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Weighted majorization inequalities for n -convex functions via extension of Montgomery identity using Green function

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Abstract New identities and inequalities are given for weighted majorization theorem for n -convex functions by using extension of the Montgomery identity and Green function. Various bounds for the reminders in new generalizations of weighted majorization formulae are provided using Čebyšev type inequalities. Mean value theorems are also discussed for functional related to new results.

المخلص

في هذا المقال نقدم متطابقات ومراجحات جديدة لمبرهنة الحد الأعلى المرجحة الخاصة بالدوال n -المحدبة وذلك باستخدام تمديدات هندسة مونت ودالة جرين. ففي هذا التعميم الجديد لصيغ الحدود العليا المرجحة، نعطي عدة حدود للبواقي وذلك باستعمال مراجحات من نوع تشيبيشيف. كما أن بعض مبرهنات القيمة المتوسطة قد نوقشت لداليات متعلقة بهذه النتائج الجديدة.

Mathematics Subject Classification 26A51 · 26D15 · 26D20

1 Introduction

Convex functions have got attention of most of the mathematicians of present era (see, e.g.) [7, 15–18, 20–22, 24]. For a standard definition of convex function and some alternate forms, we refer the reader to the following books [12, 14, 19, 23]. Now, we commence the present article by recalling related concepts and definition of generalized convex function as in (see, e.g.) [2, 6, 9, 10] and [19].

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Definition 1.1 The n th order divided difference of a real-valued function f defined on an interval I of $\mathbb{R}$ at distinct points $\xi_i, \xi_{i+1}, \dots, \xi_{i+n}$ for $i \in \mathbb{N}$ is defined recursively by

$$\begin{aligned} [\xi_j; f] &= f(\xi_j), \quad j \in \{i, \dots, i+n\} \\ [\xi_i, \dots, \xi_{i+n}; f] &= \frac{[\xi_{i+1}, \dots, \xi_{i+n}; f] - [\xi_i, \dots, \xi_{i+n-1}; f]}{\xi_{i+n} - \xi_i}. \end{aligned}$$

Remark 1.2 Let us denote $[\xi_i, \dots, \xi_{i+n}; f]$ by $\Delta_{(n)}f(\xi_i)$. The value $[\xi_i, \dots, \xi_{i+n}; f]$ is independent of the order of the points $\xi_i, \xi_{i+1}, \dots, \xi_{i+n}$. We can extend this definition by considering the cases in which two or more points coincide by taking respective limits. $\square$

Definition 1.3 A function $f : I \rightarrow \mathbb{R}$ is called *convex of order n* or *n -convex* if for all choices of $(n+1)$ distinct points $\xi_i, \dots, \xi_{i+n}$ we have $\Delta_{(n)}f(\xi_i) \geq 0$.

If n th order derivative $f^{(n)}$ exists, then f is n -convex iff $f^{(n)} \geq 0$.

For fixed $m \geq 2$, let $\mathbf{x} = (\xi_1, \dots, \xi_m)$ and $\mathbf{y} = (\eta_1, \dots, \eta_m)$ denote two real m -tuples and $\xi_{[m]} \leq \xi_{[m-1]} \leq \dots \leq \xi_{[2]} \leq \xi_{[1]}$, $\eta_{[m]} \leq \eta_{[m-1]} \leq \dots \leq \eta_{[2]} \leq \eta_{[1]}$ their ordered components.

Definition 1.4 For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$,

$$\mathbf{x} \prec \mathbf{y} \quad \text{if} \quad \begin{cases} \sum_{i=1}^k \xi_{[i]} \leq \sum_{i=1}^k \eta_{[i]}, & k \in \{1, \dots, m-1\}, \\ \sum_{i=1}^m \xi_i = \sum_{i=1}^m \eta_i, \end{cases}$$

when $\mathbf{x} \prec \mathbf{y}$, $\mathbf{x}$ is said to be majorized by $\mathbf{y}$.

This concept and notation of majorization were introduced by Hardy, Littlewood and Pólya [5]. Next, we state the well-known majorization theorem from [5].

Proposition 1.5 Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$. The inequality

$$\sum_{i=1}^m f(\xi_i) \leq \sum_{i=1}^m f(\eta_i) \quad (1.1)$$

holds for any continuous convex function $f : \mathbb{R} \rightarrow \mathbb{R}$ if and only if $\mathbf{x} \prec \mathbf{y}$.

The following weighted version of the majorization theorem was given by Fuchs in [4] (see also [11, p. 580] and [19, p. 323]).

Proposition 1.6 Let $\mathbf{p} \in \mathbb{R}^m$ and let $\mathbf{x}, \mathbf{y}$ be two nonincreasing real m -tuples such that

$$\sum_{i=1}^k p_i \xi_i \leq \sum_{i=1}^k p_i \eta_i, \quad k \in \{1, \dots, m-1\} \quad (1.2)$$

$$\text{and} \quad \sum_{i=1}^m p_i \xi_i = \sum_{i=1}^m p_i \eta_i. \quad (1.3)$$

Then, for any continuous convex function $f : \mathbb{R} \rightarrow \mathbb{R}$, the following inequality holds

$$\sum_{i=1}^m p_i f(\xi_i) \leq \sum_{i=1}^m p_i f(\eta_i). \quad (1.4)$$

Some integral inequalities in relation to majorization can be stated as follows. The following proposition is a consequence of Theorem 1 in [16] (see also [19, p. 328]).

Proposition 1.7 Let $\xi, \eta : [\alpha_0, \beta_0] \rightarrow \mathbb{R}$ be two nonincreasing continuous functions and let $p : [\alpha_0, \beta_0] \rightarrow \mathbb{R}$ be a continuous function. Then, if

$$\int_{\alpha_0}^{\gamma} p(u) \xi(u) du \leq \int_{\alpha_0}^{\gamma} p(u) \eta(u) du, \quad \text{for each } \gamma \in (\alpha_0, \beta_0), \quad (1.5)$$

$$\text{and } \int_{\alpha_0}^{\beta_0} p(u) \xi(u) du = \int_{\alpha_0}^{\beta_0} p(u) \eta(u) du, \quad (1.6)$$

hold, then for every continuous convex function $f : \mathbb{R} \rightarrow \mathbb{R}$ the following inequality holds

$$\int_{\alpha_0}^{\beta_0} p(u) f(\xi(u)) du \leq \int_{\alpha_0}^{\beta_0} p(u) f(\eta(u)) du. \quad (1.7)$$

Remark 1.8 Let $\xi, \eta : [\alpha_0, \beta_0] \rightarrow \mathbb{R}$ be two nondecreasing continuous functions and let $p : [\alpha_0, \beta_0] \rightarrow \mathbb{R}$ be a continuous function. If

$$\int_{\gamma}^{\beta_0} p(u) \xi(u) du \leq \int_{\gamma}^{\beta_0} p(u) \eta(u) du, \quad \text{for each } \gamma \in (\alpha_0, \beta_0),$$

$$\text{and } \int_{\alpha_0}^{\beta_0} p(u) \xi(u) du = \int_{\alpha_0}^{\beta_0} p(u) \eta(u) du,$$

then again inequality (1.7) holds.

In this article, we will state our results for nonincreasing x and y satisfying the assumption of Proposition 1.7, but they are still valid for nondecreasing x and y satisfying the above condition see for example [11, p. 584].

From [1], we have extracted here an extension of Montgomery identity via Taylor's formula which may be stated as follows.

Proposition 1.9 Let $n \in \mathbb{N}$, $f : I \rightarrow \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous, $I \subset \mathbb{R}$ an open interval, $a_0, b_0 \in I$, $a_0 < b_0$. Then, the following identity holds

$$f(x) = \frac{1}{b_0 - a_0} \int_{a_0}^{b_0} f(s) ds + \sum_{k=0}^{n-2} \frac{f^{(k+1)}(a_0)}{k!(k+2)} \frac{(x - a_0)^{k+2}}{b_0 - a_0} - \sum_{k=0}^{n-2} \frac{f^{(k+1)}(b_0)}{k!(k+2)} \frac{(x - b_0)^{k+2}}{b_0 - a_0} + \frac{1}{(n-1)!} \int_{a_0}^{b_0} T_n(x, s) f^{(n)}(s) ds, \quad (1.8)$$

where

$$T_n(x, s) = \begin{cases} -\frac{(x-s)^n}{n(b_0-a_0)} + \frac{x-a_0}{b_0-a_0} (x-s)^{n-1}, & a_0 \leq s \leq x, \\ -\frac{(x-s)^n}{n(b_0-a_0)} + \frac{x-b_0}{b_0-a_0} (x-s)^{n-1}, & x < s \leq b_0. \end{cases} \quad (1.9)$$

In case $n = 1$, the sum $\sum_{k=0}^{n-2} \dots$ is empty, so identity (1.8) reduces to the famous Montgomery identity (see for instance [13]),

$$f(s) = \frac{1}{b_0 - a_0} \int_{a_0}^{b_0} f(t) dt + \int_{a_0}^{b_0} P(s, t) f'(t) dt$$

where $P(x, s)$ is the Peano kernel, defined by

$$P(s, t) = \begin{cases} \frac{t-a_0}{b_0-a_0}, & a_0 \leq t \leq s, \\ \frac{t-b_0}{b_0-a_0}, & s < t \leq b_0. \end{cases}$$

Finally, we define of Green function $G : [a_0, b_0] \times [a_0, b_0]$ by

$$G(s, t) = \begin{cases} \frac{(s-b_0)(t-a_0)}{b_0-a_0} & \text{for } a_0 \leq t \leq s, \\ \frac{(t-b_0)(s-a_0)}{b_0-a_0} & \text{for } s \leq t \leq b_0. \end{cases} \quad (1.10)$$

The function G is continuous and convex with respect to both t and s .

For any function $f : [a_0, b_0] \rightarrow \mathbb{R}$, $f \in C^2[a_0, b_0]$, we can obtain the following integral identity by simply using integration by parts

$$f(x) = \frac{b_0 - x}{b_0 - a_0} f(a_0) + \frac{x - a_0}{b_0 - a_0} f(b_0) + \int_{a_0}^{b_0} G(x, s) f''(s) ds, \quad (1.11)$$

where the function G is defined as above in (1.10) (see also [24]).

The aim of the present article is to introduce new weighted majorization theorems for convex functions via extension of Montgomery identity.

2 Majorization type identities and inequalities via extension of Montgomery identity

In this section, we state results related to weighted majorization identities and inequalities. First, we define notations in terms of positive linear functional:

$$\Delta(p_i, \xi_i, \eta_i, f) = \sum_{i=1}^m p_i (f(\eta_i) - f(\xi_i)) \quad (2.1)$$

and by the notation $\Delta(p_i, \xi_i, \eta_i, G(\cdot, s))$ we would mean

$$\Delta(p_i, \xi_i, \eta_i, G(\cdot, s)) = \sum_{i=1}^m p_i (G(\eta_i, s) - G(\xi_i, s)) \quad (2.2)$$

where p_i, ξ_i, η_i and f are defined in Proposition 1.6 and G is defined in (1.10). Also

$$\Lambda(p, \xi, \eta, f) = \int_{\alpha_0}^{\beta_0} p(u) (f(\eta(u)) - f(\xi(u))) du \quad (2.3)$$

and by the notation $\Lambda(p, \xi, \eta, G(\cdot, s))$ we would mean

$$\Lambda(p, \xi, \eta, G(\cdot, s)) = \int_{\alpha_0}^{\beta_0} p(u) (G(\eta(u), s) - G(\xi(u), s)) du \quad (2.4)$$

where p, ξ, η and f are as defined in Proposition 1.7 and G is defined in (1.10). Also “id” represents identity function, i.e., $\text{id}(x) = x$ for all x .

Theorem 2.1 *Let all the assumptions of Proposition 1.6 be valid. Also let $f : I \rightarrow \mathbb{R}$ be a function such that $f^{(n-1)}$ ($n \geq 3$) is absolutely continuous, $I \subset \mathbb{R}$ an open interval, $a_0, b_0 \in I$, $a_0 < b_0$. Then for all $s \in [a_0, b_0]$ we have the following identity:*

$$\begin{aligned} \Delta(p_i, \xi_i, \eta_i, f) &= \frac{f(b_0) - f(a_0)}{b_0 - a_0} \Delta(p_i, \xi_i, \eta_i, \text{id}) \\ &+ \frac{f'(a_0) - f'(b_0)}{b_0 - a_0} \int_{a_0}^{b_0} \Delta(p_i, \xi_i, \eta_i, G(\cdot, s)) ds \\ &+ \sum_{k=2}^{n-1} \frac{k}{(k-1)!} \int_{a_0}^{b_0} \Delta(p_i, \xi_i, \eta_i, G(\cdot, s)) \frac{f^k(a_0)(s-a_0)^{k-1} - f^k(b_0)(s-b_0)^{k-1}}{b_0 - a_0} ds \\ &+ \frac{1}{(n-3)!} \int_{a_0}^{b_0} f^{(n)}(t) \left(\int_{a_0}^{b_0} \Delta(p_i, \xi_i, \eta_i, G(\cdot, s)) \tilde{T}_{n-2}(s, t) ds \right) dt \end{aligned} \quad (2.5)$$

where

$$\tilde{T}_{n-2}(s, t) = \begin{cases} \frac{1}{b_0 - a_0} \left[\frac{(s-t)^{n-2}}{(n-2)} + (s-a_0)(s-t)^{n-3} \right], & a_0 \leq t \leq s \leq b_0, \\ \frac{1}{b_0 - a_0} \left[\frac{(s-t)^{n-2}}{(n-2)} + (s-b_0)(s-t)^{n-3} \right], & a_0 \leq s < t \leq b_0. \end{cases}$$

and $G(\cdot, s)$ is as defined in (1.10). We also have the following identity:

$$\begin{aligned} \Delta(p_i, \xi_i, \eta_i, f) &= \frac{f(b_0) - f(a_0)}{b_0 - a_0} \Delta(p_i, \xi_i, \eta_i, \text{id}) \\ &+ \frac{f'(b_0) - f'(a_0)}{b_0 - a_0} \int_{a_0}^{b_0} \Delta(p_i, \xi_i, \eta_i, G(\cdot, s)) ds \\ &+ \sum_{k=3}^{n-1} \frac{k-2}{(k-1)!} \int_{a_0}^{b_0} \Delta(p_i, \xi_i, \eta_i, G(\cdot, s)) \frac{f^k(a_0)(s-a_0)^{k-1} - f^k(b_0)(s-b_0)^{k-1}}{b_0 - a_0} ds \\ &+ \frac{1}{(n-3)!} \int_{a_0}^{b_0} f^{(n)}(t) \left(\int_{a_0}^{b_0} \Delta(p_i, \xi_i, \eta_i, G(\cdot, s)) \tilde{T}_{n-2}(s, t) ds \right) dt \end{aligned} \quad (2.6)$$

where T_n is defined in Proposition 1.9.

Proof Using (1.11) in (2.1) and using linearity of $\Delta(p_i, \xi_i, \eta_i, f)$, we get

$$\Delta(p_i, \xi_i, \eta_i, f) = \frac{f(b_0) - f(a_0)}{b_0 - a_0} \Delta(p_i, \xi_i, \eta_i, \text{id}) + \int_{a_0}^{b_0} \Delta(p_i, \xi_i, \eta_i, G(\cdot, s)) f''(s) ds. \quad (2.7)$$

Differentiating (1.8) twice with respect to s , we get

$$\begin{aligned} f''(s) &= \frac{f'(a_0) - f'(b_0)}{b_0 - a_0} + \sum_{k=2}^{n-1} \frac{k}{(k-1)!} \frac{f^k(a_0)(s-a_0)^{k-1} - f^k(b_0)(s-b_0)^{k-1}}{b_0 - a_0} \\ &+ \frac{1}{(n-3)!} \int_{a_0}^{b_0} \tilde{T}_{n-2}(s, t) f^{(n)}(t) dt. \end{aligned} \quad (2.8)$$

Now using (2.8) in (2.7), we get

$$\begin{aligned} \Delta(p_i, \xi_i, \eta_i, f) &= \frac{f(b_0) - f(a_0)}{b_0 - a_0} \Delta(p_i, \xi_i, \eta_i, \text{id}) \\ &+ \frac{f'(a_0) - f'(b_0)}{b_0 - a_0} \int_{a_0}^{b_0} \Delta(p_i, \xi_i, \eta_i, G(\cdot, s)) ds \\ &+ \sum_{k=2}^{n-1} \frac{k}{(k-1)!} \int_{a_0}^{b_0} \Delta(p_i, \xi_i, \eta_i, G(\cdot, s)) \frac{f^k(a_0)(s-a_0)^{k-1} - f^k(b_0)(s-b_0)^{k-1}}{b_0 - a_0} ds \\ &+ \frac{1}{(n-3)!} \int_{a_0}^{b_0} \Delta(p_i, \xi_i, \eta_i, G(\cdot, s)) \left(\int_{a_0}^{b_0} \tilde{T}_{n-2}(s, t) f^{(n)}(t) dt \right) ds \end{aligned}$$

and then applying Fubini's theorem for the last term to get (2.5).

Also, using formula (1.8) on the function f'' , replacing n by $n-2$ ($n \geq 3$) and rearranging the indices, we get

$$\begin{aligned} f''(s) &= \frac{f'(b_0) - f'(a_0)}{b_0 - a_0} + \sum_{k=3}^{n-1} \frac{k-2}{(k-1)!} \frac{f^k(a_0)(s-a_0)^{k-1} - f^k(b_0)(s-b_0)^{k-1}}{b_0 - a_0} \\ &+ \frac{1}{(n-3)!} \int_{a_0}^{b_0} T_{n-2}(s, t) f^{(n)}(t) dt \end{aligned} \quad (2.9)$$

Similarly, using (2.9) in (2.7) and applying Fubini's theorem, we get (2.6). $\square$

Theorem 2.2 Let all the assumptions of Theorem 2.1 be valid with additional condition

$$\int_{a_0}^{b_0} \Delta(p_i, \xi_i, \eta_i, G(\cdot, s)) \tilde{T}_{n-2}(s, t) ds \geq 0, \quad \forall t \in [a_0, b_0] \quad (2.10)$$

where G is defined in (1.10) and $\tilde{T}_n$ in Theorem 2.1. Then for every n -convex function $f : I \rightarrow \mathbb{R}$, the following inequality holds:

$$\begin{aligned} \Delta(p_i, \xi_i, \eta_i, f) &\geq \frac{f(b_0) - f(a_0)}{b_0 - a_0} \Delta(p_i, \xi_i, \eta_i, \text{id}) \\ &\quad + \frac{f'(a_0) - f'(b_0)}{b_0 - a_0} \int_{a_0}^{b_0} \Delta(p_i, \xi_i, \eta_i, G(\cdot, s)) ds \\ &\quad + \sum_{k=2}^{n-1} \frac{k}{(k-1)!} \int_{a_0}^{b_0} \Delta(p_i, \xi_i, \eta_i, G(\cdot, s)) \frac{f^k(a_0)(s-a_0)^{k-1} - f^k(b_0)(s-b_0)^{k-1}}{b_0 - a_0} ds. \end{aligned} \quad (2.11)$$

Proof Function f is n -convex, so we have $f^{(n)} \geq 0$. Using this fact and (2.10) in (2.5) we obtain required result. $\square$

Theorem 2.3 Let all the assumptions of Theorem 2.1 hold with additional condition

$$\int_{a_0}^{b_0} \Delta(p_i, \xi_i, \eta_i, G(\cdot, s)) T_{n-2}(s, t) ds \geq 0, \quad \forall t \in [a_0, b_0] \quad (2.12)$$

where G is as defined in (1.10) and T_n as in Proposition 1.9. Then for every n -convex function $f : I \rightarrow \mathbb{R}$, the following inequality holds

$$\begin{aligned} \Delta(p_i, \xi_i, \eta_i, f) &\geq \frac{f(b_0) - f(a_0)}{b_0 - a_0} \Delta(p_i, \xi_i, \eta_i, \text{id}) \\ &\quad + \frac{f'(b_0) - f'(a_0)}{b_0 - a_0} \int_{a_0}^{b_0} \Delta(p_i, \xi_i, \eta_i, G(\cdot, s)) ds \\ &\quad + \sum_{k=3}^{n-1} \frac{k-2}{(k-1)!} \int_{a_0}^{b_0} \Delta(p_i, \xi_i, \eta_i, G(\cdot, s)) \frac{f^k(a_0)(s-a_0)^{k-1} - f^k(b_0)(s-b_0)^{k-1}}{b_0 - a_0} ds. \end{aligned} \quad (2.13)$$

Proof Function f is n -convex, so we have $f^{(n)} \geq 0$. Using this fact and (2.12) in (2.6), we obtain required result. $\square$

Next, we state an important consequence.

Theorem 2.4 Let all the assumptions from Theorem 2.1 hold. If f is n -convex and n is even, then inequalities (2.11) and (2.13) hold.

Proof Since Green function $G(s, t)$ is convex with respect to t for all $s \in [a_0, b_0]$ and $\mathbf{x} = (\xi_1, \dots, \xi_m)$ and $\mathbf{p} = (p_1, \dots, p_m)$ satisfy condition (1.2) and (1.3). Therefore, from Proposition 1.6 (inequality (1.4)), we have

$$\Delta(p_i, \xi_i, \eta_i, G(\cdot, s)) \geq 0 \quad \text{for } s \in [a_0, b_0]. \quad (2.14)$$

Also note that $\tilde{T}_{n-2}(s, t) \geq 0$ ($T_{n-2}(s, t) \geq 0$) if $n-2$ is even. Therefore combining this fact with (2.14), we get inequality (2.10) (inequality (2.12)). As f is n -convex, results follow from Theorem 2.2 (Theorem 2.3, respectively). $\square$

Following is the integral version of our main results. Since proofs of these results are of similar nature we omit details.

Theorem 2.5 Let all the assumptions of Proposition 1.7 be valid. Let $f : I \rightarrow \mathbb{R}$ be a function such that $f^{(n-1)}$ is absolutely continuous, $I \subset \mathbb{R}$ an open interval, $a_0, b_0 \in I$, $a_0 < b_0$. Then for all $s \in [a_0, b_0]$ we have the following identity

$$\begin{aligned} \Lambda(p, \xi, \eta, f) &= \frac{f(b_0) - f(a_0)}{b_0 - a_0} \Lambda(p, \xi, \eta, \text{id}) \\ &+ \frac{f'(a_0) - f'(b_0)}{b_0 - a_0} \int_{a_0}^{b_0} \Lambda(p, \xi, \eta, G(\cdot, s)) ds \\ &+ \sum_{k=2}^{n-1} \frac{k}{(k-1)!} \int_{a_0}^{b_0} \Lambda(p, \xi, \eta, G(\cdot, s)) \frac{f^k(a_0)(s-a_0)^{k-1} - f^k(b_0)(s-b_0)^{k-1}}{b_0 - a_0} ds \\ &+ \frac{1}{(n-3)!} \int_{a_0}^{b_0} f^{(n)}(t) \left(\int_{a_0}^{b_0} \Lambda(p, \xi, \eta, G(\cdot, s)) \tilde{T}_{n-2}(s, t) ds \right) dt \end{aligned} \quad (2.15)$$

where $\tilde{T}_n$ is as in Theorem 2.1 and G is as defined in (1.10). Moreover, we also obtain the following identity:

$$\begin{aligned} \Lambda(p, \xi, \eta, f) &= \frac{f(b_0) - f(a_0)}{b_0 - a_0} \Lambda(p, \xi, \eta, \text{id}) \\ &+ \frac{f'(b_0) - f'(a_0)}{b_0 - a_0} \int_{a_0}^{b_0} \Lambda(p, \xi, \eta, G(\cdot, s)) ds \\ &+ \sum_{k=3}^{n-1} \frac{k-2}{(k-1)!} \int_{a_0}^{b_0} \Lambda(p, \xi, \eta, G(\cdot, s)) \frac{f^k(a_0)(s-a_0)^{k-1} - f^k(b_0)(s-b_0)^{k-1}}{b_0 - a_0} ds \\ &+ \frac{1}{(n-3)!} \int_{a_0}^{b_0} f^{(n)}(t) \left(\int_{a_0}^{b_0} \Lambda(p, \xi, \eta, G(\cdot, s)) T_{n-2}(s, t) ds \right) dt \end{aligned} \quad (2.16)$$

where T_n is as in Proposition 1.9.

Theorem 2.6 Let all the assumptions of Theorem 2.5 hold with additional condition

$$\int_{a_0}^{b_0} \Lambda(p, \xi, \eta, G(\cdot, s)) \tilde{T}_{n-2}(s, t) ds \geq 0, \quad \forall t \in [a_0, b_0] \quad (2.17)$$

where G is as defined in (1.10) and $\tilde{T}_n$ is defined in Theorem 2.1. Then, for every n -convex function $f : I \rightarrow \mathbb{R}$, the following inequality holds

$$\begin{aligned} \Lambda(p, \xi, \eta, f) &\geq \frac{f(b_0) - f(a_0)}{b_0 - a_0} \Lambda(p, \xi, \eta, \text{id}) \\ &+ \frac{f'(a_0) - f'(b_0)}{b_0 - a_0} \int_{a_0}^{b_0} \Lambda(p, \xi, \eta, G(\cdot, s)) ds \\ &+ \sum_{k=2}^{n-1} \frac{k}{(k-1)!} \int_{a_0}^{b_0} \Lambda(p, \xi, \eta, G(\cdot, s)) \frac{f^k(a_0)(s-a_0)^{k-1} - f^k(b_0)(s-b_0)^{k-1}}{b_0 - a_0} ds. \end{aligned} \quad (2.18)$$

Theorem 2.7 Let all the assumptions of Theorem 2.5 hold with additional condition

$$\int_{a_0}^{b_0} \Lambda(p, \xi, \eta, G(\cdot, s)) T_{n-2}(s, t) ds \geq 0, \quad \forall t \in [a_0, b_0] \quad (2.19)$$

where G is as defined in (1.10) and T_n is as in Proposition 1.9. Then for every n -convex function $f : I \rightarrow \mathbb{R}$ the following inequality holds

$$\begin{aligned} \Lambda(p, \xi, \eta, f) &\geq \frac{f(b_0) - f(a_0)}{b_0 - a_0} \Lambda(p, \xi, \eta, \text{id}) \\ &\quad + \frac{f'(b_0) - f'(a_0)}{b_0 - a_0} \int_{a_0}^{b_0} \Lambda(p, \xi, \eta, G(\cdot, s)) ds \\ &\quad + \sum_{k=3}^{n-1} \frac{k-2}{(k-1)!} \int_{a_0}^{b_0} \Lambda(p, \xi, \eta, G(\cdot, s)) \frac{f^k(a_0)(s-a_0)^{k-1} - f^k(b_0)(s-b_0)^{k-1}}{b_0 - a_0} ds \end{aligned} \quad (2.20)$$

Here, we have another result.

Theorem 2.8 *Let all the assumptions from Theorem 2.5 hold. If f is n -convex and n is even, then inequalities (2.18) and (2.20) hold.*

3 Bounds for identities related to generalized linear inequalities

Let $g, h : [a_0, b_0] \rightarrow \mathbb{R}$ be two Lebesgue integrable functions. We consider the Čebyšev functional defined by

$$T(g, h) = \frac{1}{b_0 - a_0} \int_{a_0}^{b_0} g(u)h(u)du - \left(\frac{1}{b_0 - a_0} \int_{a_0}^{b_0} g(u)du \right) \left(\frac{1}{b_0 - a_0} \int_{a_0}^{b_0} h(u)du \right). \quad (3.1)$$

The following result can be found in [3]:

Proposition 3.1 *Let $g : [a_0, b_0] \rightarrow \mathbb{R}$ be a Lebesgue integrable function and let $h : [a_0, b_0] \rightarrow \mathbb{R}$ be an absolutely continuous function with $(\cdot - a_0)(b_0 - \cdot)[h']^2 \in L[a_0, b_0]$. Then we have the inequality*

$$|T(g, h)| \leq \frac{1}{\sqrt{2}} \left(\frac{1}{b_0 - a_0} |T(g, g)| \int_{a_0}^{b_0} (u - a_0)(b_0 - u)[h'(u)]^2 du \right)^{1/2}. \quad (3.2)$$

The constant $\frac{1}{\sqrt{2}}$ in (3.2) is the best possible.

Using aforementioned result, we are going to obtain generalizations of the result proved in previous section.

Under the assumptions of Theorems 2.2, 2.3, 2.6 and 2.7, respectively, we define the following linear functionals

$$\Omega_1(t) = \int_{a_0}^{b_0} \Delta(p_i, \xi_i, \eta_i, G(\cdot, s)) \tilde{T}_{n-2}(s, t) ds \geq 0, \quad \forall t \in [a_0, b_0] \quad (3.3)$$

$$\Omega_2(t) = \int_{a_0}^{b_0} \Delta(p_i, \xi_i, \eta_i, G(\cdot, s)) T_{n-2}(s, t) ds \geq 0, \quad \forall t \in [a_0, b_0] \quad (3.4)$$

$$\Omega_3(t) = \int_{a_0}^{b_0} \Lambda(p, \xi, \eta, G(\cdot, s)) \tilde{T}_{n-2}(s, t) ds, \quad \forall t \in [a_0, b_0] \quad (3.5)$$

$$\Omega_4(t) = \int_{a_0}^{b_0} \Lambda(p, \xi, \eta, G(\cdot, s)) T_{n-2}(s, t) ds, \quad \forall t \in [a_0, b_0] \quad (3.6)$$

Hence using these notations, we may define Čebyšev functional as follows (e.g., using Ω):

$$T(\Omega, \Omega) = \frac{1}{b_0 - a_0} \int_{a_0}^{b_0} \Omega^2(s) ds - \left(\frac{1}{b_0 - a_0} \int_{a_0}^{b_0} \Omega(s) ds \right)^2$$

Now, we state main results of this section: bounds for the reminders in new generalizations of weighted majorization identities.

Theorem 3.2 Let $f : [a_0, b_0] \rightarrow \mathbb{R}$ be such that $f^{(n)}$ is an absolutely continuous function for $n \in \mathbb{N}$ with $(\cdot - a_0)(b_0 - \cdot)[f^{(n+1)}]^2 \in L[a_0, b_0]$. Let p_i and ξ_i , $i \in \{1, \dots, m\}$ satisfy the assumptions of Proposition 1.6 and the functions G , T and Ω_1 be defined in (1.10), (3.1) and (3.3), respectively. Then, we have

$$\begin{aligned} \Delta(p_i, \xi_i, \eta_i, f) &= \frac{f(b_0) - f(a_0)}{b_0 - a_0} \Delta(p_i, \xi_i, \eta_i, \text{id}) \\ &+ \frac{f'(a_0) - f'(b_0)}{b_0 - a_0} \int_{a_0}^{b_0} \Delta(p_i, \xi_i, \eta_i, G(\cdot, s)) ds \\ &+ \sum_{k=2}^{n-1} \frac{k}{(k-1)!} \int_{a_0}^{b_0} \Delta(p_i, \xi_i, \eta_i, G(\cdot, s)) \frac{f^k(a_0)(s-a_0)^{k-1} - f^k(b_0)(s-b_0)^{k-1}}{b_0 - a_0} ds \\ &+ \frac{[f^{(n-1)}(b_0) - f^{(n-1)}(a_0)]}{(n-3)!(b_0 - a_0)} \int_{a_0}^{b_0} \Omega_1(s) ds + R_n^1(f; a_0, b_0), \end{aligned} \quad (3.7)$$

where the remainder $R_n^1(f; a_0, b_0)$ satisfies the estimation

$$|R_n^1(f; a_0, b_0)| \leq \frac{1}{(n-3)!} \left(\frac{(b_0 - a_0)}{2} \left| T(\Omega_1, \Omega_1) \int_{a_0}^{b_0} (s - a_0)(b_0 - s)[f^{(n+1)}(s)]^2 ds \right| \right)^{1/2}. \quad (3.8)$$

Proof If we apply Proposition 3.1 for $g \rightarrow \Omega_1$ and $h \rightarrow f^{(n)}$, then we obtain

$$\begin{aligned} &\left| \frac{1}{b_0 - a_0} \int_{a_0}^{b_0} \Omega_1(t) f^{(n)}(t) dt - \left(\frac{1}{b_0 - a_0} \int_{a_0}^{b_0} \Omega_1(t) dt \right) \left(\frac{1}{b_0 - a_0} \int_{a_0}^{b_0} f^{(n)}(t) dt \right) \right| \\ &\leq \frac{1}{\sqrt{2}} \left(\frac{1}{b_0 - a_0} |T(\Omega_1, \Omega_1)| \int_{a_0}^{b_0} (t - a_0)(b_0 - t)[f^{(n+1)}(t)]^2 dt \right)^{1/2}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \frac{1}{(n-3)!} \int_{a_0}^{b_0} \Omega_1(t) f^{(n)}(t) dt &= \frac{[f^{(n-1)}(b_0) - f^{(n-1)}(a_0)]}{(n-3)!(b_0 - a_0)} \int_{a_0}^{b_0} \Omega_1(t) dt \\ &+ R_n^1(f; a_0, b_0). \end{aligned}$$

where $R_n^1(f; a_0, b_0)$ satisfies inequality (3.8). Now from identity (2.5), we obtain (3.7). $\square$

Theorem 3.3 Let $f : [a_0, b_0] \rightarrow \mathbb{R}$ be such that $f^{(n)}$ is an absolutely continuous function for $n \in \mathbb{N}$ with $(\cdot - a_0)(b_0 - \cdot)[f^{(n+1)}]^2 \in L[a_0, b_0]$. Let p_i and ξ_i , $i \in \{1, \dots, m\}$ satisfy the assumptions of Proposition 1.6 and the functions G , T and Ω_2 be defined in (1.11), (3.1) and (3.4), respectively. Then, we have

$$\begin{aligned} \Delta(p_i, \xi_i, \eta_i, f) &= \frac{f(b_0) - f(a_0)}{b_0 - a_0} \Delta(p_i, \xi_i, \eta_i, \text{id}) \\ &+ \frac{f'(b_0) - f'(a_0)}{b_0 - a_0} \int_{a_0}^{b_0} \Delta(p_i, \xi_i, \eta_i, G(\cdot, s)) ds \\ &+ \sum_{k=3}^{n-1} \frac{k-2}{(k-1)!} \int_{a_0}^{b_0} \Delta(p_i, \xi_i, \eta_i, G(\cdot, s)) \frac{f^k(a_0)(s-a_0)^{k-1} - f^k(b_0)(s-b_0)^{k-1}}{b_0 - a_0} ds \\ &+ \frac{[f^{(n-1)}(b_0) - f^{(n-1)}(a_0)]}{(n-3)!(b_0 - a_0)} \int_{a_0}^{b_0} \Omega_2(s) ds + R_n^2(f; a_0, b_0), \end{aligned} \quad (3.9)$$

where the remainder $R_n^2(f; a_0, b_0)$ satisfies the estimation

$$|R_n^2(f; a_0, b_0)| \leq \frac{1}{(n-3)!} \left(\frac{(b_0 - a_0)}{2} \left| T(\Omega_2, \Omega_2) \int_{a_0}^{b_0} (s - a_0)(b_0 - s)[f^{(n+1)}(s)]^2 ds \right| \right)^{1/2}. \quad (3.10)$$

Finally, we obtain Ostrowski type inequalities for functions f whose n -th derivative belongs to L_p spaces.

The symbol $L_p[a_0, b_0]$ ($1 \leq p < \infty$) denotes the space of p -power integrable functions on the interval $[a_0, b_0]$ equipped with the norm

$$\|\phi\|_p = \left(\int_{a_0}^{b_0} |\phi(t)|^p dt \right)^{\frac{1}{p}}$$

and $L_\infty[a_0, b_0]$ denotes the space of essentially bounded functions on $[a_0, b_0]$ with the norm

$$\|\phi\|_\infty = \operatorname{ess\,sup}_{t \in [a_0, b_0]} |\phi(t)|.$$

Now, we state some Ostrowski type inequalities related to the generalized majorization inequalities.

Theorem 3.4 *Let all the assumptions of Theorem 2.1 be valid. Furthermore, let (q, r) be a pair of conjugate exponents and $f^{(n)} \in L_q[a_0, b_0]$ for some $n \in \mathbb{N}$. Then, we have*

$$\begin{aligned} & \left| \Delta(p_i, \xi_i, \eta_i, f) - \frac{f(b_0) - f(a_0)}{b_0 - a_0} \Delta(p_i, \xi_i, \eta_i, \operatorname{id}) \right. \\ & \quad - \frac{f'(a_0) - f'(b_0)}{b_0 - a_0} \int_{a_0}^{b_0} \Delta(p_i, \xi_i, \eta_i, G(\cdot, s)) ds \\ & \quad \left. - \sum_{k=2}^{n-1} \frac{k}{(k-1)!} \int_{a_0}^{b_0} \Delta(p_i, \xi_i, \eta_i, G(\cdot, s)) \frac{f^{(k)}(a_0)(s-a_0)^{k-1} - f^{(k)}(b_0)(s-b_0)^{k-1}}{b_0 - a_0} ds \right| \\ & \leq \frac{1}{(n-3)!} \|f^{(n)}\|_q \left\| \int_{a_0}^{b_0} \Delta(p_i, \xi_i, \eta_i, G(\cdot, s)) \tilde{T}_{n-2}(s, t) ds \right\|_r. \end{aligned} \quad (3.11)$$

The constant on the R.H.S. of (3.11) is sharp for $1 < q \leq \infty$ and the best possible for $q = 1$.

Proof Let us denote

$$\lambda(t) = \frac{1}{(n-3)!} \int_{a_0}^{b_0} \Delta(p_i, \xi_i, \eta_i, G(\cdot, s)) \tilde{T}_{n-2}(s, t) ds$$

Using the identity (2.5) and applying Hölder's inequality, we obtain

$$\begin{aligned} & \left| \Delta(p_i, \xi_i, \eta_i, f) - \frac{f(b_0) - f(a_0)}{b_0 - a_0} \Delta(p_i, \xi_i, \eta_i, \operatorname{id}) \right. \\ & \quad - \frac{f'(a_0) - f'(b_0)}{b_0 - a_0} \int_{a_0}^{b_0} \Delta(p_i, \xi_i, \eta_i, G(\cdot, s)) ds \\ & \quad \left. - \sum_{k=2}^{n-1} \frac{k}{(k-1)!} \int_{a_0}^{b_0} \Delta(p_i, \xi_i, \eta_i, G(\cdot, s)) \frac{f^{(k)}(a_0)(s-a_0)^{k-1} - f^{(k)}(b_0)(s-b_0)^{k-1}}{b_0 - a_0} ds \right| \\ & \leq \|f^{(n)}\|_q \|\lambda\|_r. \end{aligned} \quad (3.12)$$

To discuss the sharpness of the constant $\left(\int_{a_0}^{b_0} |\lambda(t)|^r dt \right)^{1/r}$, we find a function f for which the equality in (3.12) is obtained.

For $1 < q < \infty$ take f to be such that

$$f^{(n)}(t) = \operatorname{sgn} \lambda(t) \cdot |\lambda(t)|^{1/(q-1)}.$$

For $q = \infty$, take f such that

$$f^{(n)}(t) = \operatorname{sgn} \lambda(t).$$

The fact that (3.11) is the best possible for $q = 1$ can be proved as in [8, Thm 12]. $\square$

Theorem 3.5 *Let all the assumptions of Theorem 2.1 be valid. Furthermore, let (q, r) be a pair of conjugate exponents. Let $f^{(n)} \in L_q[a_0, b_0]$ for some $n \in \mathbb{N}$. Then, we have*

$$\begin{aligned} & \left| \Delta(p_i, \xi_i, \eta_i, f) - \frac{f(b_0) - f(a_0)}{b_0 - a_0} \Delta(p_i, \xi_i, \eta_i, \text{id}) \right. \\ & \quad - \frac{f'(b_0) - f'(a_0)}{b_0 - a_0} \int_{a_0}^{b_0} \Delta(p_i, \xi_i, \eta_i, G(\cdot, s)) ds \\ & \quad \left. - \sum_{k=3}^{n-1} \frac{k-2}{(k-1)!} \int_{a_0}^{b_0} \Delta(p_i, \xi_i, \eta_i, G(\cdot, s)) \frac{f^k(a_0)(s-a_0)^{k-1} - f^k(b_0)(s-b_0)^{k-1}}{b_0 - a_0} ds \right| \\ & \leq \frac{1}{(n-3)!} \|f^{(n)}\|_q \left\| \int_{a_0}^{b_0} \Delta(p_i, \xi_i, \eta_i, G(\cdot, s)) T_{n-2}(s, t) ds \right\|_r. \end{aligned} \quad (3.13)$$

The constant on the R.H.S. of (3.13) is sharp for $1 < q \leq \infty$ and the best possible for $q = 1$.

Finally, we state all the integral analogous of results from this section. Since proofs are of similar nature, so we omit the details.

Theorem 3.6 *Let $f : [a_0, b_0] \rightarrow \mathbb{R}$ be such that $f^{(n)}$ is an absolutely continuous function for $n \in \mathbb{N}$ with $(\cdot - a_0)(b_0 - \cdot)[f^{(n+1)}]^2 \in L[a_0, b_0]$ and let p, x and y be as defined in Proposition 1.7 and let the functions G, T and Ω_3 be defined in (1.10), (3.1) and (3.5), respectively. Then, we have*

$$\begin{aligned} \Lambda(p, \xi, \eta, f) &= \frac{f(b_0) - f(a_0)}{b_0 - a_0} \Lambda(p, \xi, \eta, \text{id}) \\ &+ \frac{f'(a_0) - f'(b_0)}{b_0 - a_0} \int_{a_0}^{b_0} \Lambda(p, \xi, \eta, G(\cdot, s)) ds \\ &+ \sum_{k=2}^{n-1} \frac{k}{(k-1)!} \int_{a_0}^{b_0} \Lambda(p, \xi, \eta, G(\cdot, s)) \frac{f^k(a_0)(s-a_0)^{k-1} - f^k(b_0)(s-b_0)^{k-1}}{b_0 - a_0} ds \\ &+ \frac{[f^{(n-1)}(b_0) - f^{(n-1)}(a_0)]}{(n-3)!(b_0 - a_0)} \int_{a_0}^{b_0} \Omega_3(s) ds + R_n^3(f; a_0, b_0), \end{aligned} \quad (3.14)$$

where the remainder $R_n^3(f; a_0, b_0)$ satisfies the estimation

$$|R_n^3(f; a_0, b_0)| \leq \frac{1}{(n-3)!} \left(\frac{(b_0 - a_0)}{2} \left| T(\Omega_3, \Omega_3) \int_{a_0}^{b_0} (s - a_0)(b_0 - s) [f^{(n+1)}(s)]^2 ds \right| \right)^{1/2}. \quad (3.15)$$

Theorem 3.7 *Let all the assumptions of Theorem 3.6 valid. Then, we have*

$$\begin{aligned} \Lambda(p, \xi, \eta, f) &= \frac{f(b_0) - f(a_0)}{b_0 - a_0} \Lambda(p, \xi, \eta, \text{id}) \\ &+ \frac{f'(b_0) - f'(a_0)}{b_0 - a_0} \int_{a_0}^{b_0} \Lambda(p, \xi, \eta, G(\cdot, s)) ds \\ &+ \sum_{k=3}^{n-1} \frac{k-2}{(k-1)!} \int_{a_0}^{b_0} \Lambda(p, \xi, \eta, G(\cdot, s)) \frac{f^k(a_0)(s-a_0)^{k-1} - f^k(b_0)(s-b_0)^{k-1}}{b_0 - a_0} ds \\ &+ \frac{[f^{(n-1)}(b_0) - f^{(n-1)}(a_0)]}{(n-3)!(b_0 - a_0)} \int_{a_0}^{b_0} \Omega_4(s) ds + R_n^4(f; a_0, b_0), \end{aligned} \quad (3.16)$$

where Ω_4 is as defined in (3.6) and the remainder $R_n^4(f; a_0, b_0)$ satisfies the estimation

$$|R_n^4(f; a_0, b_0)| \leq \frac{1}{(n-3)!} \left(\frac{(b_0 - a_0)}{2} \left| T(\Omega_4, \Omega_4) \int_{a_0}^{b_0} (s - a_0)(b_0 - s) [f^{(n+1)}(s)]^2 ds \right| \right)^{1/2}. \quad (3.17)$$

Now, we state some Ostrowski type inequalities related to the generalized majorization inequalities.

Theorem 3.8 *Let all the assumptions of Theorem 2.5 hold. Furthermore, let (q, r) be a pair of conjugate exponents. Let $f^{(n)} \in L_q[a_0, b_0]$ for some $n \in \mathbb{N}$. Then, we have*

$$\begin{aligned} & \left| \Lambda(p, \xi, \eta, f) - \frac{f(b_0) - f(a_0)}{b_0 - a_0} \Lambda(p, \xi, \eta, \text{id}) \right. \\ & \quad - \frac{f'(a_0) - f'(b_0)}{b_0 - a_0} \int_{a_0}^{b_0} \Lambda(p, \xi, \eta, G(\cdot, s)) ds \\ & \quad \left. - \sum_{k=2}^{n-1} \frac{k}{(k-1)!} \int_{a_0}^{b_0} \Lambda(p, \xi, \eta, G(\cdot, s)) \frac{f^k(a_0)(s-a_0)^{k-1} - f^k(b_0)(s-b_0)^{k-1}}{b_0 - a_0} ds \right| \\ & \leq \frac{1}{(n-3)!} \|f^{(n)}\|_q \left\| \int_{a_0}^{b_0} \Lambda(p, \xi, \eta, G(\cdot, s)) \tilde{T}_{n-2}(s, t) ds \right\|_r. \end{aligned} \quad (3.18)$$

The constant on the right hand side of (3.18) is sharp for $1 < q \leq \infty$ and the best possible for $q = 1$.

Theorem 3.9 *Let all the assumptions of Theorem 2.5 hold. Furthermore, let (q, r) be a pair of conjugate exponents. Let $f^{(n)} \in L_q[a_0, b_0]$ for some $n \in \mathbb{N}$. Then, we have*

$$\begin{aligned} & \left| \Lambda(p, \xi, \eta, f) - \frac{f(b_0) - f(a_0)}{b_0 - a_0} \Lambda(p, \xi, \eta, \text{id}) \right. \\ & \quad - \frac{f'(a_0) - f'(b_0)}{b_0 - a_0} \int_{a_0}^{b_0} \Lambda(p, \xi, \eta, G(\cdot, s)) ds \\ & \quad \left. - \sum_{k=3}^{n-1} \frac{k-2}{(k-1)!} \int_{a_0}^{b_0} \Lambda(p, \xi, \eta, G(\cdot, s)) \frac{f^k(a_0)(s-a_0)^{k-1} - f^k(b_0)(s-b_0)^{k-1}}{b_0 - a_0} ds \right| \\ & \leq \frac{1}{(n-3)!} \|f^{(n)}\|_q \left\| \int_{a_0}^{b_0} \Lambda(p, \xi, \eta, G(\cdot, s)) T_{n-2}(s, t) ds \right\|_r. \end{aligned} \quad (3.19)$$

The constant on the R.H.S. of (3.19) is sharp for $1 < q \leq \infty$ and the best possible for $q = 1$.

4 Mean value theorems

In the last section, we give mean value theorems for new functionals related to our obtained results. Under the assumptions of Theorem 2.2 using (2.11), Theorem 2.3 using (2.13), Theorem 2.6 using (2.18) and Theorem 2.7 using (2.20), we define the following functionals, respectively:

$$\begin{aligned} \Gamma_1(f) &= \Delta(p_i, \xi_i, \eta_i, f) - \frac{f(b_0) - f(a_0)}{b_0 - a_0} \Delta(p_i, \xi_i, \eta_i, \text{id}) \\ & \quad - \frac{f'(a_0) - f'(b_0)}{b_0 - a_0} \int_{a_0}^{b_0} \Delta(p_i, \xi_i, \eta_i, G(\cdot, s)) ds \\ & \quad - \sum_{k=2}^{n-1} \frac{k}{(k-1)!} \int_{a_0}^{b_0} \Delta(p_i, \xi_i, \eta_i, G(\cdot, s)) \frac{f^k(a_0)(s-a_0)^{k-1} - f^k(b_0)(s-b_0)^{k-1}}{b_0 - a_0} ds. \end{aligned} \quad (4.1)$$

$$\begin{aligned} \Gamma_2(f) &= \Delta(p_i, \xi_i, \eta_i, f) - \frac{f(b_0) - f(a_0)}{b_0 - a_0} \Delta(p_i, \xi_i, \eta_i, \text{id}) \\ & \quad - \frac{f'(a_0) - f'(b_0)}{b_0 - a_0} \int_{a_0}^{b_0} \Delta(p_i, \xi_i, \eta_i, G(\cdot, s)) ds \\ & \quad - \sum_{k=3}^{n-1} \frac{k-2}{(k-1)!} \int_{a_0}^{b_0} \Delta(p_i, \xi_i, \eta_i, G(\cdot, s)) \frac{f^k(a_0)(s-a_0)^{k-1} - f^k(b_0)(s-b_0)^{k-1}}{b_0 - a_0} ds \end{aligned} \quad (4.2)$$

$$\begin{aligned}\Gamma_3(f) &= \Lambda(p, \xi, \eta, f) - \frac{f(b_0) - f(a_0)}{b_0 - a_0} \Lambda(p, \xi, \eta, \text{id}) \\ &\quad - \frac{f'(a_0) - f'(b_0)}{b_0 - a_0} \int_{a_0}^{b_0} \Lambda(p, \xi, \eta, G(\cdot, s)) ds \\ &\quad - \sum_{k=2}^{n-1} \frac{k}{(k-1)!} \int_{a_0}^{b_0} \Lambda(p, \xi, \eta, G(\cdot, s)) \frac{f^k(a_0)(s-a_0)^{k-1} - f^k(b_0)(s-b_0)^{k-1}}{b_0 - a_0} ds.\end{aligned}\quad (4.3)$$

$$\begin{aligned}\Gamma_4(f) &= \Lambda(p, \xi, \eta, f) - \frac{f(b_0) - f(a_0)}{b_0 - a_0} \Lambda(p, \xi, \eta, \text{id}) \\ &\quad - \frac{f'(a_0) - f'(b_0)}{b_0 - a_0} \int_{a_0}^{b_0} \Lambda(p, \xi, \eta, G(\cdot, s)) ds \\ &\quad - \sum_{k=3}^{n-1} \frac{k-2}{(k-1)!} \int_{a_0}^{b_0} \Lambda(p, \xi, \eta, G(\cdot, s)) \frac{f^k(a_0)(s-a_0)^{k-1} - f^k(b_0)(s-b_0)^{k-1}}{b_0 - a_0} ds\end{aligned}\quad (4.4)$$

Now, we give mean value theorems for Γ_k , $k \in \{1, 2, 3, 4\}$. Here, $f_0(x) = \frac{x^n}{n!}$.

Theorem 4.1 *Let $f \in C^n[a_0, b_0]$ and let $\Gamma_k : C^n[a_0, b_0] \rightarrow \mathbb{R}$ for $k \in \{1, 2, 3, 4\}$ be linear functionals as defined in (A1), (A2), (A3) and (A4), respectively. Then, there exists $\xi_k \in [a_0, b_0]$ for $k \in \{1, 2, 3, 4\}$ such that*

$$\Gamma_k(f) = f^{(n)}(\xi_k) \Gamma_k(f_0), \quad k \in \{1, 2, 3, 4\} \quad (4.5)$$

Proof Let us define functions

$$F_1(x) = Mf_0(x) - f(x)$$

and

$$F_2(x) = f(x) - Lf_0(x)$$

where L and M are the minimum and maximum of the image of the n -th derivative of f $[a_0, b_0]$, i.e.,

$$f^{(n)}([a_0, b_0]) = [L, M]$$

Then, F_1 and F_2 are n -convex. Hence, $\Gamma_k(F_1) \geq 0$ and $\Gamma_k(F_2) \geq 0$ and

$$L\Gamma_k(f_0) \leq \Gamma_k(f) \leq M\Gamma_k(f_0).$$

If $\Gamma_k(f_0) = 0$, then the statement obviously holds.

If $\Gamma_k(f_0) \neq 0$, then $\frac{\Gamma_k(f)}{\Gamma_k(f_0)} \in [L, M] = f^{(n)}([a_0, b_0])$, so there exist $\xi_k \in [a_0, b_0]$ such that $\frac{\Gamma_k(f)}{\Gamma_k(f_0)} = f^{(n)}(\xi_k)$. $\square$

Applying Theorem 4.1 for function $\omega = \Gamma_k(h)f - \Gamma_k(f)h$, we get the following result.

Theorem 4.2 *Let $f, h \in C^n[a_0, b_0]$ and let $\Gamma_k : C^n[a_0, b_0] \rightarrow \mathbb{R}$ for $k \in \{1, 2, 3, 4\}$ be linear functionals as defined in (A1), (A2), (A3) and (A4), respectively. Then, there exists $\xi_k \in [a_0, b_0]$ for $k \in \{1, 2, 3, 4\}$ such that*

$$\frac{\Gamma_k(f)}{\Gamma_k(h)} = \frac{f^{(n)}(\xi_k)}{h^{(n)}(\xi_k)}$$

assuming that both the denominators are non-zero.

Remark 4.3 If the inverse of $\frac{f^{(n)}}{h^{(n)}}$ exists, then from the above mean value theorems we can give generalized means

$$\xi_k = \left(\frac{f^{(n)}}{h^{(n)}} \right)^{-1} \left(\frac{\Gamma_k(f)}{\Gamma_k(h)} \right). \quad (4.6)$$

Remark 4.4 Using the same method as in [8], we can construct new families of exponentially convex functions and Cauchy type means (see also [2]). Also, using the idea described in [8] we can obtain the results for n -convex functions at point.

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