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A new lower bound for LS-category

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Abstract Let X be a simply connected CW-complex of finite type and \mathbb{K} an arbitrary field. In this paper, we use the Eilenberg–Moore spectral sequence of $C_*(\Omega(X), \mathbb{K})$ to introduce a new homotopical invariant $R(X, \mathbb{K})$. If X is a Gorenstein space with nonzero evaluation map, then $R(X, \mathbb{K})$ turns out to interpolate $\text{depth}(H_*(\Omega(X), \mathbb{K}))$ and $e_{\mathbb{K}}(X)$. We also define for any minimal Sullivan algebra $(\Lambda V, d)$ a new spectral sequence and make use of it to associate to any 1-connected commutative differential graded algebra (A, d) a similar invariant $R(A, d)$. When $(\Lambda V, d)$ is a minimal Sullivan model of X , this invariant fulfills the relation $R(X, \mathbb{K}) = R(\Lambda V, d)$.

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المخلص

لتكن X مجموعة CW-مركب بسيط الترابط من النوع المحدود و K حقل اختياري. في هذا البحث، نستخدم متتالية إيلينبرج-مور الطيفية للمجموعة $C_*(\Omega(X), K)$ وذلك بغرض تقديم لا متغير متشوه $r(X, K)$. إذا كانت X فضاء جورينستاين ذي راسم تقييم غير معدوم، فإن $r(X, K)$ تصبح استكمالاً للعمق $(H_*(\Omega(X), K))$ و $e_K(X)$. وكذلك، لكل جبرسوليفان أصغري $(\Lambda V, d)$ ، نعرف متتالية طيف جديدة ونستخدمها لترفق بكل جبر تفاضلي تدرجي تبديلي-1 مترابط (A, d) لا متغير مشابه $r(A, d)$. عندما يكون $(\Lambda V, d)$ نموذج سوليفان ل X ، فإن هذا اللا متغير يحقق العلاقة $r(X, K) = r(\Lambda V, d)$.

1 Introduction

Lusternik–Schnirelman category (LS-category for short), originally introduced in [19], is an integer that gives a numerical measure of possible dynamics on a smooth closed manifold. In fact, it provides a lower bound on the number of critical points admitted by any smooth function on any closed manifold. If X is a topological space, $\text{cat}(X)$ is the least integer n such that X is covered by $n + 1$ open subsets U_i , $0 \leq i \leq n$, each of which is contractible in X . It is an invariant of homotopy type of the space. Though its definition seems easy, its computation turns out to be a tough task. In [23], Toomer has introduced $e_{\mathbb{K}}(X)$ (\mathbb{K} being an arbitrary field) which proves to be a lower bound of $\text{cat}(X)$. Subsequently, Félix et al. proved the famous depth theorem which states that $\text{depth}(H_*(\Omega(X), \mathbb{K})) \leq \text{cat}(X)$ [6]. Recall that when X is a simply connected CW-complex such

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that each $H_i(X, \mathbb{K})$ is finite dimensional, then $H_*(\Omega(X), \mathbb{K})$ is a positively graded connected finite type and cocommutative Hopf algebra. It is worthwhile reminding that the enveloping algebra UL of any homotopy Lie algebra L associated to a finite type minimal Sullivan algebra $(\Lambda V, d)$ is a further example of a positively graded connected finite type and cocommutative Hopf algebra. If $\text{depth}(UL) < \infty$, then UL is left Noetherian if and only if L is finite dimensional [8, Example 1.3 and Theorem C]. In this context, Bisiaux improved the depth theorem by proving that $\text{depth}(H_*(\Omega(X), \mathbb{K})) \leq e_{\mathbb{K}}(X)$ if in addition $ev_{C^*(X, \mathbb{K})} \neq 0$ [2]. Recall from [6] that for any graded \mathbb{K} -algebra G , $\text{depth}(G)$ (possibly ∞) is the largest integer n such that $\text{Ext}_G^{i,*}(\mathbb{K}, G) = 0$, $\forall i < n$. Equivalently, $\text{depth}(G) = \inf\{p, \text{Ext}_G^{p,*}(\mathbb{K}, G) \neq 0\}$ with the convention that G has infinite depth when $\text{Ext}_G^{*,*}(\mathbb{K}, G) \equiv 0$. Here $\text{Ext}^{p,*}$ stands for the $(p, *)$ component of the graded functor Ext (see §20 in [9] for instance).

The object of this paper is twofold. First, inspired by the definition of Toomer's invariant in terms of the Milnor–Moore spectral sequence, we use the Eilenberg–Moore spectral sequence

$$E_2^{p,q} = \text{Ext}_{H_*(\Omega(X), \mathbb{K})}^{p,q}(\mathbb{K}, H_*(\Omega(X), \mathbb{K})) \Rightarrow \mathcal{E}xt_{C_*(\Omega(X), \mathbb{K})}^{p+q}(\mathbb{K}, C_*(\Omega(X), \mathbb{K})) \quad (1.0.1)$$

to define the integer

$$R(X, \mathbb{K}) = \sup\{p \in \mathbb{N} \mid \mathcal{E}_\infty^{p,*} \neq 0\}$$

for any simply connected CW-complex X of finite type and any field \mathbb{K} . If there is no such integer, we put $R(X, \mathbb{K}) = \infty$. Here $\mathcal{E}xt$ is the differential-Ext of Eilenberg–Moore (see Sect. 2 for more details). Our first main theorem proves the following improvement of Bisiaux result:

Theorem 1.1 *Let X be a simply connected CW-complex such that each $H_i(X, \mathbb{K})$ is finite dimensional over an arbitrary field \mathbb{K} . If X is Gorenstein and $ev_{C^*(X, \mathbb{K})}$ is nonzero, then*

$$\text{depth}(H_*(\Omega(X), \mathbb{K})) \leq R(X, \mathbb{K}) \leq e_{\mathbb{K}}(X).$$

On the other hand, in order to make an algebraic study of $R(X, \mathbb{K})$, our second goal is to associate to any minimal Sullivan algebra $(\Lambda V, d)$ the following spectral sequence:

$$\mathcal{E}xt_{(\Lambda V, d_k)}^{p,q}(\mathbb{K}, (\Lambda V, d_k)) \Rightarrow \mathcal{E}xt_{(\Lambda V, d)}^{p+q}(\mathbb{K}, (\Lambda V, d)) \quad (1.0.2)$$

which we call the Eilenberg–Moore spectral sequence of $(\Lambda V, d)$. Throughout, the differential d_k (cf. Sect. 3.1) is the first nonzero homogeneous part of the differential d . Now, given a 1-connected commutative differential graded algebra (A, d) and $(\Lambda V, d)$ its minimal Sullivan model, we set

$$R(A, d) := \sup\{p \in \mathbb{N} \mid \mathcal{E}_\infty^{p,*}[(\Lambda V, d)] \neq 0\},$$

where $\mathcal{E}_\infty^{*,*}[(\Lambda V, d)]$ is the ∞ term of (1.0.2). Similarly, if there is no such integer, we put $R(A, d) = \infty$.

In this perspective, our second main result reads the following:

Theorem 1.2 *Let \mathbb{K} be a field whose $\text{char}(\mathbb{K}) \neq 2$. If X is a simply connected CW-complex of finite type (or else in the range of Anick) and $(\Lambda V, d)$ its minimal Sullivan model, then*

1. *the two Eilenberg–Moore spectral sequence (1.0.1) and (1.0.2) are isomorphic.*
2. *$R(X, \mathbb{K}) = R(\Lambda V, d) \leq R(\Lambda V, d_k)$ and the equality holds if $\dim(V) < \infty$.*

2 Preliminaries

This section provides the tools and notions which are useful in the sequel. All graduations are written either as superscripts (for cohomology) or as subscripts (for homology) with the convention $V^k = V_{-k}$. A commutative differential graded algebra (resp., differential graded algebra, resp., differential graded Lie algebra) will be abbreviated by cdga (resp., dga, resp., dgl). The suspension (resp., desuspension, resp., dual) of a graded \mathbb{K} -vector space V is defined by $(sV)^p = V^{p+1}$ (resp., $(s^{-1}V)^p = V^{p-1}$, resp., $V^\vee = \text{Hom}_{\mathbb{K}}(V, \mathbb{K})$).



2.1 A minimal Sullivan model

Let \mathbb{K} be a field of characteristic $p \neq 2$. A Sullivan algebra is a free cdga $(\Lambda V, d)$, where

$$\Lambda V = \text{Exterior}(V^{\text{odd}}) \otimes \text{Symmetric}(V^{\text{even}}),$$

generated by the graded \mathbb{K} -vector space $V = \bigoplus_{i=0}^{\infty} V^i$ which has a well ordered basis $\{x_\alpha\}_{\alpha \in I}$ such that $dx_\alpha \in \Lambda V_{<\alpha}$ ($V_{<\alpha} = \text{span}\{v_\gamma, \gamma < \alpha\}$). Such algebra is said to be minimal if

$$\text{deg}(x_\alpha) < \text{deg}(x_\beta) \implies \alpha < \beta, \quad \forall \alpha, \beta \in I.$$

If $V^0 = \mathbb{K}$ and $V^1 = 0$, this is equivalent to say that $d(V) \subseteq \bigoplus_{i=2}^{\infty} \Lambda^i V$, where $\Lambda^i V$ designates the subspace of ΛV spanned by elements of word-length i . A minimal Sullivan model for a cdga (A, d) is a minimal Sullivan algebra $(\Lambda V, d)$ equipped with a quasi-isomorphism $(\Lambda V, d) \xrightarrow{\sim} (A, d)$. By [12, Theorem 7.1], if $H^0(A, d) = \mathbb{K}$, $H^1(A, d) = 0$ and $\dim(H^i(A, d)) < \infty$ for all $i \geq 0$, then (A, d) always has a minimal Sullivan model.

To define such a notion for spaces, we distinguish two cases. First, suppose that \mathbb{K} has characteristic zero and let X be a finite type simply connected CW-complex. The minimal Sullivan model $(\Lambda V, d)$ of X is by definition that of the cdga $A_{PL}(X)$ of polynomial differential forms on X with coefficients in \mathbb{K} [22]. It is unique (up to quasi-isomorphism) and its generator satisfies the isomorphism $V \cong \text{Hom}_{\mathbb{Z}}(\pi_*(X), \mathbb{Q})$. Now, assume that $\text{char}(\mathbb{K}) = p > 2$ and let X be an r -connected CW-complex with $\dim(X) < rp$ (some $r \geq 1$). For such a space, said to be in the range of Anick [1], the chain algebra $(C_*(\Omega(X), \mathbb{K}))$ is quasi-isomorphic to the enveloping algebra UL of an appropriate finite type dgl $L = L_{\geq 1}$. Denote by $C^*(L) = (\Lambda(sL)^\vee, d)$ the Cartan–Chevalley–Eilenberg complex of L [12]. This is a cdga which is related to $C^*(X, \mathbb{K})$ by a sequence of chain homotopies [1]. We still call its minimal Sullivan model, the minimal Sullivan model of X .

2.2 Eilenberg–Moore functors

Given (A, d) a connected augmented \mathbb{K} -dga on an arbitrary field \mathbb{K} and denote by $A_\#$ its underlying graded algebra. Following [7] (see also §6. in [9]), an (A, d) -module (P, d) is called free if it is free as an $A_\#$ -module on a basis of cocycles. It is said (A, d) -semi-free if it is the union of (A, d) -submodules $0 = F_{-1} \subset F_0 \subset F_1 \subset \dots$ such that each F_i/F_{i-1} is free. If in addition $\mathbb{K} \otimes_A P$ is zero, (P, d) is called an (A, d) -semi-free minimal module.

Now, let (M, d) be an (A, d) -module. A morphism $(P, d) \rightarrow (M, d)$ of degree zero of (A, d) -modules inducing an isomorphism in cohomology is called a quasi-isomorphism of A -modules. We denote it by $(P, d) \xrightarrow{\sim} (M, d)$ and call it an (A, d) -semi-free resolution (resp., minimal semi-free resolution) of (M, d) if (P, d) is (A, d) -semi-free (resp., (A, d) -semi-free and minimal). If $A^1 = 0$ and $M = M^{\geq r}$ (some $r \in \mathbb{Z}$) such a resolution exists and may be chosen to be minimal [7, Lemma A.3].

We are now in a position to recall the definition of Eilenberg–Moore functors called also differential graded Tor and Ext, since they are introduced in the context of differential graded homological algebra, where semi-free resolutions replace ordinary free resolutions.

Given another (A, d) -module (N, d) and let $(P, d) \xrightarrow{\sim} (M, d)$ an (A, d) -semi-free resolution of (M, d) . The (A, d) -module $\text{Hom}_A(P, N) = \bigoplus_{p \geq 0} \text{Hom}_A^{p,*}(P, N)$ where the p -component $\text{Hom}_A^{p,*}(P, N) = \bigoplus_{i \geq 0} \text{Hom}_A(P^i, N^{i+p})$, is provided with the differential defined by:

$$\mathcal{D}(f) = d \circ f - (-1)^p f \circ d, \quad \forall f \in \text{Hom}_A^{p,*}(P, N).$$

The Eilenberg–Moore functor Ext is defined as follows:

$$\text{Ext}_{(A,d)}((M, d), (N, d)) = H^*(\text{Hom}_A(P, N), \mathcal{D}).$$

Analogously, if $P \otimes_A N = \bigoplus_{p \geq 0} (P \otimes_A N)_{p,*}$ with $(P \otimes_A N)_{p,*} = \bigoplus_{i \geq 0} (P_i \otimes_A N_{i-p})$ is endowed with the following differential

$$\mathcal{D}(p \otimes n) = d(p) \otimes n + (-1)^p p \otimes d(n), \quad \forall p \otimes n \in (P \otimes_A N)_{p,*},$$

we obtain the Eilenberg–Moore functor Tor defined by

$$\text{Tor}_{(A,d)}((M, d), (N, d)) = H_*(P \otimes_A N, \mathcal{D}).$$

Remark 2.1 Now assume that $\text{char}(\mathbb{K}) \neq 2$. Consider a dga (A, d) over \mathbb{K} endowed with an augmentation $\varepsilon : A \rightarrow \mathbb{K}$ and denote by $\bar{A} = \text{Ker}(\varepsilon)$ its ideal of augmentation. Recall that the reduced bar-construction $(A \otimes B(A), d)$ with coefficients in A , where $B(A) = \bigoplus_{n \geq 0} T^n(s\bar{A})$, is a semi-free resolution of \mathbb{K} as an (A, d) -module (cf. §19 in [9] or §2.2 in [18]). On the other hand, $H(A, d) \otimes B(H(A, d))$ being a free resolution of \mathbb{K} as an $H(A, d)$ -module [9, Proposition 20.11], there is an Eilenberg–Moore resolution $(P, d) \xrightarrow{\sim} \mathbb{K}$ of \mathbb{K} as an (A, d) -module. Therefore, there is an equivalence of (A, d) -modules: $(P, d) \xrightarrow{\sim} (A \otimes B(A), d)$ [9, Proposition 6.6]. Roughly speaking, one can suppose the E_1 term of the spectral sequence induced by the filtration $(F^q = \bigoplus_{n \leq q} A \otimes T^n(s\bar{A}))_{q \geq 0}$ on $A \otimes B(A)$ to be an $H(A, d)$ -semi-free resolution of \mathbb{K} .

2.3 Evaluation map and Gorenstein spaces

Let (A, d) be an augmented dga over an arbitrary field \mathbb{K} and $\rho : (P, d) \xrightarrow{\sim} (\mathbb{K}, 0)$ any minimal semi-free resolution of \mathbb{K} . A chain map:

$$\text{Hom}_{(A,d)}((P, d), (A, d)) \longrightarrow (A, d) \quad (2.3.1)$$

is given by $f \mapsto f(z)$, where $z \in P$ is a cocycle representing $1_{\mathbb{K}}$. Passing to the cohomology, we obtain the evaluation map of (A, d) :

$$ev_{(A,d)} : \mathcal{E}xt_{(A,d)}(\mathbb{K}, (A, d)) \longrightarrow H^*(A, d). \quad (2.3.2)$$

Note that the definition of $ev_{(A,d)}$ is independent on the choice of (P, d) and z . Moreover, it is natural with respect to (A, d) . As a particular case, $ev_{C^*(X, \mathbb{K})}$ is called the evaluation map of X over \mathbb{K} .

On the other hand, the authors of [7] introduced the concept of a Gorenstein space over \mathbb{K} . It is a space X such that $\dim \mathcal{E}xt_{C^*(X, \mathbb{K})}(\mathbb{K}, C^*(X, \mathbb{K})) = 1$. In addition, if $\dim H^*(X, \mathbb{K}) < \infty$, then X satisfies the Poincaré duality property over \mathbb{K} and its fundamental class is closely related to the evaluation map (See, [11, 16, 21] for more details).

2.4 The Toomer invariant

We assume that $\text{char}(\mathbb{K}) \neq 2$ and consider the projection

$$p_n : \Lambda V \rightarrow \Lambda V / \Lambda^{\geq n+1} V$$

of a minimal Sullivan algebra $(\Lambda V, d)$ onto the quotient dga obtained by factoring out by the differential graded ideal generated by monomials of length at least $n + 1$. Thus, we define The Toomer invariant $e_{\mathbb{K}}(\Lambda V, d)$ to be the smallest integer n (possibly ∞) such that p_n induces an injection in cohomology. By [4], if $\text{char}(\mathbb{K}) = 0$ and $(\Lambda V, d)$ is a minimal Sullivan model of a simply connected finite type CW-complex X , then $e_{\mathbb{K}}(X) = e_{\mathbb{K}}(\Lambda V, d)$, where $e_{\mathbb{K}}(X)$ denotes the classical Toomer invariant introduced in [23]. If X is taken in the range of Anick, applying a similar argument yields in odd characteristic the coincidence $e_{\mathbb{K}}(X) = e_{\mathbb{K}}(\Lambda V, d)$.

Now, consider an arbitrary field \mathbb{K} and denote by $(T(W), d)$ the free model of X introduced in [14]. In a similar way, Halperin and Lemaire defined the invariant $e_{\mathbb{K}}(T(W), d)$ with respect to the projection

$$p_n : T(W) \rightarrow T(W) / T^{\geq n+1} W$$

and showed that $e_{\mathbb{K}}(X) = e_{\mathbb{K}}(T(W), d)$.

Recall finally that an alternative version of the Toomer invariant is given in terms of the Milnor–Moore spectral sequence

$$\text{Ext}_{H_*(\Omega(X), \mathbb{K})}^{p,q}(\mathbb{K}, \mathbb{K}) \Rightarrow H^{p+q}(X, \mathbb{K}) \quad (2.4.1)$$

(cf. [23] or [14], Prop. 1.6 (iii)) by

$$e_{\mathbb{K}}(X) = \sup\{p \in \mathbb{N} \mid E_{\infty}^{p,*} \neq 0\}$$

or ∞ if such p doesn't exist.



Remark 2.2 In [4], it is shown that for any minimal Sullivan model $(\Lambda V, d)$ of X , the Milnor–Moore spectral sequence (2.4.1) and the following one:

$$H^{p,q}(\Lambda V, d_2) \Rightarrow H^{p+q}(\Lambda V, d), \tag{2.4.2}$$

are isomorphic from their second terms. Here d_2 designates the quadratic part of the differential d .

Therefore, whenever X satisfies the Poincaré duality property and denoting by ω its fundamental class,

$$e_{\mathbb{K}}(X) = e_{\mathbb{K}}(\Lambda V, d) = \sup\{p/\omega \text{ can be represented by a cocycle in } \Lambda^{\geq p} V\}$$

[4, Lemma 10.1]. Similarly, when \mathbb{K} is any field and $(T(W), d)$ is a minimal free model of X over \mathbb{K} , by [2, Lemma 2.1], we have

$$e_{\mathbb{K}}(T(W), d) = \sup\{p/\omega \text{ can be represented by a cocycle in } T^{\geq p} W\}.$$

3 Main results

In this section, we first introduce the spectral sequence (1.0.2) and then use it, in conjunction with (1.0.1), to give the proofs of our main results.

3.1 Eilenberg–Moore spectral sequence of a free cdga

Let $(\Lambda V, d)$ be a free cdga over a field \mathbb{K} whose $\text{char}(\mathbb{K}) \neq 2$ and assume that $d = \sum_{i \geq k} d_i$, with $d_i(V) \subseteq \Lambda^i V$ and $k \geq 2$. The map d is an algebra derivation defined on V ; that is,

$$d(xy) = d(x)y + (-1)^{|x|}x d(y), \quad \forall x, y \in V.$$

So, by extension, we have $d(\Lambda^i V) \subseteq \Lambda^{\geq i+k-1} V, \forall i \geq 1$. Hence each d_i is also a derivation defined on V and particularly, for any $x \in V, d_k^2(x) \in \Lambda^{2k-1} V$ is by word-length reason the $(2k - 1)^{\text{th}}$ homogeneous part of $d^2(x)$. Whence $d_k^2 = 0$ and then $(\Lambda V, d_k)$ is also a free cdga.

As $\text{char}(\mathbb{K})$ may be nonzero, in the sequel, we will use the divided power algebra $\Gamma(sV)$ (see for instance [12]). If $\{v_i\}_{i \in I}$ is a well ordered basis of V and $V_{<i}$ denotes the subspace generated by $\{v_j, j < i\}$, the differential D on the product algebra $\Lambda V \otimes \Gamma(sV)$ restricts to d_V on V and on sV it is given by:

$$D(sv_i) = v_i + \phi, \quad \phi \in \Lambda V_{<i} \otimes \Gamma(sV_{<i}), \quad \forall i \in I. \tag{3.1.1}$$

[21, Remark 1.2]. So, by extension, we have

$$D(\gamma^p(sv)) = D(sv)\gamma^{p-1}(sv); \quad \forall p \geq 1, \quad \forall sv \in (sV)^{\text{even}}. \tag{3.1.2}$$

$(\Lambda V \otimes \Gamma(sV), d)$ is a differential graded algebra and also an $(\Lambda V, d)$ -semi-free module. Therefore, the projection $(\Lambda V \otimes \Gamma(sV), D) \xrightarrow{\sim} \mathbb{K}$ is a semi-free resolution of \mathbb{K} called an acyclic closure of $(\Lambda V, d)$ (cf. section 2 in [12]). When $\mathbb{K} = \mathbb{Q}$, $\Gamma(sV)$ is replaced by the free cdga $\Lambda(sV)$.

Consider now on $\mathcal{A} = \text{Hom}_{(\Lambda V, d)}((\Lambda V \otimes \Gamma(sV), D), (\Lambda V, d))$ the filtration

$$\mathcal{F}^p = \{f \in \text{Hom}_{\Lambda V}(\Lambda V \otimes \Gamma(sV), \Lambda V) \mid f(\Gamma(sV)) \subseteq \Lambda^{\geq p} V\}, \quad \forall p \geq 0 \tag{3.1.3}$$

and the differential defined by

$$\mathcal{D}(f) = d \circ f + (-1)^{|f|+1} f \circ D, \quad \forall f \in \mathcal{A}.$$

Lemma 3.1 *The filtration (3.1.3) verifies the following:*

- (i) $(\mathcal{F}^p)_{p \geq 0}$ is decreasing,
- (ii) $\mathcal{F}^0(\mathcal{A}) = \mathcal{A}$,
- (iii) $\mathcal{D}(\mathcal{F}^p(\mathcal{A})) \subseteq \mathcal{F}^p(\mathcal{A})$.

Proof Properties (i) and (ii) are immediate. Moreover, the property (iii) follows from the definition of \mathcal{D} on \mathcal{A} , specially, the relation $D(\gamma^p(sv)) = (v + \phi)\gamma^{p-1}(sv)$ on $(sV)^{\text{even}}$ [cf. (3.1.1) and (3.1.2) above]. \square

The general term of the spectral sequence induced by the filtration (3.1.3) is given by:

$$\mathcal{E}_r^p = \frac{\{f \in \mathcal{F}^p, \mathcal{D}(f) \in \mathcal{F}^{p+r}\}}{\{f \in \mathcal{F}^{p+1}, \mathcal{D}(f) \in \mathcal{F}^{p+r}\} + \mathcal{F}^p \cap \mathcal{D}(\mathcal{F}^{p-r+1})}.$$

Moreover, $\mathcal{A}^{r,s} = (\text{Hom}_{\Lambda V}^r(\Lambda V \otimes \Gamma(sV), \Lambda V))^{r+s} = \{f \in \mathcal{A}, f(\Gamma(sV)) \subseteq \Lambda^r V\}^{r+s}$ is isomorphic to $(\mathcal{F}^r / \mathcal{F}^{r+1})^{r+s}$. A straightforward calculation permits to prove that

$$\mathcal{E}_k^{p,q} \cong \frac{\text{Ker}[\mathcal{A}^{p,q} \xrightarrow{\mathcal{D}_k} \mathcal{A}^{p+k-1,q-k+2}]}{\text{Im}[\mathcal{A}^{p-k+1,q+k-2} \xrightarrow{\mathcal{D}_k} \mathcal{A}^{p,q}]},$$

where \mathcal{D}_k stands for the differential of $\text{Hom}_{(\Lambda V, d_k)}((\Lambda V \otimes \Gamma(sV), D_k), (\Lambda V, d_k))$. Hence, we obtain the isomorphism of graded modules $\mathcal{E}_k^{*,*} \cong \bigoplus_{p,q \geq 0} \mathcal{E}xt_{(\Lambda V, d_k)}^{p,q}(\mathbb{K}, (\Lambda V, d_k))$. This yields the spectral sequence

$$\mathcal{E}xt_{(\Lambda V, d_k)}^{p,q}(\mathbb{K}, (\Lambda V, d_k)) \Rightarrow \mathcal{E}xt_{(\Lambda V, d)}^{p+q}(\mathbb{K}, (\Lambda V, d)) \tag{3.1.4}$$

which we call the Eilenberg–Moore spectral sequence of $(\Lambda V, d)$.

Remark 3.2 1. When $\dim(V) < \infty$, the filtration (3.1.3) is clearly bounded in the sense of Theorem 2.6 in [20]; that is, for each dimension n , there exists $s = s(n)$ and $t = t(n)$ such that

$$\{0\} = \mathcal{F}^s(\mathcal{A}^n) \subseteq \mathcal{F}^{s-1}(\mathcal{A}^n) \subseteq \dots \subseteq \mathcal{F}^{t+1}(\mathcal{A}^n) \subseteq \mathcal{F}^t(\mathcal{A}^n) = \mathcal{A}^n.$$

So the spectral sequence (3.1.4) is convergent. Furthermore, if V is of finite type, the convergence is a consequence of Theorem 1.2 (1).

2. Consider on $(\Lambda V, d)$ the filtration defined by

$$F^p = \Lambda^{\geq p} V = \bigoplus_{i=p}^{\infty} \Lambda^i V. \tag{3.1.5}$$

An easy calculation shows that it induces the following spectral sequence:

$$H^{p,q}(\Lambda V, d_k) \Rightarrow H^{p+q}(\Lambda V, d) \tag{3.1.6}$$

whose convergence is guaranteed if $(\Lambda V, d)$ is a minimal Sullivan algebra (cf. §9. in [4]). Moreover, the chain map (2.3.1) is filtration-preserving, so that, the evaluation map $ev_{(\Lambda V, d)} : \mathcal{E}xt_{(\Lambda V, d)}(\mathbb{K}, (\Lambda V, d)) \longrightarrow H^*(\Lambda V, d)$ is a morphism of the spectral sequences (3.1.4) and (3.1.6). Notice that if $k = 2$, we find the spectral sequence (2.4.2).

3.2 Proofs of the main results

Let us note at the outset that the proofs of our main results are consequences of the following Proposition:

Proposition 3.3 *Let X be a simply-connected CW-complex of finite type (or else in the range of Anick) and denote by $(\Lambda V, d)$ its minimal Sullivan model. Then, the cohomological Eilenberg–Moore spectral sequence of $C_*(\Omega(X), \mathbb{K})$ is convergent and it is isomorphic to the one of $(\Lambda V, d)$.*

Before giving the proof of this proposition, we recall beforehand the construction of the Eilenberg–Moore spectral sequence associated with $C_*(\Omega(X), \mathbb{K})$ as well as that of the one introduced by Bisiaux in [2]. In fact, we will show that the latter is isomorphic to each of the two spectral sequences of Eilenberg–Moore, namely (1.0.1) and (1.0.2).

Notice first that for any field \mathbb{K} and any simply connected CW-complex X of finite type, the Adams–Hilton model $A = (T(W), d) \xrightarrow{\cong} C_*(\Omega(X), \mathbb{K})$ is a finite type free model of $C_*(\Omega(X), \mathbb{K})$ [14]. Now, consider on $A \otimes B(A)$, the filtration defined by

$$IF^q = \bigoplus_{n \leq q} A \otimes T^n(s\bar{A}), \quad \forall q \geq 0$$

and endow $\text{Hom}_A(A \otimes B(A), A)$ with the following one:

$$\mathcal{IF}^q = \{f \mid f(\mathcal{IF}^k) = 0, \forall k < q\}, \quad \forall q \geq 0. \tag{3.2.1}$$

Clearly $\mathcal{IF}^0 = \text{Hom}_A(A \otimes B(A), A)$, $(\mathcal{IF}^q)_{q \geq 0}$ is decreasing and it is stable with respect to the differential of $\text{Hom}_A(A \otimes B(A), A)$. So it induces a cohomological spectral sequence which is, by Remark 2.1, the Eilenberg–Moore spectral sequence (1.0.1) (see for instance [9], §20(d)).

On the other hand, let $(B, d) = (\mathbb{K} \oplus B^{\geq 2}, d)$ be a dga quasi-isomorphic to $C^*(X, \mathbb{K})$ (\mathbb{K} being any field) and denote by $(T(Z), d) \xrightarrow{\sim} (B, d)$ its free minimal model [14]. An acyclic closure of $(T(Z), d)$ has the form $(T(Z) \otimes (sZ \oplus \mathbb{K}), D)$ [2]. Therefore, by taking on $(\text{Hom}_{T(Z)}(T(Z) \otimes (sZ \oplus \mathbb{K}), T(Z)), \mathcal{D})$ the filtration

$$\mathcal{IF}_{(T(Z),d)}^q = \{f \mid f(T(Z) \otimes (sZ \oplus \mathbb{K})) \subseteq T^{\geq q} Z\}, \forall q \geq 0, \tag{3.2.2}$$

we obtain the following convergent spectral sequence introduced by Bisiaux [2]:

$$\text{Ext}_{(T(Z),d_2)}^{p,q}(\mathbb{K}, (T(Z), d_2)) \Rightarrow \text{Ext}_{(T(Z),d)}^{p+q}(\mathbb{K}, (T(Z), d)). \tag{3.2.3}$$

Proof of Proposition 3.3 Let us denote by $\Omega(A) = (T(s^{-1}\overline{A^\vee}), d)$ the dual of $B(A)$. Thus $(A \otimes B(A))^\vee = \Omega(A) \otimes A^\vee$ is a left $\Omega(A)$ -module and then the filtration

$$\mathbb{F}'^q = \bigoplus_{n \leq q} \Omega(A) \otimes A_n^\vee, \quad \forall q \geq 0$$

exhibits it as an $\Omega(A)$ -semi-free resolution of \mathbb{K} [7]. Again, by Remark 2.1, we can assume that it is an Eilenberg–Moore semi-free resolution of \mathbb{K} . Notice that $\Omega(A)$, denoted thereafter by $(T(W'), d)$, is a finite-type free model of $C^*(X, \mathbb{K})$ [14, Proposition 1.6].

The rest of the proof falls into two steps:

Step 1. (In this step, we assume \mathbb{K} an arbitrary field). Using [7, Remark 1.3], we will replace $C_*(\Omega(X), \mathbb{K})$ and $C^*(X, \mathbb{K})$, respectively, by A and $\Omega(A)$. Thus, referring to [7, Theorem 2.1], the isomorphism:

$$\text{Ext}_{C_*(\Omega(X), \mathbb{K})}(\mathbb{K}, C_*(\Omega(X), \mathbb{K})) \cong \text{Ext}_{C^*(X, \mathbb{K})}(\mathbb{K}, C^*(X, \mathbb{K})) \tag{3.2.4}$$

is deduced from the following isomorphisms of complexes:

$$\begin{aligned} (\text{Hom}_A(A \otimes B(A), A), \mathcal{D}) &\xrightarrow{\sim \varphi_A} (\text{End}_{A \otimes B(A)}(A \otimes B(A)), [d, \]) \\ &\xrightarrow{\cong \vee} (\text{End}_{\Omega(A) \otimes A^\vee}(\Omega(A) \otimes A^\vee), [d^\vee, \]) \\ &\xleftarrow{\sim \varphi_{\Omega(A)}} (\text{Hom}_{\Omega(A)}(\Omega(A) \otimes A^\vee, \Omega(A)), \mathcal{D}). \end{aligned}$$

Here, $\varphi_A(f) = (f \otimes id_{B(A)}) \circ (id_A \otimes \Delta_{B(A)})$ and its inverse map is the projection $\psi_{A \otimes B(A)}(g) = (Id_A \otimes \varepsilon_{B(A)}) \circ g$. $\Delta_{B(A)}$ and $\varepsilon_{B(A)}$ are, respectively, the diagonal and the co-unity of BA (see for instance [18, Proposition 1.5.14]).

We endow respectively $\text{Hom}_A(A \otimes B(A), A)$ and $\text{Hom}_{\Omega(A)}(\Omega(A) \otimes A^\vee, \Omega(A))$ with the filtrations \mathcal{IF}^q and $\mathcal{IF}_{(T(W'),d)}^q = \{f \mid f(\Omega(A) \otimes A^\vee) \subseteq \Omega(A)^{\geq q}\}$. If $f \in \mathcal{IF}^q$, then

$$\varphi_A(f)(A \otimes B(A)^{<q}) = 0 \text{ and } \varphi_A(f)(A \otimes B(A)^{\geq q}) \subseteq A \otimes B(A)^{\geq q}.$$

So

$$\vee \circ \varphi_A(f)(\Omega(A)^{<q} \otimes A^\vee) = 0 \text{ and } \vee \circ \varphi_A(f)(\Omega(A)^{\geq q} \otimes A^\vee) \subseteq \Omega(A)^{\geq q} \otimes A^\vee.$$

Hence $\vee \circ \varphi_A(f)(\Omega(A) \otimes A^\vee) \subseteq \Omega(A)^{\geq q} \otimes A^\vee$.

Applying $\varphi_{\Omega A}^{-1}$, we conclude that $\varphi_{\Omega A}^{-1} \circ \vee \circ \varphi_A(f)(\Omega(A)^{\geq q} \otimes A^\vee) \subseteq \Omega(A)^{\geq q}$. Therefore,

$$\varphi_{\Omega A}^{-1} \circ \vee \circ \varphi_A(\mathcal{IF}^q) \subseteq \mathcal{IF}_{(T(W'),d)}^q$$

and then the composition isomorphism is one of filtered complexes. Consequently, the spectral sequence (1.0.1) is isomorphic to the one (3.2.3) and then it is convergent.

Step 2. (Here we assume $\mathbb{K} = \mathbb{Q}$ and notice that the same proof remains valid in odd characteristic with X in the range in Anick (cf. Sect. 2.1).

Let $(\Lambda V, d)$ be a minimal Sullivan model of X and consider its minimal free model given by a quasi-isomorphism $\varphi : (T(Z), d) \xrightarrow{\cong} (\Lambda V, d)$. Since $(T(Z), d)$ is also a free model of $C^*(X, \mathbb{Q})$, the spectral sequences of the form (3.2.3) induced by both $(T(W'), d)$ and $(T(Z), d)$ are isomorphic. Now, by degree reason, φ preserves filtrations and then by [14, Proposition 3.6] it induces a quasi-isomorphism $E_2(\varphi) : (T(Z), d_2) \xrightarrow{\cong} (\Lambda V, d_2)$, where d_2 stands for the quadratic part of d . Also, φ induces on $(\Lambda V \otimes \Gamma(sV), D)$ the structure of a $(T(Z), d)$ -module and then [9, Proposition 6.4.] the following diagram

$$\begin{array}{ccc} & & (\Lambda V \otimes \Gamma(sV), D) \\ & & \downarrow \cong \\ (T(Z) \otimes (sZ \oplus \mathbb{Q}), D) & \xrightarrow{\cong} & \mathbb{Q} \end{array}$$

is completed by a quasi-isomorphism

$$\Phi : (T(Z) \otimes (sZ \oplus \mathbb{Q}), D) \xrightarrow{\cong} (\Lambda V \otimes \Gamma(sV), D)$$

of $(T(Z), d)$ -modules between acyclic closures of $(T(Z), d)$ and $(\Lambda V, d)$. We then provide $\text{Hom}_{T(Z)}(T(Z) \otimes (sZ \oplus \mathbb{Q}), \Lambda V, D)$ with the filtration

$$\mathcal{H}^q = \{f \mid f(T(Z) \otimes (sZ \oplus \mathbb{Q})) \subseteq (\Lambda V)^{\geq q}\} \tag{3.2.5}$$

and define the differential graded morphism of complexes

$$\begin{aligned} \Psi : \text{Hom}_{T(Z)}(T(Z) \otimes (sZ \oplus \mathbb{Q}), T(Z)) &\rightarrow \text{Hom}_{T(Z)}(T(Z) \otimes (sZ \oplus \mathbb{Q}), \Lambda V) \\ (\text{resp., } \Psi' : \text{Hom}_{\Lambda V}(\Lambda V \otimes \Gamma(sV), \Lambda V)) &\rightarrow \text{Hom}_{T(Z)}(T(Z) \otimes (sZ \oplus \mathbb{Q}), \Lambda V) \end{aligned}$$

by putting $\Psi(f) = \varphi \circ f$ (resp., $\Psi'(g) = g \circ \Phi$). These morphisms preserve filtrations (3.2.2) and (3.2.5) (resp., (3.1.3) and (3.2.5)) and induce morphisms of spectral sequences. Finally, since $(\Lambda V \otimes \Gamma(sV), D)$ yields an acyclic closure $(\Lambda V \otimes \Gamma(sV), D_2)$ of $(\Lambda V, d_2)$ ([10, Prop. 3.12] and [21, Lemma 2.1]), the quasi-isomorphism $E_2(\varphi)$ aforementioned implies that the later morphisms are in fact isomorphisms of spectral sequences. Hence, the two spectral sequences (3.2.3) and (1.0.2) are isomorphic.

Composing the obtained isomorphisms, we deduce that the two spectral sequences (1.0.1) and (1.0.2) are isomorphic. □

Proof of Theorem 1.1 The inequality $\text{depth}(H_*(\Omega(X), \mathbb{K})) \leq \text{r}(X, \mathbb{K})$ comes immediately from the convergence of the Eilenberg-Moore spectral sequence (1.0.1) established in the first step of the proof of Proposition 3.3. Now, X being a Gorenstein space, we have $\dim \text{Ext}_{(T(W'), d)}(\mathbb{K}, (T(W'), d)) = 1$. There exists then an unique pair (p, q) of integers and an unique $[f]$ of bi-degree (p, q) [with respect to the filtration (3.2.2)] generating $\text{Ext}_{(T(W'), d)}(\mathbb{K}, (T(W'), d))$. It follows from the isomorphism between (1.0.1) and (3.2.3) that $p = \text{r}(X, \mathbb{K})$. The hypothesis $\text{ev}_{\mathbb{K}}(T(W'), d) \neq 0$ implies that $\text{ev}([f]) = [f(1)] \neq 0$. But $f(1) \in T^{\geq p}(W')$, so $\text{e}_{\mathbb{K}}(T(W'), d) \geq p$ by the characterization given in Remark 2.2. Consequently, $\text{r}(X, \mathbb{K}) \leq \text{e}_{\mathbb{K}}(X)$. □

Proof of Theorem 1.2 It remains to prove the second assertion. By Proposition 3.3, we have $\text{r}(X, \mathbb{K}) = \text{r}(\Lambda V, d)$. Moreover, remark that the spectral sequence (1.0.2) relative to $(\Lambda V, d_k)$ degenerates at its first term. Hence, $\text{r}(\Lambda V, d) \leq \text{r}(\Lambda V, d_k)$ is then a consequence of the convergence of (1.0.2) (relative to $(\Lambda V, d)$). Now suppose that $\dim V < \infty$. Since d is decomposable, [21, Proposition 3.1] asserts that both $(\Lambda V, d_k)$ and $(\Lambda V, d_k)$ are Gorenstein algebras. Therefore, in (1.0.2), there exists a unique pair (p, q) such that

$$\text{Ext}_{(\Lambda V, d_k)}^{*,*}(\mathbb{K}, (\Lambda V, d_k)) = \text{Ext}_{(\Lambda V, d_k)}^{p,q}(\mathbb{K}, (\Lambda V, d_k)) = \mathcal{E}_{\infty}^{p,q} = \text{Ext}_{(\Lambda V, d)}^{*,*}(\mathbb{K}, (\Lambda V, d))$$

and then the desired equality follows. □

Remark 3.4 Following the same approach of [7], we state below for any minimal Sullivan algebra $(\Lambda V, d)$ some properties for $\text{r}(\Lambda V, d)$ similar to that listed in Remark 2.2.

1. $R(\Lambda V, d)$ is the largest integer p such that some non trivial class in $\mathcal{E}xt_{(\Lambda V, d)}^*(\mathbb{Q}, (\Lambda V, d))$ is represented by a cocycle in \mathcal{F}^p . Equivalently, it is the least integer p such that the projection $\mathcal{A} \rightarrow \mathcal{A}/\mathcal{F}^{>p}$ induces an injection in cohomology.
2. Suppose that $\dim(V) < \infty$; hence, by [7], $(\Lambda V, d)$ is a Gorenstein algebra and denote by Ω the unique generator of $\mathcal{E}xt_{(\Lambda V, d)}^*(\mathbb{Q}, (\Lambda V, d))$. Therefore,

$$R(\Lambda V, d) = \sup\{p \mid \Omega \text{ can be represented by a cocycle in } \mathcal{F}^p\}.$$

We end this paper by asking the following question.

Question Recall first that a minimal Sullivan algebra $(\Lambda V, d)$ is said to be elliptic, if both V and $H(\Lambda V, d)$ are finite dimensional. The main result established by Lechuga and Murillo [17] states that if $(\Lambda V, d_k)$ is also elliptic, then $e_0(\Lambda V, d) = e_0(\Lambda V, d_{k,\sigma})$, where $d_{k,\sigma}$ designates the pure differential associated to d_k [13]. So the following natural question arises:

$$\text{Under what hypothesis, } R(\Lambda V, d) = R(\Lambda V, d_\sigma)?$$

By Theorem 1.2, since $d_{\sigma,k} = d_{k,\sigma}$, if the answer to this question is positive, we will have $R(\Lambda V, d) = R(\Lambda V, d_{\sigma,k}) = R(\Lambda V, d_{k,\sigma})$ when $\dim(V) < \infty$.

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