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Upper neighbours of Čech closure operators

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Abstract Here, we investigate the existence of upper neighbours in the lattice of T_1 Čech closure operators on a fixed set. In this paper, we prove that a first countable T_1 closure operator has no upper neighbour in the lattice of Čech closure operators.

Mathematics Subject Classification 54A05 · 03G10

المخلص

هنا نبحث وجود الجوار العلوي في شبكية T_1 مؤثر إغلاق تشيك على مجموعة ثابتة. في هذه المقالة نثبت أن أول T_1 مؤثر إغلاق قابل للعد لا يملك جوارا علويا ضمن شبكية مؤثرات إغلاق تشيك.

1 Introduction

Čech closure operators can be considered as a generalisation of Kuratowski closure operators. A Čech closure operator or simply a closure operator on a set X is a function $V : P(X) \rightarrow P(X)$ satisfying the conditions $V(\phi) = \phi$, $A \subseteq V(A)$ and $V(A \cup B) = V(A) \cup V(B)$ for all subsets A, B of X [3]. A set X together with a closure operator is called a Čech closure space or a closure space. A subset A of X in (X, V) is closed if $V(A) = A$ and is open if its complement is closed. The set of all open sets in a closure space (X, V) forms a topology on X called the topology associated with V . A closure operator V on X is said to be topological if $V(V(A)) = V(A)$ for every $A \subseteq X$. Associated with a topology on a set X , we can define a unique closure operator V on X as $V(A) = \bar{A}$, $A \subseteq X$. Then, V is a topological closure operator on X .

Set theoretical notions in this paper coincide with [15] and lattice definitions are as in [2]. Throughout this paper, we consider X as an infinite set. The properties of the lattice of topologies were studied by many authors [9, 16]. The concept of topological expansions was discussed by Hewitt E. [4]. A similar concept of simple expansions in the lattice of topologies was introduced by Levine N. [10]. A simple expansion of a topology T by a subset A of X such that $A \notin T$ is the smallest topology containing T and A . Agashe P. and Levine N. proved that an upper neighbour of a topology T is a simple expansion of T by some subset of X [1].

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Kunheenkutty M. studied properties of simple expansions of topologies in detail in [5–7]. He also introduced a similar concept in Čech closure spaces [7].

The set of all Čech closure operators on a fixed non-empty set X forms a lattice $L(X)$ under the partial order given by $V_1 \leq V_2$ if $V_2(A) \subseteq V_1(A)$ for all $A \subseteq X$ [3]. This lattice $L(X)$ is a complete lattice and it contains the lattice of topologies. The smallest element of $L(X)$ is the closure operator associated with the indiscrete topology on X and the largest element is the closure operator associated with the discrete topology on X .

Let $\{V_a; a \in \mathcal{A}\}$, where $\mathcal{A}$ is some indexing set, be a non-empty collection of closure operators on a set X . The closure operator given by $V(A) = \bigcup_{a \in \mathcal{A}} V_a(A)$ for $A \subseteq X$ is the *infimum* of the collection $\{V_a : a \in \mathcal{A}\}$ in $L(X)$. If U is the *supremum* of the non-empty collection $\{V_a : a \in \mathcal{A}\}$ in $L(X)$ and $A \in P(X)$, then $x \in U(A)$ if and only if for each finite cover $\{A_1, A_2, \dots, A_n\}$ of A , there exists an A_i such that $x \in V_a(A_i)$ for each $a \in \mathcal{A}$ [3].

The *co-finite closure operator* on X is given by the function $C_0 : P(X) \rightarrow P(X)$ defined by

$$C_0(A) = \begin{cases} A & \text{if } A \text{ is finite} \\ X & \text{otherwise.} \end{cases}$$

More generally, we define C_α on $P(X)$, where α is any infinite cardinal number such that $\alpha \leq |X|$, by,

$$C_\alpha(A) = \begin{cases} A & \text{if } |A| < \alpha \\ X & \text{otherwise} \end{cases}$$

When $\alpha = \aleph_0$, we get that $C_\alpha = C_{\aleph_0} = C_0$.

Let (X, V) be a closure space. If $V(\{x\}) = \{x\}$ for every $x \in X$, then V is said to be a T_1 closure operator on X . Let U and V be two closure operators on X such that $U < V$. Then, V is called an *upper neighbour* of U if for any closure operator W on X such that $U \leq W \leq V$, we have either $W = U$ or $W = V$ [8]. Then, U and V are said to be adjacent.

Ramachandran P. T. studied properties of the lattice $L(X)$ of closure operators [11, 13, 14]. He proved that $L(X)$ is complemented if and only if X is finite [12]. Agashe P. and Levine N. introduced the adjacency in the lattice of topologies and proved that a first countable T_1 completely normal topological space has no upper neighbour in the lattice of topologies [1]. In [8], we discussed adjacency in the lattice of Čech closure operators analogous to the adjacency in the lattice of topologies. There we distinguished upper neighbours and topological upper neighbours of a Čech closure operator. Here, we study some closure operators finer than co-finite closure operator and prove that a first countable T_1 closure operator has no upper neighbour in $L(X)$.

2 Main results

As a continuation of [8], we investigate the existence of upper neighbours of some closure operators finer than the co-finite closure operator.

Definition 2.1 [13] Let $x \in X$. Suppose A is an infinite subset of X not containing x . Define $V_{A,x} : P(X) \rightarrow P(X)$ as

$$V_{A,x}(S) = \begin{cases} S & \text{if } S \text{ is finite} \\ X \setminus \{x\} & \text{if } S \text{ is infinite, } S \setminus A \text{ finite and } x \notin S \\ X & \text{otherwise.} \end{cases}$$

Then, $V_{A,x}$ is a closure operator on X and $C_0 < V_{A,x}$.

In [8], we proved that $V_{A,x}$ has no upper neighbour in $L(X)$. Here, we generalize the Definition 2.1 for a monotonically increasing collection of infinite subsets of X as $V_{A_1, A_2, \dots, x}$ and prove that $V_{A_1, A_2, \dots, x}$ has no upper neighbour in $L(X)$. We use the following theorems in [8].

Theorem 2.2 [8] If A and B are two infinite subsets of X , $x \notin A \cup B$, then $V_{A,x} \vee V_{B,x} = V_{A \cup B, x}$.

Theorem 2.3 [8] If $\{A_1, A_2, \dots, A_n\}$ is a finite collection of infinite subsets of X such that $x \notin A_i$ for any $i \in \{1, 2, \dots, n\}$, then

$$V_{A_1, x} \vee V_{A_2, x} \vee \dots \vee V_{A_n, x} = V_{A_1 \cup A_2 \cup \dots \cup A_n, x}.$$

Lemma 2.4 [7] Let V be a closure operator on a set X . Then, $V_{A,x} \leq V$ if and only if $x \notin V(A)$.

Theorem 2.5 [7] Let A be an infinite subset of X and $x \in X \setminus A$. Then, $V_{A,x}$ has no upper neighbour in $L(X)$.

Remark 2.6 Theorem 2.3 is not true for an infinite collection of subsets of X . For example, let $X = \mathbb{Z}$ and A_k $k = 1, 2, \dots$ denotes the set of all integers n not equal to 1 and 2^k does not divide n . Then, $1 \in \bigvee_{k=1}^{\infty} V_{A_k,1}(A_2 \cup 8\mathbb{Z})$, but $1 \notin \bigvee_{k=1}^{\infty} V_{A_k,1}(A_2 \cup 8\mathbb{Z})$. Hence, $\bigvee_{k=1}^{\infty} V_{A_k,1} \neq V_{\bigcup_{k=1}^{\infty} A_k,1}$.

Now, we generalise Definition 2.1 as:

Definition 2.7 Let $A_1, A_2, \dots$ be a monotonically increasing sequence of infinite subsets of X and $x \notin A_i$ for any positive integer i , then define

$$V_{A_1, A_2, \dots, x}(S) = \begin{cases} S & ; \text{if } S \text{ is finite} \\ X \setminus \{x\} & ; \text{if } S \text{ is infinite, } x \notin S \text{ and } S \setminus A_i \text{ finite for} \\ & \text{some positive integer } i \\ X & ; \text{otherwise.} \end{cases}$$

Then, $V_{A_1, A_2, \dots, x}$ is a closure operator on X such that $C_0 < V_{A_1, A_2, \dots, x}$.

Theorem 2.8 Let $A_1, A_2, \dots$ be a monotonically increasing sequence of infinite subsets of X and $x \notin A_i$ for any positive integer i . Then, a closure operator $V_{A_1, A_2, \dots, x}$ is topological if and only if $X \setminus A_i$ is finite for some positive integer i .

Proof Suppose that $V_{A_1, A_2, \dots, x}$ is a topological closure operator. Then, for every positive integer j , $V_{A_1, A_2, \dots, x}(V_{A_1, A_2, \dots, x}(A_j)) = V_{A_1, A_2, \dots, x}(A_j)$. This implies that $V_{A_1, A_2, \dots, x}(X \setminus \{x\}) = X \setminus \{x\}$. Thus, $X \setminus A_i$ is finite for some positive integer i .

Now suppose that $V_{A_1, A_2, \dots, x}$ is not topological. Then, there exists an infinite subset S of X such that $V_{A_1, A_2, \dots, x}(V_{A_1, A_2, \dots, x}(S)) = X$ but $V_{A_1, A_2, \dots, x}(S) = X \setminus \{x\}$. Then, $X \setminus A_i$ is infinite for every i . Hence, if $X \setminus A_i$ is finite for some positive integer i , then $V_{A_1, A_2, \dots, x}$ is topological. $\square$

Theorem 2.9 $V_{A_1, A_2, \dots, x} = \bigvee_{i=1}^{\infty} V_{A_i, x}$.

Proof If S is finite, we have $V_{A_1, A_2, \dots, x}(S) = S = \bigvee_{i=1}^{\infty} V_{A_i, x}(S)$. Now suppose that S is an infinite subset of X and $x \notin V_{A_i, x}(S)$. Then, $S \setminus A_i$ is finite and $x \notin V_{A_1, A_2, \dots, x}(S)$. Hence, $V_{A_1, A_2, \dots, x}(S) \subseteq V_{A_i, x}(S)$. That is $V_{A_i, x} \leq V_{A_1, A_2, \dots, x}$. It follows that $\bigvee_{i=1}^{\infty} V_{A_i, x} \leq V_{A_1, A_2, \dots, x}$.

We have to prove that $V_{A_1, A_2, \dots, x} \leq \bigvee_{i=1}^{\infty} V_{A_i, x}$, i.e., that $\bigvee_{i=1}^{\infty} V_{A_i, x}(S) \subseteq V_{A_1, A_2, \dots, x}(S)$ for all $S \subseteq X$. This is true if S is finite or $x \in S$ or $S \setminus A_i$ is infinite for every i . Assume that S is infinite, $x \notin S$ and there exists i such that $S \setminus A_i$ is finite. Then, $V_{A_i, x}(S) = X \setminus \{x\} = V_{A_1, A_2, \dots, x}(S)$. Since $V_{A_i, x} \leq \bigvee_{j=1}^{\infty} V_{A_j, x}$, we have $\bigvee_{i=1}^{\infty} V_{A_i, x}(S) \subseteq V_{A_i, x}(S) \subseteq V_{A_1, A_2, \dots, x}(S)$. $\square$

Lemma 2.10 Let V be a closure operator on X such that $C_0 < V$. Then, $V_{A_1, A_2, \dots, x} \leq V$ if and only if $x \notin V(A_i)$ for any positive integer i .

Proof If $V_{A_1, A_2, \dots, x} \leq V$, then $V(A_i) \subseteq V_{A_1, A_2, \dots, x}(A_i) = X \setminus \{x\}$ for every positive integer i . Thus, $x \notin V(A_i)$ for any positive integer i . Conversely suppose that $x \notin V(A_i)$ for any positive integer i . If $x \notin V(A_i)$, then $V_{A_i, x} \leq V$ by Lemma 2.4. That is $V_{A_i, x} \leq V$ for $i = 1, 2, \dots$. Hence, $V_{A_1, A_2, \dots, x} \leq V$ by Theorem 2.9. $\square$

Remark 2.11 Let X be a set and $x \in X$. Suppose that $A_1, A_2, \dots$ and $B_1, B_2, \dots$ are monotonically increasing sequences of infinite subsets of X such that $x \notin A_i$ and $x \notin B_i$ for any positive integer i . Then, $V_{A_1, A_2, \dots, x} \leq V_{B_1, B_2, \dots, x}$ if and only if $x \notin V_{B_1, B_2, \dots, x}(A_i)$ for every positive integer i . That is for every A_i there exists a B_j such that $A_i \setminus B_j$ is finite. Similarly, $V_{B_1, B_2, \dots, x} \leq V_{A_1, A_2, \dots, x}$ if and only if for every B_k there exists A_l such that $B_k \setminus A_l$ is finite. Thus, $V_{A_1, A_2, \dots, x} = V_{B_1, B_2, \dots, x}$ if and only if for every A_i , there exists a B_j and for every B_j there exists A_k such that $A_i \setminus B_j$ and $B_j \setminus A_k$ are finite for $i, j, k = 1, 2, \dots$.

Theorem 2.12 Let X be any set and $A_1, A_2, \dots$ be a monotonically increasing sequence of infinite subsets of X such that $x \notin A_i$ for any positive integer i . Let B be any subset of X such that $B \Delta A_i$ is infinite for every positive integer i . Then, $V_{A_1, A_2, \dots, x} \bigvee V_{B, x} = V_{A_1 \cup B, A_2 \cup B, \dots, x}$.

Proof Let $x \notin [V_{A_1, A_2, \dots, x} \vee V_{B, x}](S)$. By the definition of join of closure operators, this implies that $x \notin V_{A_1, A_2, \dots, x}(S)$ or $x \notin V_{B, x}(S)$. We have the following implications.

$$\begin{aligned} x \notin V_{A_1, A_2, \dots, x}(S) \text{ or } x \notin V_{B, x}(S) &\Rightarrow (S \setminus A_i) \text{ is finite for some } i = 1, 2, \dots \text{ or} \\ &\quad (S \setminus B) \text{ is finite.} \\ &\Rightarrow (S \setminus A_i) \cap (S \setminus B) \text{ is finite for some} \\ &\quad i = 1, 2, \dots \\ &\Rightarrow S \setminus (A_i \cup B) \text{ is finite for some } i = 1, 2, \dots \\ &\Rightarrow x \notin V_{A_1 \cup B, A_2 \cup B, \dots, x}(S). \end{aligned}$$

Hence, $V_{A_1 \cup B, A_2 \cup B, \dots, x}(S) \subseteq [V_{A_1, A_2, \dots, x} \vee V_{B, x}](S)$ for $S \subseteq X$.

Hence, $[V_{A_1, A_2, \dots, x} \vee V_{B, x}] \leq V_{A_1 \cup B, A_2 \cup B, \dots, x}$. To prove the reverse inequality, we have to prove that $x \notin [V_{A_1, A_2, \dots, x} \vee V_{B, x}](A_i \cup B)$ for $i = 1, 2, \dots$. We have $[V_{A_1, A_2, \dots, x} \vee V_{B, x}](A_i \cup B) = [V_{A_1, A_2, \dots, x} \vee V_{B, x}](A_i) \cup [V_{A_1, A_2, \dots, x} \vee V_{B, x}](B)$. But $x \notin [V_{A_1, A_2, \dots, x} \vee V_{B, x}](A_i)$ and $x \notin [V_{A_1, A_2, \dots, x} \vee V_{B, x}](B)$. This implies that $x \notin [V_{A_1, A_2, \dots, x} \vee V_{B, x}](A_i \cup B)$ for any $i = 1, 2, \dots$. Hence by Lemma 2.10, we have $V_{A_1 \cup B, A_2 \cup B, \dots, x} \leq [V_{A_1, A_2, \dots, x} \vee V_{B, x}]$. Thus $V_{A_1, A_2, \dots, x} \vee V_{B, x} = V_{A_1 \cup B, A_2 \cup B, \dots, x}$. $\square$

Theorem 2.13 Let $A_1, A_2, \dots$ be infinite subsets of X such that $A_1 \subseteq A_2 \subseteq \dots$. Suppose $x \notin A_i$ for any positive integer i and $A_{i+1} \setminus A_i$ is infinite for every positive integer i . Then, a closure operator of the form $V_{A_1, A_2, \dots, x}$ has no upper neighbour in $L(X)$.

Proof Let $A_1, A_2, \dots$ be infinite subsets of X such that $x \notin A_i$ for any positive integer i . So $V_{A_1, x} \leq V_{A_1, A_2, \dots, x}$. Suppose that $V_{A_1, A_2, \dots, x}$ has an upper neighbour V in $L(X)$. For a finite set S , $V_{A_1, A_2, \dots, x}(S) = V(S)$. Since $V \neq V_{A_1, A_2, \dots, x}$, there exists an infinite subset A of X such that $V(A) = X \setminus \{x\}$ but $V_{A_1, A_2, \dots, x}(A) = X$. Thus, $A \setminus A_i$ is infinite for every i . We can choose a countable infinite subset B_i of the infinite sets $A \setminus A_i$ for each positive integer i . Let $B_i = \{a_{i1}, a_{i2}, \dots\}$. Now consider a subset

$C_i = \{a_{in} : n \text{ is an even positive integer}\}$. Then, $C_1 \supseteq C_2 \supseteq C_3 \supseteq \dots$.

Let $V' = V_{C_1^c \setminus \{x\}, C_2^c \setminus \{x\}, \dots, x}$. Then, $V'(C_1) = X$ but $V(C_1) \subseteq V(A) = X \setminus \{x\}$ and $V'(C_1^c \setminus \{x\}) = X \setminus \{x\}$, but $V_{A_1, A_2, \dots, x}(C_1^c \setminus \{x\}) = X$.

Note that

$$C_i \subseteq A \setminus A_i \Rightarrow X \setminus (A \cap A_i^c) \subseteq X \setminus C_i \Rightarrow A_i \subseteq C_i^c, \text{ for } i = 1, 2, \dots$$

Since $A_i \subseteq C_i^c \setminus \{x\}$, for $i = 1, 2, \dots$, $S \setminus (C_i^c \setminus \{x\}) \subseteq S \setminus A_i$ for $i = 1, 2, \dots$.

Thus, we have the following implications.

$$\begin{aligned} x \notin V_{A_1, A_2, \dots, x}(S) &\Rightarrow (S \setminus A_i) \text{ is finite for some } i. \\ &\Rightarrow S \setminus (C_i^c \setminus \{x\}) \text{ is finite for some } i. \\ &\Rightarrow x \notin V'(S). \end{aligned}$$

Hence, $V'(S) \subseteq V_{A_1, A_2, \dots, x}(S)$ for $S \subseteq X$. That is $V_{A_1, A_2, \dots, x} \leq V'$.

Also $V_{B_1^c \setminus \{x\}, B_2^c \setminus \{x\}, \dots, x} \leq V'$ and $V_{A_1, A_2, \dots, x} \leq V_{B_1^c \setminus \{x\}, B_2^c \setminus \{x\}, \dots, x}$. That is $V_{A_1, A_2, \dots, x} \leq V_{B_1^c \setminus \{x\}, B_2^c \setminus \{x\}, \dots, x} \leq V'$. Note that $V'(C_1) = X$ but $V(C_1) = X \setminus \{x\}$. Let $x \notin V_{A_1, A_2, \dots, x}(S)$. Then, $S \setminus A_i$ is finite for some i . Choose sufficiently large j such that $i \leq j$. Then, $S \setminus (A_j)$ is finite. Consider the closure operator $V' \wedge V_{A_j, x}$. Then, $V_{A_1, A_2, \dots, x} \leq V' \wedge V_{A_j, x} \leq V$. This completes the proof of the theorem. $\square$

Now, we decompose a T_1 closure operator by defining closure operators of the form V^x at each $x \in X$.

Definition 2.14 Let V be a T_1 closure operator on a set X . Let $x \in X$.

Define V^x on $P(X)$ by

$$V^x(A) = \begin{cases} A & ; \text{ if } A \text{ is finite} \\ X \setminus \{x\} & ; \text{ if } A \text{ is infinite, } x \notin V(A) \\ X & ; \text{ otherwise.} \end{cases}$$

Then, V^x is a closure operator on X .

Lemma 2.15 Let V be a T_1 closure operator on a set X . Then, $V = \bigvee_{x \in X} V^x$.

Proof Let $V' = \bigvee_{x \in X} V^x$. We have by definition of V^x , $V(A) \subseteq V^x(A)$ for every subset A of X . Then, $V^x \leq V$. Hence, $V' \leq V$.

Now we have to prove that $V \leq V'$. Suppose not. Then, there exists an infinite subset A of X such that $V'(A) \not\subseteq V(A)$. Thus, there exists $y \in V'(A)$ such that $y \notin V(A)$. Since $y \notin V(A)$, $V^y(A) = X \setminus \{y\}$. Now $y \in V'(A)$ implies that for every finite cover $\{A_1, A_2, \dots, A_n\}$ of A , there exists A_i such that $y \in V^x(A_i)$ for every $x \in X$. In particular for the cover $\{A\}$, $y \in V^y(A)$. Hence, $V^y(A) = X$. This is a contradiction to the fact that $V^y(A) = X \setminus \{y\}$. Hence, $V \leq V'$. $\square$

Now, we prove a characterization Theorem for the existence of upper neighbours of T_1 closure operators.

Theorem 2.16 *A T_1 closure operator V on a set X has an upper neighbour in $L(X)$ if and only if there exists an $x \in X$ such that V^x has an upper neighbour in $L(X)$.*

Proof Suppose V has an upper neighbour W in $L(X)$. Then, $V < W$. That is we can find some $S \subseteq X$ such that $W(S) \subset V(S)$. Then, there exists $x \in X$ such that $x \notin W(S)$ but $x \in V(S)$. We have $W = \bigvee_{x \in X} W^x$ by Lemma 2.15. We have the following sequence of implications for an infinite subset A :

$$x \in W^x(A) \Rightarrow x \in W(A) \Rightarrow x \in V(A) \Rightarrow V^x(A) = X \Rightarrow x \in V^x(A).$$

Thus, $W^x(A) \subseteq V^x(A)$ and hence $V^x \leq W^x$. Also $x \notin W(S) \Rightarrow x \notin W^x(S)$. But $x \in V(S) \Rightarrow x \in V^x(S)$. Hence, $V^x < W^x$. Now let U be a closure operator on X such that $V^x \leq U \leq W^x$. Then,

$$V = \left(\bigvee_{\substack{y \in X \\ y \neq x}} V^y \right) \bigvee V^x \leq \left(\bigvee_{\substack{y \in X \\ y \neq x}} V^y \right) \bigvee U \leq \left(\bigvee_{\substack{y \in X \\ y \neq x}} V^y \right) \bigvee W^x \leq W.$$

That is,

$$V \leq \left(\bigvee_{\substack{y \in X \\ y \neq x}} V^y \right) \bigvee U \leq W.$$

Since W is an upper neighbour of V , either $V = \left(\bigvee_{\substack{y \in X \\ y \neq x}} V^y \right) \bigvee U$ or $W = \left(\bigvee_{\substack{y \in X \\ y \neq x}} V^y \right) \bigvee U$.

Assume that $V = \left(\bigvee_{\substack{y \in X \\ y \neq x}} V^y \right) \bigvee U$. We have

$$V^x \leq U. \tag{1}$$

That is $U(A) \subseteq V^x(A)$ for each $A \subseteq X$. To prove that $U \leq V^x$, it is enough to prove that $V^x(A) \subseteq U(A)$ for every $A \subseteq X$. If A is finite, $V^x(A) = U(A) = A$. Assume that A is infinite, to show that $V^x(A) \subseteq U(A)$. Now $x \in V^x(A) \Rightarrow x \in V(A)$

$$\Rightarrow x \in U(A) \text{ since } V = \left(\bigvee_{\substack{y \in X \\ y \neq x}} V^y \right) \bigvee U.$$

Recall that $W^x(A) = \begin{cases} A & ; \text{ if } A \text{ is finite,} \\ X \setminus \{x\} & ; \text{ if } A \text{ is infinite, } x \notin W(A), \\ X & ; \text{ otherwise.} \end{cases}$

Also we have $V^x \leq U \leq W^x$. If $V^x(A) = X \setminus \{x\}$, then $x \notin W^x(A)$, so $V^x(A) = X \setminus \{x\} \subseteq W^x(A) \subseteq U(A)$.

Now if $V^x(A) = X$, then $W^x(A) = X$ which implies that $U(A) = X$. Thus, we can conclude that $V^x(A) \subseteq U(A)$ for every subset A of X . In other words

$$U \leq V^x. \quad (2)$$

From Eqs. (1) and (2), we get $V^x = U$.

Now let us assume that $W = \left(\bigvee_{\substack{y \in X \\ y \neq x}} V^y \right) \vee U$. We have

$$U \leq W^x. \quad (3)$$

When A is a finite subset of X , $U(A) = W^x(A)$. Now for an infinite subset A of X , $x \notin W^x(A) \Rightarrow x \notin W(A) \Rightarrow x \notin \left(\bigvee_{\substack{y \in X \\ y \neq x}} V^y \vee U \right)(A)$ implies that there exists a finite cover $\{A_1, A_2, \dots, A_n\}$ of A , such that $x \notin V^y(A_i)$ for some y or $x \notin U(A_i)$, $i = 1, 2, \dots, n$. But $x \notin V^y(A_i)$ for any y is not possible. Hence, $x \notin U(A)$. If $W^x(A) = X \setminus \{x\}$, then $x \notin U(A)$. That is $U(A) \subseteq W^x(A)$. Now if $W^x(A) = X$, then obviously $U(A) \subseteq W^x(A)$. Hence, $U(A) \subseteq W^x(A)$ for every subset A of X . In other words,

$$W^x \leq U. \quad (4)$$

From Eqs. (3) and (4), we get $U = W^x$.

Thus, W^x is an upper neighbour of V^x .

Conversely, assume that there exists an upper neighbour W of V^x for some $x \in X$. Consider $U = \bigvee_{y \in X \setminus \{x\}} V^y \vee W$. We have $V = \bigvee_{x \in X} V^x$ and $W = \bigvee_{x \in X} W^x$ by Lemma 2.15. Since $V^x < W$, we have $\bigvee_{y \in X \setminus \{x\}} V^y \vee V^x < \bigvee_{y \in X \setminus \{x\}} V^y \vee W$. Hence, $V < U$. Suppose that there exists a closure operator U' on X such that $V \leq U' \leq U$. Then, $x \notin V^x(S) \Rightarrow x \notin V(S) \Rightarrow x \notin U'(S) \Rightarrow x \notin U'^x(S)$. Thus

$$V^x \leq U'^x. \quad (5)$$

Also

$$\begin{aligned} x \notin U'^x(S) &\Rightarrow x \notin U'(S) \\ &\Rightarrow x \notin U(S) \\ &\Rightarrow \text{There exists a finite cover } \{S_1, S_2, \dots, S_n\} \text{ of } S \text{ such that} \\ &\quad x \notin V^y(S_i) \text{ for some } y \text{ or } x \notin W(S_i) \text{ for } i = 1, 2, \dots, n. \end{aligned}$$

Since $V^y(S) = X \setminus \{y\}$ and $y \neq x$, we have $x \in V^y(S_i)$ for every y and $i = 1, 2, \dots, n$. Therefore, $x \notin W(S)$. Also we have $U'^x(S) = X \setminus \{x\}$ or X . This implies that $W(S) \subseteq U'^x(S)$ and therefore $V^x \leq U'^x \leq W$. Since W is an upper neighbour of V^x , we have either $U'^x = V^x$ or $U'^x = W$. Also $U^y = U'^y = V^y$, for $y \neq x$. Therefore, if $U'^x = V^x$, then $U' = V$ and if $U'^x = W$, then $U' = U$. This completes the proof. $\square$

Now, we prove our main Theorem.

Theorem 2.17 *Let V be a first countable T_1 closure operator on a set X . Then, V has no upper neighbour in $L(X)$.*

Proof Let V be a first countable T_1 closure operator on X . Then, $V = \bigvee_{x \in X} V^x$, by Lemma 2.15. Let $x \in X$. Suppose $L = \{A_1, A_2, \dots\}$ denotes a countable local base at x . Let $A'_1 = A_1$, $A'_2 = A_1 \cap A_2$, $A'_3 = A_1 \cap A_2 \cap A_3, \dots$. Thus $A'_1 \supseteq A'_2 \supseteq A'_3 \supseteq \dots$. That is $L' = \{A'_1, A'_2, \dots\}$ is a monotonically decreasing countable local base at x .

Take $B_1 = A'_1$. Let B_2 be the first A'_i , $i > 1$ such that $A'_1 \setminus A'_i$ is infinite. Now choose B_3 as first A'_j , $j > i$ such that $B_2 \setminus A'_j$ is infinite. Continuing this process we arrive at one of the two cases.

Case 1: Above process terminates after a finite stage. That is there exists a positive integer k such that $A'_k \setminus A'_{k+j}$ is finite for $j = 1, 2, \dots$

Observe that $x \notin V(X \setminus A'_m)$, hence $x \notin V^x(X \setminus A'_m)$. It follows that from Lemma 2.4 that $V_{X \setminus A'_m, x} \leq V^x$. Suppose that $V^x \not\leq V_{X \setminus A'_m, x}$. Then, there exists an infinite subset $M \subseteq X$ such that $V_{X \setminus A'_m, x}(M) \not\leq V^x(M)$. Thus, $V_{X \setminus A'_m, x}(M) = X$ and $V^x(M) = X \setminus \{x\}$. From the definition of $V_{X \setminus A'_m, x}$, it follows that $M \setminus (X \setminus A'_m) = M \cap A'_m$ is infinite. Since $x \notin V^x(M)$, we obtain $x \notin V(M)$; hence, $X \setminus M$ is a neighbourhood of x . There exists $l \geq m$ such that $A'_l \subseteq X \setminus M$. Then, $A'_l \subseteq A'_m$ and $M \cap A'_m = M \cap A'_l \cup (A'_m \setminus A'_l) \subseteq (M \cap A'_l) \cup (A'_m \setminus A'_l) = A'_m \setminus A'_l$, a contradiction, since $M \cap A'_m$ is infinite and $A'_m \setminus A'_l$ is finite. Since $B_k \setminus A'_m$ is infinite and $B_k \setminus A'_m \subseteq X \setminus A'_m$, we have $X \setminus A'_m$ is infinite. Thus, $V^x = V_{X \setminus A'_m, x}$. So V^x has no upper neighbour in $L(X)$ by Theorem 2.5.

Case 2: The above process can be continued and we get an infinite collection $\{B_1, B_2, \dots\}$ of subsets of X such that $B_i \setminus B_{i+1}$ is infinite for $i = 1, 2, \dots$

In this case, we prove that $V^x = V_{B_1^c, B_2^c, \dots, x}$. We have $A'_i \in L'$. Since L' is a local base at x , $x \notin V(X \setminus A'_i)$ for $i = 1, 2, \dots$. When S is a finite subset of X , then $V^x(S) = S = V_{B_1^c, B_2^c, \dots, x}(S)$.

Now suppose A is an infinite subset of X . If $x \notin V^x(A)$ then $x \notin V(A)$ by the definition of V^x . Then, $X \setminus A$ is a neighbourhood of x in (X, V) . Hence, there exists a $A'_m \in L'$ such that $A'_m \subseteq X \setminus A$, for some positive integer m . That is $A \subseteq X \setminus A'_m$. We can choose B_k from the infinite collection $\{B_1, B_2, \dots\}$ so that $B_k \subseteq A'_m$. Then, $A \subseteq X \setminus A'_m \subseteq X \setminus B_k$. Thus, $A \subseteq B_k^c$, for the positive integer k . Thus, $A \setminus B_k^c = \emptyset$. So $V_{B_1^c, B_2^c, \dots, x}(A) = X \setminus \{x\}$ by definition. In other words, $x \notin V_{B_1^c, B_2^c, \dots, x}(A)$. So $V_{B_1^c, B_2^c, \dots, x}(A) \subseteq V^x(A)$ for every infinite subset A of X . Hence, $V^x \leq V_{B_1^c, B_2^c, \dots, x}$.

Next, we have to prove that $V_{B_1^c, B_2^c, \dots, x} \leq V^x$. We have $V_{B_1^c, B_2^c, \dots, x} \leq V^x$ if and only if $x \notin V^x(B_i^c)$ for $i = 1, 2, \dots$, by Lemma 2.10. If $x \in V^x(B_i^c)$ for some i , then $x \in V(B_i^c)$. Thus, $x \in V(X \setminus A'_j)$ for some positive integer j . This is a contradiction since A'_j is a neighbourhood of x with respect to V . Hence $V_{B_1^c, B_2^c, \dots, x} = V^x$.

By Theorem 2.13, $V_{B_1^c, B_2^c, \dots, x}$ has no upper neighbour and, therefore, V^x has no upper neighbour.

In either case, V^x has no upper neighbour in $L(X)$. Since V is first countable at each $x \in X$, V^x at each $x \in X$ has no upper neighbour in $L(X)$. Hence by Theorem 2.16, V has no upper neighbour in $L(X)$. $\square$

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