S. A. Iyase

On a third-order boundary value problem at resonance on the half-line

Received: 13 June 2017 / Accepted: 21 May 2018 / Published online: 12 June 2018
 © The Author(s) 2018

Abstract In this paper, we establish existence of solutions for the following boundary value problem on the half-line: $(q(t)u''(t))' = g(t, u(t), u'(t), u''(t))$, $t \in (0, \infty)$ subject to the boundary conditions $u'(0) = \sum_{i=1}^m \alpha_i \int_0^{\xi_i} u(t)dt$, $u(0) = 0$, $\lim_{t \rightarrow \infty} q(t)u''(t) = 0$. We establish sufficient conditions for the existence of at least one solution using coincidence degree arguments. An example is provided to validate our result.

Mathematics Subject Classification 34B10 · 34B15 · 34B45

المخلص

في هذا البحث نثبت وجود حلول للمسألة الحدية التالية في نصف مستقيم:

$$(q(t)u''(t))' = g(t, u(t), u'(t), u''(t)), \quad t \in (0, \infty)$$

تحت الشروط الحدية:

$$u'(0) = \sum_{i=1}^m \alpha_i \int_0^{\xi_i} u(t)dt, u(0) = 0, \lim_{t \rightarrow \infty} q(t)u''(t) = 0.$$

نقدم بعض شروط الكفاية لوجود حل واحد على الأقل مستخدمين حجة تطابق الدرجات.

1 Introduction

In this paper, we derive sufficient conditions for the existence of solutions for the following boundary value problem on the half-line

$$(q(t)u''(t))' = g(t, u(t), u'(t), u''(t)), \quad t \in [0, \infty) \quad (1.1)$$

$$u'(0) = \sum_{i=1}^m \alpha_i \int_0^{\xi_i} u(t)dt, u(0) = 0, \lim_{t \rightarrow \infty} q(t)u''(t) = 0 \quad (1.2)$$

S. A. Iyase (✉)

Department of Mathematics, Covenant University Ota, P.M.B 1023 Ota, Ogun State, Nigeria
 E-mail: samuel.iyase@covenantuniversity.edu.ng; driyase2011@yahoo.com

where $g : [0, \infty) \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is a Caratheodory's function, $\alpha_i \in \mathbb{R} (1 \leq i \leq m)$, $0 < \xi_1 < \xi_2 < \dots < \xi_m < 1$, $q > 0$, $q \in C[0, \infty) \cap C^1(0, \infty)$, $\frac{1}{q} \in L^1[0, \infty)$ and $\sum_{i=1}^m \alpha_i \xi_i^2 = 2$. We assume that $g : [0, \infty) \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is a Caratheodory's function with respect to $L^1[0, \infty)$. The condition $\sum_{i=1}^m \alpha_i \xi_i^2 = 2$ is critical since we require a nontrivial kernel for our differential operator.

Third-order boundary value problems at resonance on the half-line have received little attention in the literature, especially those involving integral boundary conditions. Most papers focused on second-order boundary value problems on finite intervals and a few on second-order or third-order boundary value problems on the half-line or infinite intervals see [2–10, 12] and references therein. In this work, we extend the application of coincidence degree theory to third-order differential equations of the form (1.1)–(1.2). To the best of our knowledge, there is little or no research in this direction concerning this type of equation. It is, therefore important to investigate it. The remainder of this paper is organized as follows. In Sect. 2, we provide background definitions and some preliminary Lemmas. Section 3 will be devoted to stating and proving the main existence result. In Sect. 4, we provide an example to demonstrate our result.

2 Preliminaries

For the convenience of the reader, we provide some background definitions and coincidence degree theory.

Definition 2.1 Let X and Z be two real Banach spaces. A linear mapping $L : \text{dom } L \subset X \rightarrow Z$ is said to be a Fredholm mapping if

- (i) $\ker L$ has a finite dimension
- (ii) $\text{Im } L$ is closed and has a finite codimension

If L is a Fredholm mapping then the Fredholm index is given by

$$\text{Ind } L = \dim \ker L - \text{Codim Im } L.$$

The above definition implies that there exist continuous projections

$$\begin{aligned} P : X &\longrightarrow X, \quad Q : Z \longrightarrow Z \text{ such that} \\ \text{Im } P &= \ker L, \quad \ker Q = \text{Im } L, \quad X = \ker L \oplus \ker P, \quad Z = \text{Im } L \oplus \text{Im } Q \end{aligned}$$

and

$$L|_{\text{dom } L \cap \ker P} : \text{dom } L \cap \ker P \longrightarrow \text{Im } L$$

is invertible. We denote this inverse by K_P . The generalized inverse $K_{P,Q} : Z \longrightarrow \text{sdom } L \cap \ker P$ is denoted by

$$K_{P,Q} = K_P(I - Q).$$

Definition 2.2 Let $L : \text{dom } L \subset X \longrightarrow Z$ be a Fredholm mapping, M a metric space and $N : M \longrightarrow Z$ be a mapping. N is said to be L -compact on M if $QN : M \longrightarrow Z$ and $K_P(I - Q)N : M \longrightarrow X$ are compact on M . N is L -completely continuous if it is L -compact on every boundary subset M of X .

We shall formulate (1.1)–(1.2) as an abstract equation in the form $Lu = Nu$ and employ the following existence theorem of Mawhin [8] to derive existence of solution.

Theorem 2.3 [8] Let $\Omega \subset X$ be open and bounded and let L be a Fredholm operator of index zero, and N be L -compact on $\bar{\Omega}$. Assume that the following conditions are satisfied

- (1) $Lu \neq \lambda Nu$ for every $(u, \lambda) \in [(\text{dom } L \setminus \ker L) \cap \partial\Omega] \times (0, 1)$
- (2) $Nu \notin \text{Im } L$ for $u \in \ker L \cap \partial\Omega$
- (3) $\deg(QN|_{\ker L}, \Omega \cap \ker L, 0) \neq 0$ with $Q : Z \longrightarrow Z$ a continuous projection such that $\ker Q = \text{Im } L$.

Then the equation $Lu = Nu$ has at least one solution in $\text{dom } L \cap \bar{\Omega}$.

Definition 2.4 The map $f : [0, \infty) \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is an $L^1[0, \infty)$ -Caratheodory if the following conditions are satisfied

- (i) for each $u \in \mathbb{R}^n$, $f(t, u)$ is Lebesgue measurable
- (ii) for a.e $t \in [0, \infty)$, $f(t, u)$ is continuous on $\mathbb{R}^n$
- (iii) for each $r > 0$ there exist $\varphi_r \in L^1[0, \infty)$ such that for a.e $t \in [0, \infty)$ and every u such that $\|u\| \leq r$ we have $|f(t, u)| \leq \varphi_r(t)$.

Let $AC[0, \infty)$ denote the space of absolutely continuous functions on $[0, \infty)$. We will use the following spaces

$$X = \{u \in C^2[0, \infty), u, u', qu'' \in AC[0, \infty), \lim_{t \rightarrow \infty} e^{-t}|u^{(i)}(t)| \text{ exists } 0 \leq i \leq 2, \\ (qu'')' \in L^1[0, \infty)\}$$

endowed with the norm

$$\|u\| = \max \left[\sup_{t \in [0, \infty)} e^{-t}|u(t)|, \sup_{t \in [0, \infty)} e^{-t}|u'(t)|, \sup_{t \in [0, \infty)} e^{-t}|u''(t)| \right].$$

Then X is a Banach space. Let $Z = L^1[0, \infty)$ with the norm

$$\|y\|_1 = \int_0^\infty |y(t)|dt, \quad y \in Z.$$

Theorem 2.5 [1] *Let F be a subset of $C_\infty = \{y \in C([0, \infty)), \lim_{t \rightarrow \infty} y(t) \text{ exists}\}$ that is equipped with the norm $\|y\|_\infty = \sup_{t \in [0, \infty)} |y(t)|$. Then F is relatively compact if the following conditions hold*

- (1) F is bounded in X
- (2) The functions belonging to F are equi-continuous on any compact subinterval of $[0, \infty)$.
- (3) The functions from F are equi-convergent at ∞ .

We will use the following adaptation of Theorem 2.5 to show that $K_P(I - Q)N : \bar{\Omega} \rightarrow X$ is relatively compact in X .

Lemma 2.6 [3] *Let $D \subset X$. Then D is relatively compact in X if the following conditions are satisfied*

- (i) D is bounded in X
- (ii) The family $W^i = \{\psi_i : \psi_i(t) = e^{-t}u^{(i)}(t), t \geq 0, u \in D\}$ is equicontinuous on any compact subinterval of $[0, \infty)$ for $i = 0, 1, \dots, n-1$.
- (iii) The family $W^i = \{\psi_i : \psi_i(t) = e^{-t}u^{(i)}(t), t \geq 0, u \in D\}$ is equiconvergent at infinity for $i = 0, 1, \dots, n-1$.

We define $L : \text{dom } L \subset X \rightarrow Z$ by

$$Lu = (q(t)u''(t))', \quad t \in [0, \infty) \quad \text{where} \\ \text{dom } L = \left\{ u \in X : u'(0) = \sum_{i=1}^m \alpha_i \int_0^{\xi_i} u(t)dt, u(0) = 0, \lim_{t \rightarrow \infty} (q(t)u''(t)) = 0 \right\}.$$

We define the operator $N : X \rightarrow Z$ by

$$Nu(t) = g(t, u(t), u'(t), u''(t)), \quad t \in [0, \infty).$$

Then (1.1)–(1.2) takes the form

$$Lu = Nu. \tag{2.1}$$

Lemma 2.7 *If $\sum_{i=1}^m \alpha_i \xi_i^2 = 2$ and $\sum_{i=1}^m \alpha_i \int_0^{\xi_i} \int_0^t \int_0^v \frac{e^{-s}}{q(s)} ds dv dt \neq 0$ then*

- (i) $\ker L = \{u \in \text{dom } L : u(t) = dt, d \in \mathbb{R}; t \in [0, \infty)\}$
- (ii) $\text{Im } L = \left\{ y \in Z : \sum_{i=1}^m \alpha_i \int_0^{\xi_i} \int_0^t \int_0^v \frac{1}{q(s)} \int_s^\infty y(\tau) d\tau ds dv dt = 0 \right\}.$

Proof (i) It is easily shown that (i) holds. To verify (ii) we show that the equation

$$(q(t)u''(t))' = y(t) \quad (2.2)$$

has a solution $u(t)$ if and only if

$$\sum_{i=1}^m \alpha_i \int_0^{\xi_i} \int_0^t \int_0^v \frac{1}{q(s)} \int_s^\infty y(\tau) d\tau ds dv dt = 0. \quad (2.3)$$

Suppose (2.2) has a solution $u(t)$. We show that (2.3) holds. From (2.2) and (1.2) we obtain

$$u(t) = - \int_0^t \int_0^v \frac{1}{q(s)} \int_s^\infty y(\tau) d\tau ds dv + tu'(0) \quad (2.4)$$

$$\begin{aligned} u'(0) &= \sum_{i=1}^m \alpha_i \int_0^{\xi_i} u(t) dt = - \sum_{i=1}^m \alpha_i \int_0^{\xi_i} \int_0^t \int_0^v \frac{1}{q(s)} \int_s^\infty y(\tau) d\tau ds dv dt \\ &\quad + u'(0) \sum_{i=1}^m \alpha_i \int_0^{\xi_i} t dt \end{aligned}$$

and hence

$$\sum_{i=1}^m \alpha_i \int_0^{\xi_i} \int_0^t \int_0^v \frac{1}{q(s)} \int_s^\infty y(\tau) d\tau ds dv dt = 0.$$

If (2.3) holds then

$$u(t) = dt - \int_0^t \int_0^v \frac{1}{q(s)} \int_s^\infty y(\tau) d\tau ds dv,$$

where d is an arbitrary constant, is a solution of (2.2). $\square$

Lemma 2.8 *If the conditions of Lemma 2.7 hold then*

- (i) $L : \text{dom } L \subset X \longrightarrow Z$ is a Fredholm operator of index zero and the linear continuous projection $Q : Z \longrightarrow Z$ is given by

$$(Qy)(t) = h(t) \sum_{i=1}^m \alpha_i \int_0^{\xi_i} \int_0^t \int_0^v \frac{1}{q(s)} \int_s^\infty y(\tau) d\tau ds dv dt, \quad t \in [0, \infty),$$

where

$$h(t) = \frac{e^{-t}}{\sum_{i=1}^m \alpha_i \int_0^{\xi_i} \int_0^t \int_0^v \frac{e^{-s}}{q(s)} ds dv dt} \quad (2.5)$$

- (ii) The inverse $K_P : \text{Im } L \longrightarrow \text{dom } L \cap \ker P$ can be defined by

$$K_P y(t) = - \int_0^t \int_0^v \frac{1}{q(s)} \int_s^\infty y(\tau) d\tau ds dv \quad (2.6)$$

with

$$\begin{aligned} \|K_P y\| &\leq \max \left(\sup_{t \in [0, \infty)} e^{-t} t, 1, \frac{\|q^{-1}\|_\infty}{\|q^{-1}\|_1} \right) \|q^{-1}\|_1 \|y\|_1 \\ &= D_1 \|q^{-1}\|_1 \|y\|_1 \end{aligned} \quad (2.7)$$

Proof (i) For any $y \in Z$, we define the projection Q as

$$(Qy)(t) = h(t) \sum_{i=1}^m \alpha_i \int_0^{\xi_i} \int_0^s \int_0^w \frac{1}{q(v)} \int_v^\infty y(\tau) d\tau dv dw ds$$

indeed we have

$$\begin{aligned} Q^2 y &= Q(Qy) \\ &= h(t) \sum_{i=1}^m \alpha_i \int_0^{\xi_i} \int_0^s \int_0^w \frac{1}{q(v)} \int_v^\infty y(\tau) dv dw ds \cdot \frac{\sum_{i=1}^m \alpha_i \int_0^{\xi_i} \int_0^t \int_0^v \frac{e^{-s}}{q(s)} ds dv dt}{\sum_{i=1}^m \alpha_i \int_0^{\xi_i} \int_0^t \int_0^v \frac{e^{-s}}{q(s)} ds dv dt} \\ &= Qy \end{aligned}$$

For $y \in Z$, $y - Qy \in \ker Q = \text{Im} L$ and $Qy \in \text{Im} Q$. Hence, $Z = \text{Im} L + \text{Im} Q = \text{Im} L + \mathbb{R}$. Since $\text{Im} L \cap \text{Im} \mathbb{R} = \{0\}$ we obtain $Z = \text{Im} L \oplus \text{Im} Q$. Thus $\dim \ker L = \dim \text{Im} Q = 1$. This proves that L is a Fredholm operator of index zero.

(ii) Let $P : X \rightarrow X$ be defined by

$$Pu(t) = u'(0)t, \quad t \in [0, \infty) \quad (2.8)$$

then $\text{Im} P = \ker L$ and $P^2 u = Pu$. We can write $u \in X$ in the form $u(t) = Pu(t) + (u(t) - Pu(t))$ and thus $X = \ker P + \ker L$. For $u \in \ker P \cap \ker L$ $u(t) = u'(0)t = Pu(t) = 0$. Thus $\ker L \cap \ker P = 0$ and hence $X = \ker P \oplus \ker L$.

For $y \in \text{Im} L$, we have

$$(LK_P)y(t) = [q(t)(K_P y(t))'']' = y(t), \quad t \in [0, \infty)$$

and for $u \in \text{dom} L \cap \ker P$, we have

$$\begin{aligned} (K_P L)u(t) &= K_P[q(t)u''(t)]' = - \int_0^t \int_0^v \frac{1}{q(s)} \int_s^\infty (q(\tau)u''(\tau))' d\tau ds dv \\ &= \int_0^t \int_0^v u''(\tau) d\tau dv = u(t) - u(0) - tu'(0) \end{aligned}$$

in view of $u \in \text{dom} L \cap \ker P$ and $u(0) = 0$ we obtain

$$(K_P L)u(t) = u(t).$$

From the definition of K_P , we derive

$$e^{-t}|(K_P y)(t)| = e^{-t} \left| \int_0^t \int_0^v \frac{1}{q(s)} \int_s^\infty y(\tau) d\tau ds dv \right| \leq \sup_{t \in [0, \infty)} e^{-t} \|q^{-1}\|_1 \|y\|_1 \quad (2.9)$$

$$\begin{aligned} e^{-t}|(K_P y)'(t)| &= e^{-t} \left| \int_0^t \frac{1}{q(s)} \int_s^\infty y(\tau) d\tau ds \right| \leq \sup_{t \in [0, \infty)} e^{-t} \|q^{-1}\|_1 \|y\|_1 \\ &= \|q^{-1}\|_1 \|y\|_1 \end{aligned} \quad (2.10)$$

$$\begin{aligned} e^{-t}|(K_P y)''(t)| &= e^{-t} \left| \frac{1}{q(t)} \int_t^\infty y(\tau) d\tau \right| \leq \sup_{t \in [0, \infty)} e^{-t} \|q^{-1}\|_\infty \|y\|_1 \\ &= \|q^{-1}\|_\infty \|y\|_1 \end{aligned} \quad (2.11)$$

From (2.9), (2.10) and (2.11) we obtain

$$\begin{aligned} \|K_P y\| &\leq \max \left[\sup_{t \in [0, \infty)} e^{-t} t, 1, \frac{\|q^{-1}\|_\infty}{\|q^{-1}\|_1} \right] \|q^{-1}\|_1 \|y\|_1 \\ &= D_1 \|q^{-1}\|_1 \|y\|_1 \end{aligned} \quad (2.12)$$

□

Lemma 2.9 *If g is a Caratheodory's function, then N is L -compact.*

Proof Let $M \subset X$ be bounded with $r = \sup\{\|u\| : u \in M\}$.

Since $g : [0, \infty) \times \mathbb{R}^3 \rightarrow \mathbb{R}$ satisfies Caratheodory's conditions with respect to $L^1[0, \infty)$, there exists a Lebesgue integrable function φ_r such that

$$\begin{aligned} |Nu(t)| &= |g(t, u(t), u'(t), u''(t))| \leq \varphi_r(t), \quad t \in [0, \infty), \quad u \in M. \\ \|Nu\|_1 &\leq \int_0^\infty |Nu(s)|ds \leq \int_0^\infty |\varphi_r(s)|ds = \|\varphi_r\|_1 \\ \|QNu\|_1 &\leq \int_0^\infty |QNu(s)|ds \leq \int_0^\infty |h(t)| \left[\sum_{i=1}^m |\alpha_i| \int_0^{\xi_i} \int_0^v \int_0^s \frac{1}{q(\tau)} \int_\tau^\infty |Nu(r)|dr d\tau ds dv \right] dt \\ &\leq \|h\|_1 \|\varphi_r\|_1 \left[\sum_{i=1}^m |\alpha_i| \int_0^{\xi_i} \int_0^v \int_0^s \frac{1}{q(\tau)} d\tau ds dv \right] \leq \|h\|_1 \|\varphi_r\|_1 \frac{q^{-1}}{2} \|1\| \sum_{i=1}^m |\alpha_i| \xi_i^2 \\ &= D_2 < \infty \end{aligned}$$

From (2.7) we obtain

$$\begin{aligned} \|K_P(I - Q)Nu(s)\| &\leq D_1 \|q^{-1}\|_1 \|(I - Q)Nu\|_1 \leq D_1 \|q^{-1}\|_1 \|Nu\|_1 + D_1 \|q^{-1}\|_1 \|QNu\|_1 \\ &\leq D_1 \|q^{-1}\|_1 [\|\varphi_r\|_1 + D_2]. \end{aligned}$$

This shows that $K_P(I - Q)N(M)$ is uniformly bounded in X . Let $u \in M$, and $t_1, t_2 \in [0, L_1]$, $L_1 \in (0, \infty)$ with $t_1 < t_2$. Then

$$\begin{aligned} &|(e^{-t_2}(K_P(I - Q)Nu(t_2)) - e^{-t_1}(K_P(I - Q)Nu(t_1)))| \\ &= \left| \int_{t_1}^{t_2} [e^{-s}(K_P(I - Q)Nu(s))] ds \right| \\ &= \left| \int_{t_1}^{t_2} [-e^{-s}(K_P(I - Q)Nu(s))] ds + \int_{t_1}^{t_2} e^{-s}(K_P(I - Q)Nu(s))' ds \right| \\ &\leq 2(t_2 - t_1) \|K_P(I - Q)Nu\| \\ &\leq 2(t_2 - t_1) D_1 \|q^{-1}\|_1 [\|\varphi_r\|_1 + D_2] \rightarrow 0 \text{ as } t_1 \rightarrow t_2 \end{aligned}$$

$$\begin{aligned} &|e^{-t_2}(K_P(I - Q)Nu)'(t_2) - e^{-t_1}(K_P(I - Q)Nu)'(t_1)| \\ &= \left| \int_{t_1}^{t_2} [e^{-s}(K_P(I - Q)Nu)'(s)] ds \right| \\ &= \left| \int_{t_1}^{t_2} [-e^{-s}(K_P(I - Q)Nu)'(s)] ds + e^{-s}(K_P(I - Q)Nu)''(s) ds \right| \\ &\leq 2(t_2 - t_1) \|K_P(I - Q)Nu\| \leq 2(t_2 - t_1) D_1 \|q^{-1}\|_1 [\|\varphi_r\|_1 + D_2] \\ &\rightarrow 0 \text{ as } t_1 \rightarrow t_2 \end{aligned}$$

$$\begin{aligned} &|e^{-t_2}(K_P(I - Q)Nu)''(t_2) - e^{-t_1}(K_P(I - Q)Nu)''(t_1)| \\ &= \left| \frac{e^{-t_2}}{q(t_2)} \int_{t_2}^\infty (I - Q)Nu(s) ds - \frac{e^{-t_1}}{q(t_1)} \int_{t_1}^\infty (I - Q)Nu(s) ds \right| \\ &\leq \left| \frac{e^{-t_2}}{q(t_2)} - \frac{e^{-t_1}}{q(t_1)} \right| \int_{t_2}^\infty |(I - Q)Nu(s)| ds + \frac{1}{q(t_1)} \int_{t_1}^{t_2} |(I - Q)Nu(s)| ds \\ &\leq \|q^{-1}\|_\infty^2 |q(t_1)e^{-t_2} - q(t_2)e^{-t_1}| \|(I - Q)Nu\|_1 + \|q^{-1}\|_\infty \int_{t_1}^{t_2} |(I - Q)Nu(s)| ds \\ &\leq \|q^{-1}\|_\infty^2 |q(t_1)e^{-t_2} - q(t_2)e^{-t_1}| (\|\varphi_r\|_1 + D_2) \end{aligned}$$

$$\begin{aligned}
 & + \|q^{-1}\|_{\infty} \int_{t_1}^{t_2} \left[|\varphi_r(s)| + |h(\tau)| \sum_{i=1}^m |\alpha_i| \int_0^{\xi_i} \int_0^s \int_0^v \frac{1}{q(w)} dw dv ds \|\varphi_r\|_1 \right] d\tau \\
 & \longrightarrow 0 \text{ as } t_1 \rightarrow t_2
 \end{aligned}$$

This proves that $K_P(I - Q)N(M)$ is equicontinuous on every compact subset of $[0, \infty)$.

Next, we show that $K_P(I - Q)N(M)$ is equiconvergent at infinity. For $u \in M$, we have

$$\begin{aligned}
 |e^{-t}(K_P(I - Q)Nu)(t)| &= \left| e^{-t} \int_0^t \int_0^v \frac{1}{q(s)} \int_s^{\infty} (I - Q)Nu(\tau) d\tau ds dv \right| \\
 &\leq e^{-t} t \|q^{-1}\|_1 \|(I - Q)Nu\|_1 \\
 &\leq e^{-t} t \|q^{-1}\|_1 (\|\varphi_r\|_1 + D_2) \longrightarrow 0 \text{ as } t \longrightarrow \infty
 \end{aligned}$$

$$\begin{aligned}
 |e^{-t}(K_P(I - Q)Nu)'(t)| &= \left| e^{-t} \int_0^t \frac{1}{q(s)} \int_s^{\infty} (I - Q)Nu(\tau) d\tau ds \right| \\
 &\leq e^{-t} \|q^{-1}\|_1 \|(I - Q)Nu\|_1 \\
 &\leq e^{-t} \|q^{-1}\|_1 [\|Nu\|_1 + \|QNu\|_1] \longrightarrow 0 \text{ as } t \longrightarrow \infty
 \end{aligned}$$

$$\begin{aligned}
 |e^{-t}(K_P(I - Q)Nu)''(t)| &= \left| e^{-t} \frac{1}{q(t)} \int_t^{\infty} (I - Q)Nu(\tau) d\tau \right| \\
 &\leq e^{-t} \|q^{-1}\|_{\infty} \|(I - Q)Nu\|_1 \\
 &\leq e^{-t} \|q^{-1}\|_{\infty} [\|Nu\|_1 + \|QNu\|_1] \\
 &\longrightarrow 0 \text{ as } t \rightarrow \infty
 \end{aligned}$$

Thus $K_P(I - Q)N(M)$ is equiconvergent at infinity.

The conditions of Lemma 2.6 (or Theorem 2.5) are fulfilled. Therefore, $K_P(I - Q)N(M)$ is relatively compact for every bounded $M \subset X$

□

3 Existence result

To prove the existence of solution for (1.1)–(1.2) we assume the following conditions

$$(H_0) \sum_{i=1}^m \alpha_i \xi_i^2 = 2, \text{ and } \sum_{i=1}^m \alpha_i \int_0^{\xi_i} \int_0^t \int_0^v \frac{e^{-s}}{q(s)} ds dv dt \neq 0.$$

(H₁) There exists a constant $B_1 > 0$ such that for each $u \in \text{dom } L \setminus \ker L$, satisfying $|u'(t)| > B_1$ for all $t \in [0, \infty)$ we have $QNu \neq 0$.

(H₂) There exist positive functions $a, b, c, r \in L^1[0, \infty)$ such that for all $(u_1, u_2, u_3) \in \mathbb{R}^3$, $t \in [0, \infty)$ the following inequality holds

$$|g(t, u_1, u_2, u_3)| \leq e^{-t} [a(t)|u_1| + b(t)|u_2| + c(t)|u_3|^{\theta}] + r(t), \theta \in [0, 1). \quad (3.1)$$

(H₃) There exists a constant $B_2 > 0$ such that for every $d \in \mathbb{R}$ with $|d| > B_2$ we have

$$\text{either } d \cdot \sum_{i=1}^m \alpha_i \int_0^{\xi_i} \int_0^t \int_0^v \frac{1}{q(s)} \int_0^{\infty} g(\tau, d\tau, d, 0) d\tau ds dv dt < 0 \quad (3.2)$$

or

$$d \cdot \sum_{i=1}^m \alpha_i \int_0^{\xi_i} \int_0^t \int_0^v \frac{1}{q(s)} \int_s^{\infty} g(\tau, d\tau, d, 0) d\tau ds dv dt > 0. \quad (3.3)$$

Theorem 3.1 *If (H_0) – (H_3) hold, then the boundary value problem (1.1)–(1.2) has at least one solution provided*

$$\|a\|_1 + \|b\|_1 < \frac{1}{2D_1\|q^{-1}\|_1}. \quad (3.4)$$

Proof We shall apply Theorem 2.3 by first constructing an open set $\Omega \subset X$ that will satisfy the assumptions in the theorem. Let

$$\Omega_1 = \{u \in \text{dom } L \setminus \ker L : Lu = \lambda Nu, \lambda \in (0, 1]\}$$

For $u \in \Omega_1$, $u \notin \ker L$, $\lambda \neq 0$ and hence $Nu \in \text{Im } L$. Since $\ker Q = \text{Im } L$, we obtain

$$\sum_{i=1}^m \alpha_i \int_0^{\xi_i} \int_0^t \int_0^v \frac{1}{q(s)} \int_s^\infty Nu(\tau) d\tau ds dv dt = 0, \quad y \in Z$$

We derive from (H_1) that there exists $t_1 \in [0, \infty)$ such that $|u'(t_1)| \leq B_1$. Now

$$\begin{aligned} |u'(0)| &= \left| u'(t_1) - \int_0^{t_1} u''(s) ds \right| \leq |u'(t_1)| + \int_0^{t_1} |u''(s)| ds \\ &\leq B_1 + \|u''\|_1 \end{aligned} \quad (3.5)$$

Also from (2.4) one obtains

$$u''(t) = \frac{-1}{q(t)} \int_t^\infty (q(s)u''(s))' ds = -\frac{1}{q(t)} \int_t^\infty Lu(s) ds$$

and hence

$$\|u''\|_1 = \int_0^\infty \left| -\frac{1}{q(t)} \int_t^\infty Lu(s) ds \right| dt \leq \|q^{-1}\|_1 \|Nu\|_1. \quad (3.6)$$

From (3.5) and (3.6) we get

$$|u'(0)| < B_1 + \|q^{-1}\|_1 \|Nu\|_1. \quad (3.7)$$

For $u \in \Omega_1$, $(I - P)u \in \text{dom } L \cap \ker P$.

Therefore, using (2.7) we obtain

$$\begin{aligned} \|(I - P)u\| &= \|K_P L(I - P)u\| \leq D_1 \|q^{-1}\|_1 \|L(I - P)u\|_1 \\ &\leq D_1 \|q^{-1}\|_1 \|Lu\|_1 \leq D_1 \|q^{-1}\|_1 \|Nu\|_1. \end{aligned} \quad (3.8)$$

By the definition of P , and (3.7) we have

$$Pu(t) = u'(0)t, \quad (Pu)'(t) = u'(0), \quad t \in [0, \infty)$$

$$\begin{aligned} \|Pu\| &= \max \left\{ \sup_{t \in [0, \infty)} e^{-t} t |u'(0)|, |u'(0)| \right\} \\ &= \max \left\{ \sup_{t \in [0, \infty)} e^{-t} t, 1 \right\} |u'(0)| < D_1 |u'(0)| \\ &\leq D_1 [B_1 + \|q^{-1}\|_1 \|Nu\|_1] \end{aligned} \quad (3.9)$$

$$\begin{aligned} \|u\| &= \|Pu + (I - P)u\| \leq \|Pu\| + \|(I - P)u\| \\ &\leq D_1 [B_1 + \|q^{-1}\|_1 \|Nu\|_1] + D_1 \|q^{-1}\|_1 \|Nu\|_1 \end{aligned}$$

$$\begin{aligned} &= B_1 D_1 + [D_1 \|q^{-1}\|_1 + D_1 \|q^{-1}\|_1] \|Nu\|_1 \\ &= B_1 D_1 + 2D_1 \|q^{-1}\|_1 \|Nu\|_1. \end{aligned} \quad (3.10)$$

From (3.1) we obtain

$$\begin{aligned} \|Nu\|_1 &\leq \int_0^\infty |g(s, u(s), u'(s), u''(s))| ds \\ &\leq \|a\|_1 \|u\| + \|b\|_1 \|u\| + \|d\|_1 \|u\|^\theta + \|r\|_1. \end{aligned} \quad (3.11)$$

Therefore, (3.10) and (3.11) yields

$$\|u\| \leq B_1 D_1 + 2D_1 \|q^{-1}\|_1 [\|a\|_1 \|u\| + \|b\|_1 \|u\| + \|d\|_1 \|u\|^\theta + \|r\|_1]$$

or

$$\|u\| \leq \frac{B_1 D_1}{1 - 2D_1 \|q^{-1}\|_1 [\|a\|_1 + \|b\|_1]} + \frac{2D_1 \|q^{-1}\|_1 [\|d\|_1 \|u\|^\theta + \|r\|_1]}{1 - 2D_1 \|q^{-1}\|_1 [\|a\|_1 + \|b\|_1]}.$$

Since $\theta \in [0, 1)$ and (3.4), there exist $M_1 > 0$ such that

$$\|u\| \leq M_1. \quad (3.12)$$

Therefore, Ω_1 is bounded.

We define

$$\Omega_2 = \{u \in \ker L : Nu \in \operatorname{Im} L\}.$$

Then for $d \in \mathbb{R}$, and $t \in [0, \infty)$.

$u(t) = dt$ and $Nu \in \operatorname{Im} L$ implies $Nu \in \ker Q$ and hence

$$\sum_{i=1}^m \alpha_i \int_0^{\xi_i} \int_0^t \int_0^v \frac{1}{q(s)} \int_s^\infty g(\tau, d\tau, d, 0) d\tau ds dv dt = 0.$$

By (H_3) we have

$$\begin{aligned} |d| &< B_2 \\ \|u\| &= \max \left(\sup_{t \in [0, \infty)} e^{-t} t, 1 \right) |d| < D_1 |d| < D_1 B_2 \end{aligned} \quad (3.13)$$

which implies that Ω_2 is bounded.

We define the isomorphism $J : \ker L \rightarrow \operatorname{Im} Q$ by

$$J(dt) = dh(t), \quad d \in \mathbb{R}, \quad t \in [0, \infty)$$

where $h(t)$ is as in (2.5). If (3.3) holds, let

$$\Omega_3 = \{u \in \ker L : -\lambda Ju + (1 - \lambda)QNu = 0, \lambda \in [0, 1]\}.$$

Then

$$\lambda Ju = (1 - \lambda)QNu$$

that is,

$$\lambda dh(t) = (1 - \lambda)h(t) \sum_{i=1}^m \alpha_i \int_0^{\xi_i} \int_0^t \int_0^v \frac{1}{q(s)} \int_s^\infty g(\tau, d\tau, d, 0) d\tau ds dv dt.$$

If $\lambda = 1$, then $d = 0$ and if $|d| > B_2$ then in view of (3.2) one has

$$\lambda d^2 = (1 - \lambda)d \sum_{i=1}^m \alpha_i \int_0^{\xi_i} \int_0^t \int_0^v \frac{1}{q(s)} \int_s^\infty g(\tau, d\tau, d, 0) d\tau ds dv dt < 0$$

which is a contradiction. Hence Ω_3 is bounded. If (3.3) holds, we set

$$\Omega_3 = \{u \in \ker L : \lambda Ju + (1 - \lambda)QNu = 0, \lambda \in [0, 1]\}.$$

Using a similar argument we obtain that Ω_3 is bounded.

We let Ω be an open and bounded set such that $U_{i=1}^3 \bar{\Omega}_i \subset \Omega$. Then it is easily seen that Assumptions (i) and (ii) of Theorem 2.3 are satisfied.

Lemmas 2.8 and 2.9 respectively have established that L is Fredholm of index zero and the mapping N is L -compact on $\bar{\Omega}$. We verify the third assumption.

We define

$$H(u, \lambda) = \pm \lambda Ju + (1 - \lambda)QNu.$$

It is easily checked that $H(u, \lambda) \neq 0$ for every $u \in \partial\Omega \cap \ker L$. Therefore,

$$\begin{aligned} \deg(QN|_{\ker L}, \Omega \cap \ker L, 0) &= \deg(H(\cdot, 0), \Omega \cap \ker L, 0) \\ &= \deg(H(\cdot, 1), \Omega \cap \ker L, 0) \\ &= \deg(\pm J, \Omega \cap \ker L, 0) \neq 0. \end{aligned}$$

We conclude from Theorem 2.3 that $Lu = Nu$ has at least one solution in $\text{dom } L \cap \Omega$. $\square$

Finally, we provide an example to validate our result.

4 Example

Consider the following boundary value problem.

$$(q(t)u''(t))' = e^{-t} \left(2 + \frac{|u(t)|}{8(t+1)^2} + \frac{|u'(t)|}{16(t+1)^4} + |\sin t| |u''(t)|^{\frac{1}{4}} \right) \quad (4.1)$$

$$u'(0) = \sum_{i=1}^2 \alpha_i \int_0^{3i} u(t) dt, u(0) = 0, \lim_{t \rightarrow \infty} q(t)u''(t) = 0. \quad (4.2)$$

We have

$$\begin{aligned} q(t) &= e^t, t \in [0, \infty), \quad \xi_1 = \frac{1}{\sqrt{3}}, \quad \xi_2 = \frac{1}{\sqrt{2}}, \quad \alpha_1 = 3, \quad \alpha_2 = 2, \\ g(t, u, u', u'') &= e^{-t} \left(2 + \frac{|u|}{8(t+1)^2} + \frac{|u'|}{16(t+1)^4} + |\sin t| |u''|^{\frac{1}{4}} \right) \end{aligned}$$

$$\begin{aligned} \sum_{i=1}^2 \alpha_i \xi_i^2 &= \alpha_1 \xi_1^2 + \alpha_2 \xi_2^2 = 2 \\ \sum_{i=1}^2 \alpha_i \int_0^{\xi_i} \int_0^t \int_0^v \frac{e^{-s}}{e^s} ds dv dt &\neq 0. \end{aligned}$$

Therefore, Assumption H_0 is satisfied. It is easily observed that since $g(t, x, y, z) > 0$ for all $(t, x, y, z) \in [0, \infty) \times \mathbb{R}^3$ then $QNu(t) \neq 0$ on $[0, \infty)$ for all $u \in \text{dom } L \setminus \ker L$. This satisfies Assumption H_1 .

$$\begin{aligned} |g(t, u, u', u'')| &\leq e^{-t} \left(2 + \frac{|u|}{8(t+1)^2} + \frac{|u'|}{16(t+1)^4} + |u''|^{\frac{1}{4}} \right) \\ \text{Here } a(t) &= \frac{1}{8(t+1)^2}, b(t) = \frac{1}{16(t+1)^4}. \end{aligned}$$

Hence Assumption H_2 is satisfied.

The first and second inequalities in Assumption H_3 can be estimated as follows

$$\begin{aligned} & 3d \int_0^{\frac{1}{\sqrt{3}}} \int_0^t \int_0^v \frac{1}{e^{-s}} \int_s^\infty e^{-\tau} \left(2 + \frac{|\mathrm{d}\tau|}{8(\tau+1)^2} + \frac{|d|}{16(\tau+1)^4} \right) \mathrm{d}\tau \mathrm{d}s \mathrm{d}v \mathrm{d}t \\ & + 2d \int_0^{\frac{1}{\sqrt{2}}} \int_0^t \int_0^v \frac{1}{e^{-s}} \int_s^\infty e^{-\tau} \left(2 + \frac{|\mathrm{d}\tau|}{8(1+\tau)^2} + \frac{|d|}{16(1+\tau)^4} \right) \mathrm{d}\tau \mathrm{d}s \mathrm{d}v \mathrm{d}t \\ & \geq 3d \int_0^{\frac{1}{\sqrt{3}}} \int_0^t \int_0^v \frac{1}{e^{-s}} \int_s^\infty 2e^{-\tau} \mathrm{d}\tau \mathrm{d}s \mathrm{d}v \mathrm{d}t \\ & + 2d \int_0^{\frac{1}{\sqrt{2}}} \int_0^t \int_0^v \frac{1}{e^{-s}} \int_s^\infty 2e^{-\tau} \mathrm{d}\tau \mathrm{d}s \mathrm{d}v \mathrm{d}t \\ & = d \left[3^{\frac{-3}{2}} + \frac{2^{\frac{-1}{2}}}{3} \right] \end{aligned}$$

and satisfies (3.2) or (3.3) respectively if $d < -1$ or $d > 1$, i.e., if $|d| > B_2 = 1$.

Finally,

$$D_1 = 1, \|q^{-1}\|_1 = 1, \|a\|_1 + \|b\|_1 = \frac{1}{8} + \frac{1}{32} = \frac{5}{32} < \frac{1}{2D_1\|q^{-1}\|_1} = \frac{1}{2}.$$

All the assumptions of Theorem 3.1 are satisfied. Therefore, Example 4.1–4.2 has at least one solution.

Acknowledgements The author wishes to express his gratitude to the anonymous referees for their useful suggestions and corrections.

Open Access This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

References

1. Agarwal, R.P.; O'Regan, D.: Infinite Interval Problems for Differential, Difference and Integral Equations. Kluwer, Dordrecht (2001)
2. Agarwal, R.P.; Cetin, E.: Unbounded solutions of third order three-point boundary value problems on a half-line. Adv. Nonlinear Anal. **5**(2), 105–119 (2016)
3. Frioui, A.; Guezane-Lakud, A.; Khaldi, R.: Higher order boundary value problems at resonance on an unbounded interval. Electron. J. Differ. Equ. **2016**(29), 110 (2016). <http://ejde.math.txstate.edu>
4. Kosmatov, N.: Multipoint boundary value problems on an unbounded domain at resonance. Nonlinear Anal. **65**, 2158–2171 (2008)
5. Lian, H.R.; Ge, W.G.: Solvability for second order three-point boundary value problems at resonance on a half-line. J. Math. Anal. Appl. **337**, 1171–1181 (2007)
6. Liu, Y.; Li, D.; Fang, M.: Solvability for second order m-point boundary value problems at resonance on the half-line. Electron. J. Differ. Equ. **2009**(13), 1–11 (2009). <http://ejde.math.txstate.edu>
7. Liu, B.M.; Liu, L.S.; Wu, Y.H.: Unbounded solutions for three point boundary value problems with non-linear boundary conditions on $[0, \infty)$. Nonlinear Anal. **23**, 2923–2932 (2010)
8. Mawhin, J.: Topological Degree Methods in Nonlinear Boundary Value Problems, NSF-CBMS Regional Conference Series in Mathematics. American Mathematical Society, Providence (1979).
9. O'Regan, D.; Yan, B.; Agarwal, R.P.: Solutions in weighted spaces of singular boundary value problems on the half-line. J. Comput. Appl. Math. **205**, 751–763 (2007)
10. Shi, H.; Pei, M.: Solvability of a third-order three-point boundary value problem on half-line. Bull. Malays. Math. Sci. Soc. **38**, 909–926 (2015)
11. Yan, B.Q.; Liu, Y.S.: Unbounded solutions of singular boundary value problems for second order differential equations on the half-line. Appl. Math. Comput. **147**, 629–644 (2004)
12. Zima, M.: On positive solutions of boundary value-problems on the half-line. J. Math. Anal. Appl. **259**, 127–136 (2001)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.