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Approximate solutions of impulsive integro-differential equations

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Abstract In this paper, we consider impulsive integro-differential equations in Banach space and we establish the bound on the difference between two approximate solutions. We also discuss nearness and convergence of solutions of the problem under consideration. The impulsive integral inequality of Grownwall type is used to obtain results.

Mathematical Subject Classification 34A12 · 47GXX · 47B38

الملخص

في هذا المقال، قدمنا حدودا عظمى جديدة لمتراجحات تكاملية من نوع أوستروؤسكي وذلك باستخدام التكامل الكسري المريح. وبالتوافق مع هذا الغرض، فقد استفدنا من نشر تايلور للمشتقات الكسرية المريحة والتي قُدّمت من طرف د. ر. أندرسان.

1 Introduction

In the recent decade, the study of impulsive integro-differential equations has become an important thrust area for many researchers across the world. Many real-life phenomena and processes are subject to short-term perturbations whose duration is negligible as compared to the whole phenomena. These problems mostly arise in medicine, economics, biological sciences, engineering, etc. Such type of problems can be modelled with impulsive integro-differential equations. Thus, many researchers [2,4,5,8–10,13] have opted for this research area and contributed to the development of the theory of impulsive differential equations. More information related to this can be found in monographs of Bainov and Simeonov [2] and Lkashmikantham et al. [8].

As it is difficult to provide explicit solutions to most of the physical problems, the method of approximate solution is the best analytic tool for such situation which provides the required information of solutions without finding the explicit solution. Tidke and Dhakne [14,15], Pachpatte [11], Pachpatte [12], Kucche et al. [7] used

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this technique to study the qualitative properties of solutions of different initial value problems. Since there is less information available in the literature about the approximate solutions of impulsive integro-differential equations, we apply this technique for the following impulsive integro-differential equation of the type:

$$x'(t) = Ax(t) + f(t, x(t), \int_0^t k(t,s)h(s, x(s))ds), \quad t \in (0, T], \quad t \neq \tau_k, k = 1, 2, \ldots, m \tag{1}$$

$$x(0) = x_0, \tag{2}$$

$$\Delta x(\tau_k) = I_k x(\tau_k), \quad k = 1, 2, \ldots, m, \tag{3}$$

where A is the infinitesimal generator of strongly continuous semigroup of bounded linear operators $\{T(t)\}_{t\geq 0}$ and $I_k (k = 1, 2, \ldots, m)$ are the linear operators acting in a Banach space X. Let k be a real-valued continuous function on $[0, T] \times [0, T]$ and the functions and f and h are given functions satisfying some assumptions. The impulsive moments τ_k are such that $0 \leq \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_m < \tau_{m+1} \leq T$, $m \in \mathbb{N}$, $\Delta x(\tau_k) = x(\tau_k + 0) - x(\tau_k - 0)$, where $x(\tau_k + 0)$ and $x(\tau_k - 0)$ are, respectively, the right and the left limits of x at τ_k.

In [6], Kendre and Dhakne studied the existence, uniqueness, continuation and continuous dependence of solutions of IVP:

$$x'(t) + Ax(t) = f(t, x(t), \int_0^t k(t,s)x(s)ds), t > t_0,$$
$$x(t_0) = x_0 \in X,$$

using theory of analytic semigroups and fractional power of operators. The problem of existence, uniqueness and other basic properties of IVP (1)–(3) and their special forms have been studied by several authors using different methods such as Banach fixed point theorem, semigroup approach, progressive contractions, etc. See [3,10,13]. Our aim is to find the bound on the difference between two approximate solutions, nearness, convergence and continuous dependence of solutions on parameters of mild solutions of IVP (1)–(3).

The paper is organised as follows:
Sect. 2 consists of preliminaries and hypotheses. In Sect. 3, we establish the bound on the difference between two approximate solutions, nearness and convergence properties of solutions and, finally, we give continuous dependence of solutions on parameters and functions involved therein.

2 Preliminaries and hypotheses

Let X be a Banach space with the norm $\|\cdot\|$. Let $PC([0, T], X) = \{x : [0, T] \to X | x(t)$ be piecewise continuous at $t \neq \tau_k$, left continuous at $t = \tau_k$, that is, $x(\tau_k^-) = \lim_{h\to 0^+} x(\tau_k - h) = x(\tau_k)$ and the right limit $x(\tau_k + 0)$ exists for $k = 1, 2, \ldots, m\}$. Clearly, $PC([0, T], X)$ is a Banach space with the supremum norm $\|x\|_{PC([0,T],X)} = \sup\{\|x(t)\| : t \in [0, T] \setminus \{\tau_1, \tau_2, \ldots, \tau_m\}\}$.

Definition 2.1 A function $x \in PC([0, T], X)$ satisfying the equations:

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s, x(s), \int_0^s k(s,\tau)h(\tau, x(\tau)d\tau)ds + \sum_{0<\tau_k<t} T(t-\tau_k)I_k x(\tau_k), \quad t \in (0, T],$$
$$x(0) = x_0$$

is said to be the mild solution of the initial value problem (1)–(3).

Definition 2.2 Let $x_i \in PC([0, T], X)$ (i = 1,2) be the function such that $x_i(t)$ exists for each $t \in [0, T]$ and satisfies the inequality:

$$\|x_i'(t) - Ax_i(t) - f(t, x_i(t), \int_0^t k(t,s)h(s, x_i(s)ds)\| \leq \varepsilon_i, \tag{4}$$

for a given constant $\varepsilon_i \geq 0$, where it is considered that the initial and impulsive conditions,

$$x_i(0) = x_0^i, \tag{5}$$
$$\Delta x_i(\tau_k) = I_k x_i(\tau_k) \tag{6}$$

are satisfied. Then, $x_i(t)$ are called ϵ_i -approximate solutions to the IVP (1)–(3).

Lemma 2.3 [1] *Assume the following inequality holds for $t \geq t_0$:*

$$u(t) \leq a(t) + \int_{t_0}^{t} b(t,s)u(s)ds + \int_{t_0}^{t} \left(\int_{t_0}^{s} k(t,s,\tau)u(\tau)d\tau \right) ds + \sum_{t_0<\tau_k<t} \beta_k(t)u(t_k)$$

where, $u, a \in PC([t_0,\infty), \mathbb{R}_+)$, a is nondecreasing, $b(t,s)$ and $k(t,s,\tau)$ are continuous and non negative functions for $t,s,\tau \geq t_0$ and are nondecreasing with respect to t, $\beta_k(t)(k \in \mathbb{N})$ are nondecreasing for $t \geq t_0$. Then, for $t \geq t_0$, the following inequality holds:

$$u(t) \leq a(t) \prod_{t_0<\tau_k<t} (1+\beta_k(t))exp\left(\int_{t_0}^{t} b(t,s)ds \right) + \int_{t_0}^{t} \int_{t_0}^{s} k(t,s,\tau)d\tau)ds.$$

Now, we introduce the following hypotheses:

(H_1) Let $f : [0,T] \times X \times X \to X$ and $h : [0,T] \times X \to X$ be continuous functions such that there exist continuous nondecreasing functions $p : [0,T] \to \mathbb{R}_+ = [0,\infty)$ and $q : [0,T] \to \mathbb{R}_+$ and

$$\|f(t,\psi,x) - f(t,\phi,y)\| \leq p(t)(\|\psi-\phi\| + \|x-y\|),$$
$$\|h(t,\psi) - h(t,\phi)\| \leq q(t)(\|\psi-\phi\|),$$

for every $t \in [0,T]$, $\psi \in X$ and $x \in X$.

(H_2) Let $I_k : X \to X$ be functions such that there exist positive constants L_k satisfying

$$\|I_k(x) - I_k(y)\| \leq L_k\|x-y\|, \quad x,y \in X, \quad k = 1,2,\ldots,m.$$

In this paper, we consider that there exists a constant $K_0 > 0$ such that $\|T(t)\| \leq K_0$. Also since $k : [0,T]\times[0,T] \to \mathbb{R}$ is a continuous function on compact set $[0,T]\times[0,T]$, there exists a constant $L > 0$ such that $|k(t,s)| \leq L$, for $0 \leq s \leq t \leq T$. Let $R(t) = \max\{p(t), Lq(t), h(t)\}$ and $R^* = \sup\{R(t) : t \in [0,T]\}$.

3 Main results

Theorem 3.1 *Suppose that the hypotheses (H_1) and (H_2) hold. If $x_1(t)$ and $x_2(t)$ are ε_i approximate solutions of Eq. (1) with Conditions (5) and (6) such that $\|(x_0^1 - x_0^2)\| \leq \delta$, where δ is a nonnegative constant, then the following inequality holds:*

$$\|x_1(t) - x_2(t)\| \leq [(\varepsilon_1+\varepsilon_2)t + K_0\delta] \prod_{0<\tau_k<t} (1+K_0L_k) \exp\left(K_0R^*T + K_0(R^*)^2\frac{T^2}{2} \right).$$

Proof Let $x_i (i = 1,2)$ be approximate solutions of Eq. (1) with Conditions (5) and (6). Then we get

$$\|x_i'(t) - Ax_i(t) - f(t, x_i(t), \int_0^t k(t,s)h(s,x_i(s)ds)\| \leq \varepsilon_i. \tag{7}$$

Taking $t = \xi$ in (7) and integrating with respect to ξ from 0 to t, we obtain

$$\begin{aligned}\int_0^t \varepsilon_i d\xi &\geq \int_0^t \|x_i'(\xi) - Ax_i(\xi) - f(\xi, x_i(\xi), \int_0^\xi k(\xi,s)h(s,x_i(s))ds)\|d\xi \\ &\geq \|x_i(t) - T(t)x_i(0) - \int_0^t T(t-s)f(s,x_i(s), \int_0^s k(s,\tau)h(\tau,x_i(\tau))d\tau)ds \\ &- \sum_{0<\tau_k<t} T(t-\tau_k)I_kx_i(\tau_k)\|.\end{aligned}$$

Using the inequalities $\|u_1 - v_1\| \leq \|u_1\| + \|v_1\|$ and $|\|u_1\| - \|v_1\|| \leq \|u_1 - v_1\|$, we get

$(\varepsilon_1 + \varepsilon_2)t$

$$\geq \|x_1(t) - T(t)x_0^1 - \int_0^t T(t-s) f(s, x_1(s), \int_0^s k(s,\tau)h(\tau, x_1(\tau))d\tau)ds - \sum_{0<\tau_k<t} T(t-\tau_k)I_k x_1(\tau_k)\|$$
$$+ \|x_2(t) - T(t)x_0^2 - \int_0^t T(t-s) f(s, x_2(s), \int_0^s k(s,\tau)h(\tau, x_2(\tau))d\tau)ds - \sum_{0<\tau_k<t} T(t-\tau_k)I_k x_2(\tau_k)\|$$
$$\geq \|[x_1(t) - x_2(t)] - [T(t)(x_0^1 - x_0^2)]$$
$$- \int_0^t T(t-s)[f(s, x_1(s), \int_0^s k(s,\tau)h(\tau, x_1(\tau)d\tau) - f(s, x_2(s), \int_0^s k(s,\tau)h(\tau, x_2(\tau))d\tau)]$$
$$- \sum_{0<\tau_k<t} T(t-\tau_k)[I_k x_1(\tau_k) - I_k x_2(\tau_k)]\|$$
$$\geq \|x_1(t) - x_2(t)\| - \|T(t)\| \|(x_0^1 - x_0^2)\|$$
$$- \int_0^t \|T(t-s)\| \|f(s, x_1(s), \int_0^s k(s,\tau)h(\tau, x_1(\tau))d\tau) - f(s, x_2(s), \int_0^s k(s,\tau)h(\tau, x_2(\tau))d\tau)\|$$
$$- \sum_{0<\tau_k<t} \|T(t-\tau_k)\| \|I_k x_1(\tau_k) - I_k x_2(\tau_k)\|.$$

Using hypotheses (H_1) and (H_2), we get

$$(\varepsilon_1 + \varepsilon_2)t$$
$$\geq \|x_1(t) - x_2(t)\| - \|T(t)\| \|(x_0^1 - x_0^2)\| - \int_0^t \|T(t-s)\| p(t)\big[\|x_1(s) - x_2(s)\|$$
$$+ \int_0^s |k(s,\tau)| [\|h(\tau, x_1(\tau)) - h(\tau, x_2(\tau))\| d\tau)\big]ds - \sum_{0<\tau_k<t} \|T(t-\tau_k)\| \|I_k x_1(\tau_k) - I_k x_2(\tau_k)\|$$
$$\geq \|x_1(t) - x_2(t)\| - K_0\delta - \int_0^t K_0 p(t)\big[\|x_1(s) - x_2(s)\| + \int_0^s Lq(\tau)\|x_1(\tau) - x_2(\tau)\| d\tau\big]ds$$
$$- \sum_{0<\tau_k<t} K_0 L_k \|x_1(\tau_k) - x_2(\tau_k)\|.$$

Let $u(t) = \|x_1(t) - x_2(t)\|$. Then, we have

$$(\varepsilon_1 + \varepsilon_2)t$$
$$\geq u(t) - K_0\delta - \int_0^t K_0 R(t)u(s)ds - \int_0^t \int_0^s K_0 R(t)R(\tau)u(\tau)d\tau ds - \sum_{0<\tau_k<t} K_0 L_k u(\tau_k)$$
$$u(t) \leq (\varepsilon_1 + \varepsilon_2)t + K_0\delta + \int_0^t K_0 R(t)u(s)ds + \int_0^t \int_0^s K_0 R(t)R(\tau)u(\tau)d\tau ds + \sum_{0<\tau_k<t} K_0 L_k u(\tau_k).$$

Applying Lemma 2.3, we get

$$u(t) \leq [(\varepsilon_1 + \varepsilon_2)t + K_0\delta] \prod_{0<\tau_k<t} (1 + K_0 L_k) \exp\left(\int_0^t K_0 R(t)ds + \int_0^t \int_0^s K_0 R(t)R(\tau)d\tau ds\right)$$
$$\leq [(\varepsilon_1 + \varepsilon_2)t + K_0\delta] \prod_{0<\tau_k<t} (1 + K_0 L_k) \exp\left(\int_0^t K_0 R^* ds + \int_0^t \int_0^s K_0 (R^*)^2 d\tau ds\right)$$

$$\leq [(\varepsilon_1 + \varepsilon_2)t + K_0\delta] \prod_{0<\tau_k<t} (1 + K_0 L_k) \exp\left(K_0 R^* t + K_0 (R^*)^2 \frac{t^2}{2}\right)$$

$$\leq [(\varepsilon_1 + \varepsilon_2)T + K_0\delta] \prod_{0<\tau_k<t} (1 + K_0 L_k) \exp\left(K_0 R^* T + K_0 (R^*)^2 \frac{T^2}{2}\right). \tag{8}$$

□

Remark The inequality obtained in (8) establishes the bound on the difference between the two approximate solutions of Eqs. (1)–(3). If $x_1(t)$ is a solution of Eq. (1) with $x(t) = x_0^1$, then we have $\varepsilon_1 = 0$ and, from (8), we see that $x_2(t) \to x_1(t)$ as $\varepsilon_2 \to 0$ and $\delta \to 0$. Moreover, if we put $\varepsilon_1 = \varepsilon_2 = 0$ and $x_0^1 = x_0^2$ in (8), then the uniqueness of the solutions of (1)–(3) is established.

Consider the impulsive IVP (1)–(3), along with the following initial value problem:

$$y'(t) = Ay(t) + \bar{f}\Big(t, y(t), \int_0^t k(t,s)h(s, y(s))ds\Big), \quad t \in [0, T], \tag{9}$$

$$y(0) = y_0, \tag{10}$$

$$\Delta y(\tau_k) = \bar{I}_k y(\tau_k), \quad k = 1, 2, \ldots, m, \tag{11}$$

where k, h are as given in (1)–(3) , $\bar{f} : [0, T] \times X \times X \to X$, $\bar{I}_k : X \to X$.

Theorem 3.2 *Suppose that the functions f, k, h in (1)–(3) satisfy the hypotheses (H_1) and (H_2) and there exist nonnegative constants ϵ_3, δ_k such that*

$$\|f(t,\phi,x) - \bar{f}(t,\phi,x)\| \leq \epsilon_3, \tag{12}$$

$$\|I_k(\phi) - \bar{I}_k(\phi)\| \leq \delta_k. \tag{13}$$

Let $x(t)$ and $y(t)$ be, respectively, solutions of the initial value problem (1)–(3) and (9)–(11) on [0,T]. Then, the following inequality holds:

$$\|x - y\| \leq K_0[\|x_0 - y_0\| + \epsilon_3 T + \delta_k] \prod_{0<\tau_k<t} (1 + K_0 L_k) exp\left(K_0 R^* T + K_0 (R^*)^2 \frac{T^2}{2}\right).$$

Proof Using the facts that $x(t)$ and $y(t)$ are, respectively, the solutions of the initial value problem (1)–(2) and (9)–(11) and hypotheses (H_1) and (H_2), we get

$$\begin{aligned}\|x(t) - y(t)\| \leq{}& \|T(t)\|\|x_0 - y_0\| \\ &+ \int_0^t \|T(t-s)\| \Big\| f\Big(s, x(s), \int_0^s k(s,\tau)h(\tau, x(\tau))d\tau\Big) \\ &- \bar{f}\Big(s, y(s), \int_0^s k(s,\tau)h(\tau, y(\tau))d\tau\Big)\Big\| ds + \sum_{0<\tau_k<t} \|T(t-\tau_k)\|\|I_k x(\tau_k) - \bar{I}_k y(\tau_k)\| \\ \leq{}& K_0[\|x_0 - y_0\| + \epsilon_3 t] + \int_0^t K_0 p(t)\Big[\|x(s) - y(s)\| + \int_0^s Lq(\tau)\|x(\tau) - y(\tau)\|d\tau\Big]ds \\ &+ \sum_{0<\tau_k<t} K_0 L_k \|x(\tau_k) - y(\tau_k)\| + K_0\delta_k \end{aligned} \tag{14}$$

$$\begin{aligned} \leq{}& K_0[\|x_0 - y_0\| + \epsilon_3 T + \delta_k] \\ &+ \int_0^t K_0 p(t)\|x(s) - y(s)\|ds + \int_0^t \int_0^s K_0 p(t) Lq(\tau)\|x(\tau) - y(\tau)\|d\tau ds \\ &+ \sum_{0<\tau_k<t} K_0 L_k \|x(\tau_k) - y(\tau_k)\|. \end{aligned} \tag{15}$$

Let $u(t) = \|x(t) - y(t)\|$

$$\begin{aligned} u(t) &= \|x(t) - y(t)\| \\ &\leq K_0[\|x_0 - y_0\| + \epsilon_3 T + \delta_k] \\ &\quad + \int_0^t K_0 p(t)u(s)ds + \int_0^t \int_0^s K_0 p(t) L q(\tau)u(\tau)d\tau ds + \sum_{0<\tau_k<t} K_0 L_k u(\tau_k). \end{aligned} \tag{16}$$

Now, applying the inequality given in Lemma 2.3, we get

$$u(t) \leq K_0[\|x_0 - y_0\| + \epsilon_3 t + \delta_k] \prod_{0<\tau_k<t} (1 + K_0 L_k) exp\, (\int_0^t K_0 R^* ds + \int_0^t \left(\int_0^s K_0 (R^*)^2 d\tau ds \right).$$

Consequently,

$$\|x - y\| \leq K_0[\|x_0 - y_0\| + \epsilon_3 t + \delta_k] \prod_{0<\tau_k<t} (1 + K_0 L_k) exp \left(K_0 R^* t + K_0 (R^*)^2 \frac{t^2}{2} \right),$$

which implies that

$$\|x - y\| \leq K_0[\|x_0 - y_0\| + \epsilon_3 T + \delta_k] \prod_{0<\tau_k<t} (1 + K_0 L_k) exp \left(K_0 R^* T + K_0 (R^*)^2 \frac{T^2}{2} \right).$$

This completes the proof. □

Remark If f is nearer to $\bar{f}$, x_0 to y_0, then the corresponding solutions of the initial value problem (1)–(3) and (9)–(11) are nearer to each other, and it also depends on the functions continuously involved therein. Thus, the above inequality gives the relation between the solutions of IVP (1)–(3) and (9)–(11).

Consider the initial value problem (1)–(3) with the initial value problem:

$$y_n'(t) = Ay(t) + f_n\Big(t, y(t), \int_0^t k(t,s)h(s, y(s))ds\Big), \quad t \in [0, T], \tag{17}$$

$$y(0) = y_{n0}, \tag{18}$$

$$\Delta y_n(\tau_k) = I_{kn} y(\tau_k), \quad k = 1, 2, \ldots, m, \tag{19}$$

where k, h are as given in (1), and $f_n : [0, T] \times X \times X \to X$ is a sequence in X.

As an immediate consequence of the above theorem, we have the following corollary:

Corollary 3.3 *Suppose that the functions f, k, h in (1)–(3) satisfy the hypotheses (H_1) and (H_2) and there exist nonnegative constants $\epsilon_n, \delta_n, \delta_{kn}$ such that*

$$\|f(t, \phi, x) - f_n(t, \phi, x)\| \leq \epsilon_n, \tag{20}$$

$$\|x_0 - y_{n0}\| \leq \delta_n. \tag{21}$$

$$\|I_k \phi(\tau_k) - I_{kn}\phi(\tau_k)\| \leq \delta_{kn}, \tag{22}$$

with $\epsilon_n \to 0, \delta_n \to 0, \delta_{kn} \to 0$ as $n \to \infty$. If $x(t)$ and $y_n(t), n = 1, 2, \ldots$ are, respectively, solutions of the initial value problems (1)–(3) and (17)–(19) on (0,T], then $y_n(t) \to x(t)$as $n \to \infty$ on (0, T].

Remark The result obtained in this corollary provides sufficient conditions to ensure that the solutions of the initial value problem (17)–(19) will converge to solutions of the initial value problem (1)–(3).

Here, we will study the continuous dependence of the solutions of IVP (1)–(3) on parameters and functions involved in them. Consider the following IVP:

$$x'(t) = Ax(t) + f(t, x(t), \int_0^t k(t,s)h(s, x(s))ds, \delta_2), \quad t \in (0, T], \quad t \neq \tau_k, k = 1, 2, \ldots, m, \tag{23}$$

$$x(0) = x_0, \tag{24}$$

$$\Delta x(\tau_k) = I_k x(\tau_k), \quad k = 1, 2, \ldots, m \tag{25}$$

and

$$y'(t) = Ay(t) + f(t, y(t), \int_0^t k(t,s)h(s, y(s))ds, \delta_3), \quad t \in (0, T], \quad t \neq \tau_k, k = 1, 2, \dots, m, \tag{26}$$

$$y(0) = y_0, \tag{27}$$

$$\Delta y(\tau_k) = I_k y(\tau_k), \quad k = 1, 2, \dots, m, \tag{28}$$

where $f : [0, T] \times X \times X \times R \to X$, δ_2 and δ_3 are real parameters.

Corollary 3.4 *Assume the hypotheses (H_1) and (H_2) hold. Let $f : [0, T] \times X \times X \times R \to X$ be a function satisfying*

$$\|f(t, \psi, x, \delta) - f(t, \phi, y, \delta')\| \leq h(t)(\|\psi - \phi\| + \|x - y\| + \|\delta - \delta'\|), \quad \psi, \phi, x, y \in X, \quad \delta, \delta' \in \mathbb{R}.$$

If $x(t)$ and $y(t)$ are solutions of Eqs. (23)–(25) *and* (26)–(28)*, then*

$$\|x - y\|_B \leq K_0[\|x_0 - y_0\| + R^* T \|\delta_2 - \delta_3\|] \prod_{0<\tau_k<t} (1 + K_0 L_k) exp\left(K_0 R^* T + K_0 (R^*)^2 \frac{T^2}{2}\right).$$

Proof It is an easy consequence of our main result, so we have omitted the proof. □

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