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Approximating fixed points of (λ, ρ)-firmly nonexpansive mappings in modular function spaces

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Abstract In this paper, we first introduce an iterative process in modular function spaces and then extend the idea of a λ-firmly nonexpansive mapping from Banach spaces to modular function spaces. We call such mappings as (λ, ρ)-firmly nonexpansive mappings. We incorporate the two ideas to approximate fixed points of (λ, ρ) -firmly nonexpansive mappings using the above-mentioned iterative process in modular function spaces.

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الملخص

في هذا الورقة، نقوم أولا بتقديم عملية تكرارية في فضاءات دالية مقياسية ثم نمدّد فكرة الراسم غير متوسع من نوع λ-حازم من فضاءات بناخ إلى فضاءات دالية مقياسية. نسمي مثل هذه الراسمات براسمات غير متوسعة و(λ, ρ)-حازمة. ندمج الفكرتين بغرض تقريب النقاط الثابتة لهذه الراسمات، مستخدمين العملية التكرارية المذكورة أعلاه وذلك في فضاءات دالية مقياسية.

1 Introduction

Fixed point theory has several applications in different disciplines and, therefore, it has been a flourishing area of research. The metric fixed point theory in the framework of Banach spaces usually involves a close link of geometric and topological conditions. Fixed point theory in modular function spaces and metric fixed point theory are near relatives because the former provides modular equivalents of norm and metric concepts. Modular spaces are extensions of the classical Lebesgue and Orlicz spaces, and in many instances conditions cast in this framework are more natural and more easily verified than their metric analogs. For more discussion, see, for example, Khamsi and Kozlowski [3].

Nowadays, a vigorous research activity is developed in the area of numerical reckoning fixed points for suitable classes of nonlinear operators, see, for example, [9,10], and applications to image recovery and variational inequalities, see for example, [11–14]. Existence of fixed points in modular function spaces has been studied by many researchers, for example, see Khamsi and Kozlowski [3] and the references therein. Dhompongsa et al. [2] have proved the existence of fixed point of ρ-contractions under certain conditions.

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Buthina and Kozlowski [1], for the first time, proved results on approximating fixed points in modular function spaces through Mann and Ishikawa iterative processes. Some work for multivalued mappings in modular function spaces using Mann iterative process was done by Khan and Abbas [5]. Khan [4] introduced an iterative process for approximation of fixed points of certain mappings in Banach spaces. This process is independent of both Mann and Ishikawa iterative processes in the sense that neither reduces to the other under the given conditions. Moreover, it is faster than all of Picard, Mann and Ishikawa iterative processes in case of contractions [4]. We extend this process to the framework of modular function spaces. On the other hand, firmly nonexpansive mappings play an important role in nonlinear analysis due to their correspondence with maximal monotone operators. The class of λ-firmly nonexpansive mappings in Banach spaces has attracted many researchers. For a discussion on such mappings, see, for example Ruiz et al. [6] and the references cited therein. As far as we know, no work has been done until now on this kind of mappings in modular function spaces. We thus introduce the idea of the so-called (λ, ρ)-firmly nonexpansive mappings, in short (λ, ρ)-FNEM. We approximate the fixed points of such mappings using the above-mentioned iterative process in modular function spaces. This will create new results in modular function spaces.

2 Preliminaries

Here is a brief note on modular function spaces to make the discussion self-contained. This has mainly been extracted from Khamsi and Kozlowski [3].

Let Ω be a nonempty set and Σ a nontrivial σ-algebra of subsets of Ω. Let $\mathcal{P}$ be a δ-ring of subsets of Ω, such that $E \cap A \in \mathcal{P}$ for any $E \in \mathcal{P}$ and $A \in \Sigma$. Let us assume that there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that $\Omega = \cup K_n$ (for instance, $\mathcal{P}$ can be the class of sets of finite measure in a σ-finite measure space). By 1_A, we denote the characteristic function of the set A in Ω. By $\mathcal{E}$ we denote the linear space of all simple functions with supports from $\mathcal{P}$. By $\mathcal{M}_\infty$ we will denote the space of all extended measurable functions, i.e., all functions $f : \Omega \to [-\infty, \infty]$ such that there exists a sequence $\{g_n\} \subset \mathcal{E}$, $|g_n| \leq |f|$ and $g_n(\omega) \to f(\omega)$ for all $\omega \in \Omega$.

Definition 2.1 Let $\rho : \mathcal{M}_\infty \to [0, \infty]$ be a nontrivial, convex and even function. We say that ρ is a regular convex function pseudomodular if

(1) $\rho(0) = 0$;
(2) ρ is monotone, i.e., $|f(\omega)| \leq |g(\omega)|$ for any $\omega \in \Omega$ implies $\rho(f) \leq \rho(g)$, where $f, g \in \mathcal{M}_\infty$;
(3) ρ is orthogonally sub-additive, i.e., $\rho(f 1_{A\cup B}) \leq \rho(f 1_A) + \rho(f 1_B)$ for any $A, B \in \Sigma$ such that $A \cap B \neq \phi$, $f \in \mathcal{M}_\infty$;
(4) ρ has Fatou property, i.e., $|f_n(\omega)| \uparrow |f(\omega)|$ for all $\omega \in \Omega$ implies $\rho(f_n) \uparrow \rho(f)$, where $f \in \mathcal{M}_\infty$;
(5) ρ is order continuous in $\mathcal{E}$, i.e., $g_n \in \mathcal{E}$, and $|g_n(\omega)| \downarrow 0$ implies $\rho(g_n) \downarrow 0$.

A set $A \in \Sigma$ is said to be ρ-null if $\rho(g 1_A) = 0$ for every $g \in \mathcal{E}$. A property $p(\omega)$ is said to hold ρ-almost everywhere (ρ-a.e.) if the set $\{\omega \in \Omega : p(\omega)$ does not hold$\}$ is ρ-null. As usual, we identify any pair of measurable sets whose symmetric difference is ρ-null as well as any pair of measurable functions differing only on a ρ-null set. With this in mind we define

$$\mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho) = \{f \in \mathcal{M}_\infty : |f(\omega)| < \infty \ \rho\text{-a.e.}\},$$

where $f \in \mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho)$ is actually an equivalence class of functions equal ρ-a.e. rather than an individual function. Where no confusion exists, we will write $\mathcal{M}$ instead of $\mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho)$.

It is easy to see that $\rho : \mathcal{M} \to [0, \infty]$ possess the following properties:

1. $\rho(0) = 0$ iff $f = 0$ ρ-a.e.
2. $\rho(\alpha f) = \rho(f)$ for every scalar α with $|\alpha| = 1$ and $f \in \mathcal{M}$.
3. $\rho(\alpha f + \beta g) \leq \rho(f) + \rho(g)$ if $\alpha + \beta = 1$, $\alpha, \beta \geq 0$ and $f, g \in \mathcal{M}$.
   ρ is called a convex modular if, in addition, the following property is satisfied:

3′. $\rho(\alpha f + \beta g) \leq \alpha\rho(f) + \beta\rho(g)$ if $\alpha + \beta = 1$, $\alpha, \beta \geq 0$ and $f, g \in \mathcal{M}$.

Definition 2.2 Let ρ be a regular function pseudomodular. We say that ρ is a regular convex function modular if $\rho(f) = 0$ implies $f = 0$ ρ-a.e.

The class of all nonzero regular convex function modulars defined on Ω is denoted by $\Re$. The convex function modular ρ defines the modular function space L_ρ as

$$L_\rho = \{f \in \mathcal{M} : \rho(\lambda f) \to 0 \text{ as } \lambda \to 0\}.$$

Generally, the modular ρ is not sub-additive and, therefore, does not behave as a norm or a distance. However, the modular space L_ρ can be equipped with an F-norm defined by

$$\|f\|_\rho = \inf\left\{\alpha > 0 : \rho\left(\frac{f}{\alpha}\right) \le \alpha\right\}.$$

In case ρ is convex modular,

$$\|f\|_\rho = \inf\left\{\alpha > 0 : \rho\left(\frac{f}{\alpha}\right) \le 1\right\}$$

defines a norm on the modular space L_ρ, and is called the Luxemburg norm.

Define $L_\rho^0 = \left\{f \in L_\rho : \rho(f, .) \text{ is order continuous}\right\}$ and the linear space $E_\rho = \left\{f \in L_\rho : \lambda f \in L_\rho^0 \text{ for every } \lambda > 0\right\}$.

Definition 2.3 $\rho \in \Re$ is said to satisfy the Δ_2-condition, if $\sup_{n\ge1} \rho(2f_n, D_k) \to 0$ as $k \to \infty$ whenever $\{D_k\}$ decreases to ϕ and $\sup_{n\ge1} \rho(f_n, D_k) \to 0$ as $k \to \infty$.

If ρ is convex and satisfies the Δ_2-condition, then $L_\rho = E_\rho$. Moreover, ρ satisfies the Δ_2 -condition if and only if F-norm convergence and modular convergence are equivalent.

Definition 2.4 Let $\rho \in \Re$.

(*i*) Let $r > 0,\ \varepsilon > 0$. Define

$$D_1(r, \epsilon) = \left\{(f, g) : f, g \in L_\rho, \rho(f) \le r, \rho(g) \le r, \rho(f - g) \ge \varepsilon r\right\}.$$

Let

$$\delta_1(r, \epsilon) = \inf\left\{1 - \frac{1}{r}\rho(\frac{f+g}{2}) : (f, g) \in D_1(r, \epsilon)\right\} \text{ if } D_1(r, \epsilon) \ne \phi,$$

and $\delta_1(r, \epsilon) = 1$ if $D_1(r, \epsilon) = \phi$. We say that ρ satisfies $(UC1)$ if for every $r > 0, \epsilon > 0$, $\delta_1(r, \epsilon) > 0$. Note, that for every $r > 0$, $D_1(r, \epsilon) \ne \phi$, for $\epsilon > 0$ small enough.

(*ii*) We say that ρ satisfies $(UUC1)$ if for every $s \ge 0, \epsilon > 0$, there exists $\eta_1(s, \epsilon) > 0$ depending only upon s and ϵ such that $\delta_1(r, \epsilon) > \eta_1(s, \epsilon) > 0$ for any $r > s$.

Note that $(UC1)$ implies $(UUC1)$.

Definition 2.5 Let $\rho \in \Re$. The sequence $\{f_n\} \subset L_\rho$ is called:

- ρ-convergent to $f \in L_\rho$ if $\rho(f_n - f) \to 0$ as $n \to \infty$.
- ρ-Cauchy, if $\rho(f_n - f_m) \to 0$ as n and $m \to \infty$.

Note that ρ-convergence does not imply ρ-Cauchy since ρ does not satisfy the triangle inequality. In fact, one can show that this will happen if and only if ρ satisfies the Δ_2-condition.

Definition 2.6 Let $\rho \in \Re$. A subset $D \subset L_\rho$ is called

- ρ-closed if the ρ-limit of a ρ-convergent sequence of D always belongs to D.
- ρ-a.e. closed if the ρ-a.e. limit of a ρ-a.e. convergent sequence of D always belongs to D.
- ρ-compact if every sequence in D has a ρ-convergent subsequence in D.
- ρ-a.e. compact if every sequence in D has a ρ-a.e. convergent subsequence in D.
- ρ-bounded if $diam_\rho(D) = \sup\{\rho(f - g) : f, g \in D\} < \infty$.

A sequence $\{t_n\} \subset (0, 1)$ is called bounded away from 0 if there exists $a > 0$ such that $t_n \ge a$ for every $n \in \mathbb{N}$. Similarly, $\{t_n\} \subset (0, 1)$ is called bounded away from 1 if there exists $b < 1$ such that $t_n \le b$ for every $n \in \mathbb{N}$. The following lemma can be seen as an analog of a famous lemma due to Schu [7] in Banach spaces.

Lemma 2.7 [3, Lemma 4.2] *Let $\rho \in \Re$ satisfy $(UUC1)$ and let $\{t_k\} \subset (0, 1)$ be bounded away from 0 and 1. If there exists $R > 0$ such that*

$$\limsup_{n\to\infty} \rho(f_n) \le R, \quad \limsup_{n\to\infty} \rho(g_n) \le R,$$

and

$$\lim_{n\to\infty} \rho(t_n f_n + (1 - t_n)g_n) = R,$$

then

$$\lim_{n\to\infty} \rho(f_n - g_n) = 0.$$

A function $f \in L_\rho$ is called a fixed point of $T : L_\rho \to L_\rho$ if $f = Tf$. The set of all fixed points of T is denoted by $F_\rho(T)$.

The ρ-distance from an $f \in L_\rho$ to a set $D \subset L_\rho$ is given as follows:

$$dist_\rho(f, D) = \inf\{\rho(f - h) : h \in D\}.$$

The following definition is a modular space version of the condition (I) of Senter and Dotson [8]. Let $D \subset L_\rho$. A mapping $T : D \to D$ is said to satisfy condition (I) if there exists a nondecreasing function $\ell : [0, \infty) \to [0, \infty)$ with $\ell(0) = 0,\ \ell(r) > 0$ for all $r \in (0, \infty)$ such that

$$\rho(f - Tf) \ge \ell(dist_\rho(f, F_\rho(T))$$

for all $f \in D$.

Definition 2.8 A mapping $T : D \to D$ is called ρ-nonexpansive mapping if

$$\rho(Tf - Tg) \le \rho\,(f - g) \ \text{ for all } f, g \in D.$$

The following general theorem ([3, Theorem 5.7]) confirms the existence of fixed points of ρ-nonexpansive mappings.

Theorem 2.9 *Assume $\rho \in \Re$ satisfies $(UUC1)$. Let D be a ρ-closed, ρ-bounded convex and nonempty subset of L_ρ. Then, any $T : D \to D$ pointwise asymptotically nonexpansive mapping has a fixed point. Moreover, the set of all fixed points $F(T)$ is ρ-closed and convex.*

3 Fixed point approximation of (λ, ρ)-FNEM

We first extend the idea of a λ-firmly nonexpansive mapping from Banach spaces to modular function spaces and call it (λ, ρ)-firmly nonexpansive mapping. We define the idea as follows.

Definition 3.1 Let $D \subset L_\rho$. We say that a mapping $T : D \to D$ is called (λ, ρ)-firmly nonexpansive mapping if for given $\lambda \in (0, 1)$,

$$\rho(Tf - Tg) \le \rho\,[(1 - \lambda)\,(f - g) + \lambda(Tf - Tg)] \quad \text{for all } f, g \in D\ .$$

For simplicity, we denote a (λ, ρ)-firmly nonexpansive mapping by (λ, ρ)-FNEM.

Remark 3.2 (λ, ρ)-firmly nonexpansive implies ρ-nonexpansiveness. To see this, let $T : D \to D$ be a (λ, ρ)-firmly nonexpansive mapping. Then

$$\begin{aligned}\rho(Tf - Tg) &\le \rho\,[(1 - \lambda)\,(f - g) + \lambda(Tf - Tg)]\\ &\le (1 - \lambda)\rho\,(f - g) + \lambda\rho(Tf - Tg)\end{aligned}$$

for all $f, g \in D$. This implies that $(1-\lambda)\rho(Tf - Tg) \le (1-\lambda)\rho\,(f - g)$. Since $\lambda \ne 1$, we get $\rho(Tf - Tg) \le \rho\,(f - g)$ as desired.

Lemma 3.3 *The set of fixed points $F_\rho(T)$ of a (λ, ρ)-firmly nonexpansive mapping is nonempty. Moreover, it is ρ-closed and convex.*

Proof It follows from Remark 3.2 and Theorem 2.9. □

Next we introduce the following iterative process in the setting of modular function spaces. For a mapping $T : D \to D$, we define a sequence $\{f_n\}$ by the following iterative process:

$$\begin{aligned} & f_1 \in D, \\ & f_{n+1} = Tg_n, \\ & g_n = (1-\alpha_n) f_n + \alpha_n T f_n, \; n \in \mathbb{N} \end{aligned} \tag{3.1}$$

where $\{\alpha_n\} \subset (0, 1)$ is bounded away from both 0 and 1.

For details on a similar iterative process but in Banach spaces, see [4].

In this paper, using the above two ideas together, we prove our main result for approximating fixed points in modular function spaces as follows.

Theorem 3.4 *Let $\rho \in \Re$ satisfy $(UUC1)$ and Δ_2-condition. Let D be a nonempty ρ-closed, ρ-bounded and convex subset of L_ρ. Let $T : D \to D$ be a (λ, ρ)-FNEM. Let $\{f_n\} \subset D$ be defined by the iterative process. Then*

$$\lim_{n\to\infty} \rho(f_n - w) \text{ exists for all } w \in F_\rho(T),$$

and

$$\lim_{n\to\infty} \rho(f_n - Tf_n) = 0.$$

Proof Since $F_\rho(T) \neq \varnothing$ by Lemma 3.3, let $w \in F_\rho(T)$. To prove that $\lim_{n\to\infty} \rho(f_n - w)$ exists for all $w \in F_\rho(T)$, consider

$$\begin{aligned} \rho(f_{n+1} - w) &= \rho\,(Tg_n - Tw) \\ &\le \rho\,[(1-\lambda)\,(g_n - w) + \lambda\,(Tg_n - Tw)] \\ &\le (1-\lambda)\rho\,(g_n - w) + \lambda\rho\,(Tg_n - Tw) \quad \text{by convexity of } \rho. \end{aligned}$$

This implies $\rho\,(Tg_n - Tw) \le \rho\,(g_n - w)$ and hence

$$\rho(f_{n+1} - w) \le \rho\,(g_n - w)\,. \tag{3.2}$$

Also, because T is a (λ, ρ)-FNEM,

$$\rho\,(Tf_n - Tw) \le (1-\lambda)\rho\,(f_n - w) + \lambda\rho\,(Tf_n - Tw)$$

implies $\rho\,(Tf_n - Tw) \le \rho\,(f_n - w)$; therefore,

$$\begin{aligned} \rho(f_{n+1} - w) &\le \rho\,(g_n - w) \\ &= \rho[(1-\alpha_n)\rho\,(f_n - w) + \alpha_n\rho\,(Tf_n - Tw)] \\ &\le (1-\alpha_n)\rho\,(f_n - w) + \alpha_n\rho\,(Tf_n - Tw) \\ &\le (1-\alpha_n)\rho\,(f_n - w) + \alpha_n\rho\,(f_n - w) \\ &= \rho\,(f_n - w)\,. \end{aligned}$$

Thus, $\lim_{n\to\infty} \rho(f_n - w)$ exists for each $w \in F_\rho(T)$.

Suppose that

$$\lim_{n\to\infty} \rho(f_n - w) \; = m \tag{3.3}$$

where $m \ge 0$.

Note that the above calculations also give the following inequality:

$$\rho\,(g_n - w) \le \rho\,(f_n - w)\,. \tag{3.4}$$

Next, we prove that $\lim_{n\to\infty} \rho(f_n - Tf_n) = 0$. Now using 3.4, 3.2 and 3.3, we have

$$m = \lim_{n\to\infty} \rho(f_n - w) = \lim_{n\to\infty} \rho(\,g_n - w) \le \rho\,(f_n - w) = m.$$

This gives

$$\lim_{n\to\infty} \rho(\,g_n - w) = \; m.$$

Moreover,

$$\limsup_{n\to\infty} \rho(Tf_n - w) \leq \lim_{n\to\infty} \rho(f_n - w) = m. \tag{3.5}$$

But then $\rho(f_{n+1} - w) \leq \rho(g_n - w)$ implies that

$$\begin{aligned}\lim_{n\to\infty} \rho\left[(1-\alpha_n)(f_n - w) + \alpha_n(Tf_n - w)\right] &= \lim_{n\to\infty} \rho\left[(1-\alpha_n)f_n + \alpha_n Tf_n) - w\right] \\ &= \lim_{n\to\infty} \rho(g_n - w) \\ &= m.\end{aligned}$$

Now by (3.3) , (3.5) and Lemma 2.7, we have

$$\lim_{n\to\infty} \rho(f_n - Tf_n) = 0.$$

as required. □

Using the above result, we now prove our convergence result for approximating fixed points of (λ, ρ)-firmly nonexpansive mappings in modular function spaces using our iterative process (3.1) as follows.

Theorem 3.5 *Let $\rho \in \Re$ satisfy $(UUC1)$ and Δ_2-condition. Let D be a nonempty ρ-compact and convex subset of L_ρ. Let $T : D \to D$ be a (λ, ρ) -FNEM. Let $\{f_n\}$ be as defined in Theorem 3.4. Then $\{f_n\}$ ρ-converges to a fixed point of T.*

Proof Since D is ρ-compact, there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\lim_{k\to\infty}\left(f_{n_k} - z\right) = 0$ for some $z \in D$. Since T is a (λ, ρ)-FNEM, using convexity of ρ, we have

$$\begin{aligned}\rho\left(\frac{z - Tz}{3}\right) &= \rho\left(\frac{z - f_{n_k}}{3} + \frac{f_{n_k} - Tf_{n_k}}{3} + \frac{Tf_{n_k} - Tz}{3}\right) \\ &\leq \frac{1}{3}\rho(z - f_{n_k}) + \frac{1}{3}\rho(f_{n_k} - Tf_{n_k}) + \frac{1}{3}\rho(Tf_{n_k} - Tz) \\ &\leq \rho(z - f_{n_k}) + \rho(f_{n_k} - Tf_{n_k}) + \rho(f_{n_k} - z) \\ &\leq 2\rho\left(z - f_{n_k}\right) + \rho(f_{n_k} - Tf_{n_k}).\end{aligned}$$

Applying Theorem 3.4, $\lim_{n\to\infty} \rho(f_{n_k} - Tf_{n_k}) = 0$, that is, $\rho(\frac{z-Tz}{3}) = 0$. Hence, z is a fixed point of T, that is, $\{f_n\}$ ρ-converges to a fixed point of T. □

Theorem 3.6 *Let $\rho \in \Re$ satisfy $(UUC1)$ and Δ_2-condition. Let D be a nonempty ρ-closed, ρ-bounded and convex subset of L_ρ. Let $T : D \to D$ be a (λ, ρ)-FNEM satisfying condition (I). Let $\{f_n\}$ be as defined in Theorem 3.4. Then $\{f_n\}$ ρ-converges to a fixed point of T.*

Proof By Theorem 3.4, $\lim_{n\to\infty} \rho(f_n - w)$ exists for all $w \in F_\rho(T)$. Suppose that $\lim_{n\to\infty} \rho(f_n - w) = m > 0$ because otherwise $\lim_{n\to\infty} \rho(f_n - w) = 0$ means nothing left to prove. Now by Theorem 3.4, we have $\rho(f_{n+1} - w) \leq \rho(f_n - w)$ so that

$$dist_\rho(f_{n+1}, F_\rho(T)) \leq dist_\rho(f_n, F_\rho(T)).$$

This means that $\lim_{n\to\infty} dist_\rho(f_n, F_\rho(T))$ exists. Applying condition (I) and Theorem 3.4, we have

$$\lim_{n\to\infty} \ell(dist_\rho(f_n, F_\rho(T))) \leq \lim_{n\to\infty} \rho(f_n - Tf_n) = 0.$$

Since ℓ is a nondecreasing function and $\ell(0) = 0$,

$$\lim_{n\to\infty} dist_\rho(f_n, F_\rho(T)) = 0. \tag{3.6}$$

To prove that $\{f_n\}$ is a ρ-Cauchy sequence in D,let $\varepsilon > 0$. By (3.6) , there exists a constant n_0 such that for all $n \geq n_0$,

$$dist_\rho(f_n, F_\rho(T)) < \frac{\varepsilon}{2}.$$

Hence, there exists a $y \in F_\rho(T)$ such that

$$\rho\left(f_{n_0} - y\right) < \varepsilon.$$

Now for $m, n \geq n_0$,

$$\rho\left(\frac{f_{n+m}-f_n}{2}\right) \leq \frac{1}{2}\rho\left(f_{n+m}-y\right)+\frac{1}{2}\rho\left(f_n-y\right)$$
$$\leq \rho\left(f_{n_0}-y\right)$$
$$< \varepsilon.$$

Hence, by Δ_2-condition $\{f_n\}$ is a ρ-Cauchy sequence in a ρ-closed subset D of L_ρ, and so it converges in D. Let $\lim_{n\to\infty} f_n = w$. Then $dist_\rho(w, F_\rho(T)) = \lim_{n\to\infty} dist_\rho(f_n, F_\rho(T)) = 0$ by (3.6) . Since by Lemma 3.3 $F_\rho(T)$ is closed, $w \in F_\rho(T)$, that is, $\{f_n\}$ ρ-converges to a fixed point of T. □

4 Concluding remarks

We have proved some strong convergence results using (λ, ρ)-firmly nonexpansive mappings on a faster iterative algorithm in modular function spaces. In our opinion, using the above ideas, it would be interesting to consider the following:

(1) studying the stability and data dependency problems;
(2) finding applications to general variational inequalities or equilibrium problems as well as to split feasibility problems.

We may suggest to the reader to combine the ideas studied, for example, in [9–14].

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