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Coupled nonlinear Schrödinger equations with harmonic potential

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Abstract The initial value problem for a coupled nonlinear Schrödinger system with unbounded potential is investigated. In the defocusing case, global well-posedness is obtained. In the focusing case, the existence and stability/instability of standing waves are established. Moreover, global well-posedness is discussed via the potential well method.

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المخلص

نقوم بالبحث في مسألة القيمة الابتدائية لنظام شرودينجر متزاوج وغير خطي وذو جهد (كمون) غير محدود. في حالة عدم التركيز، نحصل على صياغة جيدة شاملة. وفي حالة التركيز، نؤسس لوجود واستقرار أو عدم استقرار الموجات الدائمة. بالإضافة، فإننا نناقش الصياغة الجيدة الشاملة من خلال طريقة بئر الكمون.

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1 Introduction

Consider the initial value problem for a Schrödinger system with power-type nonlinearities

$$\begin{cases} i\dot{u}_j + \Delta u_j - |x|^2 u_j - \mu \left(\sum_{k=1}^m a_{jk} |u_k|^p \right) |u_j|^{p-2} u_j = 0; \\ u_j(0, \cdot) = \psi_j, \end{cases} \quad (1.1)$$

where $u_j : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{C}$ for some $N \geq 2$, $j \in [1, m]$, $\mu = \pm 1$ and $a_{jk} = a_{kj}$ are positive real numbers.

The nonlinear m -component coupled nonlinear Schrödinger system

$$(CNLS)_p \quad i\dot{u}_j + \Delta u_j = \pm \left(\sum_{k=1}^m a_{jk} |u_k|^p \right) |u_j|^{p-2} u_j, \quad j \in [1, m],$$

arises in many physical problems such as nonlinear optics and Bose–Einstein condensates. It models physical systems in which the field has more than one component. In nonlinear optics [2], u_j denotes the j th component of the beam in Kerr-like photo-refractive media. The coupling constant a_{jk} acts as the interaction between the j th and the k th components of the beam. This system arises also in the Hartree–Fock theory for a two component Bose–Einstein condensate. Readers are referred, for instance, to [14, 30] for the derivation and applications of this system.

Well-posedness issues in the energy space of $(CNLS)_p$ were recently investigated by many authors [19, 25, 26]. A solution $\mathbf{u} := (u_1, \dots, u_m)$ to (1.1) formally satisfies, respectively, conservation of the mass and the energy

$$\begin{aligned} M(u_j) &:= \int_{\mathbb{R}^N} |u_j(x, t)|^2 dx = M(\psi_j); \\ E(\mathbf{u}(t)) &:= \frac{1}{2} \sum_{j=1}^m \int_{\mathbb{R}^N} \left(|\nabla u_j(t)|^2 + |x u_j(t)|^2 + \frac{\mu}{p} \sum_{k=1}^m a_{jk} |u_j(t) u_k(t)|^p \right) dx = E(\mathbf{u}(0)). \end{aligned}$$

If $\mu = 1$, the energy is always positive and (1.1) is said to be defocusing, otherwise a control of the Sobolev norm of a solution with the energy is no longer possible and a local solution may blow-up in finite time, in such a case (1.1) is focusing.

Before going further let us recall some historic facts about this problem. The one component model case given by a pure power nonlinearity is of particular interest. The question of well-posedness in the energy space was widely investigated. Denote for $p > 1$ the Schrödinger problem

$$(NLS)_p \quad i\dot{u} + \Delta u - |x|^2 u \pm u|u|^{p-1} = 0, \quad u : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{C}.$$

For $1 < p < \frac{N+2}{N-2}$ if $N \geq 3$ and $1 < p < \infty$ if $N \in \{1, 2\}$, local well-posedness holds in the energy space [9, 21]. When $p < 1 + \frac{4}{N}$ or $p \geq 1 + \frac{4}{N}$ with a defocusing nonlinearity, the solution exists globally [6]. For $p = 1 + \frac{4}{N}$, there exists a sharp condition [31] to the global existence; moreover, the standing waves are stable under some sufficient conditions [12]. When $p > 1 + \frac{4}{N}$, the solution blows up in a finite time for a class of sufficiently large data and globally exists for a class of sufficiently small data [7, 8, 28]; moreover, the standing waves are unstable under suitable assumptions [13].

In two space dimensions, similar results about global well-posedness and instability of the Schrödinger equation with harmonic potential and exponential nonlinearity exist [23].

Intensive work has been done in the last few years about coupled Schrödinger systems [18, 19, 24, 29]. These works have been mainly on 2-systems or with small couplings. Moreover, most works treat the focusing case by considering the stationary associated problem [3–5, 15, 27]. Despite the partial progress made so far, many difficult questions remain open and little is known about m -systems for $m \geq 3$.

In this note, we combine in some meaning the two problems $(NLS)_p$ and $(CNLS)_p$. Thus, we have to overcome two difficulties. The first one is the presence of a potential term and the second is the existence of coupled nonlinearities.

The purpose of this manuscript is twofold. First, global well-posedness of (1.1) is obtained in the defocusing case. Second, in the focusing case, the existence of ground states and the stability/instability of standing waves are obtained; moreover, using the potential well method [22], the global existence of solutions is discussed.

The rest of the paper is organized as follows. The next section contains the main results and some technical tools needed in the sequel. The third and fourth sections are devoted to prove well-posedness of (1.1). In section five, the existence of ground states is established. The sixth section contains a discussion of global existence of solutions via the potential well method. The last section is devoted to obtain stability/instability of standing waves. Finally, a proof of the Virial identity is given in the appendix.

Denoting $H^1(\mathbb{R}^N)$ the usual Sobolev space, define the conformal space

$$\Sigma := \left\{ u \in H^1(\mathbb{R}^N) \text{ s.t. } \int_{\mathbb{R}^N} |x|^2 |u(x)|^2 dx < \infty \right\}$$

endowed with the complete norm

$$\|u\|_{\Sigma} := \left(\|u\|_{L^2(\mathbb{R}^N)}^2 + \|xu\|_{L^2(\mathbb{R}^N)}^2 + \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 \right)^{\frac{1}{2}}$$

and the product space

$$H := \Sigma \times \cdots \times \Sigma = [\Sigma]^m.$$

Denote the real numbers called, respectively, mass critical and energy critical exponents

$$p_* := 1 + \frac{2}{N} \quad \text{and} \quad p^* := \begin{cases} \frac{N}{N-2} & \text{if } N > 2; \\ \infty & \text{if } N = 2. \end{cases}$$

We mention that C will denote a constant which may vary from line to line and if A and B are non-negative real numbers, $A \lesssim B$ means that $A \leq CB$. For $1 \leq r \leq \infty$ and $(s, T) \in [1, \infty) \times (0, \infty)$, denote the Lebesgue space $L^r := L^r(\mathbb{R}^N)$ with the usual norm $\|\cdot\|_r := \|\cdot\|_{L^r}$, $\|\cdot\| := \|\cdot\|_2$ and

$$\|u\|_{L_T^s(L^r)} := \left(\int_0^T \|u(t)\|_r^s dt \right)^{\frac{1}{s}}, \quad \|u\|_{L^s(L^r)} := \left(\int_0^{+\infty} \|u(t)\|_r^s dt \right)^{\frac{1}{s}}.$$

For simplicity, denote the usual Sobolev Space $W^{s,p} := W^{s,p}(\mathbb{R}^N)$ and $H^s := W^{s,2}$. If X is an abstract space $C_T(X) := C([0, T], X)$ stands for the set of continuous functions valued in X and X_{rd} is the set of radial elements in X ; moreover, for an eventual solution to (1.1), $T^* > 0$ denotes its lifespan.

2 Main results and background

In what follows, we give the main results and some estimates needed in the sequel.

2.1 Main results

First, local well-posedness of the Schrödinger problem (1.1) is claimed.

Theorem 2.1 *Let $2 \leq N \leq 4$ and $\Psi \in H$. Assume that $2 \leq p \leq p^*$ if $N > 2$ and $2 \leq p < p^*$ if $N = 2$. Then, there exist $T^* > 0$ and a unique maximal solution to (1.1),*

$$\mathbf{u} \in C([0, T^*), H).$$

Moreover,

- (1) $\mathbf{u} \in \left(L^{\frac{4p}{N(p-1)}}([0, T^*], W^{1,2p}) \right)^{(m)}$;
- (2) $\mathbf{u}$ satisfies conservation of the energy and the mass;
- (3) $T^* = \infty$ in the defocusing subcritical case ($\mu = 1$, $2 \leq p < p^*$).

Remark 2.2 The unnatural condition $p \geq 2$ seems to be technical and yields to the restriction $N \leq 4$.

In the critical case, global existence for small data holds in the energy space.

Theorem 2.3 *Let $3 \leq N \leq 4$ and $p = p^*$. There exists $\epsilon_0 > 0$ such that if $\Psi := (\psi_1, \dots, \psi_m) \in H$ satisfies $\sum_{j=1}^m \int_{\mathbb{R}^N} (|\nabla \psi_j|^2 + |x\psi_j|^2) dx \leq \epsilon_0$; then, the system (1.1) possesses a unique global solution $\mathbf{u} \in C(\mathbb{R}, H)$.*

Now, we are interested on the focusing problem (1.1). For $\mathbf{u} := (u_1, \dots, u_m) \in H$, define the action

$$S(\mathbf{u}) := \frac{1}{2} \sum_{j=1}^m \|u_j\|_{\Sigma}^2 - \frac{1}{2p} \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |u_j u_k|^p dx.$$

If $\alpha, \beta \in \mathbb{R}$, the following quantity is called constraint

$$\begin{aligned} 2K_{\alpha,\beta}(\mathbf{u}) := & \sum_{j=1}^m ((2\alpha + (N-2)\beta) \|\nabla u_j\|^2 + (2\alpha + N\beta) \|u_j\|^2 + (2\alpha + \beta(N+2)) \|xu_j\|^2) \\ & - \frac{1}{p} \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} (2p\alpha + N\beta) |u_j u_k|^p dx. \end{aligned}$$

Definition 2.4 $\Psi := (\psi_1, \dots, \psi_m)$ is said to be a ground state solution to (1.1) if

$$\Delta \psi_j - \psi_j - |x|^2 \psi_j + \sum_{k=1}^m a_{jk} |\psi_k|^p |\psi_j|^{p-2} \psi_j = 0, \quad 0 \neq \Psi \in H_{rd} \quad (2.1)$$

and it minimizes the problem

$$m_{\alpha,\beta} := \inf_{0 \neq \mathbf{u} \in H} \{S(\mathbf{u}) \text{ s.t. } K_{\alpha,\beta}(\mathbf{u}) = 0\}. \quad (2.2)$$

Remark 2.5 If $\Psi \in H$ is a solution to (2.1), then $e^{it}\Psi$ is a global solution of (1.1) said standing wave.

Now, the existence of a ground state solution to (1.1) is claimed. Define the set $G_p := \{(\alpha, \beta) \in \mathbb{R}_+^* \times \mathbb{R}_+ \text{ s.t. } \alpha(p-1) > \beta\}$.

Theorem 2.6 *Take $N \geq 2$, $1 < p < p^*$ and two real numbers $(\alpha, \beta) \in G_p$. Then*

- (1) $m := m_{\alpha,\beta}$ is nonzero and independent of (α, β) ;
- (2) there is a minimizer of (2.2), which is some nontrivial solution to (2.1).

Now, the global existence of a solution to the focusing problem (1.1) is discussed using the potential well method [22].

Theorem 2.7 *Take $2 \leq N \leq 4$ and $1 < p < p^*$. Let $\Psi := (\psi_1, \dots, \psi_m) \in H$ and $\mathbf{u} \in C_{T^*}(H)$ the maximal solution to (1.1). If there exist $(\alpha, \beta) \in G_p$ and $t_0 \in [0, T^*)$ such that $\mathbf{u}(t_0) \in A_{\alpha,\beta}^+ := \{\mathbf{u} \in H \text{ s.t. } S(\mathbf{u}) < m \text{ and } K_{\alpha,\beta}(\mathbf{u}) \geq 0\}$, then $\mathbf{u}$ is global.*

The last result concerns stability for standing waves.

Definition 2.8 For $\varepsilon > 0$ and $\Psi \in H$, define

(1) the set

$$V_\varepsilon(\Psi) := \left\{ \mathbf{v} \in H, \quad \text{s. t.} \quad \inf_{t \in \mathbb{R}} \|\mathbf{v} - e^{it}\Psi\|_H < \varepsilon \right\};$$

(2) if $\mathbf{u}_0 \in V_\varepsilon(\Psi)$ and $\mathbf{u}$ is the solution to (1.1) given by Theorem 2.1,

$$T_\varepsilon(\mathbf{u}_0) := \sup\{T > 0, \quad \text{s. t.} \quad \mathbf{u}(t) \in V_\varepsilon(\Psi), \quad \text{for any } t \in [0, T)\};$$

(3) $e^{it}\Psi$ is said to be orbitally stable if, for any $\sigma > 0$ there exists $\varepsilon > 0$ such that if $\mathbf{u}_0 \in V_\varepsilon(\Psi)$, then $T_\sigma(\Psi) = \infty$. Otherwise, the standing wave $e^{it}\Psi$ is said to be nonlinearly unstable;

(4) the set

$$\Pi_\varepsilon(\Psi) := \{\mathbf{v} \in V_\varepsilon(\Psi), \quad \text{s. t.} \quad E(\mathbf{v}) < E(\Psi), \quad \|\mathbf{v}\| \leq \|\Psi\| \quad \text{and} \quad K_{1, -\frac{2}{N}}(\mathbf{v}) < 0\}.$$

In the case of coupled nonlinear Schrödinger systems, it seems that there is no result of uniqueness of ground states. So, we define a weaker stability as follows [9].

Definition 2.9 For $\mu > 0$, define

(1) the set

$$\mathcal{G}_\mu := \left\{ \mathbf{v} \in H, \quad \text{s. t.} \quad S(\mathbf{v}) = \inf_{\mathbf{u} \in H} \{S(\mathbf{u}), \quad \|\mathbf{u}\| = \mu\} \right\};$$

(2) $\mathcal{G}_\mu$ is said to be stable if, $\mathcal{G}_\mu \neq \emptyset$ and for any $\varepsilon > 0$ there exists $\sigma > 0$ such that

$$\inf_{\Psi \in \mathcal{G}_\mu} \|\mathbf{u}_0 - \Psi\|_H < \sigma \quad \Rightarrow \quad \left(\inf_{\Psi \in \mathcal{G}_\mu} \|\mathbf{u}(t) - \Psi\|_H < \varepsilon, \quad \forall t \geq 0 \right),$$

where $\mathbf{u} \in C(\mathbb{R}, H)$ is a global solution to (1.1) with data $\mathbf{u}_0 \in H$.

Theorem 2.10 Take $2 \leq N \leq 4$, $1 < p < p^*$ and Ψ be a ground state solution to (2.1). Then

(1) if $p < p_*$, so $\mathcal{G}_\mu$ is stable for any $\mu > 0$;

(2) if

$$\sum_{j=1}^m \left(4\|\nabla \psi_j\|^2 - \frac{N(p-1)(N(p-1)+2)}{2p} \sum_{k=1}^m \int a_{jk} |\psi_j \psi_k|^p dx \right) < 0,$$

so, the standing wave $e^{it}\Psi$ is nonlinearly unstable.

In what follows, we collect some intermediate estimates.

2.2 Tools

First, let us recall some known results [6, 10, 11] about the free propagator associated with (1.1).

Proposition 2.11 There exists a family of operators $U := U(t, s)$, $U(t) := U(t, 0)$ such that $u(t, x) := U(t, s)\phi(x)$ is solution to the linear problem

$$i\dot{u} + \Delta u = |x|^2 u, \quad u(s, \cdot) = \phi.$$

Moreover, we have the following elementary properties:

- (1) $U(t, t) = Id$;
- (2) $(t, s) \mapsto U(t, s)$ is continuous;
- (3) $U(t, s)^* = U(t, s)^{-1}$;

- (4) $U(t, \tau)U(\tau, s) = U(t, s)$;
 (5) $U(t, s)$ is unitary of L^2 .

Thanks to Duhamel formula, it yields

Proposition 2.12 *If u is a solution to the inhomogeneous Schrödinger problem*

$$i\dot{u} + \Delta u - |x|^2 u = h, \quad u(0, \cdot) = 0,$$

then

- (1) $u(t) = -i \int_0^t U(t-s)h(s, x)ds$;
 (2) $\nabla u(t) = -i \int_0^t U(t-s)[\nabla h + 2xu]ds$;
 (3) $xu(t) = -i \int_0^t U(t-s)[xh + 2\nabla u]ds$.

Remark 2.13 Taking the derivative of the equation satisfied by u , we obtain the second point. For the last one, it is sufficient to multiply the same equation with x .

A standard tool to study Schrödinger problems is the so-called Strichartz type estimate.

Definition 2.14 A pair (q, r) of positive real numbers is admissible if

$$2 \leq r < \infty \quad \text{and} \quad N \left(\frac{1}{2} - \frac{1}{r} \right) = \frac{2}{q}.$$

In order to control an eventual solution to (1.1), the following Strichartz estimate [6] will be useful.

Proposition 2.15 *Take two admissible pairs (q, r) and (α, β) . Then, for any time slab I ,*

- (1) $\|U(t)\phi\|_{L^q(I, L^r)} \leq C_q \|\phi\|, \quad \forall \phi \in L^2$;
 (2) $\|\int_0^t U(t-s)h(s, x)ds\|_{L^q(I, L^r)} \leq C_{\alpha, |I|} \|h\|_{L^{\alpha'}(I, L^{\beta'})}, \quad \forall h \in L^{\alpha'}(I, L^{\beta'}).$

Any solution to (1.1) formally enjoys the so-called Virial identity.

Proposition 2.16 *Let $\mathbf{u} := (u_1, \dots, u_m) \in H$, a solution to (1.1) such that $x\mathbf{u} \in L^2$. Then,*

$$\frac{1}{8} \left(\sum_{j=1}^m \|xu_j(t)\|^2 \right)'' = \sum_{j=1}^m (\|\nabla u_j\|^2 - \|xu_j\|^2) - \frac{N(p-1)}{2p} \sum_{j,k=1}^m a_{j,k} \int_{\mathbb{R}^N} |u_j u_k|^p dx. \quad (2.3)$$

For the reader convenience, a proof of the Virial identity is given in the Appendix. Recall the so-called generalized Pohozaev identity [16].

Proposition 2.17 $\Psi \in H$ is a solution to (2.1) if and only if $S'(\Psi) = 0$. Moreover, in such a case

$$K_{\alpha, \beta}(\Psi) = 0, \quad \text{for any } (\alpha, \beta) \in \mathbb{R}^2.$$

The following Gagliardo–Nirenberg inequality [20] will be useful.

Proposition 2.18 *Take $1 < p \leq p^*$. Then, for any $(u_1, \dots, u_m) \in H$,*

$$\sum_{j,k=1}^m \int_{\mathbb{R}^N} |u_j u_k|^p dx \leq C \left(\sum_{j=1}^m \|\nabla u_j\|^2 \right)^{\frac{(p-1)N}{2}} \left(\sum_{j=1}^m \|u_j\|^2 \right)^{\frac{N-p(N-2)}{2}}. \quad (2.4)$$

Let us list some Sobolev embeddings [1, 17].

Proposition 2.19

- (1) $W^{s,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ whenever $1 < p < q < \infty$, $s > 0$ and $\frac{1}{p} \leq \frac{1}{q} + \frac{s}{N}$;
 (2) for $2 < p < 2p^*$,

$$H_{rd}^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N); \quad (2.5)$$

(3) for $2 \leq p < 2p^*$,

$$\Sigma(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N); \quad (2.6)$$

(4) if $xu \in L^2$ and $\nabla u \in L^2$, then $u \in L^2$ and

$$\|u\|^2 \leq \frac{2}{N} \|xu\| \|\nabla u\|.$$

Remark 2.20 Using the previous inequality, we get $\|u\|_\Sigma \simeq \|xu\| + \|\nabla u\|$.

Let us close this subsection with some absorption result.

Lemma 2.21 Let $T > 0$ and $X \in C([0, T], \mathbb{R}_+)$ such that

$$X \leq a + bX^\theta \quad \text{on } [0, T],$$

where $a, b > 0$, $\theta > 1$, $a < (1 - \frac{1}{\theta}) \frac{1}{(\theta b)^{\frac{1}{\theta}}}$ and $X(0) \leq \frac{1}{(\theta b)^{\frac{1}{\theta-1}}}$. Then

$$X \leq \frac{\theta}{\theta-1} a \quad \text{on } [0, T].$$

Proof The function $f(x) := bx^\theta - x + a$ is decreasing on $[0, (b\theta)^{\frac{1}{1-\theta}}]$ and increasing on $[(b\theta)^{\frac{1}{1-\theta}}, \infty)$. The assumptions imply that $f((b\theta)^{\frac{1}{1-\theta}}) < 0$ and $f(\frac{\theta}{\theta-1}a) \leq 0$. As $f(X(t)) \geq 0$, $f(0) > 0$ and $X(0) \leq (b\theta)^{\frac{1}{1-\theta}}$, we conclude the proof by a continuity argument. $\square$

3 Local well-posedness

This section is devoted to prove Theorem 2.1. The proof contains three steps. First, the existence of a local solution to (1.1) is obtained using a classical fixed point method, second we show uniqueness and finally global existence in the subcritical case is established. In this section, the nonlinearity is assumed to be defocusing ($\mu = 1$), indeed the sign of the nonlinearity has no local effect.

3.1 Local existence

Let us discuss two cases.

• **Subcritical case:** $2 \leq p < p^*$. For $T > 0$ and $\frac{R}{2} := C\|\Psi\|_H$, we denote $B_T(R)$ the centered ball with radius R of the space

$$E_T := \left\{ \mathbf{u} \in C([0, T], H) \quad \text{s.t.} \quad \mathbf{u}, \nabla \mathbf{u}, x\mathbf{u} \in L^{\frac{4p}{N(p-1)}}([0, T], L^{2p}) \right\}$$

endowed with the complete distance

$$d(\mathbf{u}, \mathbf{v}) := \sum_{j=1}^m \|u_j - v_j\|_{L_T^\infty(L^2) \cap L_T^{\frac{4p}{N(p-1)}}(L^{2p})} := \sum_{j=1}^m \|u_j - v_j\|_T.$$

Define, for $\mathbf{u} := (u_1, \dots, u_m)$, the function

$$\phi(\mathbf{u})(t) := T(t)\Psi - i \sum_{k=1}^m \int_0^t T(t-s) (a_{1k}|u_k|^p |u_1|^{p-2} u_1, \dots, a_{mk}|u_k|^p |u_m|^{p-2} u_m) \, ds,$$

where $T(t)\Psi := (U(t)\psi_1, \dots, U(t)\psi_m)$. We prove the existence of some small $T, R > 0$ such that ϕ is a contraction of the ball $B_T(R)$. Take $\mathbf{u}, \mathbf{v} \in B_T(R)$, using Strichartz estimate, we have

$$d(\phi(\mathbf{u}), \phi(\mathbf{v})) \lesssim \sum_{j,k=1}^m \| |u_k|^p |u_j|^{p-2} u_j - |v_k|^p |v_j|^{p-2} v_j \|_{L_T^{\frac{4p}{p(4-N)+N}}(L^{\frac{2p}{2p-1}})}.$$

To derive the contraction, consider the function

$$f_{j,k} : \mathbb{C}^m \rightarrow \mathbb{C}, \quad (u_1, \dots, u_m) \mapsto |u_k|^p |u_j|^{p-2} u_j.$$

With the mean value theorem, via the fact that $p \geq 2$, it follows that

$$|f_{j,k}(\mathbf{u}) - f_{j,k}(\mathbf{v})| \lesssim \max\{|u_k|^{p-1} |u_j|^{p-1} + |u_k|^p |u_j|^{p-2}, |v_k|^p |v_j|^{p-2} + |v_k|^{p-1} |v_j|^{p-1}\} |\mathbf{u} - \mathbf{v}|.$$

Using Hölder inequality and Sobolev embedding, compute via a symmetry argument

$$\begin{aligned} (\mathcal{I}) &:= \|f_{j,k}(\mathbf{u}) - f_{j,k}(\mathbf{v})\|_{L_T^{\frac{4p}{p(4-N)+N}} \left(L^{\frac{2p}{2p-1}} \right)} \\ &\lesssim \|(|u_k|^{p-1} |u_j|^{p-1} + |u_k|^p |u_j|^{p-2}) |\mathbf{u} - \mathbf{v}|\|_{L_T^{\frac{4p}{p(4-N)+N}} \left(L^{\frac{2p}{2p-1}} \right)} \\ &\lesssim T^{\frac{2p-N(p-1)}{2p}} \|\mathbf{u} - \mathbf{v}\|_{L_T^{\frac{4p}{N(p-1)}} (L^{2p})} \left(\|u_k\|_{L_T^\infty (L^{2p})}^{p-1} \|u_j\|_{L_T^\infty (L^{2p})}^{p-1} + \|u_k\|_{L_T^\infty (L^{2p})}^p \|u_j\|_{L_T^\infty (L^{2p})}^{p-2} \right) \\ &\lesssim T^{\frac{2p-N(p-1)}{2p}} \|\mathbf{u} - \mathbf{v}\|_{L_T^{\frac{4p}{N(p-1)}} (L^{2p})} \left(\|u_k\|_{L_T^\infty (H^1)}^{p-1} \|u_j\|_{L_T^\infty (H^1)}^{p-1} + \|u_k\|_{L_T^\infty (H^1)}^p \|u_j\|_{L_T^\infty (H^1)}^{p-2} \right). \end{aligned}$$

Then,

$$d(\phi(\mathbf{u}), \phi(\mathbf{v})) \lesssim T^{\frac{2p-N(p-1)}{2p}} R^{2(p-1)} d(\mathbf{u}, \mathbf{v}). \quad (3.1)$$

Now, using Propositions 2.12–2.15, it follows that

$$\begin{aligned} \|\phi(\mathbf{u})\|_T &\leq C \left(\|\Psi\| + \|f_{j,k}(\mathbf{u})\|_{L_T^{\frac{4p}{p(4-N)+N}} \left(L^{\frac{2p}{2p-1}} \right)} \right); \\ \|\nabla(\phi(\mathbf{u}))\|_T &\leq C \left(\|\nabla\Psi\| + \|\nabla(f_{j,k}(\mathbf{u}))\|_{L_T^{\frac{4p}{p(4-N)+N}} \left(L^{\frac{2p}{2p-1}} \right)} + T \|x\phi(\mathbf{u})\|_{L_T^\infty (L^2)} \right); \\ \|x\phi(\mathbf{u})\|_T &\leq C \left(\|x\Psi\| + \|xf_{j,k}(\mathbf{u})\|_{L_T^{\frac{4p}{p(4-N)+N}} \left(L^{\frac{2p}{2p-1}} \right)} + T \|\nabla(\phi(\mathbf{u}))\|_{L_T^\infty (L^2)} \right). \end{aligned}$$

Thus, for small $T > 0$, we get

$$\begin{aligned} \|\phi(\mathbf{u})\|_T + \|\nabla(\phi(\mathbf{u}))\|_T + \|x\phi(\mathbf{u})\|_T &\leq C \left(\|\Psi\|_H + \|f_{j,k}(\mathbf{u})\|_{L_T^{\frac{4p}{p(4-N)+N}} \left(L^{\frac{2p}{2p-1}} \right)} \right. \\ &\quad \left. + \|\nabla(f_{j,k}(\mathbf{u}))\|_{L_T^{\frac{4p}{p(4-N)+N}} \left(L^{\frac{2p}{2p-1}} \right)} + \|xf_{j,k}(\mathbf{u})\|_{L_T^{\frac{4p}{p(4-N)+N}} \left(L^{\frac{2p}{2p-1}} \right)} \right) \\ &\leq \frac{R}{2} + C \left(\|f_{j,k}(\mathbf{u})\|_{L_T^{\frac{4p}{p(4-N)+N}} \left(L^{\frac{2p}{2p-1}} \right)} \right. \\ &\quad \left. + \|\nabla(f_{j,k}(\mathbf{u}))\|_{L_T^{\frac{4p}{p(4-N)+N}} \left(L^{\frac{2p}{2p-1}} \right)} + \|xf_{j,k}(\mathbf{u})\|_{L_T^{\frac{4p}{p(4-N)+N}} \left(L^{\frac{2p}{2p-1}} \right)} \right) \end{aligned}$$

Thanks to Hölder inequality and Sobolev embedding, we obtain

$$\begin{aligned} (\mathcal{I}) &:= \|f_{j,k}(\mathbf{u})\|_{L_T^{\frac{4p}{p(4-N)+N}} (W^1, \frac{2p}{2p-1})} \\ &\lesssim T^{\frac{2p-N(p-1)}{2p}} R \left(\|u_k\|_{L_T^\infty (L^{2p})}^{p-1} \|u_j\|_{L_T^\infty (L^{2p})}^{p-1} + \|u_k\|_{L_T^\infty (L^{2p})}^p \|u_j\|_{L_T^\infty (L^{2p})}^{p-2} \right) \\ &\lesssim T^{\frac{2p-N(p-1)}{2p}} R \left(\|u_k\|_{L_T^\infty (H^1)}^{p-1} \|u_j\|_{L_T^\infty (H^1)}^{p-1} + \|u_k\|_{L_T^\infty (H^1)}^p \|u_j\|_{L_T^\infty (H^1)}^{p-2} \right) \\ &\lesssim T^{\frac{2p-N(p-1)}{2p}} R^{2p-1}. \end{aligned} \quad (3.2)$$

Using Hölder inequality, Sobolev embedding, compute via a symmetry argument

$$\begin{aligned}
 (\mathcal{J}) &:= \|xf_{j,k}(\mathbf{u})\|_{L_T^{\frac{4p}{p(4-N)+N}}\left(L^{\frac{2p}{2p-1}}\right)} \\
 &\lesssim \|(|u_k|^{p-1}|u_j|^{p-1} + |u_k|^p|u_j|^{p-2})|x\mathbf{u}|\|_{L_T^{\frac{4p}{p(4-N)+N}}\left(L^{\frac{2p}{2p-1}}\right)} \\
 &\lesssim T^{\frac{2p-N(p-1)}{2p}}\|\mathbf{x}\mathbf{u}\|_{L_T^{\frac{4p}{N(p-1)}}(L^{2p})}\left(\|u_k\|_{L_T^\infty(L^{2p})}^{p-1}\|u_j\|_{L_T^\infty(L^{2p})}^{p-1} + \|u_k\|_{L_T^\infty(L^{2p})}^p\|u_j\|_{L_T^\infty(L^{2p})}^{p-2}\right) \\
 &\lesssim T^{\frac{2p-N(p-1)}{2p}}\|\mathbf{x}\mathbf{u}\|_{L_T^{\frac{4p}{N(p-1)}}(L^{2p})}\left(\|u_k\|_{L_T^\infty(H^1)}^{p-1}\|u_j\|_{L_T^\infty(H^1)}^{p-1} + \|u_k\|_{L_T^\infty(H^1)}^p\|u_j\|_{L_T^\infty(H^1)}^{p-2}\right) \\
 &\lesssim T^{\frac{2p-N(p-1)}{2p}}R^{2p-1}.
 \end{aligned}$$

Thanks to the previous inequality and (3.2), it yields

$$\|\phi(\mathbf{u})\|_T + \|\nabla(\phi(\mathbf{u}))\|_T + \|x\phi(\mathbf{u})\|_T \leq \frac{R}{2} + CT^{\frac{N-2}{2p}(p^*-p)}R^{2p-1}.$$

Since $p < p^*$, by (3.1) and the previous inequality, ϕ is a contraction of $B_T(R)$ for some $R, T > 0$ small enough. The existence of a local solution to (1.1) follows with a classical fixed point Picard argument.

• Critical case: $p = p^*$. Take the admissible couple $(q, r) := (\frac{2N}{N-2}, \frac{2N^2}{N^2-2N+4})$ and the centered ball with radius $R > 0$ of the space

$$F_T := \left\{ \mathbf{u}, \nabla \mathbf{u}, x\mathbf{u} \in (L_T^q(L^r))^m \right\}$$

endowed with the complete distance

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|_T, \quad \|\mathbf{u}\|_T := \sum_{j=1}^m \|u_j\|_{L_T^q(L^r)}.$$

Taking account of Strichartz estimate, Hölder inequality and Sobolev embedding, write for $\frac{1}{\rho} := \frac{1}{r} - \frac{1}{N}$,

$$\begin{aligned}
 d(\phi(\mathbf{u}), \phi(\mathbf{v})) &\lesssim \sum_{j,k=1}^m \|(|u_k|^{p^*-1}|u_j|^{p^*-1} + |u_k|^{p^*}|u_j|^{p^*-2})|\mathbf{u} - \mathbf{v}|\|_{L_T^{q'}(L^{r'})} \\
 &\lesssim \|\mathbf{u} - \mathbf{v}\|_{L_T^q(L^r)} \sum_{j,k=1}^m \left(\|u_k\|_{L_T^q(L^\rho)}^{p^*-1} \|u_j\|_{L_T^q(L^\rho)}^{p^*-1} + \|u_k\|_{L_T^q(L^\rho)}^{p^*} \|u_j\|_{L_T^q(L^\rho)}^{p^*-2} \right) \\
 &\lesssim \|\mathbf{u} - \mathbf{v}\|_{L_T^q(L^r)} \|\mathbf{u}\|_{L_T^q(L^\rho)}^{2(p^*-1)} \\
 &\lesssim \|\mathbf{u} - \mathbf{v}\|_{L_T^q(L^r)} \|\mathbf{u}\|_{L_T^q(W^{1,r})}^{2(p^*-1)} \\
 &\lesssim R^{2(p^*-1)} \|\mathbf{u} - \mathbf{v}\|_{L_T^q(L^r)}.
 \end{aligned} \tag{3.3}$$

Now, using Propositions 2.12–2.15,

$$\begin{aligned}
 \|T(t)\Psi\|_T &\leq C\|\Psi\|; \\
 \|\nabla(T(t)\Psi)\|_T + \|\nabla(T(t)\Psi)\|_{L_T^\infty(L^2)} &\leq C\left(\|\nabla\Psi\| + T\|xT(t)\Psi\|_{L_T^\infty(L^2)}\right); \\
 \|xT(t)\Psi\|_T + \|xT(t)\Psi\|_{L_T^\infty(L^2)} &\leq C\left(\|x\Psi\| + T\|\nabla(T(t)\Psi)\|_{L_T^\infty(L^2)}\right).
 \end{aligned}$$

This implies that

$$\lim_{T \rightarrow 0} \|T(t)\Psi\|_T = 0.$$

Taking account of the previous equality via Strichartz estimate, it follows that for small $T > 0$,

$$\begin{aligned}\|\phi(\mathbf{u})\|_T &\leq \frac{R}{6} + C\|f_{j,k}(\mathbf{u})\|_{L_T^{q'}(L^{r'})}; \\ \|\nabla(\phi(\mathbf{u}))\|_T &\leq \frac{R}{6} + C\|\nabla(f_{j,k}(\mathbf{u}))\|_{L_T^{q'}(L^{r'})} + T\|x\phi(\mathbf{u})\|_{L_T^\infty(L^2)}; \\ \|x\phi(\mathbf{u})\|_T &\leq \frac{R}{6} + \|xf_{j,k}(\mathbf{u})\|_{L_T^{q'}(L^{r'})} + T\|\nabla(\phi(\mathbf{u}))\|_{L_T^\infty(L^2)}.\end{aligned}$$

Thus, for small $T > 0$, we get

$$\|\phi(\mathbf{u})\|_T + \|\nabla(\phi(\mathbf{u}))\|_T + \|x\phi(\mathbf{u})\|_T \leq \frac{R}{2} + \|f_{j,k}(\mathbf{u})\|_{L_T^{q'}(W^{1,r'})} + \|xf_{j,k}(\mathbf{u})\|_{L_T^{q'}(L^{r'})}.$$

Thanks to Hölder inequality and Sobolev embedding, we obtain

$$\begin{aligned}\|f_{j,k}(\mathbf{u})\|_{L_T^{q'}(W^{1,r'})} &\lesssim \|\mathbf{u}\|_{L_T^q(W^{1,r})} \|\mathbf{u}\|_{L_T^q(L^\rho)}^{2(p^*-1)} \\ &\lesssim R^{2(p^*-1)} \|\mathbf{u}\|_{L_T^q(W^{1,r})} \\ &\lesssim R^{2p^*-1}.\end{aligned}\tag{3.4}$$

Similarly

$$\begin{aligned}\|xf_{j,k}(\mathbf{u})\|_{L_T^{q'}(L^{r'})} &\lesssim (|u_k|^{p^*-1}|u_j|^{p^*-1} + |u_k|^{p^*}|u_j|^{p^*-2})|x\mathbf{u}|\|_{L_T^{q'}(L^{r'})} \\ &\lesssim \|x\mathbf{u}\|_{L_T^q(L^r)} \|\mathbf{u}\|_{L_T^q(L^\rho)}^{2(p^*-1)} \\ &\lesssim R^{2p^*-1}.\end{aligned}\tag{3.5}$$

Collecting the estimates (3.4)–(3.5), it yields

$$\|\phi(\mathbf{u})\|_T + \|\nabla(\phi(\mathbf{u}))\|_T + \|x\phi(\mathbf{u})\|_T \leq \frac{R}{2} + R^{2p^*-1}.$$

By (3.3) and the previous inequality, ϕ is a contraction of $B_T(R)$ for some $R, T > 0$ small enough. The existence of a local solution to (1.1) follows with a classical fixed point Picard argument via the next result.

Lemma 3.1 *Let $\Psi \in H$ and $\mathbf{u} \in F_T$ be a solution to (1.1). Then, there exists $0 < T' \leq T$ such that*

$$\mathbf{u} \in C_{T'}(H).$$

Proof It is sufficient to write, using the previous computation via Duhamel formula,

$$\|\mathbf{u}\|_{L_T^\infty(H)} \lesssim \|\Psi\|_H + \left(\|\mathbf{u}\|_{L_T^q(W^{1,r})} + \|x\mathbf{u}\|_{L_T^q(L^r)} \right) \|\mathbf{u}\|_{L_T^q(W^{1,r})}^{2(p^*-1)}.$$

□

3.2 Uniqueness

In what follows, we prove the uniqueness of a solution to the Cauchy problem (1.1). Let $T > 0$ be a positive time, $\mathbf{u}, \mathbf{v} \in C_T(H)$ two solutions to (1.1) and $\mathbf{w} := \mathbf{u} - \mathbf{v}$. Then

$$i\dot{w}_j + \Delta w_j - |x|^2 w_j = \sum_{k=1}^m a_{jk} (|u_k|^p |u_j|^{p-2} u_j - |v_k|^p |v_j|^{p-2} v_j), \quad w_j(0, \cdot) = 0.$$

Applying Strichartz estimate with the admissible pair $(q, r) = (\frac{4p}{N(p-1)}, 2p)$ and denoting for simplicity $L_T^q(L^r)$ the norm of $(L_T^q(L^r))^{(m)}$, we have

$$\|\mathbf{u} - \mathbf{v}\|_{L_T^q(L^r)} \lesssim \sum_{j,k=1}^m \|f_{j,k}(\mathbf{u}) - f_{j,k}(\mathbf{v})\|_{L_T^{q'}(L^{r'})}.$$

Taking $T > 0$ small enough, with a continuity argument, we may assume that

$$\max_{j=1,\dots,m} \|u_j\|_{L_T^\infty(H^1)} \leq 1.$$

Using previous computation with

$$(\mathcal{I}) := \|f_{j,k}(\mathbf{u}) - f_{j,k}(\mathbf{v})\|_{L_T^{q'}(L^{r'})} = \| |u_k|^p |u_j|^{p-2} u_j - |v_k|^p |v_j|^{p-2} v_j \|_{L_T^{q'}(L^{r'})},$$

we have

$$\begin{aligned} (\mathcal{I}) &\lesssim \| (|u_k|^{p-1} |u_j|^{p-1} + |u_k|^p |u_j|^{p-2}) |\mathbf{u} - \mathbf{v}| \|_{L_T^{\frac{4p}{p(4-N)+N}}(L^{\frac{2p}{2p-1}})} \\ &\lesssim \|\mathbf{u} - \mathbf{v}\|_{L_T^{\frac{4p}{p(4-N)+N}}(L^{2p})} \| |u_k|^{p-1} |u_j|^{p-1} + |u_k|^p |u_j|^{p-2} \|_{L_T^\infty(L^{\frac{p}{p-1}})} \\ &\lesssim T^{\frac{(4-N)p+N}{4p}} \|\mathbf{u} - \mathbf{v}\|_{L_T^{\frac{4p}{N(p-1)}}(L^{2p})} \left(\|u_k\|_{L_T^\infty(H^1)}^{p-1} \|u_j\|_{L_T^\infty(H^1)}^{p-1} + \|u_k\|_{L_T^\infty(H^1)}^p \|u_j\|_{L_T^\infty(H^1)}^{p-2} \right). \end{aligned}$$

Then

$$\|\mathbf{w}\|_{L_T^q(L^r)} \lesssim T^{\frac{(4-N)p+N}{4p}} \|\mathbf{w}\|_{L_T^q(L^r)}.$$

Uniqueness follows for small time and then for all time with a translation argument.

3.3 Global existence in the subcritical case

The global existence is a consequence of the conservation laws and previous calculations. Let $\mathbf{u} \in C([0, T^*), H)$ be the unique maximal solution of (1.1). By contradiction, suppose that $T^* < \infty$. Consider for $0 < s < T^*$, the problem

$$(\mathcal{P}_s) \begin{cases} i\dot{v}_j + \Delta v_j - |x|^2 v_j = \left(\sum_{j,k=1}^m a_{jk} |v_k|^p \right) |v_j|^{p-2} v_j; \\ v_j(s, \cdot) = u_j(s, \cdot). \end{cases}$$

By the same arguments used in the local existence, we can find a real number $\tau > 0$ and a solution $\mathbf{v} = (v_1, \dots, v_m)$ to $(\mathcal{P}_s)$ on $C([s, s + \tau], H)$. Using the conservation laws, we see that τ does not depend on s . Letting s be close to T^* such that $T^* < s + \tau$, the solution can be extended after T^* ; this contradicts the maximality of T^* and finishes the proof.

4 Global existence in the critical case

In this section, $3 \leq N \leq 4$. We establish Theorem 2.3 about global existence of a solution to (1.1) in the critical case $p = p^*$, for small data.

Several norms have to be considered in the analysis of the critical case. Letting $I \subset \mathbb{R}$ a time slab, define

$$\begin{aligned} \|u\|_{M(I)} &:= \|\nabla u\|_{L^{\frac{2(N+2)}{N-2}}(I, L^{\frac{2N(N+2)}{N^2+4}})} + \|xu\|_{L^{\frac{2(N+2)}{N-2}}(I, L^{\frac{2N(N+2)}{N^2+4}})}; \\ \|u\|_{S(I)} &:= \|u\|_{L^{\frac{2(N+2)}{N-2}}(I, L^{\frac{2(N+2)}{N-2}})}. \end{aligned}$$

Let $M(\mathbb{R})$ be the completion of $C_c^\infty(\mathbb{R}^{1+N})$ endowed with the norm $\|\cdot\|_{M(\mathbb{R})}$, and $M(I)$ be the set consisting of the restrictions to I of functions in $M(\mathbb{R})$. An important quantity closely related to the mass and the energy is the functional defined for $\mathbf{u} \in H$ by

$$\xi(\mathbf{u}) := \sum_{j=1}^m \int_{\mathbb{R}^N} (|\nabla u_j|^2 + |xu_j|^2) dx.$$

Let us give an auxiliary result.

Proposition 4.1 *Let $p = p^*$, $\Psi := (\psi_1, \dots, \psi_m) \in H$ and $A := \|\Psi\|_H$. There exists $\delta := \delta_A > 0$ such that for any interval $I = [0, T]$, if*

$$\|T(t)\Psi\|_{S(I)} < \delta,$$

then there exists a unique solution $\mathbf{u} \in C(I, H)$ of (1.1) which satisfies $\mathbf{u} \in (M(I) \cap L^{\frac{2(N+2)}{N}}(I \times \mathbb{R}^N))^{(m)}$. Moreover,

$$\sum_{j=1}^m \|u_j\|_{S(I)} \leq 2\delta.$$

Besides, the solution depends continuously on the initial data in the sense that there exists δ_0 depending on δ , such that for any $\delta_1 \in (0, \delta_0)$, if $\|\Psi - \varphi\|_H \leq \delta_1$ and $\mathbf{v}$ is the local solution of (1.1) with initial data φ , then $\mathbf{v}$ is defined on I and for any admissible couple (q, r) ,

$$\|\mathbf{u} - \mathbf{v}\|_{(L^q(I, L^r) \cap H)^{(m)}} \leq C\delta_1.$$

Proof The proposition follows from a contraction mapping argument. Let the function

$$\phi(\mathbf{u})(t) := T(t)\Psi - i \sum_{k=1}^m \int_0^t T(t-s) \left(a_{1k} |u_k|^{\frac{N}{N-2}} |u_1|^{\frac{4-N}{N-2}} u_1, \dots, a_{mk} |u_k|^{\frac{N}{N-2}} |u_m|^{\frac{4-N}{N-2}} u_m \right) ds.$$

Define $A := \|\Psi\|_H$ and the set

$$X_{a,b} := \left\{ \mathbf{u} \in (M(I))^m \text{ s. t. } \sum_{j=1}^m \|u_j\|_{M(I)} \leq a \text{ and } \sum_{j=1}^m \|u_j\|_{S(I)} \leq b \right\}$$

where $a, b > 0$ are sufficiently small to fix later. Using Strichartz estimate, we get

$$\|\phi(\mathbf{u}) - \phi(\mathbf{v})\|_{M(I)} \lesssim \sum_{j,k=1}^m \left(\|\nabla(f_{j,k}(\mathbf{u}) - f_{j,k}(\mathbf{v}))\|_{L_T^2 \left(L^{\frac{2N}{N+2}} \right)} + \|x(f_{j,k}(\mathbf{u}) - f_{j,k}(\mathbf{v}))\|_{L_T^2 \left(L^{\frac{2N}{N+2}} \right)} \right).$$

Using Hölder inequality, Sobolev embedding and denoting the quantity

$$(\mathcal{K}) := \|x(f_{j,k}(\mathbf{u}) - f_{j,k}(\mathbf{v}))\|_{L_T^2 \left(L^{\frac{2N}{N+2}} \right)},$$

we compute via a symmetry argument

$$\begin{aligned} (\mathcal{K}) &\lesssim \left\| \left(|u_k|^{\frac{2}{N-2}} |u_j|^{\frac{2}{N-2}} + |u_k|^{\frac{N}{N-2}} |u_j|^{\frac{4-N}{N-2}} \right) |x(\mathbf{u} - \mathbf{v})| \right\|_{L_T^2 \left(L^{\frac{2N}{N+2}} \right)} \\ &\lesssim \|x(\mathbf{u} - \mathbf{v})\|_{L_T^{\frac{2(N+2)}{N-2}} \left(L^{\frac{2N(N+2)}{N^2+4}} \right)} \left\| |u_k|^{\frac{2}{N-2}} |u_j|^{\frac{2}{N-2}} + |u_k|^{\frac{N}{N-2}} |u_j|^{\frac{4-N}{N-2}} \right\|_{L_T^{\frac{N+2}{2}} \left(L^{\frac{N+2}{2}} \right)} \\ &\lesssim \|x(\mathbf{u} - \mathbf{v})\|_{L_T^{\frac{2(N+2)}{N-2}} \left(L^{\frac{2N(N+2)}{N^2+4}} \right)} \left(\|u_k\|_{S(I)}^{\frac{2}{N-2}} \|u_j\|_{S(I)}^{\frac{2}{N-2}} + \|u_k\|_{S(I)}^{\frac{N}{N-2}} \|u_j\|_{S(I)}^{\frac{4-N}{N-2}} \right) \\ &\lesssim \|x(\mathbf{u} - \mathbf{v})\|_{L_T^{\frac{2(N+2)}{N-2}} \left(L^{\frac{2N(N+2)}{N^2+4}} \right)} \|\mathbf{u}\|_{(S(I))^m}^{\frac{4}{N-2}}. \end{aligned}$$

Write

$$\begin{aligned}\partial_i(f_{j,k}(\mathbf{u}) - f_{j,k}(\mathbf{v})) &= (\partial_i u \partial_i(f_{j,k})(\mathbf{u}) - \partial_i v \partial_i(f_{j,k})(\mathbf{v})) \\ &= \partial_i(u - v) \partial_i(f_{j,k})(\mathbf{u}) + \partial_i v (\partial_i(f_{j,k})(\mathbf{u}) - \partial_i(f_{j,k})(\mathbf{v})).\end{aligned}$$

Thus,

$$\begin{aligned}\|\nabla(f_{j,k}(\mathbf{u}) - f_{j,k}(\mathbf{v}))\|_{L_T^2(L^{\frac{2N}{N+2}})} &\leq \left\| \sum_{i=1}^m \partial_i(u - v) \partial_i(f_{j,k})(\mathbf{u}) \right\|_{L_T^2(L^{\frac{2N}{N+2}})} \\ &\quad + \left\| \sum_{i=1}^m \partial_i v (\partial_i(f_{j,k})(\mathbf{u}) - \partial_i(f_{j,k})(\mathbf{v})) \right\|_{L_T^2(L^{\frac{2N}{N+2}})} \\ &\leq (\mathcal{I}_1) + (\mathcal{I}_2).\end{aligned}$$

Thanks to Hölder inequality and Sobolev embedding, it yields

$$\begin{aligned}(\mathcal{I}_1) &\lesssim \left\| \nabla(\mathbf{u} - \mathbf{v}) \left(|u_k|^{\frac{2}{N-2}} |u_j|^{\frac{2}{N-2}} + |u_k|^{\frac{N}{N-2}} |u_j|^{\frac{4-N}{N-2}} \right) \right\|_{L_T^2(L^{\frac{2N}{N+2}})} \\ &\lesssim \|\nabla(\mathbf{u} - \mathbf{v})\|_{L_T^{\frac{2(N+2)}{N-2}}(L^{\frac{2N(N+2)}{N^2+4}})} \left(\|u_k\|_{L_T^{\frac{2}{N-2}}(L^{\frac{2(N+2)}{N-2}})}^{\frac{2}{N-2}} \|u_j\|_{L_T^{\frac{2(N+2)}{N-2}}(L^{\frac{2(N+2)}{N-2}})}^{\frac{2}{N-2}} \right. \\ &\quad \left. + \|u_k\|_{L_T^{\frac{N}{N-2}}(L^{\frac{2(N+2)}{N-2}})}^{\frac{N}{N-2}} \|u_j\|_{L_T^{\frac{4-N}{N-2}}(L^{\frac{2(N+2)}{N-2}})}^{\frac{4-N}{N-2}} \right) \\ &\lesssim \|\mathbf{u} - \mathbf{v}\|_{(M(I))^m} \left(\|u_k\|_{S(I)}^{\frac{2}{N-2}} \|u_j\|_{S(I)}^{\frac{2}{N-2}} + \|u_k\|_{S(I)}^{\frac{N}{N-2}} \|u_j\|_{S(I)}^{\frac{4-N}{N-2}} \right) \\ &\lesssim \|\mathbf{u} - \mathbf{v}\|_{(M(I))^m} \|\mathbf{u}\|_{(S(I))^m}^{\frac{4}{N-2}}.\end{aligned}$$

Using Hölder inequality and Sobolev embedding, it yields

$$\begin{aligned}(\mathcal{I}_2) &\lesssim \|\nabla \mathbf{u}(\mathbf{u} - \mathbf{v})\| \left(|u_k|^{\frac{4-N}{N-2}} |u_j|^{\frac{2}{N-2}} + |u_k|^{\frac{N}{N-2}} |u_j|^{\frac{6-2N}{N-2}} \right)_{L_T^2(L^{\frac{2N}{N+2}})} \\ &\lesssim \|\nabla \mathbf{u}\|_{L_T^{\frac{2(N+2)}{N-2}}(L^{\frac{2N(N+2)}{N^2+4}})} \|\mathbf{u} - \mathbf{v}\|_{(S(I))^m} \left(\|u_k\|_{L_T^{\frac{4-N}{N-2}}(L^{\frac{2(N+2)}{N-2}})}^{\frac{4-N}{N-2}} \|u_j\|_{L_T^{\frac{2(N+2)}{N-2}}(L^{\frac{2(N+2)}{N-2}})}^{\frac{2}{N-2}} \right. \\ &\quad \left. + \|u_k\|_{L_T^{\frac{N}{N-2}}(L^{\frac{2(N+2)}{N-2}})}^{\frac{N}{N-2}} \|u_j\|_{L_T^{\frac{6-2N}{N-2}}(L^{\frac{2(N+2)}{N-2}})}^{\frac{6-2N}{N-2}} \right) \\ &\lesssim \|\mathbf{u}\|_{(M(I))^m} \|\mathbf{u} - \mathbf{v}\|_{(S(I))^m} \|\mathbf{u}\|_{(S(I))^m}^{\frac{6-N}{N-2}}.\end{aligned}$$

Then

$$\begin{aligned}\|\phi(\mathbf{u}) - \phi(\mathbf{v})\|_{(M(I))^m} &\lesssim a^{\frac{4}{N-2}} \|\mathbf{u} - \mathbf{v}\|_{(M(I))^m} + ba^{\frac{6-N}{N-2}} \|\mathbf{u} - \mathbf{v}\|_{(S(I))^m} \\ &\lesssim (a^{\frac{4}{N-2}} + ba^{\frac{6-N}{N-2}}) \|\mathbf{u} - \mathbf{v}\|_{(M(I))^m}.\end{aligned}$$

Moreover, taking in the previous inequality $\mathbf{v} = \mathbf{0}$, we get for small $\delta > 0$,

$$\begin{aligned}\|\phi(\mathbf{u})\|_{(S(I))^m} &\leq \delta + Ca^{\frac{4}{N-2}}; \\ \|\phi(\mathbf{u})\|_{(M(I))^m} &\leq CA + Cba^{\frac{4}{N-2}}.\end{aligned}$$

With a classical Picard argument, for small $a = 2\delta$, $b > 0$, there exists $\mathbf{u} \in X_{a,b}$ a solution to (1.1) satisfying

$$\|\mathbf{u}\|_{(S(I))^{(m)}} \leq 2\delta.$$

The rest of the proposition is a consequence of the fixed point properties. $\square$

Now, we are ready to prove Theorem 2.3.

Proof of Theorem 2.3 Using the previous proposition via the fact that

$$\|T(t)\Psi\|_{S(I)} \lesssim \|T(t)\Psi\|_{M(I)} \lesssim \|x\Psi\| + \|\nabla\Psi\|,$$

it suffices to prove that $\|x\mathbf{u}\| + \|\nabla\mathbf{u}\|$ remains small on the whole interval of existence of $\mathbf{u}$. Write with conservation of the energy and Sobolev's inequality

$$\begin{aligned} (\|x\mathbf{u}\| + \|\nabla\mathbf{u}\|)^2 &\leq 2E(\Psi) + \frac{N-2}{N} \sum_{j,k=1}^m \int_{\mathbb{R}^N} a_{jk} |u_j(x,t)|^{\frac{N}{N-2}} |u_k(x,t)|^{\frac{N}{N-2}} dx \\ &\leq C(\xi(\Psi) + \xi(\Psi)^{\frac{N}{N-2}}) + C\left(\sum_{j=1}^m \|\nabla u_j\|^2\right)^{\frac{N}{N-2}} \\ &\leq C(\xi(\Psi) + \xi(\Psi)^{\frac{N}{N-2}}) + C(\|x\mathbf{u}\| + \|\nabla\mathbf{u}\|)^{\frac{2N}{N-2}}. \end{aligned}$$

So, by Lemma 2.21, if $\xi(\Psi)$ is sufficiently small, then $\|x\mathbf{u}\| + \|\nabla\mathbf{u}\|$ stays small for any time.

5 The stationary problem

The goal of this section is to prove that the elliptic problem (2.1) has a ground state solution. Let us start with some notations. For $\mathbf{u} := (u_1, \dots, u_m) \in H$ and $\lambda, \alpha, \beta \in \mathbb{R}$, we introduce the scaling

$$(u_j^\lambda)^{\alpha,\beta} := e^{\alpha\lambda} u_j(e^{-\beta\lambda} \cdot)$$

and the differential operator

$$\mathfrak{L}_{\alpha,\beta} : H^1 \rightarrow H^1, \quad u_j \mapsto \partial_\lambda((u_j^\lambda)^{\alpha,\beta})|_{\lambda=0}.$$

We extend the previous operator as follows, if $A : H^1(\mathbb{R}^N) \rightarrow \mathbb{R}$, then

$$\mathfrak{L}_{\alpha,\beta} A(u_j) := \partial_\lambda(A((u_j^\lambda)^{\alpha,\beta}))|_{\lambda=0}.$$

Denote also the constraint

$$\begin{aligned} K_{\alpha,\beta}(\mathbf{u}) &:= \partial_\lambda(S((\mathbf{u}^\lambda)^{\alpha,\beta}))|_{\lambda=0} \\ &= \frac{1}{2} \sum_{j=1}^m \left((2\alpha + (N-2)\beta) \|\nabla u_j\|^2 + (2\alpha + N\beta) \|u_j\|^2 + (2\alpha + \beta(N+2)) \|x u_j\|^2 \right) \\ &\quad - \frac{1}{2p} \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} (2p\alpha + N\beta) |u_j u_k|^p dx \\ &:= \frac{1}{2} \sum_{j=1}^m K_{\alpha,\beta}^Q(u_j) - \frac{1}{2p} \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} (2p\alpha + N\beta) |u_j u_k|^p dx. \end{aligned}$$

Finally, we introduce the quantity

$$\begin{aligned} H_{\alpha,\beta}(\mathbf{u}) &:= S(\mathbf{u}) - \frac{1}{2\alpha + \beta(N+2)} K_{\alpha,\beta}(\mathbf{u}) \\ &= \frac{1}{2\alpha + (N+2)\beta} \left[\sum_{j=1}^m \beta (\|u_j\|^2 + 2\|\nabla u_j\|^2) + \frac{1}{p} (\alpha(p-1) - \beta) \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |u_j u_k|^p dx \right]. \end{aligned}$$

Now, we prove Theorem 2.6 about the existence of a ground state solution to the stationary problem (2.1).

Remark 5.1 (i) The proof of the Theorem 2.6 is based on several lemmas;

(ii) we write, for easy notation, $u_j^\lambda := (u_j^\lambda)^{\alpha, \beta}$, $K := K_{\alpha, \beta}$, $K^Q := K_{\alpha, \beta}^Q$, $\mathfrak{L} := \mathfrak{L}_{\alpha, \beta}$ and $H := H_{\alpha, \beta}$.

Lemma 5.2 Let $(\alpha, \beta) \in G_p$. Then

(1) $\min(\mathfrak{L}H(\mathbf{u}), H(\mathbf{u})) \geq 0$ for all $\mathbf{u} \in H$;

(2) $\lambda \mapsto H(\mathbf{u}^\lambda)$ is increasing.

Proof With a direct computation

$$\begin{aligned} \mathfrak{L}H(\mathbf{u}) &= \mathfrak{L} \left(1 - \frac{\mathfrak{L}}{2\alpha + (N+2)\beta} \right) S(\mathbf{u}) \\ &= \frac{-1}{2\alpha + (N+2)\beta} (\mathfrak{L} - (2\alpha + (N+2)\beta)) (\mathfrak{L} - (2\alpha + (N-2)\beta)) S(\mathbf{u}) \\ &\quad + (2\alpha + (N-2)\beta) \left(1 - \frac{\mathfrak{L}}{2\alpha + (N+2)\beta} \right) S(\mathbf{u}) \\ &= \frac{-1}{2\alpha + (N+2)\beta} (\mathfrak{L} - (2\alpha + (N+2)\beta)) (\mathfrak{L} - (2\alpha + (N-2)\beta)) S(\mathbf{u}) + (2\alpha + (N-2)\beta) H(\mathbf{u}). \end{aligned}$$

Since $(\mathfrak{L} - (2\alpha + (N-2)\beta)) \|\nabla u_j\|^2 = (\mathfrak{L} - (2\alpha + (N+2)\beta)) \|xu_j\|^2 = 0$, we have $(\mathfrak{L} - (2\alpha + (N-2)\beta)) (\mathfrak{L} - (2\alpha + N\beta)) (\|\nabla u_j\|^2 + \|xu_j\|^2) = 0$ and

$$\begin{aligned} \mathfrak{L}H(\mathbf{u}) &\geq \frac{-1}{2\alpha + (2+N)\beta} (\mathfrak{L} - (2\alpha + (N-2)\beta)) (\mathfrak{L} - (2\alpha + (2+N)\beta)) \left(\frac{-1}{2p} \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |u_j u_k|^p dx \right) \\ &\geq \frac{1}{2p} \frac{2\alpha(p-1) - 2\beta}{2\alpha + (2+N)\beta} (2\alpha(p-1) + 2\beta) \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |u_j u_k|^p dx \geq 0. \end{aligned}$$

The last point is a consequence of the equality $\partial_\lambda H(\mathbf{u}^\lambda) = \mathfrak{L}H(\mathbf{u}^\lambda)$. □

The next intermediate result is the following.

Lemma 5.3 Let $(\alpha, \beta) \in \mathbb{R}^2$ satisfying $2\alpha + (N-2)\beta > 0$, $2\alpha + N\beta \geq 0$, $2\alpha + (N+2)\beta \geq 0$ and $0 \neq (u_1^n, \dots, u_m^n)$ be a bounded sequence of H such that

$$\lim_n \left(\sum_{j=1}^m K^Q(u_j^n) \right) = 0.$$

Then, there exists $n_0 \in \mathbb{N}$ such that $K(u_1^n, \dots, u_m^n) > 0$ for all $n \geq n_0$.

Proof Write

$$K^Q(u_j^n) = \left((2\alpha + (N-2)\beta) \|\nabla u_j^n\|^2 + (2\alpha + N\beta) \|u_j^n\|^2 + (2\alpha + (N+2)\beta) \|xu_j^n\|^2 \right) \rightarrow 0,$$

Using Proposition 2.18, via the fact that $p_* < p < p^*$, it yields

$$\sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |u_j^n u_k^n|^p dx = o \left(\sum_{j=1}^m \|\nabla u_j^n\|^2 \right) = o \left(\sum_{j=1}^m K^Q(u_j^n) \right).$$

Thus,

$$\begin{aligned} K(u_1^n, \dots, u_m^n) &= \frac{1}{2} \sum_{j=1}^m K^Q(u_j^n) - \frac{(2p\alpha + N\beta)}{2p} \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |u_j^n u_k^n|^p dx \\ &\simeq \frac{1}{2} \sum_{j=1}^m K^Q(u_j^n) \geq 0. \end{aligned}$$

□

Let us read an auxiliary result.

Lemma 5.4 *Let $(\alpha, \beta) \in G_p$. Then*

$$m_{\alpha, \beta} = \inf_{0 \neq \mathbf{u} \in H} \{H(\mathbf{u}) \text{ s. t. } K(\mathbf{u}) \leq 0\}.$$

Proof Denoting by a the right hand side of the previous equality, it is sufficient to prove that $m_{\alpha, \beta} \leq a$. Take $\mathbf{u} \in H$ such that $K(\mathbf{u}) < 0$. Because $\lim_{\lambda \rightarrow -\infty} K^Q(\mathbf{u}^\lambda) = 0$, by the previous lemma, there exists some $\lambda < 0$ such that $K(\mathbf{u}^\lambda) > 0$. With a continuity argument, there exists $\lambda_0 \leq 0$ such that $K(\mathbf{u}^{\lambda_0}) = 0$, then since $\lambda \mapsto H(\mathbf{u}^\lambda)$ is increasing, we get

$$m_{\alpha, \beta} \leq H(\mathbf{u}^{\lambda_0}) \leq H(\mathbf{u}).$$

This closes the proof. $\square$

Proof of Theorem 2.6 Let $(\phi_n) := (\phi_1^n, \dots, \phi_m^n)$ be a minimizing sequence, namely

$$0 \neq (\phi_n) \in H, \quad K(\phi_n) = 0 \quad \text{and} \quad \lim_n H(\phi_n) = \lim_n S(\phi_n) = m. \quad (5.1)$$

With a rearrangement argument via Lemma 5.4, we can assume that (ϕ_n) is radial decreasing and satisfies (5.1).

• First step: (ϕ_n) is bounded in H .

First subcase $\alpha \neq 0$. Write

$$\begin{aligned} \alpha \left(\sum_{j=1}^m \|\phi_j^n\|_\Sigma^2 - \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |\phi_j^n \phi_k^n|^p dx \right) &= \frac{\beta}{2} \left(2 \sum_{j=1}^m (\|\nabla \phi_j^n\|^2 - \|x \phi_j^n\|^2) \right. \\ &\quad \left. - N \sum_{j=1}^m \|\phi_j^n\|_\Sigma^2 + \frac{N}{p} \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |\phi_j^n \phi_k^n|^p dx \right); \\ &\times \sum_{j=1}^m \|\phi_j^n\|_\Sigma^2 - \frac{1}{p} \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |\phi_j^n \phi_k^n|^p dx \rightarrow 2m. \end{aligned}$$

Denoting $\lambda := \frac{\beta}{2\alpha}$, it yields

$$\begin{aligned} \sum_{j=1}^m \|\phi_j^n\|_\Sigma^2 - \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |\phi_j^n \phi_k^n|^p dx \\ = \lambda \left(2 \sum_{j=1}^m (\|\nabla \phi_j^n\|^2 - \|x \phi_j^n\|^2) - N \sum_{j=1}^m \|\phi_j^n\|_\Sigma^2 + \frac{N}{p} \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |\phi_j^n \phi_k^n|^p dx \right). \end{aligned}$$

So the following sequences are bounded

$$\begin{aligned} 2\lambda \sum_{j=1}^m (\|\nabla \phi_j^n\|^2 - \|x \phi_j^n\|^2) - \sum_{j=1}^m \|\phi_j^n\|_\Sigma^2 + \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |\phi_j^n \phi_k^n|^p dx; \\ 2\lambda \sum_{j=1}^m \|\nabla \phi_j^n\|^2 + 2\lambda \sum_{j=1}^m \|\phi_j^n\|_{H^1}^2 + \left(1 - \frac{1+2\lambda}{p}\right) \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |\phi_j^n \phi_k^n|^p dx; \\ \sum_{j=1}^m \|\phi_j^n\|_\Sigma^2 - \frac{1}{p} \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |\phi_j^n \phi_k^n|^p dx. \end{aligned}$$

Thus, for any real number a , the following sequence is also bounded

$$2\lambda \sum_{j=1}^m \|\nabla \phi_j^n\|^2 + 2\lambda \sum_{j=1}^m \|\phi_j^n\|_{H^1}^2 + a \sum_{j=1}^m \|\phi_j^n\|_{\Sigma}^2 + \left(1 - \frac{1+a+2\lambda}{p}\right) \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |\phi_j^n \phi_k^n|^p dx.$$

Choosing $a > 0$ near to zero, via the fact that $2\lambda < p - 1$, it follows that (ϕ_n) is bounded in H .

• Second step: the limit of (ϕ_n) is nonzero and $m > 0$.

Taking account of the compact injection (2.5), take

$$(\phi_1^n, \dots, \phi_m^n) \rightharpoonup \phi = (\phi_1, \dots, \phi_m) \quad \text{in } H$$

and

$$(\phi_1^n, \dots, \phi_m^n) \rightarrow (\phi_1, \dots, \phi_m) \quad \text{in } (L^{2p})^{(m)}.$$

The equality $K(\phi_n) = 0$ implies that

$$\begin{aligned} & \sum_{j=1}^m \left((2\alpha + (N-2)\beta) \|\nabla \phi_j^n\|^2 + (2\alpha + N\beta) \|\phi_j^n\|^2 + (2\alpha + \beta(N+2)) \|x \phi_j^n\|^2 \right) \\ &= \frac{1}{p} \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} (2p\alpha + N\beta) |\phi_j^n \phi_k^n|^p dx. \end{aligned}$$

Assume that $\phi = 0$. Using Hölder inequality

$$\|\phi_j^n \phi_k^n\|_p^p \leq \|\phi_j^n\|_{2p}^p \|\phi_k^n\|_{2p}^p \rightarrow \|\phi_j\|_{2p}^p \|\phi_k\|_{2p}^p = 0.$$

Now, by Lemma 5.3, it yields $K(\phi_n) > 0$ for large n . This contradiction implies that

$$\phi \neq 0.$$

With lower semicontinuity of the H norm,

$$\begin{aligned} 0 &= \liminf_n K(\phi_n) \\ &\geq \frac{2\alpha + (N-2)\beta}{2} \liminf_n \sum_{j=1}^m \|\nabla \phi_j^n\|^2 + \frac{2\alpha + (N+2)\beta}{2} \liminf_n \sum_{j=1}^m \|x \phi_j^n\|^2 \\ &\quad + \frac{2\alpha + N\beta}{2} \liminf_n \sum_{j=1}^m \|\phi_j^n\|^2 - \frac{2\alpha p + N\beta}{2p} \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |\phi_j \phi_k|^p dx \\ &\geq K(\phi). \end{aligned}$$

Similarly, we have $H(\phi) \leq m$. Moreover, thanks to Lemma 5.4, we can assume that $K(\phi) = 0$ and $S(\phi) = H(\phi) \leq m$. So that ϕ is a minimizer satisfying (5.1) and

$$m = H(\phi) = \frac{1}{2\alpha + (N+2)\beta} \left[\sum_{j=1}^m \beta (\|\phi_j\|^2 + 2\|\nabla \phi_j\|^2) + \frac{1}{p} (\alpha(p-1) - \beta) \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |\phi_j \phi_k|^p dx \right] > 0.$$

• Third step: the limit ϕ is a solution to (2.1).

There is a Lagrange multiplier $\eta \in \mathbb{R}$ such that $S'(\phi) = \eta K'(\phi)$. Thus

$$0 = K(\phi) = \mathbb{E} S(\phi) = \langle S'(\phi), \mathbb{E}(\phi) \rangle = \eta \langle K'(\phi), \mathbb{E}(\phi) \rangle = \eta \mathbb{E} K(\phi) = \eta \mathbb{E}^2 S(\phi).$$

With a previous computation, for $(A) := -\mathbb{E}^2 S(\phi) - (2\alpha + (N-2)\beta)(2\alpha + (N+2)\beta) S(\phi)$, we have

$$\begin{aligned} (A) &= -(\mathbb{E} - (2\alpha + (N-2)\beta))(\mathbb{E} - (2\alpha + (N+2)\beta)) S(\phi) \\ &= \frac{2\alpha(p-1) - 2\beta}{2p} (2\alpha(p-1) + 2\beta) \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |\phi_j \phi_k|^p dx \\ &> 0. \end{aligned}$$

Therefore, $\mathbb{E}^2 S(\phi) < 0$. Thus, $\eta = 0$ and $S'(\phi) = 0$. So, ϕ is a ground state and m is independent of (α, β) .

6 Invariant sets and applications

This section is devoted to obtain the existence of global solutions to the system (1.1). Precisely, we prove Theorem 2.7. We start with a classical result about stable sets under the flow of (1.1). Define the sets

$$\begin{aligned} A_{\alpha,\beta}^- &:= \{\mathbf{u} \in H \text{ s. t. } S(\mathbf{u}) < m \text{ and } K_{\alpha,\beta}(\mathbf{u}) < 0\}; \\ A_{\alpha,\beta}^+ &:= \{\mathbf{u} \in H \text{ s. t. } S(\mathbf{u}) < m \text{ and } K_{\alpha,\beta}(\mathbf{u}) \geq 0\}. \end{aligned}$$

Lemma 6.1 For $(\alpha, \beta) \in G_p$, the sets $A_{\alpha,\beta}^+$ and $A_{\alpha,\beta}^-$ are invariant under the flow of (1.1).

Proof Let $\Psi \in A_{\alpha,\beta}^+$ and $\mathbf{u} \in C_{T^*}(H)$ be the maximal solution to (1.1). Assume that $\mathbf{u}(t_0) \notin A_{\alpha,\beta}^+$ for some $t_0 \in (0, T^*)$. Since $S(\mathbf{u})$ is conserved, we have $K_{\alpha,\beta}(\mathbf{u}(t_0)) < 0$. So, with a continuity argument, there exists a positive time $t_1 \in (0, t_0)$ such that $K_{\alpha,\beta}(\mathbf{u}(t_1)) = 0$ and $S(\mathbf{u}(t_1)) < m$. This contradicts the definition of m . The proof is similar in the case of $A_{\alpha,\beta}^-$. $\square$

The previous stable sets are independent of the parameter (α, β) .

Lemma 6.2 For $(\alpha, \beta) \in G_p$, the sets $A_{\alpha,\beta}^+$ and $A_{\alpha,\beta}^-$ are independent of (α, β) .

Proof Let (α, β) and $(\alpha', \beta') \in G_p$. By Theorem 2.6, the reunion $A_{\alpha,\beta}^+ \cup A_{\alpha,\beta}^-$ is independent of (α, β) . So, it is sufficient to prove that $A_{\alpha,\beta}^+$ is independent of (α, β) . The rescaling $\mathbf{u}^\lambda := e^{\alpha\lambda}\mathbf{u}(e^{-\beta\lambda}\cdot)$ implies that a neighborhood of zero is in $A_{\alpha,\beta}^+$. If $S(\mathbf{u}) < m$ and $K_{\alpha,\beta}(\mathbf{u}) = 0$, then $\mathbf{u} = 0$. So, $A_{\alpha,\beta}^+$ is open. Moreover, this rescaling with $\lambda \rightarrow -\infty$ gives that $A_{\alpha,\beta}^+$ is contracted to zero and so it is connected. Now, write

$$A_{\alpha,\beta}^+ = A_{\alpha,\beta}^+ \cap (A_{\alpha',\beta'}^+ \cup A_{\alpha',\beta'}^-) = (A_{\alpha,\beta}^+ \cap A_{\alpha',\beta'}^+) \cup (A_{\alpha,\beta}^+ \cap A_{\alpha',\beta'}^-).$$

Since by the definition, $A_{\alpha,\beta}^-$ is open and $0 \in A_{\alpha,\beta}^+ \cap A_{\alpha',\beta'}^+$, using a connectivity argument, we have $A_{\alpha,\beta}^+ = A_{\alpha',\beta'}^+$. $\square$

Now, we prove Theorem 2.7. With a translation argument, we assume that $t_0 = 0$. Thus, $S(\Psi) < m$ and with Lemma 6.1, $\mathbf{u}(t) \in A_{1,1}^+$ for any $t \in [0, T^*)$. Moreover,

$$\begin{aligned} m &\geq \left(S - \frac{1}{2+N} K_{1,1} \right) (\mathbf{u}) \\ &= H_{1,1}(\mathbf{u}) \\ &= \frac{1}{N+4} \left[\sum_{j=1}^m (\|u_j\|^2 + 2\|\nabla u_j\|^2) + \frac{1}{p}(p-2) \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |u_j u_k|^p dx \right] \\ &\geq \frac{2}{4+N} \sum_{j=1}^m \|\nabla u_j\|^2. \end{aligned}$$

Then, since the L^2 norm is conserved, we have

$$\sup_{0 \leq t \leq T^*} \sum_{j=1}^m \|u_j\|_{H^1}^2 < \infty.$$

Moreover, using the energy identity and Proposition 2.18, it yields

$$\begin{aligned} \sum_{j=1}^m \int_{\mathbb{R}^N} (|\nabla u_j|^2 + |x u_j|^2) dx &= 2E(\Psi) + \frac{1}{p} \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |u_j u_k|^p dx \\ &\lesssim E(\Psi) + \left(\sum_{j=1}^m \|\nabla u_j\|^2 \right)^{\frac{(p-1)N}{2}} \left(\sum_{j=1}^m \|u_j\|^2 \right)^{\frac{N-p(N-2)}{2}}. \end{aligned}$$

Finally, $T^* = \infty$ because

$$\sup_{0 \leq t \leq T^*} \sum_{j=1}^m \|u_j\|_{\Sigma}^2 < \infty.$$

7 Orbital stability

This section is devoted to prove Theorem 2.10 about stability of standing waves. Denote $K := K_{1, -\frac{2}{N}}$ and $I := K_{1,0}$. First, let us do some computations.

Proposition 7.1 For $\mathbf{v} \in H$, $\lambda \in \mathbb{R}$, $\mathbf{v}_\lambda := \lambda^{\frac{N}{2}} \mathbf{v}(\lambda \cdot)$ and $\Psi \in H$ a ground state solution to (2.1) yields

$$\begin{aligned} \partial_\lambda E(\mathbf{v}_\lambda) &= \sum_{j=1}^m \left(\lambda \|\nabla v_j\|^2 - \lambda^{-3} \|x v_j\|^2 - \frac{N(p-1)\lambda^{N(p-1)-1}}{2p} \sum_{k=1}^m \int a_{jk} |u_j u_k|^p dx \right); \\ \frac{N}{2} K(\mathbf{v}) &= \partial_\lambda E(\mathbf{v}_\lambda)|_{\lambda=1}; \\ \partial_\lambda^2 E(\mathbf{v}_\lambda)|_{\lambda=1} &= \sum_{j=1}^m \left(\|\nabla v_j\|^2 + 3 \|x v_j\|^2 - \frac{N(p-1)(N(p-1)-1)}{2p} \sum_{k=1}^m \int a_{jk} |u_j u_k|^p dx \right); \\ \partial_\lambda^2 E(\Psi_\lambda)|_{\lambda=1} &= \sum_{j=1}^m \left(4 \|\nabla \psi_j\|^2 - \frac{N(p-1)(N(p-1)+2)}{2p} \sum_{k=1}^m \int a_{jk} |\psi_j \psi_k|^p dx \right). \end{aligned}$$

7.1 Stable ground state

In this subsection, taking $1 < p < p_*$ and $m := \inf_{\mathbf{u} \in H} \{S(\mathbf{u}), \|\mathbf{u}\| = \mu\}$, we establish the first point of Theorem 2.10. Let us start by proving that $\mathcal{G}_\mu \neq \emptyset$.

Take a minimizing sequence

$$S(\mathbf{v}_n) \longrightarrow m \quad \text{and} \quad \|\mathbf{v}_n\| = \mu.$$

So, for some $\varepsilon > 0$ and large n , thanks to Gagliardo–Nirenberg inequality (2.4),

$$\begin{aligned} m + \varepsilon &\geq \frac{1}{2} \|\mathbf{v}_n\|_H^2 - \frac{1}{2p} \sum_{j,k=1}^m a_{jk} \int |v_j^n v_k^n|^p dx \\ &\geq \frac{1}{2} \|\mathbf{v}_n\|_{(H^1)^m}^2 \left(1 - C \left(\sum_{j=1}^m \|\nabla v_j^n\|^2 \right)^{\frac{(p-1)N}{2}-1} \left(\sum_{j=1}^m \|v_j^n\|^2 \right)^{\frac{N-p(N-2)}{2}-1} \right) \\ &\geq \frac{1}{2} \|\mathbf{v}_n\|_{(H^1)^m}^2 \left(1 - C \mu^{(N-2)(1-p)} \|\mathbf{v}_n\|_{(\dot{H}^1)^m}^{\frac{N}{2}(p-p_*)} \right). \end{aligned} \quad (7.1)$$

Since $p < p_*$, it follows that $\mathbf{v}_n$ is bounded in H . Thanks to the compact Sobolev injections (2.6), this implies that there exists $\mathbf{v} \in H$ such that

$$\begin{aligned} \mathbf{v}_n &\longrightarrow \mathbf{v} \quad \text{in } L^q \quad \text{for any } q \in [2, 2p^*); \\ \mathbf{v}_n &\rightharpoonup \mathbf{v} \quad \text{in } H. \end{aligned}$$

Now, with the lower semicontinuity of the H norm, it follows that

$$m = \liminf_n S(\mathbf{v}_n) \geq S(\mathbf{v}) \quad \text{and} \quad \|\mathbf{v}\| = \mu.$$

So $\mathbf{v} \in \mathcal{G}_\mu$. This achieves the proof.

Now, we prove that $\mathcal{G}_\mu$ is stable. The proof proceeds by contradiction. Suppose that there exists a sequence $\mathbf{u}_0^n \in H$ such that, when n goes to infinity

$$\inf_{\Phi \in \mathcal{G}_\mu} \|\mathbf{u}_0^n - \Phi\|_H < \frac{1}{n} \quad \text{and} \quad \inf_{\Phi \in \mathcal{G}_\mu} \|\mathbf{u}_n(t_n) - \Phi\|_H > \varepsilon_0 \quad (7.2)$$

for some sequence of positive real numbers (t_n) and $\varepsilon_0 > 0$, where $\mathbf{u}_n \in C(\mathbb{R}, H)$ is the global solution to (1.1) with data $\mathbf{u}_0^n$. Let us denote $\Phi_n := \mathbf{u}_n(t_n)$. Taking account of the definition of $\mathcal{G}_\mu$, there exists $\mathbf{v}_n \in H$ such that for a subsequence

$$\|\mathbf{u}_0^n - \mathbf{v}_n\|_H < \frac{2}{n}, \quad S(\mathbf{v}_n) = m \quad \text{and} \quad \|\mathbf{v}_n\| \rightarrow \mu.$$

Then, arguing as in (7.1), it follows that $(\mathbf{v}_n)$ is bounded in H . Thus,

$$\max(\sup_n \|\mathbf{v}_n\|_H, \sup_n \|\mathbf{u}_0^n\|_H) \lesssim 1.$$

So, taking account of the compact Sobolev injection (2.6), there exists $\mathbf{v} \in H$ such that

$$\begin{aligned} \mathbf{v}_n &\longrightarrow \mathbf{v} \quad \text{in } L^q \quad \text{for any } q \in [2, 2p^*); \\ \mathbf{v}_n &\rightharpoonup \mathbf{v} \quad \text{in } H. \end{aligned}$$

This implies, via the lower semicontinuity of the H norm, that

$$m = \liminf_n S(\mathbf{v}_n) \geq S(\mathbf{v}) \quad \text{and} \quad \|\mathbf{v}\| = \mu.$$

Now, since

$$\begin{aligned} m &= \liminf_n S(\mathbf{v}_n) \\ &\geq \frac{1}{2} \left(\liminf_n \|\nabla \mathbf{v}_n\|^2 + \liminf_n \|x \mathbf{v}_n\|^2 + \mu^2 - \frac{1}{p} \sum_{j,k=1}^p a_{jk} \|v_j v_k\|_p^p \right), \end{aligned}$$

we get

$$\liminf_n \|\nabla \mathbf{v}_n\| = \|\nabla \mathbf{v}\| \quad \text{and} \quad \liminf_n \|x \mathbf{v}_n\| = \|x \mathbf{v}\|.$$

So, we have the strong convergence $\mathbf{v}_n \longrightarrow \mathbf{v}$ in H . Thus, $\mathbf{u}_0^n \longrightarrow \mathbf{v}$ in H . Then

$$\|\mathbf{u}_0^n\| \rightarrow \mu \quad \text{and} \quad S(\mathbf{u}_0^n) \rightarrow m.$$

Using the conservation laws, it follows that

$$\|\Phi_n\| \rightarrow \mu \quad \text{and} \quad S(\Phi_n) \rightarrow m.$$

Arguing as previously, there exists $\Psi \in \mathcal{G}_\mu$ such that for a subsequence

$$\Phi_n \longrightarrow \Psi \quad \text{in } H.$$

This contradicts (7.2) and finishes the proof.

7.2 Unstable ground state

The proof of the second part of Theorem 2.10 is based on several lemmas.

Lemma 7.2 Assume that $\partial_\lambda^2 E(\Psi_\lambda)|_{\lambda=1} < 0$. Then, there exist two real numbers $\varepsilon_0 > 0$, $\sigma_0 > 0$ and a mapping $\lambda : V_{\varepsilon_0}(\Psi) \rightarrow (1 - \sigma_0, 1 + \sigma_0)$ such that $I(\mathbf{v}_\lambda) = 0$ for any $\mathbf{v} \in V_{\varepsilon_0}(\Psi)$.

Proof If $\langle I'(\Psi), \partial_\lambda(\Psi_\lambda)|_{\lambda=1} \rangle = 0$, then $\partial_\lambda(\Psi_\lambda)|_{\lambda=1}$ would be the tangent to $\{\mathbf{v} \in H, I(\mathbf{v}) = 0\}$ at Ψ . Therefore, $\langle S''(\Psi) \partial_\lambda(\Psi_\lambda)|_{\lambda=1}, \partial_\lambda(\Psi_\lambda)|_{\lambda=1} \rangle \geq 0$ because Ψ is a minimizer. This contradicts the fact that

$$\partial_\lambda^2 E(\Psi_\lambda)|_{\lambda=1} = \partial_\lambda^2 S(\Psi_\lambda)|_{\lambda=1} = \langle S''(\Psi) \partial_\lambda(\Psi_\lambda)|_{\lambda=1}, \partial_\lambda(\Psi_\lambda)|_{\lambda=1} \rangle < 0.$$

So, $\partial_\lambda I(\mathbf{v}_\lambda)|_{\lambda=1, \mathbf{v}=\Psi} \neq 0$ and $I(\mathbf{v}_\lambda)|_{\lambda=1, \mathbf{v}=\Psi} = 0$. A direct application of implicit theorem concludes the proof. $\square$

The next auxiliary result reads.

Lemma 7.3 Assume that $\partial_\lambda^2 E(\Psi_\lambda)|_{\lambda=1} < 0$. Then, there exist two real numbers $\varepsilon_1 > 0$, $\sigma_1 > 0$ such that for any $\mathbf{v} \in V_{\varepsilon_1}(\Psi)$ satisfying $\|\mathbf{v}\| \leq \|\Psi\|$, we have

$$E(\Psi) < E(\mathbf{v}) + (\lambda - 1)K(\mathbf{v}), \quad \text{for some } \lambda \in (1 - \sigma_1, 1 + \sigma_1).$$

Proof With a continuity argument, there exist $\varepsilon_1 > 0$ and $\sigma_1 > 0$ such that

$$\partial_\lambda^2 E(\mathbf{v}_\lambda) < 0, \quad \forall (\lambda, \mathbf{v}) \in (1 - \sigma_1, 1 + \sigma_1) \times V_{\varepsilon_1}(\Psi).$$

Thus, with Taylor expansion

$$E(\mathbf{v}_\lambda) < E(\mathbf{v}) + (\lambda - 1)K(\mathbf{v}), \quad \forall (\lambda, \mathbf{v}) \in (1 - \sigma_1, 1 + \sigma_1) \times V_{\varepsilon_1}(\Psi).$$

By the previous lemma,

$$\forall \mathbf{v} \in V_{\varepsilon_1}(\Psi), \quad \exists \lambda \in (1 - \sigma_1, 1 + \sigma_1) \quad \text{s.t.} \quad I(\mathbf{v}_\lambda) = 0.$$

Thus, $\forall \mathbf{v} \in V_{\varepsilon_1}(\Psi)$ there exists $\lambda \in (1 - \sigma_1, 1 + \sigma_1)$ such that

$$S(\mathbf{v}_\lambda) \geq S(\Psi).$$

It follows that

$$\begin{aligned} E(\mathbf{v}_\lambda) &= S(\mathbf{v}_\lambda) - M(\mathbf{v}_\lambda) \\ &\geq S(\Psi) - M(\mathbf{v}_\lambda) \\ &\geq S(\Psi) - M(\Psi) = E(\Psi). \end{aligned}$$

The proof is finished. $\square$

Lemma 7.4 Assume that $\partial_\lambda^2 E(\Psi_\lambda)|_{\lambda=1} < 0$. Then, for $\mathbf{u}_0 \in \Pi_{\varepsilon_1}$ there exists a real number $\sigma_0 > 0$ such that the solution $\mathbf{u}$ to (1.1) given by Theorem 2.1 satisfies

$$K(\mathbf{u}(t)) < -\sigma_0, \quad \text{for all } t \in [0, T(\mathbf{u}_0)).$$

Proof Let $\mathbf{u}_0 \in \Pi_{\varepsilon_1}$, then $E(\mathbf{u}_0) < E(\Psi)$, $\|\mathbf{u}_0\| \leq \|\Psi\|$ and $K(\mathbf{u}_0) < 0$. Put $\sigma_2 := E(\Psi) - E(\mathbf{u}_0) > 0$. With the previous lemma, there exists $\lambda \in (1 - \sigma_1, 1 + \sigma_1)$ such that

$$(\lambda - 1)K(\mathbf{u}(t)) + E(\mathbf{u}(t)) > E(\Psi), \quad \forall t \in [0, T(\mathbf{u}_0)).$$

By conservation of the energy, there exists $\lambda \in (1 - \sigma_1, 1 + \sigma_1)$ such that

$$(\lambda - 1)K(\mathbf{u}(t)) > \sigma_2, \quad \forall t \in [0, T(\mathbf{u}_0)).$$

So, by a continuity argument via $K(\mathbf{u}_0) < 0$, we have $K(\mathbf{u}(t)) < 0$ for all $t \in [0, T(\mathbf{u}_0))$. Then, $\lambda - 1 < 0$ and $-\sigma_0 := -\frac{\sigma_2}{1-\lambda} < 0$ for any $t \in [0, T(\mathbf{u}_0))$. The proof is closed. $\square$

Now, we are ready to prove the crucial result of this subsection. By Proposition 7.1, it follows that $\partial_\lambda^2 E(\Psi_\lambda)|_{\lambda=1} < 0$ and $\partial_\lambda E(\Psi_\lambda) = \frac{N}{2\lambda} K(\Psi_\lambda)$. Then, 1 is a maximum for $\lambda \mapsto E(\Psi_\lambda)$ and $\frac{1}{\lambda}(K(\Psi_\lambda)) < K(\Psi) = 0$ as $\lambda > 1$ near to one (we denote $\lambda = 1^+$). Thus, $\Psi_\lambda \in \Pi_\varepsilon$ for $\varepsilon = \varepsilon(\lambda) > 0$ and $\lambda = 1^+$. Take $\mathbf{u}_0 = \Psi_\lambda$, for $\lambda = 1^+$, then

$$\mathbf{u}_0 \in \Pi_\varepsilon \quad \text{and} \quad \lim_{\lambda=1} \|\mathbf{u}_0 - \Psi\|_H = 0.$$

By the previous lemma, there exists $\sigma_0 > 0$ such that

$$K(\mathbf{u}(t)) < -\sigma_0 \quad \text{for all } t \in [0, T(\mathbf{u}_0)).$$

Now, if $e^{it}\Psi$ is orbitally stable, $T(\mathbf{u}_0) = \infty$ (for some positive real number ε) and $K(\mathbf{u}) < -\sigma_0$ on $\mathbb{R}_+$. With virial identity, $\sum_{j=1}^m \|xu_j(t)\|^2$ becomes negative for long time. This absurdity finishes the proof.

8 Appendix

We give a proof of Proposition 2.16 about Virial identity. Let $\mathbf{u} \in H$, a solution to (1.1). Denote the quantity

$$V(t) := \sum_{j=1}^m \|xu_j(t)\|^2.$$

Multiplying Eq. (1.1) by $2u_j$ and examining the imaginary parts,

$$\partial_t(|u_j|^2) = -2\Im(\bar{u}_j \Delta u_j).$$

Thus, for $a(x) := |x|^2$, we get

$$\begin{aligned} V'(t) &= -2 \sum_{j=1}^m \int_{\mathbb{R}^N} |x|^2 \Im(\bar{u}_j \Delta u_j) \, dx \\ &= 4 \sum_{j=1}^m \Im \int_{\mathbb{R}^N} (x \cdot \nabla u_j) \bar{u}_j \, dx \\ &= 2 \sum_{j=1}^m \Im \int_{\mathbb{R}^N} (\partial_k a \partial_k u_j) \bar{u}_j \, dx. \end{aligned}$$

Compute, for g the nonlinearity in (1.1),

$$\begin{aligned} \partial_t \Im(\partial_k u_j \bar{u}_j) &= \Im(\partial_k \dot{u}_j \bar{u}_j) + \Im(\partial_k u_j \bar{\dot{u}}_j) \\ &= \Re(i \dot{u}_j \partial_k \bar{u}_j) - \Re(i \partial_k \dot{u}_j \bar{u}_j) \\ &= \Re(\partial_k \bar{u}_j (-\Delta u_j + |x|^2 u_j - f_{j,k}(\mathbf{u}))) - \Re(\bar{u}_j \partial_k (-\Delta u_j + |x|^2 u_j - f_{j,k}(\mathbf{u}))) \\ &= \Re(\bar{u}_j \partial_k \Delta u_j - \partial_k \bar{u}_j \Delta u_j) - \Re(\bar{u}_j \partial_k (|x|^2 u_j) - \partial_k \bar{u}_j |x|^2 u_j) + \Re(\bar{u}_j \partial_k f_{j,k}(\mathbf{u}) - \partial_k \bar{u}_j f_{j,k}(\mathbf{u})). \end{aligned}$$

Recall the identity

$$\frac{1}{2} \partial_k \Delta(|u_j|^2) - 2\partial_l \Re(\partial_k u_j \partial_l \bar{u}_j) = \Re(\bar{u}_j \partial_k \Delta u_j - \partial_k \bar{u}_j \Delta u_j).$$

Then,

$$\begin{aligned} \int_{\mathbb{R}^N} \partial_k a \Re(\bar{u}_j \partial_k \Delta u_j - \partial_k \bar{u}_j \Delta u_j) \, dx &= \int_{\mathbb{R}^N} \partial_k a \left(\frac{1}{2} \partial_k \Delta(|u_j|^2) - 2\partial_l \Re(\partial_k u_j \partial_l \bar{u}_j) \right) \, dx \\ &= 2 \int_{\mathbb{R}^N} \partial_l \partial_k a \Re(\partial_k u_j \partial_l \bar{u}_j) \, dx \\ &= 4 \|\nabla u_j\|^2. \end{aligned}$$

Moreover,

$$\begin{aligned}\int_{\mathbb{R}^N} \partial_k a \Re(\bar{u}_j \partial_k (a u_j) - \partial_k \bar{u}_j a u_j) \, dx &= \int_{\mathbb{R}^N} (\partial_k a)^2 |u_j|^2 \, dx \\ &= 4 \|x u_j\|^2.\end{aligned}$$

On the other hand

$$\begin{aligned}\int_{\mathbb{R}^N} \partial_k a \Re(\bar{u}_j \partial_k f_{j,k}(\mathbf{u}) - \partial_k \bar{u}_j f_{j,k}(\mathbf{u})) \, dx &= \int_{\mathbb{R}^N} \partial_k a \Re(\partial_k [\bar{u}_j f_{j,k}(\mathbf{u})] - 2 \partial_k \bar{u}_j f_{j,k}(\mathbf{u})) \, dx \\ &= - \int_{\mathbb{R}^N} (\Delta a \bar{u}_j f_{j,k}(\mathbf{u}) - 2 \Re(\partial_k a \partial_k \bar{u}_j f_{j,k}(\mathbf{u}))) \, dx \\ &= -2N \sum_{k=1}^m a_{jk} \int_{\mathbb{R}^N} |u_k u_j|^p \, dx - 2 \int_{\mathbb{R}^N} \partial_k a \Re(\partial_k \bar{u}_j f_{j,k}(\mathbf{u})) \, dx.\end{aligned}$$

Write

$$\begin{aligned}\Re(\partial_k \bar{u}_j f_{j,k}(\mathbf{u})) &= \sum_{l=1}^m a_{jl} \Re(\partial_k \bar{u}_j |u_l|^p |u_j|^{p-2} u_j) \\ &= \frac{1}{p} \sum_{l=1}^m a_{jl} \partial_k (|u_j|^p) |u_l|^p.\end{aligned}$$

Then

$$\begin{aligned}\sum_{j=1}^m \Re(\partial_k \bar{u}_j f_{j,k}(\mathbf{u})) &= \frac{1}{p} \sum_{j,l=1}^m a_{jl} \partial_k (|u_j|^p) |u_l|^p \\ &= \frac{1}{2p} \sum_{j,l=1}^m a_{jl} \partial_k (|u_j u_l|^p).\end{aligned}$$

Finally

$$\frac{1}{2} V''(t) = 4 \sum_{j=1}^m (\|\nabla u_j\|^2 - \|x u_j\|^2) - 2N \left(1 - \frac{1}{p}\right) \sum_{j,l=1}^m a_{jl} \int_{\mathbb{R}^N} |u_j u_l|^p \, dx.$$

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