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Hyponormality of generalised slant weighted Toeplitz operators with polynomial symbols

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Abstract For a sequence of positive numbers $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$, the space $L^2(\beta)$ consists of all $f(z) = \sum_{-\infty}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$ for which $\sum_{-\infty}^{\infty} |a_n|^2 \beta_n^2 < \infty$. For a bounded function $\varphi(z) = \sum_{-\infty}^{\infty} a_n z^n$, the slant weighted Toeplitz operator $A_\varphi^{(\beta)}$ is an operator on $L^2(\beta)$ defined as $A_\varphi^{(\beta)} = W M_\varphi^{(\beta)}$, where $M_\varphi^{(\beta)}$ is the weighted multiplication operator on $L^2(\beta)$ and W is an operator on $L^2(\beta)$ such that $W z^{2n} = z^n$, $W z^{2n-1} = 0$ for all $n \in \mathbb{Z}$. In this paper we show that for a trigonometric polynomial $\varphi(z) = \sum_{n=-p}^{q} a_n z^n$, $A_\varphi^{(\beta)}$ cannot be hyponormal unless $\varphi \equiv 0$. We also show that, for $k \geq 2$ the k^{th} order slant weighted Toeplitz operator $U_{k,\varphi}^{(\beta)}$ cannot be hyponormal unless $\phi \equiv 0$. Also the compression of $U_{k,\varphi}^{(\beta)}$ to $H^2(\beta)$, denoted by $V_{k,\varphi}^{(\beta)}$, cannot be hyponormal unless $\phi \equiv 0$.

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الملخص

لكل متتالية أعداد موجبة $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$، يتكون الفضاء $L^2(\beta)$ من كل الدوال $f(z) = \sum_{-\infty}^{+\infty} a_n z^n$، $a_n \in \mathbb{C}$ بحيث أن $\sum_{-\infty}^{+\infty} |a_n|^2 \beta_n^2 < \infty$. ولدالة محدودة $\varphi(z) = \sum_{-\infty}^{+\infty} a_n z^n$، نعرف مؤثر توبليتز المثقل والمائل $A_\varphi^{(\beta)}$ على $L^2(\beta)$ كالآتي $A_\varphi^{(\beta)} = W M_\varphi^{(\beta)}$، بحيث أن $M_\varphi^{(\beta)}$ هو مؤثر الضرب المثقل على $L^2(\beta)$ و W هو المؤثر المعرف كما يلي $Wz^{2n} = z^n,\ Wz^{2n-1} = 0$، لكل $n \in \mathbb{Z}$. في هذا البحث نبين أن لكل كثير حدود مثلثي $\varphi(z) = \sum_{n=-p}^{q} a_n z^n$، لن تكون $A_\varphi^{(\beta)}$ موجبة شبه محددة إلا إذا كانت $\varphi \equiv 0$. كما نبين أيضا أنه، لكل $k \geq 2$، فإن مؤثر توبليتز المثقل والمائل $U_{k,\varphi}^{(\beta)}$ من الرتبة k لن يكون موجبا شبه محدد إلا إذا كانت $\varphi \equiv 0$. وأيضا، فإن انضغاط $U_{k,\varphi}^{(\beta)}$ إلى $H^2(\beta)$، والمرمّز ب $V_{k,\varphi}^{(\beta)}$، لن يكون موجبا شبه محدد إلا إذا كانت $\varphi \equiv 0$.

1 Introduction and preliminaries

The class of Toeplitz operators was first defined by Toeplitz in 1911 [11]. Since then this class of non-self-adjoint operators has been widely studied. By eliminating every other row of a doubly infinite Toeplitz matrix, Ho defined slant Toeplitz operators [7]. The spectral properties of this class of operators have a connection to the smoothness of wavelets, and as such, appear frequently in wavelet analysis. Various properties of slant

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Toeplitz operators have been studied in [7,12]. Motivated by the multidirectional applications of the slant Toeplitz operators, Arora and Kathuria [2] introduced the notion of slant weighted Toeplitz operators. The study of this new class has also gained momentum as it is expected to be meaningful not only to specialists in the theory of Toeplitz operators, but would also be of interest to physicists, probabilists, and computer scientists. Properties of these classes of operators are detailed in [2,3,5,6]. To define and understand slant weighted Toeplitz operators we need to begin with a few preliminaries.

Let $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ be a sequence of positive numbers with $\beta_0 = 1$, $r \leq \frac{\beta_n}{\beta_{n+1}} \leq 1$ for $n \geq 0$, and $r \leq \frac{\beta_n}{\beta_{n-1}} \leq 1$ for $n \leq 0$, for some $r > 0$. Let $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$ be the formal Laurent series (whether or not the series converges for any values of z). Define $\|f\|_\beta^2 = \sum_{n=-\infty}^{\infty} |a_n|^2\beta_n^2$. We consider the weighted sequence space

$$L^2(\beta) = \left\{ f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \mid a_n \in \mathbb{C}, \|f\|_\beta^2 = \sum_{n=-\infty}^{\infty} |a_n|^2\beta_n^2 < \infty \right\}$$

$(L^2(\beta), \|\cdot\|_\beta)$ is a Hilbert space with an orthonormal basis given by $\{e_k(z) = \frac{z^k}{\beta_k}\}_{k\in\mathbb{Z}}$ and with inner product defined by

$$\left\langle \sum_{n=-\infty}^{\infty} a_n z^n, \sum_{n=-\infty}^{\infty} b_n z^n \right\rangle = \sum_{n=-\infty}^{\infty} a_n \bar{b}_n \beta_n^2.$$

Let $L^\infty(\beta)$ denote the set of formal Laurent series $\varphi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ having the following properties:

(i) $\varphi L^2(\beta) \subseteq L^2(\beta)$, and
(ii) there exists some $c > 0$ satisfying $\|\varphi f\|_\beta \leq c\|f\|_\beta$ for each $f \in L^2(\beta)$.

For $\varphi \in L^\infty(\beta)$, define the norm $\|\varphi\|_\infty$ as

$$\|\varphi\|_\infty = \inf\{c > 0 : \|\varphi f\|_\beta \leq c\|f\|_\beta \text{ for each } f \in L^2(\beta)\}.$$

$L^\infty(\beta)$ is a Banach space with respect to $\|\cdot\|_\infty$.
We refer to [8,10] for details of the spaces $L^2(\beta)$ and $L^\infty(\beta)$.
Let $\varphi \in L^\infty(\beta)$ and $\varphi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$. $M_\varphi^{(\beta)}$ is the weighted multiplication operator on $L^2(\beta)$ defined as

$$M_\varphi^{(\beta)}(f) = \varphi f.$$

The slant weighted Toeplitz operator $A_\varphi^{(\beta)}$ on $L^2(\beta)$ is defined [2] as

$$A_\varphi^{(\beta)} = W M_\varphi^{(\beta)}$$

where W is the operator on $L^2(\beta)$ given by

$$We_{2n}(z) = \frac{\beta_n}{\beta_{2n}} e_n(z), \quad We_{2n-1}(z) = 0 \text{ for } n \in \mathbb{Z}.$$

If $\beta \equiv 1$, then $A_\varphi^{(\beta)}$ is the slant Toeplitz operator as defined by Ho [7], and denoted simply as A_φ.

2 Slant weighted Toeplitz operator

We begin with a few algebraic properties of $A_\varphi^{(\beta)}$ that are found in [2,3]. For $\varphi(z) = \sum_{n=-\infty}^{\infty} a_n z^n \in L^\infty(\beta)$ we have:

(1) $A_\varphi^{(\beta)}(e_n) = \sum_{k=-\infty}^{\infty} \frac{\beta_k}{\beta_n} a_{2k-n} e_k$ for each $n \in \mathbb{Z}$. Here $\{e_k(z) = \frac{z^k}{\beta_k}\}_{k\in\mathbb{Z}}$ is an orthonomal basis for $L^2(\beta)$.

(2) The matrix representation of $A_\varphi^{(\beta)}$ is a two way matrix. If $[\cdot]$ denotes the central $(0,0)^{th}$ entry, then the matrix of $A_\varphi^{(\beta)}$ with respect to orthonormal basis $\{e_k\}_{k\in\mathbb{Z}}$ is

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \frac{\beta_{-1}}{\beta_{-1}}a_{-1} & \frac{\beta_{-1}}{\beta_0}a_{-2} & \frac{\beta_{-1}}{\beta_1}a_{-3} & \cdots \\ \cdots & \frac{\beta_0}{\beta_{-1}}a_1 & [\frac{\beta_0}{\beta_0}a_0] & \frac{\beta_0}{\beta_1}a_{-1} & \cdots \\ \cdots & \frac{\beta_1}{\beta_{-1}}a_3 & \frac{\beta_1}{\beta_0}a_2 & \frac{\beta_1}{\beta_1}a_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

(3) A bounded operator T on $L^2(\beta)$ is a slant weighted Toeplitz operator iff $M_z^{(\beta)}T = TM_{z^2}^{(\beta)}$.
(4) The sum of two slant weighted Toeplitz operator is a slant weighted Toeplitz operator.
(5) $A_\varphi^{(\beta)}A_\psi^{(\beta)}$ is a slant weighted Toeplitz operator iff $A_\varphi^{(\beta)}A_\psi^{(\beta)} = 0$.
(6) $A_\varphi^{(\beta)*}$ is a slant weighted Toeplitz operator iff $\varphi = 0$.

In this paper we consider a trigonometric polynomial $\varphi(z) = \sum_{n=-p}^{q} a_n z^n$ and show that $A_\varphi^{(\beta)}$ cannot be hyponormal unless $\varphi \equiv 0$. It may be mentioned that in Theorem 5 [12] it is shown that A_φ is hyponormal iff $\varphi \equiv 0$.
Let $\varphi \in L^\infty(\beta)$ and $\varphi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$. The slant weighted Toeplitz operator $A_\varphi^{(\beta)}$ is hyponormal if and only if

$$[A_\varphi^{(\beta)*}, A_\varphi^{(\beta)}] = A_\varphi^{(\beta)*}A_\varphi^{(\beta)} - A_\varphi^{(\beta)}A_\varphi^{(\beta)*} \geq 0$$

Let $[\lambda_{i,j}]$ be the matrix representation of $[A_\varphi^{(\beta)*}, A_\varphi^{(\beta)}]$ with respect to orthonormal basis $\{e_k(z) = \frac{z^k}{\beta_k}\}_{k\in\mathbb{Z}}$ of $L^2(\beta)$. Then for $i, j \in \mathbb{Z}$,

$$\lambda_{i,j} = \sum_{k=-\infty}^{\infty}\left[\frac{\beta_k^2}{\beta_i\beta_j}\bar{a}_{2k-i}a_{2k-j} - \frac{\beta_i\beta_j}{\beta_k^2}a_{2i-k}\bar{a}_{2j-k}\right]$$

$\lambda_{i,i}$ are the diagonal entries of this matrix. If we denote $\lambda_{i,i}$ by d_i, then for each $i \in \mathbb{Z}$,

$$d_i = \sum_{k=-\infty}^{\infty}\left[\frac{\beta_k^2}{\beta_i^2}|a_{2k-i}|^2 - \frac{\beta_i^2}{\beta_k^2}|a_{2i-k}|^2\right]$$

As $d_i = \langle [A_\varphi^{(\beta)*}, A_\varphi^{(\beta)}]e_i, e_i\rangle$, so $d_i < 0$ would imply that $A_\varphi^{(\beta)}$ is not hyponormal. Hence for $A_\varphi^{(\beta)}$ to be hyponormal it is necessary that $d_i \geq 0\ \forall i \in \mathbb{Z}$.
Here we will show that for a non-zero trigonometric polynomial $\varphi(z) = \sum_{n=-p}^{q} a_n z^n$, there will exist $i \in \mathbb{Z}$ such that $d_i < 0$, so that $A_\varphi^{(\beta)}$ cannot be hyponormal.

3 Hyponormality of $A_\varphi^{(\beta)}$

For $\varphi \in L^\infty(\beta)$ given by $\varphi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, we consider the operator $A_\varphi^{(\beta)}$ on $L^2(\beta)$. If $\{d_n\}_{n\in\mathbb{Z}}$ represent the sequence of diagonal entries in the matrix representation of $[A_\varphi^{(\beta)*}, A_\varphi^{(\beta)}]$ with respect to orthonormal basis $\{e_k(z) = \frac{z^k}{\beta_k}\}_{k\in\mathbb{Z}}$, then for each $n \in \mathbb{Z}$ we have,

$$d_n = \sum_{k=-\infty}^{\infty}\left[\frac{\beta_k^2}{\beta_n^2}|a_{2k-n}|^2 - \frac{\beta_n^2}{\beta_k^2}|a_{2n-k}|^2\right] = \sum_{p=-\infty}^{\infty} C_p^{(n)}|a_p|^2, \qquad \text{where}$$

$$C_p^{(n)} = \begin{cases} \frac{\beta_{\frac{n+p}{2}}^2}{\beta_n^2} - \frac{\beta_n^2}{\beta_{2n-p}^2} & \text{if } (n+p) \text{ is even,} \\ -\frac{\beta_n^2}{\beta_{2n-p}^2} & \text{if } (n+p) \text{ is odd.} \end{cases}$$

We observe the following:

(1) If $p = n$ then $C_p^{(n)} = 0$, for $n \in \mathbb{Z}$.

(2) If for $n \in \mathbb{Z}$ we have $p \in \mathbb{Z}$ such that $(n + p)$ is even and $\beta_{\frac{n+p}{2}} \beta_{2n-p} = \beta_n^2$, then $C_p^{(n)} = 0$.

Let $\eta = \{(p, n) \in \mathbb{Z} \times \mathbb{Z} \mid p \neq n\}$. Clearly, for $(p, n) \in \mathbb{Z} \times \mathbb{Z} - \eta$, we have $C_p^{(n)} = 0$. For $(p, n) \in \eta$, we define order of (p, n), denoted as $o(p, n)$, and a (p, n)-induced set denoted by $[p : n]$ as follows:

Definition 3.1 For $(p, n) \in \eta$, let $u_0 := n$ and $u_m := \frac{p+u_{m-1}}{2}$ $\forall$ $m \in \mathbb{N}$. Let r be the smallest non-negative integer such that $p + u_r$ is odd. Then $o(p, n) := r$ and $[p : n] := \{u_j : 0 \leq j \leq o(p, n)\}$.

Remark 3.2 Let $(p, n) \in \eta$, with $u_0 = n$ and $u_m = \frac{p+u_{m-1}}{2}$ for $0 < m \leq o(p, n)$. Then from Definition 3.1 we have the following:

(i) If $p < n$ then $p < u_{i+1} < u_i \leq n$ $\forall$ $0 \leq i < o(p, n)$,
(ii) If $n < p$ then $n \leq u_i < u_{i+1} < p$ $\forall$ $0 \leq i < o(p, n)$.

Theorem 3.3 *If $(p, n) \in \eta$, then $\sum_{i\in[p:n]} C_p^{(i)} = -\frac{\beta_n^2}{\beta_{2n-p}^2} < 0$.*

Proof Let $r = o(p, n)$ and $[p : n] = \{u_j : 0 \leq j \leq o(p, n)\}$ where $u_0 = n$ and $u_j = \frac{p+u_{j-1}}{2}$ for $0 < j \leq r$.
If $r = 0$, then $C_p^{(n)} = -\frac{\beta_n^2}{\beta_{2n-p}^2} < 0$.
If $r > 0$, then

$$C_p^{(u_0)} = \frac{\beta_{u_1}^2}{\beta_{u_0}^2} - \frac{\beta_n^2}{\beta_{2n-p}^2},$$

$$C_p^{(u_j)} = \frac{\beta_{u_{j+1}}^2}{\beta_{u_j}^2} - \frac{\beta_{u_j}^2}{\beta_{u_{j-1}}^2} \quad \text{for} \quad 0 < j < r,$$

$$\text{and} \quad C_p^{(u_r)} = -\frac{\beta_{u_r}^2}{\beta_{u_{r-1}}^2}.$$

$$\therefore \sum_{i\in[p:n]} C_p^{(i)} = \sum_{j=0}^{r} C_p^{(u_j)} = -\frac{\beta_n^2}{\beta_{2n-p}^2} < 0.$$

□

Theorem 3.4 *Let $(p, q) \in \eta$. Then $\sum_{n=q}^{p} C_p^{(n)} < 0$ if $q < p$, and $\sum_{n=p}^{q} C_p^{(n)} < 0$ if $p < q$.*

Proof Without loss of generality, we assume that $p < q$.
Let $q_0 = q$. For $i > 0$ let q_i be the greatest integer such that $p < q_i < q_{i-1}$ and $q_i \notin [p : q_0] \cup \cdots \cup [p : q_{i-1}]$. Thus there exist some finite $q_0 > q_1 > \cdots > q_t$ such that $[p : q_i] \cap [p : q_j] = \emptyset$ for $i \neq j$, and $\cup_{i=0}^{t}[p : q_i] = \{p + 1, \ldots, q\}$.
By Theorem 3.3, $\sum_{n\in[p:q_i]} C_p^{(n)} < 0$ $\forall$ $0 \leq i \leq t$.
Therefore $\sum_{n=p}^{q} C_p^{(n)} = C_p^{(p)} + \sum_{i=0}^{t}(\sum_{n\in[p:q_i]} C_p^{(n)}) < 0$. □

Theorem 3.5 *Let $\varphi(z) = \sum_{n=m}^{q} a_n z^n$ where $m, q \in \mathbb{Z}$ and $m \leq q$. If $\varphi \not\equiv 0$, then $\sum_{n=m}^{q} d_n < 0$.*

Proof For $n \in \mathbb{Z}$, $d_n = \sum_{p=m}^{q} C_p^{(n)} |a_p|^2$. Therefore,

$$\sum_{n=m}^{q} d_n = \sum_{p=m}^{q} \left(\sum_{n=m}^{q} C_p^{(n)} \right) |a_p|^2 = \sum_{p=m}^{q} \left(\sum_{n=m}^{p} C_p^{(n)} + \sum_{n=p}^{q} C_p^{(n)} \right) |a_p|^2 < 0, \quad \text{by Theorem 3.4.}$$

□

Theorem 3.6 *Let $\varphi(z) = \sum_{n=m}^{q} a_n z^n$ where $m, q \in \mathbb{Z}$ and $m \leq q$. For $\varphi \not\equiv 0$, $A_\varphi^{(\beta)}$ cannot be hyponormal.*

Proof By Theorem 3.5, $\sum_{n=m}^{q} d_n < 0$. Thus, there exist d_n, $m \leq n \leq q$ such that $d_n < 0$. Hence $A_\varphi^{(\beta)}$ can not be hyponormal. □

4 Generalised slant weighted Toeplitz operator

We now discuss the hyponormality of the generalised slant weighted Toeplitz operator which was first defined in [4]. We consider the space $L^2(\beta)$ as introduced in Sect. 1 and for an integer $k \geq 2$, let $W_k : L^2(\beta) \to L^2(\beta)$ be defined as

$$W_k e_n(z) = \begin{cases} \frac{\beta_{\frac{n}{k}}}{\beta_n} e_{\frac{n}{k}}(z) & \text{if } n \text{ is divisible by } k, \\ 0 & \text{otherwise.} \end{cases}$$

We assume that $\{\frac{\beta_n}{\beta_{kn}}\}_{n\in\mathbb{Z}}$ is bounded so that W_k is bounded. For $\varphi \in L^\infty(\beta)$ the k^{th} order slant weighted Toeplitz operator $U^{(\beta)}_{k,\varphi} : L^2(\beta) \to L^2(\beta)$ is defined as $U^{(\beta)}_{k,\varphi} = W_k M^{(\beta)}_\varphi$, where $M^{(\beta)}_\varphi$ is the weighted multiplication operator on $L^2(\beta)$, already mentioned in Sect. 1.

The effect of $U^{(\beta)}_{k,\varphi}$ on the orthonormal basis $\{e_i(z) = \frac{z^i}{\beta_i}\}_{i\in\mathbb{Z}}$ can be given by

$U^{(\beta)}_{k,\varphi} e_i(z) = \frac{1}{\beta_i}\sum_{n=-\infty}^{\infty} a_{nk-i}\beta_n e_n(z)$, where $\varphi(z) = \sum_{n=-\infty}^{\infty} a_n z^n \in L^\infty(\beta)$.

The adjoint of $U^{(\beta)}_{k,\varphi}$, denoted by $U^{(\beta)*}_{k,\varphi}$ is given by $\langle U^{(\beta)*}_{k,\varphi} e_j, e_i\rangle = \bar{a}_{kj-i}\frac{\beta_j}{\beta_i}$.

For $k = 2$, $U^{(\beta)}_{k,\varphi}$ is the slant weighted Toeplitz operator $A^{(\beta)}_\varphi$ discussed earlier in Sects. 2 and 3. Properties of the k^{th} order slant weighted Toeplitz operator can be found in [4,6]. If $\beta_n = 1 \ \forall \ n$, then $U^{(\beta)}_{k,\varphi}$ is the k^{th} order slant Toeplitz operator denoted by $U_{k,\varphi}$ and discussed in [1,9].

Here we consider a trigonometric polynomial $\varphi(z) = \sum_{n=-p}^{q} a_n z^n$ and show that $U^{(\beta)}_{k,\varphi}$ cannot be hyponormal unless $\varphi \equiv 0$. It may be mentioned that if $\beta_n = 1 \ \forall \ n$ then the k^{th} order slant Toeplitz operator $U_{k,\varphi}$ is hyponormal iff $\varphi \equiv 0$, as shown in Theorem 5 [1].

5 Hyponormality of $U^{(\beta)}_{k,\varphi}$

Let $k \geq 2$ and $\varphi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ be in $L^\infty(\beta)$. If $\{d_n\}_{n\in\mathbb{Z}}$ represents the sequence of diagonal entries in the matrix representation of $[U^{(\beta)*}_{k,\varphi}, U^{(\beta)}_{k,\varphi}]$ with respect to orthonormal basis $\{e_i(z) = \frac{z^i}{\beta_i}\}_{i\in\mathbb{Z}}$, then for each $n \in \mathbb{Z}$ we have,

$$d_n = \sum_{i=-\infty}^{\infty}\left[\frac{\beta_i^2}{\beta_n^2}|a_{ki-n}|^2 - \frac{\beta_n^2}{\beta_i^2}|a_{kn-i}|^2\right] = \sum_{p=-\infty}^{\infty} C^{(n)}_p |a_p|^2, \quad \text{where}$$

$$C^{(n)}_p = \begin{cases} \frac{\beta^2_{\frac{p+n}{k}}}{\beta_n^2} - \frac{\beta_n^2}{\beta_{kn-p}^2} & \text{if } (p+n) \text{ is divisible by } k, \\ -\frac{\beta_n^2}{\beta_{kn-p}^2} & \text{otherwise.} \end{cases} \tag{5.1}$$

We observe the following:

(1) If $p, n \in \mathbb{Z}$ are such that $p = (k-1)n$, then $C^{(n)}_p = 0$.

(2) If $(p+n)$ is divisible by k and $\beta_{\frac{p+n}{k}}\beta_{kn-p} = \beta_n^2$, then $C^{(n)}_p = 0$.

Let $\eta_k := \{(p, n) \in \mathbb{Z}\times\mathbb{Z} \mid p \neq (k-1)n\}$. Then for $(p, n) \in \mathbb{Z}\times\mathbb{Z} - \eta_k$, we have $C^{(n)}_p = 0$.

Definition 5.1 For $(p, n) \in \eta_k$, let $u_0 := n$ and $u_m := \frac{p+u_{m-1}}{k} \ \forall \ m \in \mathbb{N}$. Let r be the smallest non-negative integer such that $p + u_r$ is not divisible by k. We define order of (p, n) as r, and denote it by $o(p, n)$. Also we define $[p : n]$ to be the set $\{u_j : 0 \leq j \leq o(p, n)\}$.

For $k = 3$ we look at the following examples:

(i) $[3 : 15] = \{15, 6, 3, 2\}$.
(ii) $[1 : -4] = \{-4, -1, 0\}$.
(iii) $[9 : 3] = \{3, 4\}$.
(iv) $[9 : 6] = \{6, 5\}$.
(v) $[-4 : -11] = \{-11, -5, -3\}$.
(vi) $[-4 : 1] = \{1, -1\}$.
(vii) $[6 : 6] = \{6, 4\}$.
(viii) $[-3 : -3] = \{-3, -2\}$.

In view of the examples above, we first try to determine values $a < b$ such that the following conditions will hold:

(i) $[p : n] = \{u_j : 0 \leq j \leq o(p, n)\}$ is contained in the interval $[a, b]$, and
(ii) for $[p : n] \subseteq [a, b]$ we have either $u_j < u_{j+1}\ \forall\ j$, or $u_{j+1} < u_j\ \forall\ j$.

For this, we first define the functions ψ_g and ψ_s on $\mathbb{R}$ as follows:

For $x \in \mathbb{R}$, let $\psi_s(x)$ be the smallest integer greater than or equal to x. Thus, $\psi_s(x) = m + 1$ if $m < x \leq m + 1,\ m \in \mathbb{Z}$.
Again let $\psi_g(x)$ be the greatest integer less than or equal to x. So, $\psi_g(x) = m$ if $m \leq x < m + 1,\ m \in \mathbb{Z}$.
Hence for $x \in \mathbb{R}$, $\psi_g(x) < x < \psi_s(x)$ and $\psi_s(x) - \psi_g(x) = 1$ if $x \notin \mathbb{Z}$, and $\psi_g(x) = x = \psi_s(x)$ if $x \in \mathbb{Z}$.
Thus for $p \in \mathbb{Z}$ and $k \geq 2$, we have

$$p = (k-1)\psi_s\left(\frac{p}{k-1}\right) - \delta, \quad 0 \leq \delta < k - 1.$$

$$p = (k-1)\psi_g\left(\frac{p}{k-1}\right) + \mu, \quad 0 \leq \mu < k - 1.$$

Following these notations, we can now state the following results:

Theorem 5.2 *For $k \geq 2$, let $(p, n) \in \eta_k$, and $p < n(k-1)$. Also let $[p : n] = \{u_i : 0 \leq i \leq o(p, n)\}$ where $u_0 = n$ and $u_{i+1} = \frac{p+u_i}{k}\ \forall\ 0 \leq i < o(p, n)$. Then*

(1) $u_{i+1} < u_i\ \forall\, 0 \leq i < o(p, n)$.
(2) $[p : n] \subseteq (\frac{p}{k-1}, n] \cap \mathbb{Z}$.

Proof (1) As $\frac{p}{k-1} < n$, so $\psi_s(\frac{p}{k-1}) \leq n$.
Since $\psi_s(\frac{p}{k-1}) = \frac{p+\delta}{k-1}$ for $0 \leq \delta < k - 1$, so
$\psi_s(\frac{p}{k-1}) \leq n \implies \frac{p+\delta}{k-1} \leq n \implies \frac{p+n}{k} \leq n - \frac{\delta}{k} \leq n \implies u_1 \leq u_0$.
However, $u_1 = u_0 \implies \frac{p+n}{k} = n \implies p = (k-1)n$, which is not possible as $(p, n) \in \eta_k$. Thus we must have $u_1 < u_0$.
Therefore, by induction, $u_{i+1} < u_i\ \forall\ 0 \leq i < o(p, n)$.
(2) If $o(p, n) = 0$ then $[p : n] = \{n\} \subset (\frac{p}{k-1}, n] \cap \mathbb{Z}$.
If $o(p, n) > 0$ then for $0 \leq i < o(p, n)$,

$$u_{i+1} < u_i \implies ku_{i+1} < ku_i \implies p + u_i < ku_i \implies \frac{p}{k-1} < u_i$$

In particular, if $r = o(p, n)$, then $\frac{p}{k-1} < u_{r-1} \implies u_r = \frac{p+u_{r-1}}{k} > \frac{p}{k-1}$.
Therefore $[p : n] = \{u_i : 0 \leq i \leq o(p, n)\} \subseteq (\frac{p}{k-1}, n] \cap \mathbb{Z}$. □

Theorem 5.3 *For $k \geq 2$, let $(p, n) \in \eta_k$ and $p > n(k-1)$. Also let $[p : n] = \{u_i : 0 \leq i \leq o(p, n)\}$ where $u_0 = n$ and $u_{i+1} = \frac{p+u_i}{k}\ \forall\ 0 \leq i < o(p, n)$. Then*

(1) $u_i < u_{i+1}\ \forall\, 0 \leq i < o(p, n)$.
(2) $[p : n] \subseteq [n, \frac{p}{k-1}) \cap \mathbb{Z}$.

Proof (1) As $n < \frac{p}{k-1}$ so $n \leq \psi_g(\frac{p}{k-1})$. Also $\psi_g(\frac{p}{k-1}) = \frac{p-\mu}{k-1}$ for $0 \leq \mu < k-1$.
Therefore $\frac{p-\mu}{k-1} \geq n \implies \frac{p+n}{k} \geq n + \frac{\mu}{k} \geq n \implies u_1 \geq u_0$.
But $u_1 = u_0 \implies p = (k-1)n$, contradicting the fact that $(p, n) \in \eta_k$.
Thus $u_0 < u_1$, and so by induction we get $u_i < u_{i+1}$ $\forall$ $0 \leq i < o(p, n)$.
(2) If $o(p, n) = 0$ then $[p : n] = \{n\} \subset [n, \frac{p}{k-1}) \cap \mathbb{Z}$.
If $o(p, n) > 0$ then for $0 \leq i < o(p, n)$,

$$u_i < u_{i+1} \implies ku_i < ku_{i+1} = p + u_i \implies u_i < \frac{p}{k-1}.$$

Also if $r = o(p, n)$, then $u_{r-1} < \frac{p}{k-1} \implies u_r = \frac{p+u_{r-1}}{k} < \frac{p}{k-1}$.
Therefore $[p : n] = \{u_i : 0 \leq i \leq o(p, n)\} \subseteq [n, \frac{p}{k-1}) \cap \mathbb{Z}$. □

Theorem 5.4 *For $k \geq 2$, let $(p, m), (p, n) \in \eta_k$. If $m \notin [p : n]$ and $n \notin [p : m]$, then $[p : n] \cap [p : m] = \emptyset$.*

Proof Let $o(p, m) = r$ and $[p : m] = \{u_0, u_1, \cdots, u_r\}$ where $u_0 = m$ and $u_j = \frac{p+u_{j-1}}{k}$ for $1 \leq j \leq r$. As $m \notin [p : n]$, so $u_0 \notin [p : n]$.

Claim: For $0 \leq j < r$, $u_j \notin [p : n] \implies u_{j+1} \notin [p : n]$.
Let, if possible, the claim does not hold. Then there exists some $0 \leq j < r$ such that $u_{j+1} \in [p : n]$ but $u_j \notin [p : n]$.
$\therefore$ either $u_{j+1} = n$, or $u_{j+1} = \frac{p+y}{k}$ for $y \in [p : n]$.
Now $u_{j+1} = n \implies n \in [p : m]$, a contradiction.
Hence $u_{j+1} = \frac{p+y}{k}$ for some $y \in [p : n]$.
But $\frac{p+y}{k} = u_{j+1} = \frac{p+u_j}{k} \implies u_j = y \in [p : n]$, which is also a contradiction. Thus the claim is established.
Therefore,

$$\begin{aligned} u_0 \notin [p : n] &\implies u_j \notin [p : n] \quad \forall\, 0 \leq j \leq r \\ &\implies [p : n] \cap [p : m] = \emptyset \end{aligned}$$

□

Theorem 5.5 *For $k \geq 2$ if $(p, n) \in \eta_k$, then $\sum_{i\in[p:n]} C_p^{(i)} = -\frac{\beta_n^2}{\beta_{kn-p}^2} < 0$.*

Proof being identical to that of Theorem 3.3, is omitted.

Theorem 5.6 *For $k \geq 2$, let $(p, m) \in \eta_k$ and $t = \psi_g(\frac{p}{k-1})$.*

(1) *If $t < m$, then $\sum_{n=t+1}^{m} C_p^{(n)} < 0$.*
(2) *If $t \geq m$, then $\sum_{n=m}^{t} C_p^{(n)} < 0$.*

Proof (1) Let $t < m$.

Claim: $p < (k-1)m$.
If $\frac{p}{k-1} \in \mathbb{Z}$ then $t = \frac{p}{k-1}$, and so $t < m \implies p < (k-1)m$.
Again if $\frac{p}{k-1} \notin \mathbb{Z}$ then $t < \frac{p}{k-1} < t + 1$.
Also $t < m \Rightarrow t + 1 \leq m$. Thus $p < m(k-1)$, and our claim is established.
Let $q_0 = m$. By Theorem 5.2 we have $[p : q_0] \subseteq (\frac{p}{k-1}, q_0] \cap \mathbb{Z} = [t+1, q_0] \cap \mathbb{Z}$. For $i > 0$ let q_i be the greatest integer such that $t < q_i < q_{i-1}$ and $q_i \notin \cup_{j=0}^{i-1}[p : q_j]$. Clearly $(p, q_i) \in \eta_k$, $[p : q_i] \subseteq [t+1, q_i]$, and $[p : q_i] \cap [p : q_j] = \emptyset$ $\forall$ $0 \leq j < i$.
Since there exist only finite number of integers in the interval $[t+1, q_0]$, so there exist distinct integers $q_0 > q_1 > \cdots > q_\xi$ such that $\cup_{i=0}^{\xi}[p : q_i] = [t+1, m] \cap \mathbb{Z}$.
Therefore $\sum_{n=t+1}^{m} C_p^{(n)} = \sum_{i=0}^{\xi}(\sum_{n\in[p:q_i]} C_p^{(n)}) < 0$, by Theorem 5.5.
(2) Let $t \geq m$.

Claim: $p > (k-1)m$.
If $\frac{p}{k-1} \in \mathbb{Z}$ then $t = \frac{p}{k-1} \implies \frac{p}{k-1} \geq m \implies p \geq m(k-1)$.
But $(p, m) \in \eta_k \implies p \neq m(k-1)$. So $p > m(k-1)$.

If $\frac{p}{k-1} \notin \mathbb{Z}$ then $t < \frac{p}{k-1} < t+1 \implies m < \frac{p}{k-1} \implies m(k-1) < p$.
Thus our claim is established.
Let $q_0 = m$. By Theorem 5.3 we have $[p : q_0] \subseteq [q_0, \frac{p}{k-1}) \cap \mathbb{Z} = [q_0, t] \cap \mathbb{Z}$. For $i > 0$ let q_i be the smallest integer such that $q_{i-1} < q_i \leq t$ and $q_i \notin \cup_{j=0}^{i-1}[p : q_j]$. Then there exist distinct integers $q_0 < q_1 < \cdots < q_\tau$ such that $[p : q_i] \cap [p : q_j] = \emptyset$ for $i \neq j$, $[p : q_i] \subseteq [q_i, t]$ $\forall\, i$, and $\cup_{i=0}^{\tau}[p : q_i] = [m, t] \cap \mathbb{Z}$.
Therefore $\sum_{n=m}^{t} C_p^{(n)} = \sum_{i=0}^{\tau}(\sum_{n\in[p:q_i]} C_p^{(n)}) < 0$, by Theorem 5.5. □

Theorem 5.7 *If $\varphi(z) = a_m z^m$ then $d_n < 0$ for each $n \in \mathbb{Z}$ such that $m+n$ is not divisible by k.*

Proof If $\varphi(z) = a_m z^m$ then for each $n \in \mathbb{Z}$, $d_n = C_m^{(n)}|a_m|^2$. If $n \in \mathbb{Z}$ such that $m+n$ is not divisible by k, then $C_m^{(n)} < 0 \implies d_n < 0$. □

Lemma 5.8 *If $p, s \in \mathbb{Z}$ and $(p, s) \notin \eta_k$, then $s = \psi_g(\frac{p}{k-1})$.*

Proof $(p, s) \notin \eta_k \implies p = (k-1)s \implies \frac{p}{k-1} = s \in \mathbb{Z}$.
Therefore, $s = \psi_g(\frac{p}{k-1})$. □

Theorem 5.9 *Let $\varphi(z) = \sum_{n=m}^{q} a_n z^n$ where $m, q \in \mathbb{Z}$ and $m < q$. If $l := \psi_g(\frac{m}{k-1}) - 1$ and $w := \psi_g(\frac{q}{k-1}) + 1$, then $\sum_{n=l}^{w} d_n < 0$.*

Proof For $p \in \mathbb{Z} \cap [m, q]$, let $t_p := \psi_g(\frac{p}{k-1})$, $l_p := t_p - 1$ and $w_p := t_p + 1$. Then $l \leq l_p < t_p < w_p \leq w$. Also by Lemma 5.8 we must have $(p, l) \in \eta_k$ and $(p, w) \in \eta_k$.
Thus $\sum_{n=l}^{w} C_p^{(n)} = \sum_{n=l}^{t_p} C_p^{(n)} + \sum_{n=t_p+1}^{w} C_p^{(n)} < 0$ by Theorem 5.6. Therefore,

$$\sum_{n=l}^{w} d_n = \sum_{n=l}^{w}\left(\sum_{p=m}^{q} C_p^{(n)}|a_p|^2\right) = \sum_{p=m}^{q}\left(\sum_{n=l}^{w} C_p^{(n)}\right)|a_p|^2 < 0.$$

□

Theorem 5.10 *Let $\varphi(z) = \sum_{n=m}^{q} a_n z^n$ where $m, q \in \mathbb{Z}$ and $m \leq q$. For $\varphi \not\equiv 0$, $U_{k,\varphi}^{(\beta)}$ cannot be hyponormal.*

Proof By Theorems 5.7 and 5.9, there exist integers $l \leq w$ such that $\sum_{n=l}^{w} d_n < 0$. This implies that $d_n < 0$ for some $l \leq n \leq w$.
Hence $U_{k,\varphi}^{(\beta)}$ can not be hyponormal. □

6 Hyponormality of $V_{k,\varphi}^{(\beta)}$

Let $V_{k,\varphi}^{(\beta)}$ be the compression of the k^{th} order generalised slant weighted Toeplitz operator $U_{k,\varphi}^{(\beta)}$ to $H^2(\beta)$. Here $H^2(\beta) = \{f(z) = \sum_{n=0}^{\infty} a_n z^n |\ a_n \in \mathbb{C}, \|f\|_\beta^2 = \sum_{n=0}^{\infty} |a_n|^2\beta_n^2 < \infty\}$. It is a Hilbert subspace of $L^2(\beta)$ with an orthonormal basis given by $\{e_n(z) = \frac{z^n}{\beta_n}\}_{n\in\mathbb{Z}_0}$ and with inner product defined by

$$\left\langle \sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{\infty} b_n z^n \right\rangle = \sum_{n=0}^{\infty} a_n \bar{b}_n \beta_n^2.$$

Note that we use the notation $\mathbb{N}_0$ to represent the set $\{0, 1, 2, \ldots\}$. For $\varphi \in L^\infty(\beta)$ given by $\varphi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ and $k \geq 2$,

$$V_{k,\varphi}^{(\beta)}(e_n) = \sum_{i=0}^{\infty} \frac{\beta_i}{\beta_n} a_{ki-n} e_i \quad \text{for each } \ n \in \mathbb{N}_0.$$

If $\{d_n\}_{n\in\mathbb{N}_0}$ represent the sequence of diagonal entries in the matrix representation of $[V_{k,\varphi}^{(\beta)*}, V_{k,\varphi}^{(\beta)}]$ with respect to orthonormal basis $\{\frac{z^i}{\beta_i}\}_{i\in\mathbb{N}_0}$, then for each $n \in \mathbb{N}_0$ we have,

$$d_n = \sum_{l=0}^{\infty}\left[\frac{\beta_l^2}{\beta_n^2}|a_{kl-n}|^2 - \frac{\beta_n^2}{\beta_l^2}|a_{kn-l}|^2\right] = \sum_{p=-\infty}^{\infty} C_p^{(n)}|a_p|^2, \quad \text{where}$$

$$C_p^{(n)} = \begin{cases} 0 & \text{if } p > kn \text{ and } (n+p) \text{ is not divisible by } k, \\ \dfrac{\beta^2_{\frac{n+p}{k}}}{\beta_n^2} & \text{if } p > kn \text{ and } (n+p) \text{ is divisible by } k, \\ -\dfrac{\beta_n^2}{\beta_{kn-p}^2} & \text{if } -n \le p \le kn \text{ and } (n+p) \text{ is not divisible by } k, \\ \dfrac{\beta^2_{\frac{n+p}{k}}}{\beta_n^2} - \dfrac{\beta_n^2}{\beta_{kn-p}^2} & \text{if } -n \le p \le kn \text{ and } (n+p) \text{ is divisible by } k, \\ -\dfrac{\beta_n^2}{\beta_{kn-p}^2} & \text{if } p < -n. \end{cases}$$

We observe the following:

(1) If for $p \in \mathbb{Z}$ there exists $n \in \mathbb{N}_0$ such that $p = (k-1)n$, then $C_p^{(n)} = 0$.
(2) If $p > kn$ and $n + p$ is not divisible by k, then $C_p^{(n)} = 0$, for $n \in \mathbb{N}_0$.
(3) If for $n \in \mathbb{N}_0$ and $p \in \mathbb{Z}$ with $-n \le p \le kn$, we have $(n+p)$ must be divisible by k and $\beta_{\frac{n+p}{k}}\beta_{kn-p} = \beta_n^2$, then $C_p^{(n)} = 0$.

Let $\zeta_k = \{(p, n) \in \mathbb{Z} \times \mathbb{N}_0 \mid p \ne (k-1)n$, and if $p > kn$ then $(p+n)$ must be divisible by $k\}$. Clearly, for $(p, n) \in \mathbb{Z} \times \mathbb{N}_0 - \zeta_k$, we have $C_p^{(n)} = 0$. For $(p, n) \in \zeta_k$, we define order of (p, n), denoted as $o(p, n)$, and a (p, n)-induced set denoted by $[p : n]$.

Definition 6.1 For $(p, n) \in \zeta_k$ let $u_0 := n$ and $u_m := \frac{p+u_{m-1}}{k}$ $\forall\, m \in \mathbb{N}$. Then

(1) For $p \ge 0$ we define $o(p, n)$ to be the smallest non-negative integer r such that $p + u_r$ is not divisible by k.
(2) For $p < -n$ we define $o(p, n)$ to be zero.
(3) For $-n \le p < 0$ let $o(p, n)$ be the smallest non-negative integer for which either $p < -u_r$ or $p + u_r$ is not divisible by k. Thus for each integer j with $0 \le j < r$, we must have $-u_j \le p$ and $p + u_j$ is divisible by k.

In all the above cases, $[p : n] := \{u_j : 0 \le j \le o(p, n)\}$.

Theorem 6.2 *Let $(p, n) \in \zeta_k$. Then the following must hold:*

(1) *If $kn < p$, then $\sum_{i \in [p:n]} C_p^{(i)} = 0$.*
(2) *If $p \le kn$, then $\sum_{i \in [p:n]} C_p^{(i)} = -\frac{\beta_n^2}{\beta_{kn-p}^2} < 0$.*

Proof Let $r = o(p, n)$ and $[p : n] = \{u_j : 0 \le j \le r\}$ where $u_0 = n$ and $u_j = \frac{p+u_{j-1}}{k}$ for $0 < j \le r$.
(1) Let $p > kn$. Then by definition of ζ_k, we must have $r > 0$. Again,

$$C_p^{(u_0)} = C_p^{(n)} = \frac{\beta^2_{\frac{n+p}{k}}}{\beta_n^2} = \frac{\beta_{u_1}^2}{\beta_{u_0}^2},$$
$$C_p^{(u_j)} = \frac{\beta_{u_{j+1}}^2}{\beta_{u_j}^2} - \frac{\beta_{u_j}^2}{\beta_{u_{j-1}}^2} \text{ for } 0 < j < r,$$
$$\text{and } C_p^{(u_r)} = -\frac{\beta_{u_r}^2}{\beta_{u_{r-1}}^2}.$$

Therefore $\sum_{i \in [p:n]} C_p^{(i)} = \sum_{j=0}^{r} C_p^{(u_j)} = 0$.
(2) This part of the proof is exactly similar to Theorem 3.3, and is therefore omitted. □

Remark 6.3 For $k \ge 2$, since $\zeta_k \subseteq \eta_k$ so Theorems 5.2–5.4 also hold for $(p, n) \in \zeta_k$.

Theorem 6.4 *If $(p, n) \in \zeta_k$ such that $-n < p < 0$, then $[p : n] \subseteq [0, n] \cap \mathbb{Z}$.*

Proof By Theorem 5.2(2) we have $[p : n] \subseteq (\frac{p}{k-1}, n] \cap \mathbb{Z}$. Now let $r = o(p, n)$, $u_0 = n$ and $u_j = \frac{p+u_{j-1}}{k}$ for $1 \leq j \leq r$. Then by Definition 6.1(3), $-u_j \leq p \; \forall \; 0 \leq j < r$. Also, $-u_{r-1} \leq p \implies u_r \geq 0$. Thus $[p : n] \subseteq [0, n] \cap \mathbb{Z}$. □

Theorem 6.5 *If $p \in \mathbb{Z}$ and $0 \leq p < q$, then $\sum_{n=0}^{q} C_p^{(n)} < 0$.*

Proof Let $t = \psi_g(\frac{p}{k-1})$ and $q_0 = q$.
Clearly as $p < (k-1)q_0$, so $(p, q_0) \in \zeta_k$ and by Theorem 5.2 $[p : q_0] \subseteq [t+1, q_0]$. Also by Theorem 6.2(2) we have $\sum_{n \in [p:q_0]} C_p^{(n)} < 0$. If $[t+1, q] \cap \mathbb{Z} \neq [p : q_0]$, then let q_1 be the greatest integer such that $t < q_1 < q_0$ and $q_1 \notin [p : q_0]$. As $q_1 \geq t + 1 > \frac{p}{k-1}$, so $p < (k-1)q_1$ and hence $(p, q_1) \in \zeta_k$, $[p : q_1] \subseteq [t+1, q_1]$ and $\sum_{n \in [p:q_1]} C_p^{(n)} < 0$. Moreover, by Theorem 5.4 $[p : q_0] \cap [p : q_1] = \emptyset$. Continuing this process we get finite distinct integers $q_0 > q_1 > \cdots > q_\xi$ in $[t+1, q_0]$ such that $[p : q_i] \cap [p : q_j] = \emptyset$ for $i \neq j$, $\sum_{n \in [p:q_j]} C_p^{(n)} < 0 \; \forall j$ and $\cup_{i=0}^{\xi} [p : q_i] = [t+1, q] \cap \mathbb{Z}$.

$$\therefore \sum_{n=t+1}^{q} C_p^{(n)} = \sum_{i=0}^{\xi} \left(\sum_{n \in [p:q_i]} C_p^{(n)} \right) < 0. \tag{6.1}$$

Case I: If $p = 0$ then $t = 0$. Also $C_0^{(0)} = 0$, since $C_p^{(n)} = 0$ if $p = (k-1)n$. Therefore, $\sum_{n=0}^{q} C_p^{(n)} = \sum_{n=t+1}^{q} C_p^{(n)} < 0$, by 6.1.

Case II: If $p > 0$, then assume $s_0 = 0$. By Theorem 5.3(2), $[p : s_0] \subseteq [s_0, t] \cap \mathbb{Z}$. For $i > 0$ let s_i be the smallest integer such that $s_{i-1} < s_i \leq t$ and $s_i \notin \cup_{j=0}^{i-1} [p : s_j]$. So, there exist distinct integers $s_0 < s_1 < \cdots < s_\tau$ such that $[p : s_i] \cap [p : s_j] = \emptyset$ for $i \neq j$, and $\cup_{i=0}^{\tau} [p : s_i] = [0, t] \cap \mathbb{Z}$. Also by Theorem 6.2 $\sum_{n \in [p:s_i]} C_p^{(n)} \leq 0 \; \forall \; i$.
Therefore $\sum_{n=0}^{t} C_p^{(n)} = \sum_{i=0}^{\tau} (\sum_{n \in [p:s_i]} C_p^{(n)}) \leq 0$.
So, $\sum_{n=0}^{q} C_p^{(n)} = \sum_{n=0}^{t} C_p^{(n)} + \sum_{n=t+1}^{q} C_p^{(n)} < 0$. □

Theorem 6.6 *If $p, q \in \mathbb{Z}$ and $p < 0 \leq q$, then $\sum_{n=0}^{q} C_p^{(n)} < 0$.*

Proof (1) Let $q < -p$ and $0 \leq n \leq q$. Then $p < -q \implies p < -n \implies [p : n] = \{n\}$ and $C_p^{(n)} < 0$. Therefore $\sum_{n=0}^{q} C_p^{(n)} < 0$.
(2) Suppose $-p \leq q$.
Let $q_0 = q$. As $p < 0 < q$, so $(p, q_0) \in \zeta_k$ and by Theorem 6.4 $[p : q_0] \subseteq [0, q_0] \cap \mathbb{Z}$. Moreover, by Theorem 6.2(2) $\sum_{n \in [p:q_0]} C_p^{(n)} < 0$. If $[p : q_0] = [0, q_0] \cap \mathbb{Z}$, then $\sum_{n=0}^{q} C_p^{(n)} = \sum_{i \in [p:q_0]} C_p^{(i)} < 0$. If $[p : q_0] \subsetneq [0, q_0] \cap \mathbb{Z}$, then let q_1 be the largest integer such that $0 \leq q_1 < q_0$ and $q_1 \notin [p : q_0]$. Continuing this process we get $q_0 > q_1 > \cdots > q_\xi \geq 0$ such that $[p : q_i] \cap [p : q_j] = \emptyset$ for $i \neq j$, $[0, q_0] \cap \mathbb{Z} = \cup_{j=0}^{\xi} [p : q_j]$ and $\sum_{n \in [p:q_j]} C_p^{(n)} < 0 \;\; \forall \; j \in \{0, 1, \ldots, \xi\}$.
Hence, $\sum_{n=0}^{q} C_p^{(n)} < 0$. □

Theorem 6.7 *Let $\varphi(z) = \sum_{n=m}^{q} a_n z^n$ where $m, q \in \mathbb{Z}$ and $m \leq q$. If $\varphi \not\equiv 0$, then either $\sum_{n=0}^{q+1} d_n < 0$, or $d_0 < 0$.*

Proof (1) Let $q \geq 0$
For $n \geq 0$, $d_n = \sum_{p=m}^{q} C_p^{(n)} |a_p|^2$, and so $\sum_{n=0}^{q+1} d_n = \sum_{p=m}^{q} (\sum_{n=0}^{q+1} C_p^{(n)}) |a_p|^2$.
Again, for $m \leq p \leq q$,

(i) if $p \geq 0$ then $\sum_{n=0}^{q+1} C_p^{(n)} < 0$ by Theorem 6.5,
(ii) if $p < 0$ then $\sum_{n=0}^{q+1} C_p^{(n)} < 0$ by Theorem 6.6.

Thus, $\sum_{n=0}^{q+1} d_n = \sum_{p=m}^{q}(\sum_{n=0}^{q+1} C_p^{(n)})|a_p|^2 < 0$.
(2) If $q < 0$, then $a_n = 0\ \forall\ n \geq 0$, and so

$$d_0 = \sum_{l=0}^{\infty}\left[\frac{\beta_l^2}{\beta_0^2}|a_{kl}|^2 - \frac{\beta_0^2}{\beta_l^2}|a_{-l}|^2\right] = -\beta_0^2\sum_{l=0}^{\infty}\frac{|a_{-l}|^2}{\beta_l^2} < 0.$$

□

Theorem 6.8 *Let $\varphi(z) = \sum_{n=m}^{q} a_n z^n$ where $m, q \in \mathbb{Z}$ and $m \leq q$. For $\varphi \not\equiv 0$, $V_{k,\varphi}^{(\beta)}$ cannot be hyponormal.*

Proof By Theorem 6.7, either $d_0 < 0$ or $\sum_{n=0}^{q+1} d_n < 0$. Thus, there exists d_n, $0 \leq n \leq q+1$ such that $d_n < 0$.

Hence $V_{k,\varphi}^{(\beta)}$ cannot be hyponormal. □

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