty) \right\}.$$

By “continuable”, we mean that the solution as it exists locally we may prolongate it to infinity. It will exist globally in time.

We suppose that the functions f and k satisfy the following hypotheses:

- (H₁) $f(t, u, v)$ is a continuous function in $D = \{(t, u, v) : t \geq 0, u, v \in \mathbb{R}\}$.
 (H₂) $k(t, s, u)$ is continuous in $E = \{(t, s, u) : 0 \leq s < t < \infty, u \in \mathbb{R}\}$.
 (H₃) There are functions h_1, h_2 of type $\mathcal{H}_{\alpha q-1, q}$, h_3 of type $\mathcal{H}_{0,1}$; $q \geq 1$, and $g_i, i = 1, 2, 3$, are of type \mathcal{G} with $g_1^q \propto g_2^q \propto g_3$ such that,

$$\begin{aligned} |f(t, u, v)| &\leq h_1(t)g_1(t^\beta |u|) + h_2(t)g_2(|v|), & (t, u, v) \in D, \\ |k(t, s, u)| &\leq h_3(s)g_3(s^\gamma |u|), & (t, s, u) \in E, \end{aligned}$$

$$\int_{t_0}^\infty \frac{\tau^{q-1} d\tau}{g_1^q(\tau)} = \infty, \quad \int_{t_0}^\infty \frac{d\tau}{g_2^q(\tau)} = \infty, \quad \int_{t_0}^\infty \frac{\tau^{q-1} d\tau}{g_3(\tau)} = \infty, \quad t_0 > 0.$$

The main result of this section is stated in the following theorem.

Theorem 3.4 Suppose that the functions f and k satisfy (H₁), (H₂) and (H₃). Then, there exists a positive constant $c \in \mathbb{R}$ such that any solution $x \in C^\alpha[0, \infty)$ of problem (16) satisfies:

$$|x(t)| \leq c, \quad \left| ({}^C\mathbf{D}_{0+}^\beta x)(t) \right| \leq ct^{-\beta} \quad \text{and} \quad \left| ({}^C\mathbf{D}_{0+}^\gamma x)(t) \right| \leq ct^{-\gamma} \quad \text{for all } t > 0.$$

Proof Applying \mathbf{I}_{0+}^α to both sides of the equation in (16), we obtain from Lemma 2.12,

$$x(t) = c_0 + \left(\mathbf{I}_{0+}^\alpha f \left(s, ({}^C\mathbf{D}_{0+}^\beta x)(s), \int_0^s k(s, \tau, ({}^C\mathbf{D}_{0+}^\gamma x)(\tau)) d\tau \right) \right) (t), \quad (17)$$

for all $t > 0$. Taking the derivatives ${}^C\mathbf{D}_{0+}^\beta$ and ${}^C\mathbf{D}_{0+}^\gamma$ of (17), gives in view of Lemmas 2.8 and 2.10,

$$({}^C\mathbf{D}_{0+}^\beta x)(t) = \mathbf{I}_{0+}^{\alpha-\beta} f \left(s, ({}^C\mathbf{D}_{0+}^\beta x)(s), \int_0^s k(s, \tau, ({}^C\mathbf{D}_{0+}^\gamma x)(\tau)) d\tau \right) (t),$$

$$({}^C\mathbf{D}_{0+}^\gamma x)(t) = \left(\mathbf{I}_{0+}^{\alpha-\gamma} f \left(s, ({}^C\mathbf{D}_{0+}^\beta x)(s), \int_0^s k(s, \tau, ({}^C\mathbf{D}_{0+}^\gamma x)(\tau)) d\tau \right) \right) (t).$$



By virtue of (\mathbf{H}_3) , we have:

$$|x(t)| \leq |c_0| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^r \left[\tilde{h}_1(s) g_1 \left(s^\beta \left| ({}^C\mathbf{D}_{0+}^\beta x)(s) \right| \right) + \tilde{h}_2(s) g_2 \left(\int_0^s h_3(\tau) g_3 \left(\tau^\gamma \left| ({}^C\mathbf{D}_{0+}^\gamma x)(\tau) \right| \right) d\tau \right) \right] ds, \tag{18}$$

for all $t > 0$, where $\tilde{h}_i(t) = t^{-r} h_i(t)$, $i = 1, 2$, $r = 1 - \alpha - \frac{1}{p}$, $1 - \alpha + \beta < \frac{1}{p}$, $1 - \alpha + \gamma < \frac{1}{p}$ and $p = \frac{q}{q-1}$. Similarly, we get for all $t > 0$,

$$\left| ({}^C\mathbf{D}_{0+}^\beta x)(t) \right| \leq \frac{1}{\Gamma(\alpha - \beta)} \int_0^t (t-s)^{\alpha-\beta-1} s^r \left[\tilde{h}_1(s) g_1 \left(s^\beta \left| ({}^C\mathbf{D}_{0+}^\beta x)(s) \right| \right) + \tilde{h}_2(s) g_2 \left(\int_0^s h_3(\tau) g_3 \left(\tau^\gamma \left| ({}^C\mathbf{D}_{0+}^\gamma x)(\tau) \right| \right) d\tau \right) \right] ds,$$

$$\left| ({}^C\mathbf{D}_{0+}^\gamma x)(t) \right| \leq \frac{1}{\Gamma(\alpha - \gamma)} \int_0^t (t-s)^{\alpha-\gamma-1} s^r \left[\tilde{h}_1(s) g_1 \left(s^\beta \left| ({}^C\mathbf{D}_{0+}^\beta x)(s) \right| \right) + \tilde{h}_2(s) g_2 \left(\int_0^s h_3(\tau) g_3 \left(\tau^\gamma \left| ({}^C\mathbf{D}_{0+}^\gamma x)(\tau) \right| \right) d\tau \right) \right] ds.$$

Using Hölder’s inequality, we find:

$$\int_0^t (t-s)^{\alpha-1} s^r \tilde{h}_1(s) g_1 \left(s^\beta \left| ({}^C\mathbf{D}_{0+}^\beta x)(s) \right| \right) ds \leq \left(\int_0^t (t-s)^{p(\alpha-1)} s^{pr} ds \right)^{\frac{1}{p}} \left(\int_0^t \tilde{h}_1^q(s) g_1^q \left(s^\beta \left| ({}^C\mathbf{D}_{0+}^\beta x)(s) \right| \right) ds \right)^{\frac{1}{q}}, \quad t > 0.$$

Notice that,

$$\int_0^t (t-s)^{p(\alpha-1)} s^{pr} ds = \Gamma(p(\alpha-1) + 1) \left(\mathbf{I}_{0+}^{p(\alpha-1)+1} s^{pr} \right) (t).$$

Since $p(\alpha - 1) + 1 > 0$ and $pr + 1 > 0$, we see, from Lemma 2.8, that

$$\int_0^t (t-s)^{p(\alpha-1)} s^{pr} ds = \frac{\Gamma(pr + 1)}{\Gamma(p(\alpha + r - 1) + 2)} t^{p(\alpha+r-1)+1}, \quad t > 0.$$

Therefore,

$$\int_0^t (t-s)^{\alpha-1} s^r \tilde{h}_1(s) g_1 \left(s^\beta \left| ({}^C\mathbf{D}_{0+}^\beta x)(s) \right| \right) ds \leq B_1 t^{\alpha+r-1+\frac{1}{p}} \left(\int_0^t \tilde{h}_1^q(s) g_1^q \left(s^\beta \left| ({}^C\mathbf{D}_{0+}^\beta x)(s) \right| \right) ds \right)^{\frac{1}{q}},$$

where $B_1 = \left(\frac{\Gamma(pr+1)}{\Gamma(p(\alpha+r-1)+2)} \right)^{\frac{1}{p}}$. Also,

$$\int_0^t (t-s)^{\alpha-1} s^r \tilde{h}_2(s) g_2 \left(\int_0^s h_3(\tau) g_3 \left(\tau^\gamma \left| ({}^C\mathbf{D}_{0+}^\gamma x)(\tau) \right| \right) d\tau \right) ds \leq B_1 t^{\alpha+r-1+\frac{1}{p}} \left(\int_0^t \tilde{h}_2^q(s) g_2^q \left(\int_0^s h_3(\tau) g_3 \left(\tau^\gamma \left| ({}^C\mathbf{D}_{0+}^\gamma x)(\tau) \right| \right) d\tau \right) ds \right)^{\frac{1}{q}}.$$

Recalling that $\alpha + r - 1 + \frac{1}{p} = 0$, so (18) becomes,

$$|x(t)| \leq |c_0| + \frac{B_1}{\Gamma(\alpha)} \left(\int_0^t \tilde{h}_1^q(s) g_1^q \left(s^\beta \left| ({}^C\mathbf{D}_{0^+}^\beta x)(s) \right| \right) ds \right)^{\frac{1}{q}} \\ + \frac{B_1}{\Gamma(\alpha)} \left(\int_0^t \tilde{h}_2^q(s) g_2^q \left(\int_0^s h_3(\tau) g_3 \left(\tau^\gamma \left| ({}^C\mathbf{D}_{0^+}^\gamma x)(\tau) \right| \right) d\tau \right) ds \right)^{\frac{1}{q}} \quad (19)$$

for all $t > 0$. Similarly, we get the following estimates on ${}^C\mathbf{D}_{0^+}^\beta x$ and ${}^C\mathbf{D}_{0^+}^\gamma x$ for all $t > 0$,

$$\left| ({}^C\mathbf{D}_{0^+}^\beta x)(t) \right| \leq B_2 t^{-\beta} \left[\left(\int_0^t \tilde{h}_1^q(s) g_1^q \left(s^\beta \left| ({}^C\mathbf{D}_{0^+}^\beta x)(s) \right| \right) ds \right)^{\frac{1}{q}} \right. \\ \left. + \left(\int_0^t \tilde{h}_2^q(s) g_2^q \left(\int_0^s h_3(\tau) g_3 \left(\tau^\gamma \left| ({}^C\mathbf{D}_{0^+}^\gamma x)(\tau) \right| \right) d\tau \right) ds \right)^{\frac{1}{q}} \right], \quad (20)$$

$$\left| ({}^C\mathbf{D}_{0^+}^\gamma x)(t) \right| \leq B_3 t^{-\gamma} \left[\left(\int_0^t \tilde{h}_1^q(s) g_1^q \left(s^\beta \left| ({}^C\mathbf{D}_{0^+}^\beta x)(s) \right| \right) ds \right)^{\frac{1}{q}} \right. \\ \left. + \left(\int_0^t \tilde{h}_2^q(s) g_2^q \left(\int_0^s h_3(\tau) g_3 \left(\tau^\gamma \left| ({}^C\mathbf{D}_{0^+}^\gamma x)(\tau) \right| \right) d\tau \right) ds \right)^{\frac{1}{q}} \right], \quad (21)$$

where $B_2 = \frac{1}{\Gamma(\alpha-\beta)} \left(\frac{\Gamma(pr+1)}{\Gamma(p(\alpha-\beta+r-1)+2)} \right)^{\frac{1}{p}}$ and $B_3 = \frac{1}{\Gamma(\alpha-\gamma)} \left(\frac{\Gamma(pr+1)}{\Gamma(p(\alpha-\gamma+r-1)+2)} \right)^{\frac{1}{p}}$.

Let

$$z(t) = |c_0| + B_4 \left(\int_0^t \tilde{h}_1^q(s) g_1^q \left(s^\beta \left| ({}^C\mathbf{D}_{0^+}^\beta x)(s) \right| \right) ds \right)^{\frac{1}{q}} \\ + B_4 \left(\int_0^t \tilde{h}_2^q(s) g_2^q \left(\int_0^s h_3(\tau) g_3 \left(\tau^\gamma \left| ({}^C\mathbf{D}_{0^+}^\gamma x)(\tau) \right| \right) d\tau \right) ds \right)^{\frac{1}{q}}, \quad (22)$$

for all $t > 0$, where

$$B_4 = \max \left\{ \frac{B_1}{\Gamma(\alpha)}, B_2, B_3 \right\}.$$

It is obvious from the inequalities (19)–(22) that,

$$|x(t)| \leq z(t), \quad t^\beta \left| ({}^C\mathbf{D}_{0^+}^\beta x)(t) \right| \leq z(t), \quad t^\gamma \left| ({}^C\mathbf{D}_{0^+}^\gamma x)(t) \right| \leq z(t), \quad t > 0. \quad (23)$$

As g_1 , g_2 and g_3 are nondecreasing functions, it is clear that,

$$z(t) = |c_0| + B_4 \left(\int_0^t \tilde{h}_1^q(s) g_1^q(z(s)) ds \right)^{\frac{1}{q}} \\ + B_4 \left(\int_0^t \tilde{h}_2^q(s) g_2^q \left(\int_0^s h_3(\tau) g_3(z(\tau)) d\tau \right) ds \right)^{\frac{1}{q}}, \quad t > 0.$$

Using Lemma 2.14, we obtain:

$$(z(t))^q \leq B_5 + B_6 \int_0^t \tilde{h}_1^q(s) g_1^q(z(s)) ds \\ + B_6 \int_0^t \tilde{h}_2^q(s) g_2^q \left(\int_0^s h_3(\tau) g_3(z(\tau)) d\tau \right) ds, \quad t > 0,$$



where $B_5 = 3^{q-1} |c_0|^q$ and $B_6 = 3^{q-1} B_4^q$. Let $u(t) = (z(t))^q$. Then we have for all $t > 0$,

$$u(t) \leq B_5 + B_6 \int_0^t \tilde{h}_1^q(s) g_1^q(u^{\frac{1}{q}}(s)) ds + B_6 \int_0^t \tilde{h}_2^q(s) g_2^q \left(\int_0^s h_3(\tau) g_3(u^{\frac{1}{q}}(\tau)) d\tau \right) ds.$$

From Lemma 2.16, with

$$\lambda_1(t) = B_6 \tilde{h}_1^q(t), \quad \lambda_2(t) = B_6 \tilde{h}_2^q(t), \quad \lambda_3(t) = h_3(t),$$

$$w_1(u) = g_1^q \left(u^{\frac{1}{q}} \right), \quad w_3(u) = g_3 \left(u^{\frac{1}{q}} \right),$$

$$w_2 \left(\int_0^s h_3(\tau) w_3(u(\tau)) d\tau \right) = g_2^q \left(\int_0^s h_3(\tau) g_3(u^{\frac{1}{q}}(\tau)) d\tau \right),$$

we deduce that,

$$u(t) \leq W_3^{-1} \left(W_3(c_2) + \int_0^t \lambda_3(s) ds \right)$$

$$\leq W_3^{-1} \left(W_3(c_2) + \int_0^\infty \lambda_3(s) ds \right) := M \quad \text{for all } t > 0,$$

where M is a positive constant. The desired result follows in virtue of (23). □

Remark 3.5 As special case of Theorem 3.4 with $\beta = \gamma = 0$, there exists a positive constant $c \in \mathbb{R}$ such that any continuable solution $x \in C^\alpha [0, \infty)$ of the problem:

$$\begin{cases} ({}^C \mathbf{D}_{0+}^\alpha x)(t) = f \left(t, x(t), \int_0^t k(t, s, x(s)) ds \right), & t \geq 0, \\ x(0) = c_0, & c_0 \in \mathbb{R}, \end{cases}$$

satisfies $|x(t)| \leq c$, for all $t > 0$.

Remark 3.6 It is not hard to see that the conclusion of Theorem 3.4 is still valid if we replace the condition (\mathbf{H}_3) by the following condition: There are functions k_1 of type $\mathcal{H}_{\alpha q-1, q}$, k_2 of type $\mathcal{H}_{0,1}$, $q \geq 1$ and f_i of type \mathcal{G} with $f_1^q f_2^q \propto f_3$ such that,

$$|f(t, u, v)| \leq k_1(t) f_1(t^\beta |u|) f_2(|v|), \quad (t, u, v) \in D,$$

$$|k(t, s, u)| \leq k_2(s) f_3(s^\gamma |u|), \quad (t, s, u) \in E,$$

$$\int_{t_0}^\infty \frac{\tau^{q-1} d\tau}{f_1^q(\tau) f_2^q(\tau^q)} = \infty, \quad \int_{t_0}^\infty \frac{\tau^{q-1} d\tau}{f_3(\tau)} = \infty, \quad t_0 > 0.$$

Example 3.7 Consider the fractional integro-differential equation,

$$\left({}^C \mathbf{D}_{0+}^\alpha x \right)(t) = t^{\mu_1} e^{-t} \left| {}^C \mathbf{D}_{0+}^\beta x(t) \right|^{\sigma_1} + t^{\mu_2} e^{-t} \left(\int_0^t s^{\mu_3} e^{-(s+t)} \left| {}^C \mathbf{D}_{0+}^\gamma x(s) \right|^{\sigma_3} ds \right)^{\sigma_2}, \quad t > 0, \tag{24}$$

where

$$0 < \alpha \leq 1, \quad \mu_1, \mu_2, \mu_3 > 0 \quad \text{and} \quad 0 \leq \sigma_1 \leq \sigma_2 q \leq \frac{\sigma_3}{q} \leq 1.$$

Let

$$h_i(t) = t^{\mu_i} e^{-\rho_i t}, \quad g_i(t) = t^{\sigma_i}, \quad 0 < \rho_i \leq 1, \quad i = 1, 2, 3, \quad t > 0.$$

All the hypotheses of Theorem 3.4 are fulfilled,

$$\begin{aligned} \int_0^\infty t^{\alpha q-1} h_1(t) dt &= \int_0^\infty t^{\alpha q-1+\mu_1} e^{-\rho_1 t} dt = \frac{\Gamma(\alpha q + \mu_1)}{\rho_1^{\alpha q + \mu_1}} < \infty, \\ \int_0^\infty t^{\alpha q-1} h_2(t) dt &= \int_0^\infty t^{\alpha q-1+\mu_2} e^{-\rho_2 t} dt = \frac{\Gamma(\alpha q + \mu_2)}{\rho_2^{\alpha q + \mu_2}} < \infty, \\ \int_0^\infty h_3(t) dt &= \int_0^\infty t^{\mu_3} e^{-\rho_3 t} dt = \frac{\Gamma(\mu_3 + 1)}{\rho_3^{\mu_3 + 1}} < \infty, \\ \int_{t_0}^\infty \frac{t^{q-1} dt}{g_1^q(t)} &= \int_{t_0}^\infty \frac{t^{q-1}}{t^{\sigma_1 q}} dt = \int_{t_0}^\infty t^{q-\sigma_1 q-1} dt = \infty, \\ \int_{t_0}^\infty \frac{dt}{g_2^q(t)} &= \int_{t_0}^\infty \frac{dt}{t^{\sigma_2 q}} = \infty, \quad \int_{t_0}^\infty \frac{t^{q-1}}{g_3(t)} dt = \int_{t_0}^\infty t^{q-1-\sigma_3} dt = \infty, \quad t_0 > 0. \end{aligned}$$

3.2 Equations with Riemann–Liouville fractional derivatives

Here we consider the following problem:

$$\begin{cases} (\mathbf{D}_{0^+}^\alpha x)(t) = f\left(t, (\mathbf{D}_{0^+}^\beta x)(t), \int_0^t k(t, s, (\mathbf{D}_{0^+}^\gamma x)(s)) ds\right), & t > 0, \\ (\mathbf{I}_{0^+}^{1-\alpha} x)(0^+) = c_1, & c_1 \in \mathbb{R}, \end{cases} \tag{25}$$

where $\mathbf{D}_{0^+}^\alpha, \mathbf{D}_{0^+}^\beta, \mathbf{D}_{0^+}^\gamma$ are the Riemann–Liouville fractional derivative of orders α, β and γ , respectively, with $0 \leq \beta < \alpha \leq 1$ and $0 \leq \gamma < \alpha \leq 1$.

We study the power-type decay of continuable solutions for the problem (25) in the space $C_{1-\alpha}^{\alpha+1}[0, b]$, $0 < b \leq \infty$, defined by:

$$C_{1-\alpha}^{\alpha+1}[0, b] = \left\{ x : (0, b] \rightarrow \mathbb{R} \mid x \in C_{1-\alpha}[0, b], \mathbf{D}_{0^+}^{\alpha+1} x \in C_{1-\alpha}[0, b] \right\}, \tag{26}$$

where the space $C_{1-\alpha}[0, b]$ is defined in (9). Before stating and proving our next theorem, we assume that the functions f and k satisfy the following:

- ($\tilde{\mathbf{H}}_1$) $f(t, u, v)$ is a $C_{1-\alpha}$ function in $D = \{(t, u, v) : t \geq 0, u, v \in \mathbb{R}\}$, (\mathbf{H}_2) and
- (\mathbf{H}_4) There are functions h_1, h_2 of type ${}_q\mathcal{H}_{r,\eta}$, h_3 of type $\mathcal{H}_{0,1}$ and g_i of type \mathcal{G} , $i = 1, 2, 3$ with $g_1^q \propto g_2^q \propto g_3$, $0 \leq r < \frac{q-1}{q}$, $\eta > 0$ and $q \geq 1$ such that,

$$\begin{aligned} |f(t, u, v)| &\leq h_1(t) g_1 \left(\frac{|u|}{t^{\alpha-\beta-1}} \right) + h_2(t) g_2(|v|), \quad (t, u, v) \in D, \\ |k(t, s, u)| &\leq h_3(s) g_3 \left(\frac{|u|}{s^{\alpha-\gamma-1}} \right), \quad (t, s, u) \in E, \quad 0 \leq \beta, \quad \gamma < \alpha < 1, \\ \int_{t_0}^\infty \frac{\tau^{q-1} d\tau}{g_1^q(\tau)} &= \infty, \quad \int_{t_0}^\infty \frac{d\tau}{g_2^q(\tau)} = \infty, \quad \int_{t_0}^\infty \frac{\tau^{q-1} d\tau}{g_3(\tau)} = \infty, \quad t_0 > 0. \end{aligned}$$

Theorem 3.8 Suppose that the functions f and k satisfy the conditions ($\tilde{\mathbf{H}}_1$), (\mathbf{H}_2) and (\mathbf{H}_4). Then, there exists a positive constant c such that any solution $x \in C_{1-\alpha}^{\alpha+1}[0, \infty)$ of the problem (25) satisfies:

$$|x(t)| \leq ct^{\alpha-1}, \quad \left| (\mathbf{D}_{0^+}^\beta x)(t) \right| \leq ct^{\alpha-\beta-1} \quad \text{and} \quad \left| (\mathbf{D}_{0^+}^\gamma x)(t) \right| \leq ct^{\alpha-\gamma-1} \quad \text{for all } t > 0.$$

Proof Applying \mathbf{I}_{0+}^α to both sides of the equation in (25) gives, with help of Lemma 2.11,

$$x(t) = \frac{c_1 t^{\alpha-1}}{\Gamma(\alpha)} + \left(\mathbf{I}_{0+}^\alpha f \left(s, (\mathbf{D}_{0+}^\beta x)(s), \int_0^s k(s, \tau, (\mathbf{D}_{0+}^\gamma x)(\tau)) d\tau \right) \right) (t), \quad t \geq 0. \tag{27}$$

Lemmas 2.8 and 2.9 allow us to write,

$$(\mathbf{D}_{0+}^\beta x)(t) = \frac{c_1 t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} + \left(\mathbf{I}_{0+}^{\alpha-\beta} f \left(s, (\mathbf{D}_{0+}^\beta x)(s), \int_0^s k(s, \tau, (\mathbf{D}_{0+}^\gamma x)(\tau)) d\tau \right) \right) (t), \tag{28}$$

$$(\mathbf{D}_{0+}^\gamma x)(t) = \frac{c_1 t^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} + \left(\mathbf{I}_{0+}^{\alpha-\gamma} f \left(s, (\mathbf{D}_{0+}^\beta x)(s), \int_0^s k(s, \tau, (\mathbf{D}_{0+}^\gamma x)(\tau)) d\tau \right) \right) (t), \tag{29}$$

for all $t > 0$. In virtue of the condition (\mathbf{H}_4) , we observe that for all $t > 0$,

$$\begin{aligned} |x(t)| \leq & \frac{|c_1| t^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{-r} e^{-\eta s} \left[\tilde{h}_1(s) g_1 \left(\frac{|(\mathbf{D}_{0+}^\beta x)(s)|}{s^{\alpha-\beta-1}} \right) \right. \\ & \left. + \tilde{h}_2(s) g_2 \left(\int_0^s h_3(\tau) g_3 \left(\frac{|(\mathbf{D}_{0+}^\gamma x)(\tau)|}{\tau^{\alpha-\gamma-1}} \right) d\tau \right) ds \right]. \end{aligned} \tag{30}$$

Furthermore,

$$\begin{aligned} |(\mathbf{D}_{0+}^\beta x)(t)| \leq & \frac{|c_1| t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} + \frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} s^{-r} e^{-\eta s} \tilde{h}_1(s) \\ & \times g_1 \left(\frac{|(\mathbf{D}_{0+}^\beta x)(s)|}{s^{\alpha-\beta-1}} \right) ds + \frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} s^{-r} e^{-\eta s} \tilde{h}_2(s) \\ & \times g_2 \left(\int_0^s h_3(\tau) g_3 \left(\frac{|(\mathbf{D}_{0+}^\gamma x)(\tau)|}{\tau^{\alpha-\gamma-1}} \right) d\tau \right) ds, \end{aligned} \tag{31}$$

and

$$\begin{aligned} |(\mathbf{D}_{0+}^\gamma x)(t)| \leq & \frac{|c_1| t^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} + \frac{1}{\Gamma(\alpha-\gamma)} \int_0^t (t-s)^{\alpha-\gamma-1} s^{-r} e^{-\eta s} \tilde{h}_1(s) \\ & \times g_1 \left(\frac{|(\mathbf{D}_{0+}^\beta x)(s)|}{s^{\alpha-\beta-1}} \right) ds + \frac{1}{\Gamma(\alpha-\gamma)} \int_0^t (t-s)^{\alpha-\gamma-1} s^{-r} e^{-\eta s} \tilde{h}_2(s) \\ & \times g_2 \left(\int_0^s h_3(\tau) g_3 \left(\frac{|(\mathbf{D}_{0+}^\gamma x)(\tau)|}{\tau^{\alpha-\gamma-1}} \right) d\tau \right) ds, \end{aligned} \tag{32}$$

where $\tilde{h}_i(t) = t^r e^{\eta t} h_i(t)$, $i = 1, 2$, $\eta > 0$, $1 - pr > 0$, $\frac{1}{p} > \max \{1 - \alpha + \beta, 1 - \alpha + \gamma\}$ and $p = \frac{q}{q-1}$.

Using Hölder’s inequality and Lemma 2.13, the estimates (30)–(32) become,

$$\begin{aligned} \frac{|x(t)|}{t^{\alpha-1}} \leq & \frac{|c_1|}{\Gamma(\alpha)} + \frac{C^{\frac{1}{p}}}{\Gamma(\alpha)} \left(\int_0^t \tilde{h}_1^q(s) g_1^q \left(\frac{|(\mathbf{D}_{0+}^\beta x)(s)|}{s^{\alpha-\beta-1}} \right) ds \right)^{\frac{1}{q}} \\ & + \frac{C^{\frac{1}{p}}}{\Gamma(\alpha)} \left(\int_0^t \tilde{h}_2^q(s) g_2^q \left(\int_0^s h_3(\tau) g_3 \left(\frac{|(\mathbf{D}_{0+}^\gamma x)(\tau)|}{\tau^{\alpha-\gamma-1}} \right) d\tau \right) ds \right)^{\frac{1}{q}}, \end{aligned}$$

$$\begin{aligned} \frac{|(\mathbf{D}_{0^+}^\beta x)(t)|}{t^{\alpha-\beta-1}} &\leq \frac{|c_1|}{\Gamma(\alpha-\beta)} + \frac{C_1^{\frac{1}{p}}}{\Gamma(\alpha-\beta)} \left(\int_0^t \tilde{h}_1^q(s) g_1^q \left(\frac{|(\mathbf{D}_{0^+}^\beta x)(s)|}{s^{\alpha-\beta-1}} \right) ds \right)^{\frac{1}{q}} \\ &\quad + \frac{C_1^{\frac{1}{p}}}{\Gamma(\alpha-\beta)} \left(\int_0^t \tilde{h}_2^q(s) g_2^q \left(\int_0^s h_3(\tau) g_3 \left(\frac{|(\mathbf{D}_{0^+}^\gamma x)(\tau)|}{\tau^{\alpha-\gamma-1}} \right) d\tau \right) ds \right)^{\frac{1}{q}}, \\ \frac{|(\mathbf{D}_{0^+}^\gamma x)(t)|}{t^{\alpha-\gamma-1}} &\leq \frac{|c_1|}{\Gamma(\alpha-\gamma)} + \frac{C_2^{\frac{1}{p}}}{\Gamma(\alpha-\gamma)} \left(\int_0^t \tilde{h}_1^q(s) g_1^q \left(\frac{|(\mathbf{D}_{0^+}^\beta x)(s)|}{s^{\alpha-\beta-1}} \right) ds \right)^{\frac{1}{q}} \\ &\quad + \frac{C_2^{\frac{1}{p}}}{\Gamma(\alpha-\gamma)} \left(\int_0^t \tilde{h}_2^q(s) g_2^q \left(\int_0^s h_3(\tau) g_3 \left(\frac{|(\mathbf{D}_{0^+}^\gamma x)(\tau)|}{\tau^{\alpha-\gamma-1}} \right) d\tau \right) ds \right)^{\frac{1}{q}}, \end{aligned}$$

where

$$\begin{aligned} C &= \max \left\{ 1, 2^{p(1-\alpha)} \right\} \Gamma(1-pr) \left(1 + \frac{(1-pr)(2-pr)}{p(\alpha-1)+1} \right) (p\eta)^{pr-1}, \\ C_1 &= \max \left\{ 1, 2^{p(1-\alpha-\beta)} \right\} \Gamma(1-pr) \left(1 + \frac{(1-pr)(2-pr)}{p(\alpha-\beta-1)+1} \right) (p\eta)^{pr-1}, \\ C_2 &= \max \left\{ 1, 2^{p(1-\alpha-\gamma)} \right\} \Gamma(1-pr) \left(1 + \frac{(1-pr)(2-pr)}{p(\alpha-\gamma-1)+1} \right) (p\eta)^{pr-1}. \end{aligned}$$

Defining:

$$\begin{aligned} z(t) &= A_1 + A_2 \left(\int_0^t \tilde{h}_1^q(s) g_1^q \left(\frac{|(\mathbf{D}_{0^+}^\beta x)(s)|}{s^{\alpha-\beta-1}} \right) ds \right)^{\frac{1}{q}} \\ &\quad + A_2 \left(\int_0^t \tilde{h}_2^q(s) g_2^q \left(\int_0^s h_3(\tau) g_3 \left(\frac{|(\mathbf{D}_{0^+}^\gamma x)(\tau)|}{\tau^{\alpha-\gamma-1}} \right) d\tau \right) ds \right)^{\frac{1}{q}}, \quad t > 0, \end{aligned}$$

and using:

$$\frac{|x(t)|}{t^{\alpha-1}}, \frac{|(\mathbf{D}_{0^+}^\beta x)(t)|}{t^{\alpha-\beta-1}}, \frac{|(\mathbf{D}_{0^+}^\gamma x)(t)|}{t^{\alpha-\gamma-1}} \leq z(t), \quad \text{for all } t > 0, \quad (33)$$

we arrive at:

$$\begin{aligned} z(t) &= A_1 + A_2 \left(\int_0^t \tilde{h}_1^q(s) g_1^q(z(s)) ds \right)^{\frac{1}{q}} \\ &\quad + A_2 \left(\int_0^t \tilde{h}_2^q(s) g_2^q \left(\int_0^s h_3(\tau) g_3(z(\tau)) d\tau \right) ds \right)^{\frac{1}{q}}, \quad t > 0, \end{aligned}$$

where

$$\begin{aligned} A_1 &= |c_1| \max \left\{ \frac{1}{\Gamma(\alpha)}, \frac{1}{\Gamma(\alpha-\beta)}, \frac{1}{\Gamma(\alpha-\gamma)} \right\}, \\ A_2 &= |c_1| \max \left\{ \frac{C^{\frac{1}{p}}}{\Gamma(\alpha)}, \frac{C_1^{\frac{1}{p}}}{\Gamma(\alpha-\beta)}, \frac{C_2^{\frac{1}{p}}}{\Gamma(\alpha-\gamma)} \right\}. \end{aligned}$$

Taking the power $q \geq 1$ of both side and using Lemma 2.14 with $m = 3$, we obtain:

$$u(t) \leq A_3 + A_4 \int_0^t \tilde{h}_1^q(s) g_1^q(u^{\frac{1}{q}}(s)) ds + A_4 \int_0^t \tilde{h}_2^q(s) g_2^q \left(\int_0^s h_3(\tau) g_3(u^{\frac{1}{q}}(\tau)) d\tau \right) ds,$$



for all $t > 0$, where $u(t) = z^q(t)$, $A_3 = 3^{q-1}A_1^q$, $A_4 = 3^{q-1}A_2^q$.

Now, we conclude from Lemma 2.16, that there exists a positive constant M_1 such that,

$$u(t) \leq M_1 \quad \text{for all } t > 0.$$

Hence, $z(t) \leq c := M_1^{\frac{1}{q}}$ and as a result of inequality (33), the assertion of the theorem is established. □

Example 3.9 Consider the equation,

$$(\mathbf{D}_{0+}^\alpha x)(t) = e^{-t} (x(t))^{\frac{1}{3}} + e^{-t} \left(\int_0^t \frac{s^\sigma e^{-t}}{(t^2 + 1) e^s} x(s) ds \right)^{\frac{1}{3}}, \quad t > 0, \tag{34}$$

where $0 < \alpha < 1$ and $\sigma > -\alpha - 1$. The right-hand side of Eq. (34) can be rewritten as:

$$t^{\frac{1}{3}(\alpha-1)} e^{-t} \frac{(x(t))^{\frac{1}{3}}}{t^{\frac{1}{3}(\alpha-1)}} + e^{-t} \left(\int_0^t s^{\sigma+\alpha-1} e^{-(s+t)} \frac{x(s)}{s^{\alpha-1}} ds \right)^{\frac{1}{3}}, \quad t > 0.$$

Let

$$h_1(t) = t^{\frac{1}{3}(\alpha-1)} e^{-\eta_1 t}, \quad h_2(t) = e^{-\eta_2 t}, \quad h_3(t) = t^{\sigma+\alpha-1} e^{-\eta_3 t}, \quad \sigma > -\alpha, \quad t > 0,$$

$$0 < \eta < \eta_i \leq 1, \quad i = 1, 2, 3, \quad g_1(t) = g_2(t) = t^{\frac{1}{3}} \quad \text{and} \quad g_3(t) = t, \quad t > 0.$$

Clearly, these functions satisfy the condition (\mathbf{H}_4) with $\beta = \gamma = 0$.

4 Conclusion

In this study, we considered some classes of fractional integro-differential equations with the Caputo or Riemann–Liouville derivatives in both sides of the equations depending on the solution, its fractional derivatives as well as an integral of a kernel involving the solution or its fractional derivatives. We assumed the continuity of the nonlinearities and the boundedness of these nonlinearities by sums or products of continuous functions of time, in certain Lebesgue spaces, and nondecreasing functions of the states. Under these nonlinear growth conditions on the nonlinearities, we treated initial value problems for which, in general, solutions cannot be found explicitly. We found that their solutions, under these conditions, behave like the solutions of the associated linear fractional differential equations with zero right-hand sides.

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