Arab. J. Math. (2018) 7:91–99
https://doi.org/10.1007/s40065-017-0194-0 **Arabian Journal of Mathematics** CrossMark

Mohamed Elouafi

On formulae for the determinant of symmetric pentadiagonal Toeplitz matrices

Received: 12 February 2016 / Accepted: 29 November 2017 / Published online: 11 December 2017

Abstract We show that the characteristic polynomial of a symmetric pentadiagonal Toeplitz matrix is the product of two polynomials given explicitly in terms of the Chebyshev polynomials.

Mathematics Subject Classification 15B05 · 65F40 · 33C45

الملخص

نبين أن كثيرة الحدود المميزة لمصفوفة توبليتز خماسية الأقطار هي ضرب لكثيرتي حدود، معطيين بصورة صريحة بدلالة كثيرات حدود تشيبي تشيف.

1 Introduction

We consider here the problem of finding the determinant of the $m \times m$ symmetric pentadiagonal Toeplitz matrix

$$\mathbf{P}_m = \mathbf{P}_m(a,b,c) = \begin{pmatrix} a & b & c & 0 & \cdots & 0 \\ b & a & b & \ddots & \ddots & \vdots \\ c & b & a & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & b & c \\ \vdots & \ddots & c & b & a & b \\ 0 & \cdots & 0 & c & b & a \end{pmatrix}.$$

This class of matrices arises naturally in many applications, such as signal processing, trigonometric moment problems, integral equations and elliptic partial differential equations with boundary conditions [9]. Computing the determinant of the matrix $\mathbf{P}_m$ have intrigued the researchers for decades. If $c = 0$, then $\mathbf{P}_m$ is reduced to a tridiagonal matrix and there exists a closed form of det $(\mathbf{P}_m)$ from which the eigenvalues of the matrix are explicitly given. It is becoming a challenge to find similar formulae for the general case and so far, little is known about the eigenvlaues of $\mathbf{P}_m$ [1,2,5,8]. In [5,7], det $(\mathbf{P}_m)$ is explicitly computed using the kernel of the Chebyshev polynomials $\{T_n\}$, $\{U_n\}$, $\{V_n\}$ and $\{W_n\}$ [11] and, as a consequence, the eigenvalues

M. Elouafi (✉)
Classes Préparatoites aux Grandes Ecoles d'Ingénieurs, Lyc ée Moulay Alhassan, Tangier, Morocco
E-mail: med3elouafi@gmail.com

of the matrix $\mathbf{P}_m$ are localized by means of explicitly given rational functions. The formulae are simplified to give $\det(\mathbf{P}_m)$ as polynomials of the parameters a, b, c [6].

In the new formula presented here, $\det(\mathbf{P}_m)$ is given as the product of two polynomials given in a standard form. Here is our main result:

Theorem 1.1 *We have*

$$\det(\mathbf{P}_{2n+1}) = 2\left(\sum_{k=0}^{n+1} \gamma_{n,k} c^{n+1-k} b^k T_k\left(\frac{a+2c}{2b}\right)\right)\left(\sum_{k=1}^{n+1} \gamma_{n,k} c^{n+1-k} b^{k-1} U_{k-1}\left(\frac{a+2c}{2b}\right)\right), \quad (1)$$

and

$$\det(\mathbf{P}_{2n}) = \left(\sum_{k=0}^{n} \mu_{n,k} c^{n-k} b^k V_k\left(\frac{a+2c}{2b}\right)\right)\left(\sum_{k=0}^{n} \mu_{n,k} c^{n-k} b^k W_k\left(\frac{a+2c}{2b}\right)\right), \quad (2)$$

where

$$\gamma_{n,k} = (-1)^k \binom{n+1+k}{n+1-k}, \quad \mu_{n,k} = (-1)^k \binom{n+1+k}{n-k}.$$

2 Proof of the main result

Since $\det(\mathbf{P}_m(a,b,c)) = c^m \det\left(\mathbf{P}_m\left(\frac{a}{c}, \frac{b}{c}, 1\right)\right)$, then we can assume for simplicity that $c = 1$. We denote by $\zeta_j, \frac{1}{\zeta_j}$, $j = 1, 2$, the roots of the polynomial $g(x) = x^4 + bx^3 + ax^2 + bx + 1$ assumed pairwise distinct and different of ± 1.

Recall that the Chebyshev polynomials $\{T_n\}, \{U_n\}, \{V_n\}$ and $\{W_n\}$ are orthogonal polynomials over $(-1, 1)$ with respect to the weight $\frac{1}{\sqrt{1-x^2}}, \sqrt{1-x^2}, \sqrt{\frac{1+x}{1-x}}$ and $\sqrt{\frac{1-x}{1+x}}$, respectively, and we have for $\zeta \in \mathbb{C}^*$

$$\begin{cases} T_n\left(\frac{1}{2}\left(\zeta + \frac{1}{\zeta}\right)\right) = \frac{1}{2}\left(\zeta^n + \zeta^{-n}\right), & U_n\left(\frac{1}{2}\left(\zeta + \frac{1}{\zeta}\right)\right) = \frac{\zeta^{n+1} - \zeta^{-n-1}}{\zeta - \zeta^{-1}}, \\ V_n\left(\frac{1}{2}\left(\zeta + \frac{1}{\zeta}\right)\right) = \frac{\zeta^{n+1/2} + \zeta^{-n-1/2}}{\zeta^{1/2} + \zeta^{-1/2}}, & W_n\left(\frac{1}{2}\left(\zeta + \frac{1}{\zeta}\right)\right) = \frac{\zeta^{n+1/2} - \zeta^{-n-1/2}}{\zeta^{1/2} - \zeta^{-1/2}}. \end{cases} \quad (3)$$

We shall use the following formula for $\det(\mathbf{P}_m)$:

Lemma 2.1 *For* $J \subset \{1, 2\}$, *let* $I_J(k) = \begin{cases} 1 & if \;\; k \in J \\ -1 & if \;\; k \notin J \end{cases}$, *and*

$$\omega_J = \prod_{k=1}^{2} \zeta_k^{I_J(k)}, \quad \gamma_J = \prod_{1 \le j < k \le 2} (\zeta_k^{I_J(k)} - \zeta_j^{I_J(j)}).$$

We have

$$\det(\mathbf{P}_m) = \frac{1}{d^2 \prod_{k=1}^{2} (\zeta_k - \zeta_k^{-1})} \left(\sum_J (-1)^{|J|} \gamma_J \omega_J^{\frac{m+1}{2}}\right) \times \left(\sum_J \gamma_J \omega_J^{\frac{m+1}{2}}\right),$$

where $d = \left(\zeta_2 + \frac{1}{\zeta_2} - \zeta_1 - \frac{1}{\zeta_1}\right)$.

Proof See [4]. □

Let us put $\alpha = \zeta_1 \zeta_2$, $\beta = \zeta_1 \zeta_2^{-1}$ and $u = \frac{1}{2}\left(\alpha + \alpha^{-1}\right)$, $v = \frac{1}{2}\left(\beta + \beta^{-1}\right)$. We have by the Vieta' formulae:

$$\begin{aligned} u + v &= \frac{1}{2}(\alpha + \alpha^{-1} + \beta + \beta^{-1}) \\ &= \frac{a}{2} - 1, \end{aligned}$$

and

$$\begin{aligned} uv &= \frac{1}{4}(\zeta_1^2 + \zeta_1^{-2} + \zeta_2^2 + \zeta_2^{-2}) \\ &= \frac{1}{4}((\zeta_1 + \zeta_1^{-1} + \zeta_2 + \zeta_2^{-1})^2 - 2(2 + \alpha + \alpha^{-1} + \beta + \beta^{-1})) \\ &= \frac{b^2}{4} - \frac{a}{2}. \end{aligned}$$

This implies that $(u+1)(v+1) = (\frac{b}{2})^2$.

Lemma 2.2 *We have*

$$\det(\mathbf{P}_m) = \frac{U_{m+1}^2\left(\sqrt{\frac{1+u}{2}}\right) - U_{m+1}^2\left(\sqrt{\frac{1+v}{2}}\right)}{2\,(u-v)}. \tag{F1}$$

Proof Using the notations from Lemma 2.1, we obtain

$$\begin{aligned} \sum_J \gamma_J \omega_J^{\frac{m+1}{2}} &= (\zeta_2 - \zeta_1)\alpha^{\frac{m+1}{2}} + (\zeta_2^{-1} - \zeta_1)\beta^{\frac{m+1}{2}} + (\zeta_2^{-1} - \zeta_1^{-1})\alpha^{-\frac{m+1}{2}} + (\zeta_2 - \zeta_1^{-1})\beta^{-\frac{m+1}{2}} \\ &= (\zeta_2 - \zeta_1)\left(\alpha^{\frac{m+1}{2}} - \alpha^{-\frac{m+1}{2}-1}\right) + (\zeta_2^{-1} - \zeta_1)\left(\beta^{\frac{m+1}{2}} - \beta^{-\frac{m+1}{2}-1}\right). \end{aligned}$$

Remark that

$$\begin{aligned} \frac{(\zeta_2 - \zeta_1)}{(\zeta_2^{-1} - \zeta_1)} &= \frac{\zeta_2\zeta_1^{-1} - 1}{\zeta_1^{-1}\zeta_2^{-1} - 1} \\ &= \frac{\beta^{-1} - 1}{\alpha^{-1} - 1} \\ &= \frac{\alpha}{\alpha - 1} \times \frac{\beta - 1}{\beta}, \end{aligned}$$

and hence

$$\begin{aligned} \sum_J \gamma_J \omega_J^{\frac{m+1}{2}} &= (\zeta_2^{-1} - \zeta_1)\left[\frac{(\zeta_2 - \zeta_1)}{(\zeta_2^{-1} - \zeta_1)}\left(\alpha^{\frac{m+1}{2}} - \alpha^{-\frac{m+1}{2}-1}\right) + \left(\beta^{\frac{m+1}{2}} - \beta^{-\frac{m+1}{2}-1}\right)\right] \\ &= (\zeta_2^{-1} - \zeta_1)\left[\frac{\alpha}{\alpha - 1} \times \frac{\beta - 1}{\beta}\left(\alpha^{\frac{m+1}{2}} - \alpha^{-\frac{m+1}{2}-1}\right) + \left(\beta^{\frac{m+1}{2}} - \beta^{-\frac{m+1}{2}-1}\right)\right] \\ &= \frac{\left(\zeta_2^{-1} - \zeta_1\right)(\beta - 1)}{\beta}\left(\frac{\alpha^{\frac{m+1}{2}+1} - \alpha^{-\frac{m+1}{2}}}{\alpha - 1} + \frac{\beta^{\frac{m+1}{2}+1} - \beta^{-\frac{m+1}{2}}}{\beta - 1}\right). \end{aligned}$$

On the other hand

$$\begin{aligned} \frac{\alpha^{\frac{m+1}{2}+1} - \alpha^{-\frac{m+1}{2}}}{\alpha - 1} &= \frac{\alpha^{\frac{m+2}{2}} - \alpha^{-\frac{m+2}{2}}}{\alpha^{1/2} - \alpha^{-1/2}} \\ &= U_{m+1}\left(\frac{1}{2}\left(\alpha^{1/2} + \alpha^{-1/2}\right)\right) \\ &= U_{m+1}\left(\sqrt{\frac{1+u}{2}}\right). \end{aligned}$$

Similarly, we obtain that

$$\frac{\beta^{\frac{m+1}{2}+1} - \beta^{-\frac{m+1}{2}}}{\beta - 1} = U_{m+1}\left(\sqrt{\frac{1+v}{2}}\right).$$

Consequently

$$\sum_J \gamma_J \omega_J^{n+1} = \frac{(\zeta_2^{-1} - \zeta_1)(\beta - 1)}{\beta} \left(U_{m+1}\left(\sqrt{\frac{1+u}{2}}\right) + U_{m+1}\left(\sqrt{\frac{1+v}{2}}\right)\right).$$

By the same method, we get

$$\sum_J (-1)^{|J|} \gamma_J \omega_J^{n+1} = \frac{(\zeta_2^{-1} - \zeta_1)\,(\beta - 1)}{\beta} \left(U_{m+1}\left(\sqrt{\frac{1+u}{2}}\right) - U_{m+1}\left(\sqrt{\frac{1+v}{2}}\right)\right).$$

Finally

$$\begin{aligned}\det\left(\mathbf{P}_m\,(a,b,1)\right) &= \frac{1}{d^2 \prod_{k=1}^{2}\left(\zeta_k - \zeta_k^{-1}\right)} \left(\sum_J (-1)^{|J|}\, \gamma_J \omega_J^{\frac{m+1}{2}}\right) \times \left(\sum_J \gamma_J \omega_J^{\frac{m+1}{2}}\right) \\ &= C\left(U_{m+1}^2\left(\sqrt{\frac{1+u}{2}}\right) - U_{m+1}^2\left(\sqrt{\frac{1+v}{2}}\right)\right),\end{aligned}$$

where

$$C = \frac{\left(\zeta_2^{-1} - \zeta_1\right)^2 (\beta - 1)^2}{\beta^2 d^2 \prod_{k=1}^{2}\left(\zeta_k - \zeta_k^{-1}\right)}.$$

A straightforward computation (using the Maple software for example) shows that

$$C = \frac{1}{2\,(u - v)},$$

and this completes the proof of the Lemma. □

Remark 2.3 We have $u + v + 2 = \frac{a}{2} + 1$ and $(u + 1)\,(v + 1) = \left(\frac{b}{2}\right)^2$. Then, $u + 1$ and $v + 1$ are the zeros of the second-order equation $x^2 - \left(\frac{a}{2} + 1\right) x + \left(\frac{b}{2}\right)^2 = 0$. This gives for example

$$u + 1 = \frac{1}{2}\left(\frac{a}{2} + 1 - \sqrt{\left(\frac{a}{2} + 1\right)^2 - b^2}\right),$$

and

$$v + 1 = \frac{1}{2}\left(\frac{a}{2} + 1 + \sqrt{\left(\frac{a}{2} + 1\right)^2 - b^2}\right).$$

The term $\frac{U_{m+1}^2\left(\sqrt{\frac{1+u}{2}}\right) - U_{m+1}^2\left(\sqrt{\frac{1+v}{2}}\right)}{2(u-v)}$ is a symmetric polynomial of $u + 1$ and $v + 1$ and, consequently, it can be expressed in terms of the elementary symmetric polynomials $u + 1 + v + 1 = \frac{a}{2} + 1$ and $(u + 1)\,(v + 1) = \left(\frac{b}{2}\right)^2$. For this, we distinguish two cases:

Case 1: $m = 2n + 1$. Using the following expression of $U_{2n+2}\,(x)$[3]:

$$\begin{aligned}U_{2n+2}\,(x) &= \sum_{k=0}^{n+1} (-1)^k \binom{2n+2-k}{k} (2x)^{2n+2-2k} \\ &= (-1)^{n+1} \sum_{k=0}^{n+1} \gamma_{n,k}\,(2x)^{2k}, \quad \gamma_{n,k} = (-1)^k \binom{n+1+k}{n+1-k},\end{aligned}$$

we obtain

$$U_{2n+2}\left(\sqrt{\frac{1+u}{2}}\right)+U_{2n+2}\left(\sqrt{\frac{1+v}{2}}\right)=(-1)^{n+1}\sum_{k=0}^{n+1}\gamma_{n,k}2^k\left((1+u)^k+(1+v)^k\right),$$
$$U_{2n+2}\left(\sqrt{\frac{1+u}{2}}\right)-U_{2n+2}\left(\sqrt{\frac{1+v}{2}}\right)=(-1)^{n+1}\sum_{k=1}^{n+1}\gamma_{n,k}2^k\left((1+u)^k-(1+v)^k\right).$$

On the other hand, we have for x, y :

$$\begin{aligned}x^{2k}+y^{2k}&=(xy)^k\left(\left(\frac{x}{y}\right)^k+\left(\frac{x}{y}\right)^{-k}\right)\\&=2\,(xy)^k\,T_k\left(\frac{x}{2y}+\frac{y}{2x}\right),\end{aligned}$$

and for $k\geq 1$

$$\begin{aligned}x^{2k}-y^{2k}&=(xy)^k\left(\left(\frac{x}{y}\right)^k-\left(\frac{x}{y}\right)^{-k}\right)\\&=(xy)^k\left(\frac{x}{y}-\frac{y}{x}\right)U_{k-1}\left(\frac{x}{2y}+\frac{y}{2x}\right)\\&=\left(x^2-y^2\right)(xy)^{k-1}\,U_{k-1}\left(\frac{x}{2y}+\frac{y}{2x}\right).\end{aligned}$$

Applying those formulae for $x=\sqrt{1+u}$ and $y=\sqrt{1+v}$ where

$$xy=\frac{b}{2},\quad x^2-y^2=u-v,$$

and

$$\frac{x}{2y}+\frac{y}{2x}=\frac{x^2+y^2}{2xy}=\frac{2+u+v}{b}=\frac{a+2}{2b},$$

gives

$$(1+u)^k+(1+v)^k=2\left(\frac{b}{2}\right)^k T_k\left(\frac{a+2}{2b}\right),$$

and for $k\geq 1$:

$$(1+u)^k-(1+v)^k=\ (u-v)\left(\frac{b}{2}\right)^{k-1}U_{k-1}\left(\frac{a+2}{2b}\right)$$

Case 2: $m=2n$. We have [3]:

$$\begin{aligned}U_{2n+1}\left(x\right)&=\sum_{k=0}^{n}(-1)^k\binom{2n+1-k}{k}(2x)^{2n+1-2k}\\&=(-1)^n\sum_{k=0}^{n}\mu_{n,k}\,(2x)^{2k+1},\ \ \mu_{n,k}=(-1)^k\binom{n+1+k}{n-k},\end{aligned}$$

and thus

$$U_{2n+1}\left(\sqrt{\frac{1+u}{2}}\right)+U_{2n+1}\left(\sqrt{\frac{1+v}{2}}\right)=(-1)^n\sum_{k=0}^{n}\mu_{n,k}2^{k+1/2}\left((1+u)^{k+1/2}+(1+v)^{k+1/2}\right),$$

$$U_{2n+1}\left(\sqrt{\frac{1+u}{2}}\right)-U_{2n+1}\left(\sqrt{\frac{1+v}{2}}\right)=(-1)^n\sum_{k=1}^{n}\mu_{n,k}2^{k+1/2}\left((1+u)^{k+1/2}-(1+v)^{k+1/2}\right).$$

As for the odd case, we have for x, y :

$$\begin{aligned}x^{2k+1}+y^{2k+1}&=(xy)^{k+1/2}\left(\left(\frac{x}{y}\right)^{k+1/2}+\left(\frac{x}{y}\right)^{-k-1/2}\right)\\&=(xy)^{k+1/2}\left(\left(\frac{x}{y}\right)^{1/2}+\left(\frac{x}{y}\right)^{-1/2}\right)V_k\left(\frac{x}{2y}+\frac{y}{2x}\right),\end{aligned}$$

and

$$\begin{aligned}x^{2k+1}-y^{2k+1}&=(xy)^{k+1/2}\left(\left(\frac{x}{y}\right)^{k+1/2}-\left(\frac{x}{y}\right)^{-k-1/2}\right)\\&=(xy)^{k+1/2}\left(\left(\frac{x}{y}\right)^{1/2}-\left(\frac{x}{y}\right)^{-1/2}\right)W_k\left(\frac{x}{2y}+\frac{y}{2x}\right).\end{aligned}$$

This implies

$$(1+u)^{k+1/2}+(1+v)^{k+1/2}=\left(\sqrt{1+u}+\sqrt{1+v}\right)\left(\frac{b}{2}\right)^k V_k\left(\frac{a+2}{2b}\right),$$

and

$$(1+u)^{k+1/2}-(1+v)^{k+1/2}=\left(\sqrt{1+u}-\sqrt{1+v}\right)\left(\frac{b}{2}\right)^k W_k\left(\frac{a+2}{2b}\right),$$

which completes the proof of Theorem 1.1.

3 Numerical computation of det $(\mathbf{P}_m)$

In this section, we shall derive from the formulae (1) and (2) an efficient algorithm for computing det $(\mathbf{P}_m)$. We are lead to evaluate sums of the form

$$S_N=\sum_{k=0}^{N}\alpha_k P_k(x)$$

where $x=\frac{a+2c}{2b}$, and $\{P_r\}$ are polynomials that satisfy the three-term recurrence

$$P_r(x)-2xP_{r-1}(x)+P_{r-2}(x)=0. \tag{4}$$

Such sums can be computed efficiently through the following method described in [11]:

Equation (4) may be written in matrix notation as $M\mathbf{p}=\mathbf{q}$, where M is the $(N+1)\times(N+1)$ matrix

$$\begin{pmatrix} 1 & 0 & & & & 0 \\ -2x & 1 & 0 & \ddots & & \\ 1 & -2x & 1 & \ddots & & \\ 0 & \ddots & \ddots & \ddots & 0 & \\ \vdots & \ddots & 1 & -2x & 1 & 0 \\ 0 & & 0 & 1 & -2x & 1 \end{pmatrix},$$

$$\mathbf{p} = \begin{pmatrix} P_0(x) \\ P_1(x) \\ \vdots \\ \vdots \\ P_N(x) \end{pmatrix} \quad \text{and } \mathbf{q} = \begin{pmatrix} P_0(x) \\ -2xP_0(x) + P_1(x) \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Let

$$\mathbf{y}^T = (y_0, y_1, \ldots y_N)$$

be the row vector such that

$$\mathbf{y}^T M = \mathbf{u}^T = (\alpha_0, \alpha_1, \ldots \alpha_N).$$

Thus, y_k are computed by putting $y_{N+1} = y_{N+2} = 0$ and performing the three-term recurrence

$$y_k = 2xy_{k+1} - y_{k+2} + \alpha_k, \quad \text{for } k = N, \ldots, 0.$$

It follows that

$$S_N = \mathbf{u}^T\mathbf{p} = \mathbf{y}^T M\mathbf{p} = \mathbf{y}^T\mathbf{q} = y_0 P_0(x) + (P_1(x) - 2xP_0(x))\, y_1.$$

For $P_k = T_k$ and $P_k = \frac{1}{b}U_{k-1}$, with $U_{-1} = 0$, respectively, we obtain

$$\sum_{k=0}^{n+1} \gamma_{n,k} c^{n+1-k} b^k T_k(x) = y_0 - xy_1$$

and

$$\sum_{k=1}^{n+1} \gamma_{n,k} c^{n+1-k} b^{k-1} U_{k-1}(x) = \frac{1}{b} y_1,$$

where $y_{n+2} = y_{n+3} = 0$ and

$$y_k = 2xy_{k+1} - y_{k+2} + \gamma_{n,k} c^{n+1-k} b^k, \quad \text{for } k = n+1, \ldots, 0.$$

For $P_k = V_k$ and $P_k = W_k$, respectively, we obtain

$$\sum_{k=0}^{n} \mu_{n,k} c^{n-k} b^k V_k(x) = y_0 - y_1$$

and

$$\sum_{k=0}^{n} \mu_{n,k} c^{n-k} b^k W_k(x) = y_0 + y_1,$$

where $y_{n+1} = y_{n+2} = 0$ and

$$y_k = 2xy_{k+1} - y_{k+2} + \mu_{n,k} c^{n-k} b^k, \quad \text{for } k = n, \ldots, 0.$$

Here is the implementation of the algorithm in **Maple** (To accelerate the algorithm, the terms $\gamma_{n,k}c^{n+1-k}b^k$ and $\mu_{n,k}c^{n-k}b^k$ are computed recursively at the same time as y_k. Implementation details are omitted):

```
## Computing det(P_2n+1)
##
detP1:=proc(n,a,b,c)
local i,j,r,s,x,k,t,z;
i := 0;
j := 0;
r := (-1)^(n+1)*b^(n+1);
x:=(a+2*c)/b;
t:=2*n;
z:=-c/b;
for k from 0 to n+1 do
      s:=i;
      i:=r+x*i-j;             ##   i:=simplify(r+x*i-j); if the purpose
                             ##  is to compute the characteristic
                             ##  polynomial with variable a
      j:=s;
      r:=r*z*((t+2 )*(t+1 ))/((t+k+2 )*(k+1 ));
      t:=t-2;
 od;
 return 2*j*(i-(j*x/2)/b); ##  return simplify(2*j*(i-(j*x/2)/b);
                              ## if the purpose is to compute the
                              ##  characteristic polynomial with
                              ##  variable a
 end;

## Computing det(P_2n)
##
detP2:=proc(n,a,b,c)
local i,j,r,s,x,k;
i := 0;
j := 0;
r := (-1)^n*b^n;
x:=(a+2*c)/b;
t:=2*n;
z:=-c/b;
for k from 0 to n do
      s:=i;
      i:=r+x*i-j;       ##   i:=simplify(r+x*i-j); if the purpose
                       ##   is to compute the characteristic
                        ##   polynomial with variable a
      j:=s;
      r:=r*z*((t+1 )*t))/(t+k+1 )*(k+1 ));
      t:=t-2;
 od;
 return i^(2)-j^(2);   ##   return simplify(i^(2)-j^(2));
                          ## if the purpose is to compute
                          ## the characteristic  polynomial
                          ## with variable a
 end;
```

One can easily check that the complexity of the algorithm is about $7N$, where N is the size of the matrix. Thus, the algorithm is the fastest among many other recently proposed (we exclude those based on the roots of certain polynomials which are approximative) [10]. Moreover, subject to minor modifications as explained in Algorithm 1, the algorithm is suitable for computing the characteristic polynomial of a symmetric pentadiagonal Toeplitz matrix using computer algebra systems such as MAPLE, MATHEMATICA, MATLAB and MACSYMA.

References

1. Barrera, M.; Grudsky, S.M.: Asymptotics of eigenvalues for pentadiagonal symmetric Toeplitz matrices. Oper. Theory Adv. Appl. **259**, 179–212 (2017)
2. Chu, M.T.; Diele, F.; Ragnion, S.: On the inverse problem of constructing symmetric pentadiagonal Toeplitz matrices from three largest eigenvalues. Inverse Probl. **21**, 1879–1894 (2005)
3. Doman, B.G.S.: The Classical Orthogonal Polynomials. World Scientific Publishing Company, Singapore (2015)
4. Elouafi, M.: A widom like formula for some Toeplitz plus Hankel determinants. J. Math. Anal. Appl. **422**(1), 240–249 (2015)
5. Elouafi, M.: An eigenvalue localization theorem for pentadiagonal symmetric Toeplitz matrices. Linear Algebra Appl. **435**, 2986–2998 (2011)
6. Elouafi, M.: A note for an explicit formula for the determinant of pentadiagonal and heptadiagonal symmetric Toeplitz matrices. Appl. Math. Comput. **219**(9), 4789–4791 (2013)
7. Elouafi, M.: On a relationship between Chebyshev polynomials and Toeplitz determinants. Appl. Math. Comput. **229**(25), 27–33 (2014)
8. Fasino, D.: Spectral and structural properties of some pentadiagonal symmetric matrices. Calcolo **25**, 301–310 (1988)
9. Grenander, U.; Szeg ö, G.: Toeplitz Forms and their Applications. Chelsea, New York (1984)
10. Jia, J.T.; Yang, B.T.; Li, S.M.: On a homogeneous recurrence relation for the determinants of general pentadiagonal Toeplitz matrices. Comput. Math. Appl. **71**, 1036–1044 (2016)
11. Mason, J.C.; Handscomb, D.: Chebyshev Polynomials. Chapman & Hall, New York (2003)

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.