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Najoua Gamara

Methods of the theory of critical points at infinity on Cauchy Riemann manifolds

Dedicated to the Memory of Professor Abbas Bahri

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Abstract Sub-Riemannian spaces are spaces whose metric structure may be viewed as a constrained geometry, where motion is only possible along a given set of directions, changing from point to point. The simplest example of such spaces is given by the so-called Heisenberg group. The characteristic constrained motion of sub-Riemannian spaces has numerous applications in robotic control in engineering and neurobiology where it arises naturally in the functional magnetic resonance imaging (FMRI). It also arises naturally in other branches of pure mathematics as Cauchy Riemann geometry, complex hyperbolic spaces, and jet spaces. In this paper, we review the use of the relationship between Heisenberg geometry and Cauchy Riemann (CR) geometry. More precisely, we focus on the problem of the prescription of the scalar curvature using techniques related to the theory of critical points at infinity. These techniques were first introduced by Bahri, Bahri and Brezis for the Yamabe conjecture in the Riemannian settings.

Mathematics Subject Classification 58E05 · 57R70 · 53C21 · 53C17

الملخص

الفضاءات تحت الريمانية هي فضاءات يمكن أن ينظر إلى بنيتها المترية كقيد هندسي، حيث أن الحركة ممكنة فقط وفق مجموعة معينة من الاتجاهات تتغيرمن نقطة إلى نقطة. ويعطى أبسط مثال على هذه الفضاءات من قبل ما يسمى بمجموعة هيزنبرغ. لمميزة الحركة المقيدة للفضاءات تحت الريمانية العديد من التطبيقات في مراقبة الروبوتات في الهندسة وعلم الأعصاب حيث تنشأ بشكل طبيعي في دالي التصوير بالرنين المغناطيسي (FMRI). كما نشأ أيضا بشكل طبيعي في فروع أخرى من الرياضيات البحتة مثل هندسة كوشي ريمان، الفضاءات المركبة الزائدية، وفضاءات جت. في هذا المقال، نراجع استخدام العلاقة بين هندسة هايزنبرغ وهندسة كوشي ريمان (CR). وأكثر دقة، سوف نركز على مسألة وصفة الانحناء العددية باستخدام التقنيات المتعلقة بنظرية النقاط الحرجة عند اللانهاية. لقد تم تقديم هذه التقنيات، لأول مرة، من قبل عباس بحري وعباس بحري وحاييم بريزيس لتخمين يامابي ضمن الإعدادات الريمانية.

N. Gamara (✉)
University Tunis El Manar, University Campus 2092, Tunis, Tunisia
E-mail: najoua.gamara@fst.rnu.tn

N. Gamara
College of Science, Taibah University, Medina, Saudi Arabia

1 Introduction

In 1995, Professor Bahri proposed to Yacoub and Gamara, to solve the remaining cases left open by Jerison and Lee of the Cauchy Riemann Yamabe Conjecture [56–58]. In 1987, Jerison and Lee formulated in [56] the CR Yamabe conjecture and developed the analogy with the Yamabe problem in conformal Riemannian geometry, which had already been solved by Aubin [2] and Schoen [66]. Besides the proof of Aubin and Schoen, another proof by Bahri [6], Bahri and Brezis [17] was available using methods related to the theory of critical points at infinity. This theory has been widely developed by Bahri at mid-1980s. Bahri introduced and performed the theory of critical points at infinity by establishing several methods which are a fundamental step in the calculus of variation. Based on his experience and his numerous works in that direction [5–8,15–24,26], Bahri was convinced that these topological methods are well adapted to solve the Yamabe problem in the Cauchy Riemann settings. Furthermore, Bahri used this theory to solve non compact variational problems as Yamabe type equations, the prescribed scalar curvature equations, n-body equations in celestial mechanics, fundamental problems in contact and conformal geometries, mean field equations etc. , we refer here to some of his work [8–14,25].

Given an orientable manifold M of odd dimension $2n+1$, a Cauchy Riemann structure on M is given by a complex n-dimensional subbundle $\mathrm{T}_{1,0}(M)$ of the complexified tangent bundle $\mathrm{T}(M)\otimes\mathbb{C}$ satisfying $\mathrm{T}_{1,0}(M)\cap\overline{\mathrm{T}_{1,0}}(M)=0$. A Cauchy Riemann in short CR manifold is such a manifold with an integrable Cauchy Riemann structure. The geometry of CR manifolds, the abstract models of real hypersurfaces in complex manifolds, has recently attracted much attention. This is in particular due to the fact that, in the strictly pseudo-convex case, there are many parallels with conformal Riemannian geometry. Indeed, a CR manifold carries a natural Hermitian metric on its holomorphic tangent bundle. The Levi form, which is, like a metric on a conformal manifold, determined only up to multiplication by a smooth function. The multiple is fixed by choosing a contact form θ (a real 1-form) annihilating the holomorphic tangent bundle. A CR manifold together with a choice of a contact form is called a pseudo-Hermitian manifold. The simplest scalar invariant for a pseudo-Hermitian manifold is the pseudo-Hermitian scalar curvature, which we denote by R_θ, defined independently by Webster [70] and Tanaka [68]. If the Levi form is positive definite the pseudo-Hermitian manifold is called a strictly pseudo-convex CR manifold.

Consider (M,θ) a strictly pseudo-convex compact CR manifold of dimension $2n+1$ and K a smooth function. The prescribed Webster scalar curvature problem consists on finding a contact form $\tilde{\theta}$ conformal to θ for which the pseudo-Hermitian scalar curvature is equal to K. If we set $\tilde{\theta}=u^{\frac{2}{n}}\theta$, where u is a smooth positive function on M, then the above problem is equivalent to solve the following equation:

$$(P_K)\begin{cases}-L_\theta u=K\,u^{1+\frac{2}{n}} & \text{on } M,\\ u>0,\end{cases}$$

where $-L_\theta=-\frac{2(n+1)}{n}\Delta_\theta+R_\theta$, is the conformal Laplacian, Δ_θ is the sub-Laplacian operator on (M,θ), and R_θ is the Webster scalar curvature of (M,θ).
Problem (P_K) is the analogue of the prescribed scalar curvature problem on Riemannian manifolds. Comparing to the scalar curvature problem in the Riemannian framework, which was extensively studied (see for example the monograph [4] and the references therein), only few authors (most of them are students of Bahri) have been interested in solving the problem (P_K) (see [40,42,45–50,53–55,63]). On the contrary, the Yamabe problem on CR manifolds (the case where K is assumed to be constant) was widely studied by various authors (see among others [43,44,56–58]).

The main difficulty one encounters in solving problem (P_K), appears when we consider it from a variational viewpoint. Indeed, the Euler functional associated with (P_K) does not satisfy the Palais–Smale condition, that is, there exist noncompact sequences along which the associated functional to (P_K) is bounded and the gradient of this functional goes to zero. Moreover, as in the Riemannian settings, there are topological obstructions of Kazdan–Warner condition type to solve (P_K), see [51]. Hence, we do not expect to solve problem (P_K) for all functions K, and so it is natural to ask the following: under which conditions on K does a positive solution exist for (P_K)? In [63], Malchiodi and Uguzzoni considered the case where $M=\mathbb{S}^{2n+1}$ the unit sphere of $\mathbb{C}^{n+1}$ and gave a perturbation result for problem (P_K), that is K is assumed to be a small perturbation of a constant (see also [40]). Their approach is based on a perturbation method due to Ambrosetti [1]. In [42], Gamara noticed an analogy between the 3-dimensional CR case with the 4-dimensional Riemannian case, see for example [18] and [27]: there is a balance phenomenon between the self-interactions and the mutual interactions of the functions failing to satisfy the Palais–Smale condition. In [42], the case where M is locally

conformally CR equivalent to the sphere $\mathbb{S}^3$ of $\mathbb{C}^2$ was considered (thus when $n = 1$), an Euler-Hopf type criterion for K was provided to find solutions for (P_K). The existence results of Gamara have been generalized by Chtioui, Ahmedou and Yacoub, see [54], where multiplicity results were also be given.
In [40,42,45,53–55,63], to prove the existence or multiplicity results for problem (P_K), the authors use a non degeneracy condition.

In this paper, we focus on the prescribed Webster scalar curvature problem on a Cauchy Riemann manifold M locally conformally equivalent to the unit sphere $\mathbb{S}^3$ of $\mathbb{C}^2$ endowed with its standard contact form. This manuscript consists of an introduction and three sections organized as follows:

- **Section 2:** We present the problem of prescribing scalar curvature in the Riemannian settings, as an example, we review the Yamabe problem.
- **Section 3:** This chapter is an introduction to Cauchy Riemann geometry: we give some useful definitions and results.

In Sect. 3.1, we define the CR structure on a orientable real $2n+1$-dimensional manifold M, which we denoted by $\mathrm{T}_{1,0}(M)$. It is a subbundle of dimension n of the complexified tangent bundle $\mathrm{T}(M) \otimes \mathbb{C}$ satisfying

$$\mathrm{T}_{1,0}(M) \cap \overline{\mathrm{T}_{1,0}}(M) = 0.$$

If $\mathrm{T}_{1,0}(M)$ is integrable, then we define the distribution of the CR manifold by

$$\mathrm{H}(M) := \mathrm{T}_{1,0}(M) \oplus \overline{\mathrm{T}_{1,0}}(M).$$

The distribution $\mathrm{H}(M)$ is a 1-codimensional subbundle of $\mathrm{T}(M)$, so there is a 1-form θ on M such that $\mathrm{H}(M) = Ker(\theta)$. The form θ defines a pseudo-Hermitian structure on M. For a given pseudo-Hermitian structure θ on M, we define in Sect. 3.2, the Levi form l_θ. It is a symmetric bilinear form on $\mathrm{H}(M)$. We extend l_θ to the hole $\mathrm{T}(M)$ which we denoted also by l_θ, so we say that (M, θ) is strictly pseudo-convex (resp. non degenerate) if l_θ is positive definite (resp. non degenerate). In Sect. 3.3, we define the Reeb vector field T (or characteristic direction) for a non degenerate CR manifold (M, θ) by $\theta(\mathrm{T}) = 1$ and T is $d\theta$-orthogonal to $\mathrm{H}(M)$, we obtain the decomposition $\mathrm{T}(M) = \mathrm{H}(M) \oplus \mathbb{R}\mathrm{T}$. In Sect. 3.4, we define the Webster metric g_θ for a non degenerate CR manifold (M, θ) via the Levi form l_θ. If (M, θ) is strictly pseudo-convex then g_θ is a Riemannian metric. Sect. 3.5 is devoted to define the Tanaka-Webster connection of a non degenerate CR manifold. In Sect. 3.6, we define the Christofell symbols and the pseudo-Hermitian torsion. In Sect. 3.7, we define the curvature tensors: the global curvature R, the pseudo-Hermitian Ricci curvature and the scalar curvature (called the Webster scalar curvature) of the Tanaka-Webster connection ∇. In Sect. 3.8, we define the divergence and the adjoint of a vector field, we define also the horizontal gradient $\nabla^{\mathrm{H}} f = \pi_{\mathrm{H}} \nabla f$ where $\pi_{\mathrm{H}} : \mathrm{T}(M) = \mathrm{H}(M) \oplus \mathbb{R}\mathrm{T} \longrightarrow \mathrm{H}(M)$ is the natural projection. We define then the sub-Laplacian operator Δ_b by $\Delta_b f = -div(\nabla^{\mathrm{H}} f),\ f \in C^2(M)$. The Sect. 3.9 is devoted to some examples, we study the Heisenberg group $\mathbb{H}^n$, the Heisenberg group of dimension $n = 1$ and the unit sphere $\mathbb{S}^{2n+1}$ of $\mathbb{C}^{2n}$. In the last section of this chapter, we introduce normal coordinates for a CR strictly pseudo-convex manifold and give the definition of the Folland–Stein spaces $\mathcal{S}_p^k(M)$ of M.

- **Section 4:** In this section, we expose the problem of the prescription of a scalar curvature on Cauchy Riemann manifolds and give a quick review on the CR Yamabe problem. Finally, we announce and prove our main result: an existence theorem for problem (P_K).

In Sect. 4.1, we give some preliminaries which are useful for the understanding of the problem. In Sect. 4.2, we formulate the Euler–Lagrange functional J associated with (P_K). Section 4.3 is devoted to the natural change of the functional. In Sect. 4.4, we review the definitions of the Hessian and the Morse Lemma. In Sect. 4.5, we introduce the notions of a homotopy and homotopy equivalence, while the definition of a retract of a topological space and the notion of deformation retract are the object of Sect. 4.6.
In Sect. 4.7, we introduce the Palais–Smale condition and in Sect. 4.8, we give a quick review on the Cauchy Riemann Yamabe Problem. The case of a CR spherical pseudo-Hermitian Manifold of dimension 3 is the object of Sect. 4.9. We define the almost solutions, then we study the properties of the functional related to the associated variational formulation.
In Sect. 4.10, we expand the functional near the sets of its critical points at infinity. The Morse Lemma is displayed in Sect. 4.11: we construct a pseudo-gradient for our functional; then, we localize the critical point at infinity of J. Sect. 4.12 is devoted to the proof of our main result an existence theorem for problem (P_K) using a topological argument.

2 Prescribing the scalar curvature on Riemannian manifolds

2.1 Prescription of a scalar curvature

Let (M, g) be a compact Riemannian manifold without boundary of dimension $n \geq 3$ and K a positive function of class C^2. The prescribed scalar curvature problem consists to find a metric $\widetilde{g}$ conformal to g for which the scalar curvature $R_{\widetilde{g}} = K$. We write $\widetilde{g} = u^{\frac{4}{n-2}} g$, $u > 0$. If we denote by Δ the Laplace–Beltrami operator of the metric g, we obtain the following transformation law for the scalar curvature of the metrics g and $\widetilde{g}$:

$$R_{\widetilde{g}} = u^{-\frac{n+2}{n-2}}(-c_n \Delta u + Ru)$$

with $c_n = 4\frac{n-1}{n-2}$. Hence, finding a solution for the prescribed scalar curvature problem is equivalent to solve the following partial differential equation

$$(P_K) \begin{cases} -c_n \Delta u + Ru = R_{\widetilde{g}} u^{\frac{n+2}{n-2}} \\ u > 0 \end{cases}$$

Let $p = \frac{2n}{n-2}$ and $-L = -c_n \Delta + R$ be the conformal Laplacian. The last equation can be rewritten as

$$(P_K) \begin{cases} -Lu = K u^{p-1} \\ u > 0. \end{cases} \tag{2.1}$$

The prescription of the scalar curvature for Riemannian manifolds is known to be the Kazdan–Warner problem; it has been extensively studied by many authors for dimensions 2, 3 and 4 as well as in higher dimensions.There is a big number of papers devoted to this problem as well as for the multiplicity of its solutions, we can mention [32–36,39,52,60,62].

Here, we will merely refer to the references [5,18,27,29] and recently [31] which are the most directly related works to our since based on the use of methods related to the theory of critical points at infinity due to Bahri.

2.2 The Yamabe problem

The Yamabe problem is the case where the function K to prescribe is constant, $K = \lambda$ for some constant $\lambda \in \mathbb{R}$. The Yamabe problem goes back to Yamabe himself [71], who claimed in 1960 to have a solution, but in 1968, Trudinger [69] discover an error in his proof and corrected Yamabe's proof. In [2], Aubin improved Trudinger's result, using variational methods and Weyl's tensor characteristics. In 1984, Schoen [66] solved the remaining cases using variational methods and the positive mass Theorem. We have also to point out the work of Lee and Parker in [61], which is a detailed discussion on the Yamabe problem unifying the work of Aubin [2] with that of Schoen [66]. Besides the proof by Aubin and Schoen for the Riemannian Yamabe conjecture, another proof by Bahri [6], Bahri and Brezis [17] was available by techniques related to the theory of critical points at infinity.

Yamabe observed that the Yamabe equation (P_λ) is the Euler–Lagrange equation of the functional

$$Q_0(\widetilde{g}) = \frac{\int_M R_{\widetilde{g}} \mathrm{d}v_{\widetilde{g}}}{\left(\int_M \mathrm{d}v_{\widetilde{g}}\right)^{\frac{2}{p}}} \tag{2.2}$$

when restricted to a conformal class $[g] = \{hg \,/\, h \in C^\infty(M),\ h > 0\}$, where $\mathrm{d}v_{\widetilde{g}}$ is the volume form of $(M, \widetilde{g})$ and $h = u^{p-2}$, $u > 0$. In fact, on $[g]$, we can write $Q_0(\widetilde{g}) = Q_0(u^{p-2}g) = J(u)$, where

$$J(u) = \frac{\int_M -Lu\ u dv_g}{\left(\int_M u^p \mathrm{d}v_g\right)^{\frac{2}{p}}} = \frac{\int_M (c_n |\nabla u|^2 + Ru^2) dv_g}{\|u\|_p^2}. \tag{2.3}$$

We call $J(u)$ the Yamabe quotient of (M, g). So for a given Riemannian manifold (M, g), it is natural to define the following constrained extremal problem:

$$\lambda(M, g) = \inf_{u \in W^{1,2}(M,g)} \{A_\theta(u)/B_\theta(u) = 1\}$$

$A_\theta(u) = \int_M -Lu\ u\ \mathrm{d}v_g; \quad B_\theta(u) = \int_M \mid u \mid^p \mathrm{d}v_g$ Therefore

$$\lambda(M, g) = \inf \left\{ Q_0(\widetilde{g}) \,/\, \widetilde{g} \in [g] \right\} \tag{2.4}$$

$\lambda(M, g)$ is a conformal invariant, which means that it is determined by the conformal class and is independent of the choice of the initial metric g in the conformal class. It is called the Yamabe invariant of (M, g).

Calculus of variation methods has been used to prove that the Yamabe problem can be solved on a general compact Riemannian manifold (M, g) of dimension n, provided that its Yamabe invariant $\lambda(M, g) < \lambda(S^n, g_0)$, where g_0 is the standard metric on the sphere. This is due to Yamabe, Trudinger and Aubin. The modification by Trudinger of Yamabe's proof works whenever $\lambda(M, g) \leq 0$. In fact, Trudinger showed that there is a positive constant Λ such that the proof works when $\lambda(M, g) < \Lambda$. In 1976, Aubin extended Trudinger's result by showing that in fact $\Lambda = \lambda(S^n, g_0)$.

We have the following results:

Theorem 2.1 ([2,3,69,71]) *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$ without boundary. The Yamabe problem has a solution if $\lambda(M, g) < \lambda(S^n, g_0)$.*

Theorem 2.2 ([2]) *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$ without boundary. Then $\lambda(M, g) \leq \lambda(S^n, g_0)$.*

Hence, the extremal problem given above is attained by a positive smooth solution of the Yamabe equation (P_λ). Thus the metric $\widetilde{g} = u^{\frac{4}{n-2}} g$ has a constant scalar curvature $R_{\widetilde{g}} = \lambda(M, g)$.

Theorem 2.1 is proved by considering a minimizing sequence $u_i \in W^{1,2}(M, g)$ satisfying $B(u_i) = 1$ to minimize functional A. Since the sequence u_i is bounded in $W^{1,2}(M, g)$, there exists a subsequence denoted again by u_i which converges weakly to $u \in W^{1,2}(M, g)$. Then, $A(u) = \lambda(M, g)$, but $B(u)$ could be different from 1 because the embedding of $W^{1,2}(M, g)$ in L^p is continuous but not compact. In particular the limit function u may be identically zero. To overcome this difficulty, a perturbed extremal problem has been considered to lead to a solution of the first one.

An alternative proof of Theorem 2.1 has been given by Uhlenbeck, which used Riemannian normal coordinates and blow-up analysis to transplant the minimizing sequences for the perturbed problem. This kind of blow-up analysis was first introduced in 1981 by Sacks and Uhlenbeck in [65]

Theorem 2.1 reduces the resolution of Yamabe problem to the estimate of the invariant $\lambda(M, g)$. In this way, Aubin [2] proved the following result:

Theorem 2.3 *If (M, g) is a compact Riemannian manifold of dimension $n \geq 6$ not conformally flat, then $\lambda(M, g) < \lambda(S^n, g_0)$.*

In [66], Schoen solved all the remaining cases of the Yamabe problem using the positive mass theorem. The proof by Bahri [6] and Bahri and Brézis [17] of the Yamabe problem is available using the theory of critical points at infinity. This proof is completely different in spirit as well as in techniques and details from the proof of Aubin and Schoen. It does not require the use of any theory of minimal surfaces neither the use of the positive mass theorem.

We remark that for the case (M, g) conformal to S^n, the Yamabe problem clearly has a solution. If $\Phi : M \to S^n$ is a conformal diffeomorphism then $\Phi^*(g_0) = f\, g$, where g_0 is the standard metric of S^n and f a positive function in $C^\infty(M)$, clearly fg has constant scalar curvature.

3 Cauchy–Riemann manifolds

3.1 CR structures

Let M be a real $2n + 1$-dimensional $\mathcal{C}^\infty$ differentiable manifold. Let $\mathrm{T}(M) \otimes \mathbb{C}$ be the complexified tangent bundle over M ($\mathrm{T}(M) \otimes \mathbb{C} = \{u + iv;\ u, v \in \mathrm{T}(M)\}$, where $i = \sqrt{-1}$).

Definition 3.1 Let us consider a complex subbundle $\mathrm{T}_{1,0}(M)$ of the complexified tangent bundle $\mathrm{T}(M) \otimes \mathbb{C}$, of complex rank n. We say that $\mathrm{T}_{1,0}(M)$ is a CR structure on M if

1) $\mathrm{T}_{1,0}(M) \cap \mathrm{T}_{0,1}(M) = 0$,
2) $[\Gamma^\infty(\mathrm{T}_{1,0}(M)), \Gamma^\infty(\mathrm{T}_{1,0}(M))] \subset \Gamma^\infty(\mathrm{T}_{1,0}(M))$ (integrability condition).

Where $T_{0,1}(M) = \overline{T_{1,0}(M)}$, over bar denotes complex conjugation, $\Gamma^\infty(T_{1,0}(M))$ denotes the set of vector fields $X : M \longrightarrow T_{1,0}(M)$ and [,] is the Lie bracket.
A pair $(M, T_{1,0}(M))$ is called a CR manifold.

Definition 3.2 Let $(M, T_{1,0}(M))$ and $(N, T_{1,0}(N))$ be two CR manifolds. A $\mathcal{C}^\infty$ map $f : M \longrightarrow N$ is a CR map if

$$(d_x f) T_{1,0}(M)_x \subset T_{1,0}(N)_{f(x)}, \tag{3.1}$$

for any $x \in M$, where $d_x f$ is the $\mathbb{C}$-linear extension to $T_x(M) \otimes \mathbb{C}$ of the differential of f at x.

Let $(M, T_{1,0}(M))$ be a CR manifold. Its Levi distribution is the real subbundle of rank $2n$ $H(M) \subset T(M)$ given by

$$H(M) = \mathrm{Re}\{T_{1,0}(M) \oplus T_{0,1}(M)\}.$$

It carries the complex structure $J : H(M) \longrightarrow H(M)$ given by

$$J(V + \overline{V}) = i(V - \overline{V}), \quad \forall\, V \in T_{1,0}(M).$$

Definition 3.3 A function $f : M \longrightarrow N$ is a CR isomorphism (or a CR equivalence) if f is both a $\mathcal{C}^\infty$ diffeomorphism and a CR map.

Standard examples of CR manifolds are those of real hypersurfaces of complex manifolds. For example if M is a hypersurface of $\mathbb{C}^{n+1}$), the CR structure is the one induced by the ambient space, for any $x \in M$

$$T_{1,0}(M)_x = \left(T(M) \otimes \mathbb{C} \bigcap T_{1,0}(\mathbb{C}^{n+1})\right)_x;$$

here, $T_{1,0}(\mathbb{C}^{n+1})$ is the tangent holomorphic subbundle having local generator system $\frac{\partial}{\partial z^j}$, $1 \le j \le n+1$ where, $(z^1, z^2 \ldots z^{n+1})$ are the complex cartesian coordinates of $\mathbb{C}^{n+1}$.
Recall that the Heisenberg group $\mathbb{H}^n$, $(n \ge 1)$, is the homogeneous Lie group whose underlying manifold is $\mathbb{C}^n \times \mathbb{R} = \mathbb{R}^{2n+1}$ and whose group law is given by

$$\tau_{(z',t')}(z,t) = (z',t') \cdot (z,t) = (x + x', y + y', t + t' + 2(< x, y' > - < x', y >)),$$

where $< ., . >$ denotes the inner product in the Euclidian space $\mathbb{R}^n$, $(z,t) = (x_1, \ldots, x_n, y_1, \ldots, y_n, t)$ and $(z',t') = (x'_1, \ldots, x'_n, y'_1, \ldots, y'_n, t')$.

Following the geometrical interpretation due to I. Piatetski-Shapiro [59], one can introduce the Heisenberg group $\mathbb{H}^n$ using its identification with the boundary M_n of the Siegel Domain:

$$D_{n+1} = \left\{(\xi_0, \xi_1, \ldots, \xi_n) = (\xi_0, \xi) \in \mathbb{C} \times \mathbb{C}^n;\ \sum_1^n |\xi_j|^2 - Im\xi_0 < 0\right\}$$

$$M_n = \partial D_{n+1} = \left\{(\xi_0, \xi) \in \mathbb{C} \times \mathbb{C}^n;\ \sum_1^n |\xi_j|^2 = Im\xi_0\right\}.$$

The Siegel domain D_{n+1} is holomorphically equivalent to the unit ball in $\mathbb{C}^{n+1}$. The Heisenberg group $\mathbb{H}^n$ acts on $\mathbb{C}^{n+1}$ by holomorphic affine transformation which preserves D_{n+1} and M_n as follows: if $(z,t) \in \mathbb{H}^n$ and $\xi \in \mathbb{C}^{n+1}$, $(z,t) \bullet \xi = \xi'$ where

$$\xi'_0 = \xi_0 + t + i|z|^2 + 2i \sum_1^n \xi_j \bar{z}_j$$

$$\xi'_j = \xi_j + z_j, \quad 1 \le j \le n.$$

Since this action is transitive on M_n, the group $\mathbb{H}^n$ is identified with M_n via the correspondence:

$$(z,t) \leftrightarrow (z,t) \bullet 0 = (t + i|z|^2, z_1, \ldots, z_n).$$

Under this identification the CR structure on $\mathbb{H}^n$ described above coincides with the CR structure on M_n induced from $\mathbb{C}^{n+1}$.

3.2 The Levi form

Let M be an orientable connected CR manifold.
Let $\mathrm{E}(M) = \mathrm{H}(M)^{\perp} := \{\alpha \in \mathrm{T}^*(M);\ \forall x \in M, \mathrm{H}(M)_x \subset \ker(\alpha_x)\}$.
$\mathrm{E}(M)$ is a real line subbundle of the cotangent bundle $\mathrm{T}^*(M)$ and

$$\mathrm{E}(M) \simeq \mathrm{T}(M)/\mathrm{H}(M)$$

(a vector bundle isomorphism).
Since M is orientable and $\mathrm{H}(M)$ is oriented by its complex structure J, it follows that $\mathrm{E}(M)$ is orientable.
Since $\mathrm{E}(M)$ is an orientable real line bundle over a connected manifold, then $\mathrm{E}(M)$ has a nowhere vanishing $\mathcal{C}^\infty$ section $\theta : M \longrightarrow \mathrm{E}(M)$.
The section θ is a 1-form and we have

$$\mathrm{H}(M) = \ker(\theta).$$

Any such section θ is referred to as a pseudo-Hermitian structure on M.

Definition 3.4 Given a pseudo-Hermitian structure θ on M, the Levi form l_θ is the symmetric bilinear form defined by

$$l_\theta(V, W) = \mathrm{d}\theta(V, J(W)) \qquad \forall\ V, W\ \in \mathrm{H}(M). \tag{3.2}$$

The $\mathbb{C}$-linear extension to $\mathbb{C}\mathrm{H}(M)$ gives an Hermitian form on $\mathrm{T}_{1,0}(M)$ defined by

$$l_\theta(V, \overline{W}) = -i\mathrm{d}\theta(V, \overline{W}) \qquad \forall\ V, W\ \in \mathrm{T}_{1,0}(M). \tag{3.3}$$

Since $\mathrm{E}(M)$ is a real line bundle, then for any two pseudo-Hermitian structures θ and $\tilde{\theta}$ there exists a nowhere-zero $\mathcal{C}^\infty$ function $\lambda : M \longrightarrow \mathbb{R}$ such that

$$\tilde{\theta} = \lambda\theta. \tag{3.4}$$

Let us apply the exterior differentiation operator d to (3.4); we get

$$\mathrm{d}\tilde{\theta} = \mathrm{d}\lambda \wedge \theta + \lambda \mathrm{d}\theta.$$

Since $\ker(\theta) = \mathrm{H}(M)$, the $\mathbb{C}$-linear extension of θ vanishes on $\mathrm{T}_{1,0}(M)$ and $\mathrm{T}_{0,1}(M)$ as well. Consequently, the Levi form changes according to

$$l_{\tilde{\theta}} = \lambda l_\theta. \tag{3.5}$$

Definition 3.5 Let $(M, \mathrm{T}_{1,0}(M))$ be a CR manifold and θ a pseudo-Hermitian structure on M.

1) We say that $(M, \mathrm{T}_{1,0}(M))$ (or (M, θ)) is non degenerate if the Levi form l_θ is nondegenerate.
2) We say that (M, θ) is strictly pseudo-convex if l_θ is positive definite.

3.3 The Reeb field

Proposition 3.6 (See [37]) *Let $(M, \mathrm{T}_{1,0}(M))$ be a nondegenerate CR manifold, θ a pseudo-Hermitian structure on M. Then, there exists a unique globally defined nowhere zero tangent vector field* T *on M such that*

$$\theta(\mathrm{T}) = 1, \quad \textit{and} \quad \mathrm{d}\theta(\mathrm{T}, .) = 0. \tag{3.6}$$

T *is transverse to the Levi distribution* $\mathrm{H}(M)$.
Moreover, the tangent bundle decomposes as

$$\mathrm{T}(M) = \mathrm{H}(M) \oplus \mathbb{R}\mathrm{T}. \tag{3.7}$$

Definition 3.7 This vector field T is called the characteristic direction or Reeb field of (M, θ).

3.4 Webster metric

Definition 3.8 Let $(M, \mathrm{T}_{1,0}(M))$ be a non degenerate CR manifold and θ a pseudo-Hermitian structure on M. Let g_θ be the semi-Riemannian metric given by

$$\begin{cases} g_\theta(X, Y) = l_\theta(X, Y) \\ g_\theta(X, \mathrm{T}) = 0 \\ g_\theta(\mathrm{T}, \mathrm{T}) = 1 \end{cases} \qquad \forall X, Y \in \mathrm{H}(M).$$

g_θ is called the Webster metric of (M, θ).

Remark 3.9

$$g_\theta = l_\theta + \theta \odot \theta,$$

where $\odot$ denotes the symmetric tensor product defined by

$$\theta \odot \theta(X, Y) = \theta(X)\theta(JY), \quad X, Y \in \mathrm{T}(M)$$

Proposition 3.10 *Let $(M, \mathrm{T}_{1,0}(M))$ be a non degenerate CR manifold and θ a pseudo-Hermitian structure on M. If l_θ is positive definite ((M, θ) is strictly pseudo-convex), then g_θ is a Riemannian metric on M.*

3.5 The Tanaka-Webster connection

Let $(M, \mathrm{T}_{1,0}(M))$ be a non degenerate CR manifold and θ a fixed pseudo-Hermitian structure on M. Let T be the Reeb field of (M, θ). If ∇ is a linear connection on M, we denote T_∇ the associated torsion tensor field.

Definition 3.11 ([37]) We say that T_∇ is pure if

$$\mathrm{T}_\nabla(Z, W) = 0, \tag{3.8}$$

$$\mathrm{T}_\nabla(Z, \overline{W}) = 2il_\theta(Z, \overline{W})\mathrm{T}, \tag{3.9}$$

$$\tau \circ J + J \circ \tau = 0, \tag{3.10}$$

for any $Z, W \in \mathrm{T}_{1,0}(M)$. Here

$$\begin{aligned} \tau : \mathrm{T}(M) &\longrightarrow \mathrm{T}(M) \\ X &\longmapsto \mathrm{T}_\nabla(T, X). \end{aligned}$$

On each non degenerate CR manifold on which a pseudo-Hermitian structure has been fixed, there is a canonical linear connection compatible with both the complex structure of the Levi distribution and the Levi form. Precisely, we have the following result:

Theorem 3.12 *Let $(M, \mathrm{T}_{1,0}(M))$ be a non degenerate CR manifold and θ a pseudo-Hermitian structure on M. Let* T *be the Reeb field of (M, θ) and J the complex structure in $\mathrm{H}(M)$ (extended to an endomorphism of $\mathrm{T}(M)$ by requiring that $J\mathrm{T} = 0$). Let g_θ be the Webster metric of (M, θ). There is a unique linear connection ∇ on M satisfying the following axioms:*

(i) $\mathrm{H}(M)$ *is parallel with respect to ∇, that is*

$$\nabla_X \Gamma^\infty(\mathrm{H}(M)) \subset \Gamma^\infty(\mathrm{H}(M)),$$

for any $X \in \mathcal{X}(M)$.

(ii) $\nabla J = 0, \quad \nabla g_\theta = 0.$

(iii) The torsion T_∇ *of* ∇ *is pure.*

Proof Since we have the following direct sum decomposition,

$$\mathrm{T}(M) \otimes \mathbb{C} = \mathrm{T}_{1,0}(M) \oplus \mathrm{T}_{0,1}(M) \oplus \mathrm{T} \otimes \mathbb{C},$$

we can define the natural projections

$$\pi_+ : \mathrm{T}(M) \otimes \mathbb{C} \longrightarrow \mathrm{T}_{1,0}(M)$$

and

$$\pi_- : \mathrm{T}(M) \otimes \mathbb{C} \longrightarrow \mathrm{T}_{0,1}(M).$$

Then, for any $Z \in \mathrm{T}_{1,0}(M), \quad \pi_-(Z) = \overline{\pi_+(\overline{Z})}$.

We establish first the uniqueness of a linear connection ∇ on M obeying the axioms (i),(ii) and (iii). Since the torsion T_∇ is pure then for any $Y, Z \in \mathrm{T}_{1,0}(M)$, we have

$$[\overline{Y}, Z] = \nabla_{\overline{Y}} Z - \nabla_Z \overline{Y} + 2il_\theta(Z, \overline{Y})\mathrm{T},$$

where (as $\nabla_{\overline{Y}} Z \in \Gamma^\infty(\mathrm{T}_{1,0}(M))$ and $\nabla_Z \overline{Y} \in \Gamma^\infty(\mathrm{T}_{0,1}(M))$), we obtain

$$\pi_+[\overline{Y}, Z] = \nabla_{\overline{Y}} Z \tag{3.11}$$

Let Ω be the 2-form defined by $\Omega(X, Y) = g_\theta(X, JY) = -\mathrm{d}\theta(X, Y), \quad X, Y \in \mathrm{T}(M)$. It satisfies $\Omega(\mathrm{T}, .) = 0$. Since $\nabla g_\theta = 0$, we obtain

$$X(g_\theta(Y, Z)) = g_\theta(\nabla_X Y, Z) + g_\theta(Y, \nabla_X Z)$$

for any $X, Y, Z \in \mathrm{T}(M)$. In particular, for $Y = T$, it yields

$$X(\theta(Z)) = g_\theta(\nabla_X T, Z) + \theta(\nabla_X Z) \tag{3.12}$$

We distinguish two cases: $Z \in \mathrm{H}(M)$ and $Z = \mathrm{T}$.

- If $Z \in \mathrm{H}(M)$, then (3.12) yields $g_\theta(\nabla_X T, Z) = 0$ or $\pi_\mathrm{H}(\nabla_X \mathrm{T}) = 0$, where $\pi_\mathrm{H} : \mathrm{T}(M) \longrightarrow \mathrm{H}$ is the natural projection associated with the direct sum decomposition (3.7).
- Let $Z = \mathrm{T}$; we use (3.12), we obtain

  $$\theta(\nabla_X T) = 0$$

  since $\nabla_X T$ is parallel to T, we deduce that $\nabla_X T = 0$. By using (3.12) with $X = T$, we obtain that $\nabla_T T = 0$; hence we conclude that

  $$\nabla \mathrm{T} = 0 \tag{3.13}$$

Note that

$$\nabla \Omega = 0,$$

as a consequence of axiom (ii) in Theorem 3.12. Therefore,

$$X(\Omega(Y, \overline{Z})) = \Omega(\nabla_X Y, \overline{Z}) + \Omega(Y, \nabla_X \overline{Z})$$

for any $X, Y, Z \in \mathrm{T}_{1,0}(M)$. Using (3.11) we may rewrite this identity as

$$\Omega(\nabla_X Y, \overline{Z}) = X(\Omega(Y, \overline{Z})) - \Omega(Y, \pi_-[X, \overline{Z}]), \tag{3.14}$$

which, in view of the non degeneracy of Ω on $\mathrm{H}(M)$, determines $\nabla_X Y$ for any $X, Y \in \mathrm{T}_{1,0}(M)$. We shall need the bundle endomorphism K_T given by

$$K_\mathrm{T} = -\frac{1}{2} J \circ (\mathcal{L}_\mathrm{T} J),$$

where $\mathcal{L}$ denotes the Lie derivative. On the other hand, (by $\nabla \mathrm{T} = 0$)

$$\nabla_\mathrm{T} X = \tau X + \mathcal{L}_\mathrm{T} X, \quad X \in \mathrm{T}(M). \tag{3.15}$$

Note that as a consequence of property (3.10), τ is $\mathrm{H}(M)$-valued. We may use $\nabla J = 0$ and (3.15) to perform the following calculation:

$$\begin{aligned} 0 &= (\nabla_{\mathrm{T}} J)X = \nabla_{\mathrm{T}} JX - J\nabla_{\mathrm{T}} X \\ &= \tau(JX) + \mathcal{L}_{\mathrm{T}}(JX) - J(\tau X + \mathcal{L}_{\mathrm{T}} X) \\ &= -J\tau X + \mathcal{L}_{\mathrm{T}}(JX) - J(\tau X + \mathcal{L}_{\mathrm{T}} X) = -2J\tau X + (\mathcal{L}_{\mathrm{T}} J)X. \end{aligned}$$

Let us apply J in both members of this identity and use the fact that τ is $\mathrm{H}(M)$-valued to obtain

$$\tau = K_{\mathrm{T}}.$$

Therefore, (3.15) may be rewritten as

$$\nabla_{\mathrm{T}} X = K_{\mathrm{T}} X + \mathcal{L}_{\mathrm{T}} X, \quad X \in \mathrm{T}(M). \tag{3.16}$$

We use the identities (3.11), (3.13), (3.14) and (3.16) to prove uniqueness statement in Theorem 3.12. To prove existence, we consider

$$\nabla : \Gamma^{\infty}(\mathrm{T}(M) \otimes \mathbb{C}) \times \Gamma^{\infty}(\mathrm{T}(M) \otimes \mathbb{C}) \longrightarrow \Gamma^{\infty}(\mathrm{T}(M) \otimes \mathbb{C})$$

be the differential operator defined by

$$\begin{aligned} \nabla_{\overline{X}} Y &= \pi_+[\overline{X}, Y] & \nabla_X \overline{Y} &= \overline{\nabla_{\overline{X}} Y}, \\ \nabla_X Y &= U_{XY}, & \nabla_{\overline{X}} \overline{Y} &= \overline{\nabla_X Y}, \\ \nabla_{\mathrm{T}} X &= \mathcal{L}_{\mathrm{T}} X + K_{\mathrm{T}} X, & \nabla_{\mathrm{T}} \overline{X} &= \overline{\nabla_{\mathrm{T}} X}, \\ \nabla \mathrm{T} &= 0, \end{aligned}$$

for any $X, Y \in \mathrm{T}_{1,0}(M)$. Here

$$U : \Gamma^{\infty}(\mathrm{T}(M)) \times \Gamma^{\infty}(\mathrm{T}(M)) \longrightarrow \Gamma^{\infty}(\mathrm{T}(M))$$

is defined by

$$\Omega(U_{XX}, \overline{Z}) = X\Omega(U_Y, \overline{Z}) - \Omega(U_Y, \pi_-[X, \overline{Z}]), \quad X, Y, Z \in \mathrm{T}_{1,0}(M).$$

Then we can verify that ∇ satisfies axioms (i), (ii) and (iii) in Theorem 3.12. □

Definition 3.13 The connection ∇ given by Theorem 3.12 is the Tanaka-Webster connection of $(M, \mathrm{T}_{1,0}(M), \theta)$. The vector-valued 1-form τ on M is the pseudo-Hermitian torsion of ∇.

3.6 Expressions in local coordinates

3.6.1 Christoffel symbols ([37])

Let $\left\{\mathrm{T}_\alpha : \alpha \in \{1, 2 \ldots, n\}\right\}$ be a local frame of $\mathrm{T}_{1,0}(M)$ defined on a given open set $U \subset M$. Since the Tanaka-Webster connection parallelizes the eigenbundles of J, there exist uniquely defined complex 1-forms $\omega_\beta^\alpha \in \Gamma^{\infty}(\mathrm{T}^*(M) \otimes \mathbb{C})$ (locally defined on U) such that

$$\nabla \mathrm{T}_\beta = \omega_\alpha^\beta \otimes \mathrm{T}_\alpha.$$

These are the connection 1-forms. Let us set $\mathrm{T}_{\overline{\alpha}} = \overline{\mathrm{T}_\alpha}$. Then,

$$\left\{\mathrm{T}_1, \ldots, \mathrm{T}_n, \mathrm{T}_{\overline{1}}, \ldots, \mathrm{T}_{\overline{n}}\right\}$$

is a frame of $\mathrm{T}(M) \otimes \mathbb{C}$ on U. Let us set $\omega_{\overline{\beta}}^{\overline{\alpha}} = \overline{\omega_{\beta}^{\alpha}}$. Then (since ∇ is a real operator) we have

$$\nabla \mathrm{T}_{\overline{\beta}} = \omega_{\overline{\beta}}^{\overline{\alpha}} \otimes \mathrm{T}_{\overline{\alpha}}$$

For any $\alpha, \beta \in \{1, \ldots, n\}$ and $A \in \{0, 1, \ldots, n, \overline{1}, \ldots, \overline{n}\}$, we define the Christoffel symbols $\Gamma_{A\beta}^{\alpha} : U \longrightarrow \mathbb{C}$ by

$$\Gamma_{A\beta}^{\alpha} = \omega_{\beta}^{\alpha}(\mathrm{T}_A),$$

with the convention $\mathrm{T}_0 = \mathrm{T}$. Therefore,

$$\nabla_{\mathrm{T}_\gamma} \mathrm{T}_\beta = \Gamma_{\gamma\beta}^{\alpha} \mathrm{T}_\alpha, \qquad \nabla_{\mathrm{T}_{\overline{\gamma}}} \mathrm{T}_\beta = \Gamma_{\overline{\gamma}\beta}^{\alpha} \mathrm{T}_\alpha, \qquad \nabla_{\mathrm{T}} \mathrm{T}_\beta = \Gamma_{0\beta}^{\alpha} \mathrm{T}_\alpha.$$

We denote by

$$h_{\alpha\overline{\beta}} = l_\theta(\mathrm{T}_\alpha, \mathrm{T}_{\overline{\beta}})$$

the components of the Levi form. Recall that

$$\nabla_X Y = U_{XY}, \quad X, Y \in \mathrm{T}_{1,0}(M).$$

Let us set for simplicity

$$U_{\alpha\beta} = U_{\mathrm{T}_\alpha \mathrm{T}_\beta}.$$

Then on the one hand,

$$U_{\alpha\beta} = \Gamma_{\alpha\beta}^{\gamma} \mathrm{T}_\gamma.$$

On the other hand, taking into account that

$$\Omega(\mathrm{T}_\alpha, \mathrm{T}_{\overline{\beta}}) = -i h_{\alpha\overline{\beta}},$$

where $h_{\alpha\overline{\beta}} = l_\theta(\mathrm{T}_\alpha, \mathrm{T}_{\overline{\beta}})$. We have

$$-i\Gamma_{\gamma\beta}^{\alpha} h_{\alpha\overline{\sigma}} = -i\mathrm{T}_\gamma(h_{\beta\overline{\sigma}}) - \Omega(\mathrm{T}_\beta, [T_\gamma, T_{\overline{\sigma}}]))$$

and contraction by $h^{\overline{\sigma}\alpha}$ leads to

$$\Gamma_{\gamma\beta}^{\alpha} = h^{\overline{\sigma}\alpha} \left(\mathrm{T}_\gamma(h_{\beta\overline{\sigma}}) - g_\theta(\mathrm{T}_\beta, [T_\gamma, T_{\overline{\sigma}}])\right) \tag{3.17}$$

Using the equality $[h^{\overline{\alpha}\beta}] = [h_{\overline{\alpha}\beta}]^{-1}$, we obtain

$$h^{\overline{\alpha}\beta} h_{\beta\overline{\sigma}} = \delta_{\sigma}^{\overline{\alpha}}.$$

Next, we will give the computation of the Christoffel symbols $\Gamma_{\overline{\gamma}\beta}^{\alpha}$. Since

$$\pi_+[\mathrm{T}_{\overline{\gamma}}, \mathrm{T}_\beta] = \nabla_{\mathrm{T}_{\overline{\gamma}}} \mathrm{T}_\beta = \Gamma_{\overline{\gamma}\beta}^{\alpha} \mathrm{T}_{\overline{\alpha}},$$

it yields

$$\Gamma_{\overline{\gamma}\beta}^{\alpha} = h^{\overline{\mu}\alpha} g_\theta([\mathrm{T}_{\overline{\gamma}}, \mathrm{T}_\beta], \mathrm{T}\overline{\mu}) \tag{3.18}$$

3.6.2 Pseudo-Hermitian torsion

Lemma 3.14 $\tau(\mathrm{T}_{1,0}(M)) \subset \mathrm{T}_{0,1}(M)$.

The proof follows from (3.10).

By Lemma 3.14, there exist uniquely defined $\mathcal{C}^\infty$ functions $A_\beta^{\overline{\alpha}} : U \longrightarrow \mathbb{C}$, such that

$$\tau \mathrm{T}_\beta = A_\beta^{\overline{\alpha}} \mathrm{T}_{\overline{\alpha}}.$$

Since

$$\tau \mathrm{T}_\alpha = \mathrm{T}\nabla(\mathrm{T}, \mathrm{T}_\alpha) = \nabla_{\mathrm{T}} \mathrm{T}_\alpha - [\mathrm{T}, \mathrm{T}_\alpha],$$

then

$$A_\alpha^{\overline{\beta}} \mathrm{T}_{\overline{\beta}} = -\pi_-[\mathrm{T}, \mathrm{T}_\alpha].$$

Set

$$A(X, Y) = g_\theta(\tau X, Y), \quad X, Y \in \mathrm{T}(M)$$

and let

$$A_{\alpha\beta} = A(\mathrm{T}_\alpha, \mathrm{T}_\alpha).$$

Then

$$A_{\alpha\beta} = A_\alpha^{\overline{\gamma}} h_{\overline{\gamma}\beta}.$$

If $\{\mathrm{T}_1, \ldots, \mathrm{T}_n\}$ is a local frame of $\mathrm{T}_{1,0}(M)$ and $\{\theta^1, \ldots, \theta^n\}$ the dual coframe in $\mathrm{T}^*_{1,0}(M)$, that is,

$$\theta^\alpha(\mathrm{T}_\beta) = \delta_\beta^\alpha, \quad \theta^\alpha(\mathrm{T}_{\overline{\beta}}) = 0, \quad \theta^\alpha(T) = 0.$$

We define τ^α by

$$\tau^\alpha = A_{\overline{\beta}}^\alpha \overline{\theta}.$$

Then,

$$\tau = \tau^\alpha \otimes \mathrm{T}_\alpha + \tau^{\overline{\alpha}} \otimes \mathrm{T}_{\overline{\alpha}},$$

where $\tau^{\overline{\alpha}} = \overline{\tau^\alpha}$.
For any $X = x^\alpha \mathrm{T}_\alpha$, we have

$$\mathrm{T}_\nabla(X, X) = i \left(A_{\overline{\alpha}\overline{\beta}} x^{\overline{\alpha}} x^{\overline{\beta}} - A_{\alpha\beta} x^\alpha x^\beta \right).$$

We have also

$$\mathrm{d}\theta = 2i h_{\alpha\overline{\beta}} \theta^\alpha \wedge \theta^{\overline{\beta}},$$
$$\mathrm{d}\theta^\alpha = \theta^\beta \wedge \omega_\beta^\alpha + \theta \wedge \xi^\alpha.$$

Lemma 3.15 *Let $(M, \mathrm{T}_{1,0}(M))$ be a nondegenerate CR manifold and θ a fixed pseudo-Hermitian structure on M. Let ∇ be the Tanaka-Webster connection of (M, θ). Then the torsion tensor field T_∇ of ∇ is given by*

$$\mathrm{T}_\nabla = 2\left(\theta \wedge \tau - \Omega \otimes \mathrm{T}\right). \tag{3.19}$$

Moreover, the Levi-Civita connection ∇^θ of the semi-Riemannian manifold (M, g_θ) is related to ∇ by

$$\nabla^\theta = \nabla + (\Omega - A) \otimes \mathrm{T} + \tau \otimes \theta + 2\theta \odot J, \tag{3.20}$$

where $\odot$ denotes the symmetric tensor product defined by

$$2\theta \odot J(X, Y) = \theta(X) JY + \theta(Y) JX, \quad X, Y \in \mathrm{T}(M).$$

Conformal transformation:
Let $\tilde{\theta} = \mathrm{e}^u \theta$. Then

$$\widetilde{A}_{\alpha\beta} = A_{\alpha\beta} + i \nabla_{\mathrm{T}_\alpha} u_\alpha - 2i u_\alpha u_\beta.$$

3.7 Curvature tensors

3.7.1 The curvature tensor field

We refer here to [70]:

Definition 3.16 Let $(M, \mathrm{T}_{1,0}(M))$ be a non degenerate CR manifold and θ a fixed pseudo-Hermitian structure on M. The curvature tensor field R of the Tanaka-Webster connection ∇ of (M, θ) is defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \quad X, Y, Z \in \mathcal{X}(M).$$

Notation: We denote by

$$R(X,Y,Z,W) = g_\theta(R(X,Y)Z, W), \quad X, Y, Z, W \in \mathrm{T}(M).$$

Proposition 3.17 *1) For any $X, Y, Z, W \in \mathrm{T}(M)$, we have*
a) $R(X,Y,Z,W) = R(Y,X,Z,W)$
b) $R(X,Y,Z,W) = R(Y,X,W,Z)$
c) $R(X,Y,Z,W) = R(Z,W,X,Y)$
2) R satisfies the Bianchi identity:

$$\sum_{XYZ} R(X,Y)Z = \sum_{XYZ} \left(\mathrm{T}_\nabla(\mathrm{T}_\nabla(X,Y),Z) + (\nabla_X \mathrm{T}_\nabla)(Y,Z)\right),$$

for any $X, Y, Z \in \mathrm{T}(M)$. Here$\sum_{XYZ}$ denotes the cyclic sum over X, Y, Z.

Remark 3.18 For any $X, Y, Z \in \mathrm{H}(M)$, we have

$$\begin{aligned} \mathrm{T}_\nabla(\mathrm{T}_\nabla(X,Y),Z) &= -2\Omega(X,Y)\xi(Z), \\ (\nabla_X \mathrm{T}_\nabla)(Y,Z) &= -2\left(\nabla_X \Omega\right)(Y,Z)\mathrm{T} = 0. \end{aligned}$$

Then,

$$\sum_{XYZ} R(X,Y)Z = -2\sum_{XYZ} \Omega(X,Y)\tau(Z), \quad X, Y, Z \in \mathrm{H}(M).$$

Let $\{\mathrm{T}_1, \ldots, \mathrm{T}_n\}$ be a local frame of $\mathrm{T}_{1,0}(M)$. Then

$$\begin{aligned} R(X,Y)\mathrm{T}_\alpha &= \nabla_X \nabla_Y \mathrm{T}_\alpha - \nabla_Y \nabla_X \mathrm{T}_\alpha - \nabla_{[X,Y]}\mathrm{T}_\alpha \\ &= \nabla_X \left(\omega_\alpha^\beta(Y)\mathrm{T}_\beta\right) - \nabla_Y \left(\omega_\alpha^\beta(X)\mathrm{T}_\beta\right) + \omega_\alpha^\beta([X,Y])\mathrm{T}_\beta \\ &= (\mathrm{d}\omega_\alpha^\beta)(X,Y)\mathrm{T}_\beta - \left(\omega_\alpha^\beta(Y)\omega_\alpha^\gamma(X) - \omega_\alpha^\beta(X)\omega_\alpha^\gamma(Y)\right)\mathrm{T}_\gamma. \end{aligned}$$

Denote by

$$R_{\beta\overline{\alpha}\rho\overline{\sigma}} = R(\mathrm{T}_\beta, \mathrm{T}_{\overline{\alpha}}, \mathrm{T}_\rho, \mathrm{T}_{\overline{\sigma}}).$$

Then,

$$R_{\beta\overline{\alpha}\rho\overline{\sigma}} = \mathrm{T}_\rho(\Gamma^\alpha_{\beta\overline{\sigma}}) - \mathrm{T}_{\overline{\sigma}}(\Gamma^\alpha_{\beta\rho}) + \Gamma^\alpha_{\beta\gamma}\Gamma^\gamma_{\rho\overline{\sigma}} - \Gamma^\alpha_{\beta\overline{\gamma}}\Gamma^{\overline{\gamma}}_{\overline{\sigma}\rho} + i\delta^\rho_{\overline{\sigma}}\Gamma^\alpha_{\beta 0}.$$

3.7.2 Pseudo-Hermitian Ricci and scalar curvatures

Let $\{\mathrm{T}_1, \ldots, \mathrm{T}_n\}$ be a local frame of $\mathrm{T}_{1,0}(M)$.

Definition 3.19 The Ricci tensor of the Webster connection is defined by

$$\mathrm{Ric}(Y, Z) = \mathrm{trace} X \longmapsto R(X, Z)Y, \quad Y, Z \in \mathrm{T}(M).$$

The pseudo-Hermitian Ricci tensor is then given by $\mathrm{Ric}_{\alpha\overline{\beta}} = \mathrm{Ric}(\mathrm{T}_\alpha, \mathrm{T}_{\overline{\beta}})$..

Definition 3.20 The Webster scalar curvature is defined by

$$R_\theta = h^{\alpha\overline{\beta}} \mathrm{Ric}_{\alpha\overline{\beta}}$$

Proposition 3.21 $R_\theta = \dfrac{1}{2} trace(\mathrm{Ric})$

Conformal transformation:
Let $\tilde{\theta} = u^{\frac{2}{n}}\theta$. Then

$$R_{\tilde{\theta}} = u^{-\left(1+\frac{2}{n}\right)} \left(\left(2 + \frac{2}{n}\right)(u_{\overline{\gamma}}^{\overline{\gamma}} + u_\gamma^\gamma) + R_\theta u \right),$$

where $u_\alpha = \mathrm{T}_\alpha(u), \quad u_{\overline{\alpha}} = \mathrm{T}_{\overline{\alpha}}(u),\ u^\gamma = h^{\gamma\overline{\beta}} u_{\overline{\beta}}, \quad$ and $u^{\overline{\gamma}} = \overline{u^\gamma}$.

3.8 The sub-Laplacian operator ([37])

3.8.1 Divergence of a vector field

Definition 3.22 Let $(M, \mathrm{T}_{1,0}(M))$ be a nondegenerate CR manifold, θ a fixed pseudo-Hermitian structure on M and ∇ be the Tanaka-Webster connection of (M, θ). The divergence of a vector field X, $\mathrm{div}(X)$, is defined by

$$\mathrm{div}(X) = \mathrm{trace}\,\{Y \in \mathrm{T}(M) \longmapsto \nabla_Y X\}.$$

Let $\{\mathrm{T}_1, \ldots, \mathrm{T}_n\}$ be a local frame of $\mathrm{T}_{1,0}(M)$ on an open set $U \subset M$. Then for any $Z = Z^\alpha \mathrm{T}_\alpha$, we have

$$\mathrm{div}(Z) = \mathrm{T}_\alpha(Z^\alpha) + Z^\beta \Gamma_{\alpha\beta}^\alpha.$$

3.8.2 The adjoint of a vector field

Let M be a $2n+1$-dimensional non degenerate CR manifold and θ a pseudo-Hermitian structure on M.

Proposition 3.23 *The $2n+1$-form $\theta \wedge (\mathrm{d}\theta)^n$ is a volume form on M.*

In other words, θ is a contact form on M.
We define the $L^2(M)$ inner product

$$(u, v) = \int_M u\overline{v}\,\theta \wedge (\mathrm{d}\theta)^n, \quad u, v \in L^2(M).$$

Definition 3.24 The adjoint X^* of a vector field X is defined by

$$(Xu, v) = (u, X^* v), \quad u, v \in L^2(M).$$

Let $(M, \mathrm{T}_{1,0}(M))$ be a strictly pseudo-convex CR manifold, of real dimension $2n+1$ and θ be a pseudo-Hermitian structure.
Let $\{X_a,\ 1 \le a \le 2n\}$ be a local g_θ-orthonormal frame of $\mathrm{H}(M)$ (i.e., $l_\theta(X_a, X_b) = \delta_{ab}$) defined on an open set $U \subset M$ and $(U, x_1, \ldots, x_{2n+1})$ be a local coordinate system on U. Then $X_a = b_a^i \dfrac{\partial}{\partial x_i}$, where $b_a^i \in \mathcal{C}^\infty(U)$.

Proposition 3.25 *For any* $1 \le a \le n$, *the adjoint of the vector field* X_a *is given by*

$$X_a^* u = -\frac{\partial}{\partial x_i}(b_a^i u) - b_a^j \Gamma_{ij}^i u, \quad u \in \mathcal{C}_0^\infty(U),$$

where Γ^i_{jk} *are the local coefficients of* ∇ *with respect to the local frame* $\{\frac{\partial}{\partial x_i},\ 1 \le i \le 2n+1\}$.

Proposition 3.26 *Let* $\{\mathrm{T}_\alpha,\ 1 \le \alpha \le n\}$ *be a local frame of* $\mathrm{T}_{1,0}(M)$ *on an open set* U. *Then*

$$\mathrm{T}_\alpha^* = -\mathrm{T}_\alpha + \sum_\alpha^n \Gamma^\alpha_{\beta\overline{\beta}}.$$

3.8.3 The sub-Laplacian operator

Let $(M, \mathrm{T}_{1,0}(M))$ be a strictly pseudo-convex CR manifold, of real dimension $2n+1$ and θ be a pseudo-Hermitian structure. Let $\pi_{\mathrm{H}} : \mathrm{T}(M) \longrightarrow \mathrm{H}(M)$ the natural projection associated with the direct sum decomposition $\mathrm{T}(M) = \mathrm{H}(M) \oplus \mathbb{R}\mathrm{T}$ (T be the characteristic direction of (M, θ)).

Definition 3.27 Let $f \in \mathcal{C}^2(M)$. We define the Hessian of f, $\nabla^2 f$, by

$$(\nabla^2 f)(X, Y) = (\nabla_X \mathrm{d} f)Y = X(Y(f)) - (\nabla_X Y)(f) - (\nabla_Y X)(f), \quad X, Y \in \mathcal{X}(M).$$

Let $\{\mathrm{T}_\alpha,\ 1 \le \alpha \le n\}$ be a local frame of $\mathrm{T}_{1,0}(M)$ on an open set U. We denote by

$$\begin{aligned} f_\alpha &= \mathrm{T}_\alpha(f), \\ f_{\overline{\alpha}} &= \mathrm{T}_{\overline{\alpha}}(f), \\ f_0 &= \mathrm{T}(f), \\ f_{AB} &= (\nabla^2 f)(\mathrm{T}_A, \mathrm{T}_B), \quad A, B \in \{0, 1, \ldots, n, \overline{1}, \ldots, \overline{n}\} \end{aligned}$$

Then,

$$\begin{aligned} f_{\alpha\beta} &= \mathrm{T}_\alpha(f_\beta) - \Gamma^\gamma_{\alpha\beta} f_\gamma, \\ f_{\alpha\overline{\beta}} &= \mathrm{T}_\alpha(f_{\overline{\beta}}) - \Gamma^{\overline{\gamma}}_{\alpha\beta} f_{\overline{\gamma}}, \\ f_{0\beta} &= \mathrm{T}(f_\beta) - \Gamma^\gamma_{0\beta} f_\gamma, \\ f_{\alpha 0} &= \mathrm{T}_\alpha(f). \end{aligned}$$

Definition 3.28 The horizontal gradient ∇^H is defined by

$$\nabla^{\mathrm{H}} f = \pi_{\mathrm{H}} \nabla f,$$

where ∇f is the ordinary gradient of f with respect to the Webster metric i.e., $g_\theta(\nabla f, X) = X(f)$, for any $X \in \mathcal{X}(M)$.

Definition 3.29 The sub-Laplacian operator of M is the operator Δ_b defined by

$$\Delta_b f = -\mathrm{div}(\nabla^{\mathrm{H}} f), \quad f \in \mathcal{C}^2(M).$$

Proposition 3.30

$$\begin{aligned} \Delta_b f &= f^{\overline{\alpha}}_{\overline{\alpha}} + f^\alpha_\alpha \\ &= f_{\overline{\alpha}\,\overline{\alpha}} + \sum_{\alpha=1}^n \Big(f_{\alpha\alpha}\Big) \\ &= -\sum_{\alpha=1}^n \Big(\mathrm{T}_\alpha^* \mathrm{T}_\alpha(f) + \mathrm{T}_{\overline{\alpha}}^* \mathrm{T}_{\overline{\alpha}}(f)\Big). \end{aligned}$$

Proposition 3.31 *For any* $u, v \in \mathcal{C}^2(U)$ *we have*

$$\int_U (\Delta_b u) v\, \theta \wedge (\mathrm{d}\theta)^n = -\int_U g_\theta\left(\nabla^{\mathrm{H}} u, \nabla^{\mathrm{H}} v\right). \tag{3.21}$$

3.9 Examples

3.9.1 The Heisenberg group

The Heisenberg group $\mathbb{H}^n$ is the homogeneous Lie group whose underlying manifold is $\mathbb{C}^n \times \mathbb{R} = \mathbb{R}^{2n+1}$ with coordinates

$$(z^1, \dots, z^n, t) = (x_1^1, x_2^1, \dots, x_1^n, x_2^n, t),$$

where for all $1 \leq j \leq n,\ z^j = x_1^j + ix_2^j$, and whose group law is defined by
$\forall (z^1, \dots, z^n, t), (w^1, \dots, w^n, s) \in \mathbb{H}^n$

$$(z^1, \dots, z^n, t)(w^1, \dots, w^n, s) = \left(z^1 + w^1, \dots, z^n + w^n, t + s + 2\mathrm{Im} \sum_{j=1}^n z^j \overline{w^j}\right)$$

We consider the complex vector fields on $\mathbb{H}^n$:

$$T_j = \frac{\partial}{\partial z^j} + i\overline{z^j}\frac{\partial}{\partial t}, \qquad \overline{T_j} = \frac{\partial}{\partial \overline{z^j}} - iz^j\frac{\partial}{\partial t}, \tag{3.22}$$

where

$$\frac{\partial}{\partial z^j} = \frac{1}{2}\left(\frac{\partial}{\partial x_1^j} - i\frac{\partial}{\partial x_2^j}\right), \quad \frac{\partial}{\partial \overline{z^j}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1^j} + i\frac{\partial}{\partial x_2^j}\right),$$

and $z^j = x_1^j + ix_2^j,\ 1 \leq j \leq n.$
Define $\mathrm{T}_{1,0}(\mathbb{H}^n)$ as the space spanned by the $T_j\,'s$, i.e.,

$$\mathrm{T}_{1,0}(\mathbb{H}^n) = \sum_{j=1}^n \mathbb{C}T_j. \tag{3.23}$$

Since

$$[T_j, T_k] = 0, \qquad \forall\ 1 \leq j, k \leq n,$$

it follows that $(\mathbb{H}^n, \mathrm{T}_{1,0}(\mathbb{H}^n))$ is a CR manifold.
Next, we consider the real 1-form θ_0 on $\mathbb{H}^n$ defined by

$$\theta_0 = \mathrm{d}t + i\sum_{j=1}^n \left(z^j \mathrm{d}\overline{z^j} - \overline{z^j}\mathrm{d}z^j\right), \tag{3.24}$$

it is a pseudoHermitian structure on $(\mathbb{H}^n, \mathrm{T}_{1,0}(\mathbb{H}^n))$.
By differentiating (3.24) we obtain

$$\mathrm{d}\theta_0 = 2i\sum_{j=1}^n \mathrm{d}z^j \wedge \mathrm{d}\overline{z^j};$$

by taking into account (3.3), it follows that

$$l_{\theta_0}(T_j, T_{\overline{k}}) = \delta_{jk},$$

where $T_{\overline{j}} = \overline{T_j}, \quad 1 \leq j \leq n.$

Our choice of θ_0 shows that $(\mathbb{H}^n, \theta_0)$ is a strictly pseudo-convex CR manifold. Its Levi distribution $\mathrm{H}(\mathbb{H}^n)$ is spanned by the (left-invariant) tangent vector fields $\{X_1, \dots, X_n, X_{n+1}, \dots, X_{2n}\}$, where

$$X_j = \frac{\partial}{\partial x_1^j} + 2ix_2^j \frac{\partial}{\partial t}, \quad X_{n+j} = \frac{\partial}{\partial x_2^j} - 2ix_1^j \frac{\partial}{\partial t}, \quad 1 \leq j \leq n.$$

The Reeb field of $(\mathbb{H}^n, \theta_0)$ is $\mathrm{T} = \dfrac{\partial}{\partial t}$.

The Horizontal gradient of $(\mathbb{H}^n, \theta_0)$ is given by

$$\nabla^{\mathrm{H}} = (X_1, \ldots, X_n, X_{n+1}, \ldots, X_{2n})$$

and the sub-Laplacian operator of $(\mathbb{H}^n, \theta_0)$ is given by

$$\Delta_b = -\sum_{\alpha=1}^{n} \left(X_\alpha^2 + X_{\overline{\alpha}}^2 \right).$$

Definition 3.32 The map $\delta_\lambda : \mathbb{H}^n \longrightarrow \mathbb{H}^n$ given by $\delta_\lambda(z, t) = (\lambda z, \lambda^2 t)$, for any $(z, t) \in \mathbb{H}^n$, is called the dilation by the factor $\lambda > 0$.

Proposition 3.33 *Each dilation is a group homomorphism and a CR isomorphism.*

Definition 3.34 The Heisenberg norm is

$$\begin{aligned} \rho : \mathbb{H}^n &\longrightarrow \mathbb{R}_+ \\ (z^1, \ldots, z^n, t) &\longmapsto \left(\sum_{j=1}^{n} |z^j|^4 + t^2 \right)^{\frac{1}{4}} \end{aligned}$$

3.9.2 The case n = 1

The Heisenberg group $\mathbb{H}^1$ is the Lie group $\mathbb{R}^3$ equipped with the law defined for all $\xi = (x; y; t)$, $\xi_0 = (x_0; y_0; t_0) \in \mathbb{R}^3$ by

$$\xi_0 \xi = (x_0 + x;\ y_0 + y;\ t_0 + t + 2(xy_0 - yx_0))$$

$$\rho(\xi) = ((x^2 + y^2)^2 + t^2)^{\frac{1}{4}} \text{ and } \delta_\lambda(\xi) = (\lambda x; \lambda y; \lambda^2 t), \text{ for } \xi = (x; y; t) \in \mathbb{H}^1.$$

Proposition 3.35 *A basis of the corresponding Lie algebra: the Heisenberg algebra $\mathfrak{h} = T_0\mathbb{H}^1$ is given by the following vector fields:*

$$\begin{aligned} X(x, y, t) &= \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial t}; \\ Y(x, y, t) &= \frac{\partial}{\partial y} - 2 \frac{\partial}{\partial t}; \\ T(x, y, t) &= \frac{\partial}{\partial t}. \end{aligned}$$

Proof Let f be a differentiable function defined on $\mathbb{H}^1$; we have, respectively,

$$\begin{aligned} X(f) &= \lim_{\epsilon \to 0} \frac{f(\xi(\epsilon, 0, 0)) - f(\xi)}{\epsilon} \\ &= \lim_{\epsilon \to 0} \frac{f(x + \epsilon, y, t + 2y\epsilon) - f(x, y, t)}{\epsilon} \\ &= \left(\frac{\partial}{\partial x} + 2\,y \frac{\partial}{\partial t} \right) f, \\ Y(f) &= \lim_{\epsilon \to 0} \frac{f(\xi(0, \epsilon, 0)) - f(\xi)}{\epsilon} \end{aligned}$$

$$
\begin{aligned}
&= \lim_{\epsilon\to 0} \frac{f(x, y+\epsilon, t-2x\epsilon) - f(x,y,t)}{\epsilon}\\
&= \left(\frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}\right) f,\\
T(f) &= \lim_{\epsilon\to 0} \frac{f(\xi \circ (0,0,\epsilon)) - f(\xi)}{\epsilon}\\
&= \lim_{\epsilon\to 0} \frac{f(x,y,t+\epsilon) - f(x,y,t)}{\epsilon}\\
&= \frac{\partial}{\partial t}(f).
\end{aligned}
$$

□

Definition 3.36 A tangent vector is left invariant if for all $f \in C^{\infty}(\mathbb{H}^1)$, we have

$$V(f_h) = (Vf)_h,$$

where f_h is the left translation of f in $\mathbb{H}^1$ given by

$$f_h(g) = f(hg), \qquad \forall h, g, \in \mathbb{H}^1.$$

Proposition 3.37 *The vector fields X, Y and T are left invariant.*

Proof Let f be a left translation on $\mathbb{H}^1$, for example $f = L_{(s,t,u)}$, $(s,t,u) \in \mathbb{H}^1$; then ,we have

$$f(x,y,z) = L_{(s,t,u)}(x,y,z) = (s,t,u)\circ(x,y,z) = (x+s, y+t, z+u+2(tx-sy)), \forall (x,y,z) \in \mathbb{H}^1$$

The derivative of f is

$$\mathrm{d}f = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 2t & -2s & 1; \end{pmatrix}$$

hence

$$\mathrm{d}f X = \frac{\partial}{\partial x} + 2(t+y)\frac{\partial}{\partial z}$$

On the other hand, we have

$$X \circ f = \frac{\partial}{\partial x} + (t+y)\frac{\partial}{\partial z};$$

therefore, $Xf = X \circ f$; hence X is left invariant.
It is the case for the vector fields Y and Z, since we have:

$$Yf = \frac{\partial}{\partial y} - 2(s+x)\frac{\partial}{\partial z} = Y \circ f$$

and

$$Zf = \frac{\partial}{\partial z} = Z \circ f$$

The Lie brackets are given by

$$[X,Y] = 4Z, \quad [X,Z] = [Y,Z] = 0.$$

□

Lemma 3.38 *The left invariant metric g on* $\mathbb{H}^1$ *is*

$$g_\theta = 4(\mathrm{d}x^2 + \mathrm{d}y^2) + (\mathrm{d}z + 2x\mathrm{d}y - 2y\mathrm{d}x)^2.$$

Proof Let g_θ denote the Riemannian metric associated with the Levi form of $\mathbb{H}^1$

$$g_\theta(u,v)=\sum_{1\le i,j\le 3} g_{\theta,ij}\mathrm{d}x_i\otimes \mathrm{d}y_j$$

where $g_{\theta,ij}=g_\theta\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right)$.

$$g_\theta(X,X)=\mathrm{d}\theta(X,JX)=\mathrm{d}\theta(X,J(Z+\bar{Z})=\mathrm{d}\theta(X,i(Z-\bar{Z})=\mathrm{d}\theta(X,-Y)==\theta(4T)=4.$$
$$g_\theta(Y,Y)=\mathrm{d}\theta(Y,JY)=\mathrm{d}\theta(X,J\left(\frac{Z-\bar{Z}}{i}\right)=-\mathrm{d}\theta(Y,Z+\bar{Z})=\mathrm{d}\theta(X,Y)=\theta[X,Y]=\theta(4T)=4,$$

where from (3.22)

$$Z=\frac{\partial}{\partial z^j}+i\overline{z^j}\frac{\partial}{\partial t},\qquad \overline{Z}=\frac{\partial}{\partial\overline{z^j}}-iz^j\frac{\partial}{\partial t}; \tag{3.25}$$

hence, $Z=\frac{1}{2}(X+iY)$ and $\bar{Z}=\frac{1}{2}(X-iY)$

$$\begin{aligned}g_\theta(X,T)&=L_\theta(X,T)=\mathrm{d}\theta(T,-Y)=0.\\ g_\theta(Y,T)&=L_\theta(Y,T)=\mathrm{d}\theta(T,X)=0.\\ g_\theta(X,Y)&=L_\theta(X,Y)=\mathrm{d}\theta(X,-X)=0.\end{aligned}$$

Finally,

$$\begin{aligned}g_\theta(T,T)&=L_\theta(T,T)=1.\\ 4&=g_\theta(X,X)\\ &=g_\theta\left(\frac{\partial}{\partial x}-2y\frac{\partial}{\partial t}\,,\,\frac{\partial}{\partial x}-2y\frac{\partial}{\partial t}\right)\\ &=g_\theta\left(\frac{\partial}{\partial x},\frac{\partial}{\partial x}\right)+4yg_\theta\left(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)+4y^2g_\theta\left(\frac{\partial}{\partial t},\frac{\partial}{\partial t}\right)\end{aligned}$$

Hence,

$$4=g_\theta\left(\frac{\partial}{\partial x},\frac{\partial}{\partial x}\right)-2yg_\theta\left(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)+4y^2. \tag{3.26}$$

We have also

$$\begin{aligned}0&=g_\theta(X,Y)\\ &=g_\theta\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)-2xg_\theta\left(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)+2yg_\theta\left(\frac{\partial}{\partial t},\frac{\partial}{\partial y}\right)-4xy;\end{aligned} \tag{3.27}$$

a simple computation gives

$$g_{\theta,12}=-4xy\implies g_{\theta,13}=g_\theta\left(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)=-2y, \tag{3.28}$$
$$g_{\theta,23}=2x.$$
$$g_{\theta,11}=4(1+y^2)\ g_{\theta,22}=4(1+x^2) \tag{3.29}$$

The result follows since

$$\begin{aligned}g=g(u,v)&=\sum_{1\le i,j\le 3}g_{\theta,ij}\mathrm{d}x_i\otimes\mathrm{d}y_j\\ &=g_{\theta,11}\mathrm{d}x^2+g_{\theta,12}\mathrm{d}x\mathrm{d}y+g_{\theta,21}\mathrm{d}y\mathrm{d}x+g_{\theta,13}\mathrm{d}x\mathrm{d}t+g_{\theta,31}\mathrm{d}t\mathrm{d}x\\ &\quad+g_{\theta,22}\mathrm{d}y^2+g_{\theta,23}\mathrm{d}y\mathrm{d}t+g_{\theta,32}\mathrm{d}t\mathrm{d}y+g_{\theta,33}\mathrm{d}t^2.\end{aligned}$$

The dual basis associated with $\varepsilon = (e_1 = X, e_2 = Y, e_3 = T)$ is the triplet of 1-forms $(\theta^1, \theta^2, \theta^3)$ satisfying the following conditions:

$$\theta^i(e_j) = \delta_{ij}$$

δ_{ij} is the Kronecker symbol. This base is given by

$$\theta^1 = \mathrm{d}x$$
$$\theta^2 = \mathrm{d}y$$
$$\theta^3 = \mathrm{d}t + 2(x\mathrm{d}y - y\mathrm{d}x)$$

$\theta^3 = \mathrm{d}t + 2(x\mathrm{d}y - y\mathrm{d}x)$ is the contact form θ_0 (see (3.24)) of the CR structure of $\mathbb{H}^1$. □

The Tanaka Webster connection

The Tanaka Webster connection ∇ associated with the contact form θ_0 expressed on the basis (X, Y, T) and using (3.11)is given by

$$\nabla_X Y = \pi_{\mathrm{H}}([X, Y]) = \pi_+(4T) = 0 \tag{3.30}$$
$$\nabla_X T = \pi_{\mathrm{H}}([X, T]) = 0 \tag{3.31}$$
$$\nabla_Y T = \pi_{\mathrm{H}}([Y, T]) = 0, \tag{3.32}$$

where π_{H} is the projection on the distribution H for the decomposition of the tangent space, see (3.7). Hence, ∇ is identically zero since T is parallel to the connection ((3.13)).

The torsion tensor

The torsion tensor associated with ∇ is given by

$$\mathrm{T}_\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]) = 4T \tag{3.33}$$
$$\mathrm{T}_\nabla(X, T) = \nabla_X T - \nabla_T X - [X, T]) = 0 \tag{3.34}$$
$$\mathrm{T}_\nabla(Y, T) = \nabla_Y T - \nabla_T Y - [Y, T]) = 0 \tag{3.35}$$

The pseudo-Hermitian torsion

We have $\tau(W) = \mathrm{T}_\nabla(T, W),\ W \in TM$. So

$$\tau(X) = \mathrm{T}_\nabla(T, X) = 0 \tag{3.36}$$
$$\tau(Y) = \mathrm{T}_\nabla(T, Y) = 0 \tag{3.37}$$
$$\tau(T) = \mathrm{T}_\nabla(T, T) = 0 \tag{3.38}$$

Hence, the pseudo-Hermitian torsion is identically zero.

Curvature tensors

Using the expression of the curvature tensor R given in (3.18)

$$R(U, V)W = -2\Omega(U, V)\tau(W) = 0 \quad \forall U, V, W \in T\mathbb{H}^1.$$

Hence, the tensor curvature R is identically zero; therefore, both the Ricci Tensor Ric and the Webster scalar curvature R_θ are zero.

3.9.3 The CR manifold $\mathbb{S}^{2n+1}$

Let $\mathbb{S}^{2n+1}$ be the unit sphere of $\mathbb{C}^{n+1}$. The standard CR structure of $\mathbb{S}^{2n+1}$ is given by

$$\mathrm{T}_{1,0}(\mathbb{S}^{2n+1}) = \mathrm{T}^{1,0}(\mathbb{C}^{n+1}) \cap \left(\mathrm{T}(\mathbb{S}^{2n+1}) \otimes \mathbb{C}\right),$$

where

$$\mathrm{T}^{1,0}(\mathbb{C}^{n+1}) = \mathrm{span}\left\{\frac{\partial}{\partial \zeta^j},\ 1 \le j \le n+1\right\}.$$

The standard contact form on $\mathbb{S}^{2n+1}$ is

$$\theta = j^*[i(\bar{\partial} - \partial)|\zeta|^2],$$

where $j : \mathbb{S}^{2n+1} \hookrightarrow \mathbb{C}^{n+1}$ is the inclusion map and $\partial,\ \overline{\partial}$ are defined by

$$\partial f = \sum_{j=1}^{n} \frac{\partial f}{\partial \zeta^j} \mathrm{d}\zeta^j, \quad \overline{\partial} f = \sum_{j=1}^{n} \frac{\partial f}{\partial \overline{\zeta}^j} \mathrm{d}\overline{\zeta}^j, \quad f \in \mathcal{C}^1(\mathbb{C}^{n+1}).$$

The standard CR structure $T_{1,0}(\mathbb{S}^{2n+1})$ admits the (local) frame

$$\left\{ \mathrm{T}_\alpha := \frac{\partial}{\partial \zeta^\alpha} - \frac{\overline{\zeta}^\alpha}{\overline{\zeta}^{n+1}} \frac{\partial}{\partial \zeta^{n+1}},\ 1 \leq \alpha \leq n \right\}$$

defined on the open set $\mathbb{S}^{2n+1} \cap \{\zeta;\ \zeta^{n+1} \neq 0\}$. Moreover, the Reeb field T is

$$\mathrm{T} = \frac{i}{2} \sum_{j=1}^{n} \left(\zeta^j \frac{\partial}{\partial \zeta^j} - \overline{\zeta}^j \frac{\partial}{\partial \overline{\zeta}^j} \right).$$

The Cayley transform is the mapping

$$\begin{aligned} F\ :\ & \mathbb{S}^{2n+1} \backslash (0, \ldots, -1) \longrightarrow \mathbb{H}^n \\ & \zeta = (\zeta_1, \ldots, \zeta_{n+1}) \longmapsto \left(\frac{\zeta_1}{1 + \zeta_{n+1}}, \ldots, \frac{\zeta_n}{1 + \zeta_{n+1}}, \frac{2Im\zeta_{n+1}}{|1 + \zeta_{n+1}|^2} \right). \end{aligned}$$

Its inverse map is

$$\begin{aligned} F^{-1}\ :\ \qquad & \mathbb{H}^n \qquad \longrightarrow \mathbb{S}^{2n+1} \backslash (0, \ldots, -1) \\ & (z^1, \ldots, z^n, t) \longmapsto \left(\frac{2z^1}{1 + |z|^2 - it}, \ldots, \frac{2z^n}{1 + |z|^2 - it},\ i\ \frac{1 - |z|^2 + it}{1 + |z|^2 - it} \right), \end{aligned}$$

where

$$|z|^2 = \sum_{j=1}^{n} |z^j|^2.$$

F is a CR equivalence. It gives a pseudo-Hermitian normal coordinates. In this coordinates we have

$$\begin{aligned} X_\alpha &= \frac{\partial}{\partial x_1^\alpha} + 2x_2^\alpha \frac{\partial}{\partial t}, \\ X_{\alpha+n} &= \frac{\partial}{\partial x_2^\alpha} - 2x_1^\alpha \frac{\partial}{\partial t}, \\ \mathrm{T} &= \frac{\partial}{\partial t}. \end{aligned}$$

For any $\zeta = (\zeta_1, \ldots, \zeta_{n+1}) \ \in\ \mathbb{S}^{2n+1} \backslash \{(0, \ldots, -1)\}$, we have

$$(F^*\theta_0)_\zeta = \frac{1}{|1 + \zeta_{n+1}|^2} \theta_\zeta.$$

Let us set

$$b(z, t) := \frac{4}{|1 + |z|^2 - it|}, \qquad (z, t) \in \mathbb{H}^1.$$

Then, we have

$$F^*(b\theta_0) = \theta$$

and

$$\theta \wedge (\mathrm{d}\theta)^n = |1 + \zeta_{n+1}|^{2(n+1)} F^*[\theta_0 \wedge \mathrm{d}\theta_0].$$

It is also useful to note that for any $u \in C^1(\mathbb{H}^n)$ and $v(\zeta) = \frac{u(F(\zeta))}{|1+\zeta_{n+1}|}$, we have

$$\int_{\mathbb{S}^{2n+1}} \left(b_n |\nabla^{\mathrm{H}} v|^2_\theta + R_n v^2\right) \theta \wedge (\mathrm{d}\theta)^n = \int_{\mathbb{H}^n} |\nabla^{\mathrm{H}} u|^2_{\theta_0} \theta_0 \wedge (\mathrm{d}\theta_0)^n$$

and

$$\int_{\mathbb{S}^{2n+1}} |v|^p \theta \wedge (\mathrm{d}\theta)^n = \int_{\mathbb{H}^n} |u|^p \theta_0 \wedge (\mathrm{d}\theta_0)^n,$$

where $p = b_n = 2 + \frac{n}{2}$, $R_\theta = \frac{n(n+1)}{2}$ is the pseudo-Hermitian scalar curvature of the sphere,

$$|\nabla^{\mathrm{H}} v|^2_\theta = g_\theta(\nabla^{\mathrm{H}} v, \nabla^{\mathrm{H}} v), \quad \text{and} \quad |\nabla^{\mathrm{H}} u|^2_{\theta_0} = g_{\theta_0}(\nabla^{\mathrm{H}} u, \nabla^{\mathrm{H}} u).$$

3.10 Normal coordinates and Folland–Stein spaces

In ([41]), Folland and Stein have constructed normal coordinates which show how closely the Heisenberg group can approximate a pseudo-Hermitian manifold.

Let $(M, \mathrm{T}_{1,0}(M))$ be a strictly pseudo-convex CR manifold of real dimension $2n+1$, on which one has fixed a contact 1-form θ, such that the Levi form l_θ is positive definite. Let T be the Reeb vector field of (M, θ).

Definition 3.39 Let $\{\mathrm{T}_\alpha,\ 1 \le \alpha \le n\}$ be a local orthonormal (i.e., $l_\theta(\mathrm{T}_\alpha, \mathrm{T}_{\overline{\beta}}) = \delta_{\alpha\beta}$) frame of $\mathrm{T}_{1,0}(M)$ defined on the open subset $U \subset M$. Such a $\{\mathrm{T}_\alpha,\ 1 \le \alpha \le n\}$ is referred to as a pseudo-Hermitian frame.

Let $\{\mathrm{T}_\alpha,\ 1 \le \alpha \le n\}$ be a pseudo-Hermitian frame. Let us set

$$\begin{aligned} X_0 &= \mathrm{T}, \\ X_\alpha &= \mathrm{T}_\alpha + \mathrm{T}_{\overline{\alpha}}, \\ X_{\alpha+n} &= i\left(\mathrm{T}_{\overline{\alpha}} - \mathrm{T}_\alpha\right), \quad \alpha \in \{1, \ldots, n\}. \end{aligned}$$

Let also $\{\theta^j,\ 0 \le j \le 2n\}$ be the dual frame with respect to $\{X^j,\ 0 \le j \le 2n\}$ (here $\theta^0 = \theta$).

Let $x \in M$ be fixed. Given $\eta \in \mathbb{R}^{2n+1}$ ($\eta = (\eta^1, \ldots, \eta^{2n}, \eta^0)$), let us consider the tangent vector field

$$X_\eta = \eta^j X_j \in \mathcal{X}(U)$$

For η sufficiently close to the origin in $\mathbb{R}^{2n+1}$, let $E_x(\eta)$ be the endpoint $C(1)$ of the integral curve $C : [0,1] \longrightarrow M$ of X_η issuing from x, i.e.,

$$\begin{cases} \dfrac{\mathrm{d}C}{\mathrm{d}t}(t) = X_\eta(C(t)), \\ C(0) = x. \end{cases} \tag{3.39}$$

E_x is a smooth map of a star-shaped neighborhood $\widetilde{U}_x$ of $0 \in \mathbb{R}^{2n+1}$ into M. Also

$$(\mathrm{d}_0 E_x)\frac{\partial}{\partial \eta_j} = X_j(x);$$

hence E_x is a diffeomorphism of a perhaps smaller neighborhood $U_x \subset \widetilde{U}_x$ of $0 \in \mathbb{R}^{2n+1}$ (which may be assumed to be star-shaped, too) onto a neighborhood V_x of x in M. Then $E_x^{-1} : V_x \longrightarrow U_x$ is the local chart.

Definition 3.40 The resulting local coordinates are referred to as the Folland-Stein (normal) coordinates (or pseudo-Hermitian normal coordinates) at x.

For $x \in M$ fixed let

$$E_x^{-1} = (x^j) = (x_1^\alpha, x_2^\alpha, t) : V_x \longrightarrow U_x \subset \mathbb{R}^{2n+1}$$

be Folland–Stein coordinates at x.

Definition 3.41 A function f on V_x is said to be O^1 and we write $f = O^1$ (resp. O^2 and we write $f = O^2$) if

$$f(y) = O\left(\sum_{\alpha=1}^{n} |x_1^\alpha(y)| + |x_2^\alpha(y)| + |t(y)|^{\frac{1}{2}}\right)\left(\text{resp. } f(y) = O\left(\sum_{\alpha=1}^{n} |x_1^\alpha(y)|^2 + |x_2^\alpha(y)|^2 + |t(y)|\right)\right)$$

as $y \longrightarrow x$ in V_x.

Theorem 3.42 (Folland–Stein) *With respect to the Folland–Stein normal coordinates*

$$E_x^{-1} = (x^j) = (x_1^\alpha, x_2^\alpha, t)$$

on V_x, one has

$$X_\alpha = \frac{\partial}{\partial x_1^\alpha} + 2x_2^\alpha \frac{\partial}{\partial t} + \sum_{\beta=1}^{n}\left(O^1 \frac{\partial}{\partial x_1^\beta} + O^1 \frac{\partial}{\partial x_2^\beta}\right) + O^2 \frac{\partial}{\partial t},$$
$$X_{\alpha+n} = \frac{\partial}{\partial x_2^\alpha} - 2x_1^\alpha \frac{\partial}{\partial t} + \sum_{\beta=1}^{n}\left(O^1 \frac{\partial}{\partial x_1^\beta} + O^1 \frac{\partial}{\partial x_2^\beta}\right) + O^2 \frac{\partial}{\partial t},$$
$$\mathrm{T} = \frac{\partial}{\partial t} + \sum_{\beta=1}^{n}\left(O^1 \frac{\partial}{\partial x_1^\beta} + O^1 \frac{\partial}{\partial x_2^\beta}\right) + O^2 \frac{\partial}{\partial t}.$$

Theorem 3.43 [56] *Let M be a strictly pseudo-convex pseudo-Hermitian manifold of dimension $2n+1$ with a contact form θ and let $V \subset M$ be an open set on which there is given a pseudo-Hermitian frame W_α. There is a neighborhood of the diagonal $\Omega \subseteq V \times V$ and a $\mathcal{C}^\infty$ mapping $\Theta : \Omega \longrightarrow \mathbb{H}^n$ satisfying*

(1) $\Theta(\zeta, \eta) = -\Theta(\eta, \zeta) = \Theta(\eta, \zeta)^{-1}$, for any $(\zeta, \eta) \in \Omega$. (In particular $\Theta(\zeta, \zeta) = 0$).
(2) Denote $\Theta_\zeta(\eta)$. thus Θ_ζ is a diffeomorphism of a neighborhood Ω_ζ of ζ onto a neighborhood of the origin in $\mathbb{H}^n$.
(3)

$$(\Theta_\zeta^{-1})^* \left(\theta \wedge (\mathrm{d}\theta)^n\right) = (1 + O^1)\left(\theta_0 \wedge (\mathrm{d}\theta_0)^n\right).$$

For more details and explanations, we refer to [56].

Let U be a relatively compact open subset of a normal coordinate neighborhood Ω_ζ, as in Theorem 3.43. With these notations, if $X_j = \mathrm{Re}(W_j)$, $X_{j+n} = \mathrm{Im}(W_j)$. We set

$$X^\alpha = (X_{\alpha_1}, \ldots, X_{\alpha_k}), \quad \alpha = (\alpha_1, \ldots, \alpha_k), \quad 1 \le j \le 2n.$$

If $\alpha = (\alpha_1, \ldots, \alpha_l)$ set $l(\alpha) = l$. Consider the norms

$$\|f\|_{\mathcal{S}_k^r(U} = \sup_{l(\alpha) \le k} \|X^\alpha f\|_{L^r(U)}, \tag{3.40}$$

where

$$\|g\|_{L^r(U)} = \left(\int_U |g|^r \, \theta \wedge (\mathrm{d}\theta)^n\right)^{\frac{1}{r}}.$$

Definition 3.44 The Folland–Stein space $\mathcal{S}_k^r(U)$ is the completion of $C_0^\infty(U)$ for the norm (3.40).

For a compact strictly pseudo-convex CR manifold M, we choose a finite open covering of M, $(U_1, U_2 \dots U_m)$ for which the property displayed in the definition above is satisfied for each open of this covering. Then, we choose a partition of unity $\{\phi_j, 1 \leq j \leq m\}$ subjected to this covering.

Definition 3.45 The Folland–Stein space of M of indices (k,p) is

$$\mathcal{S}_k^r(M) = \{f \in L^1(M) \ \ s.t \ \ \phi_j f \in \mathcal{S}_r^p(U_j),\ 1 \leq j \leq m\}$$

Remark 3.46 Folland–Stein spaces are the Cauchy Riemann counterparts of Sobolev Spaces for Riemannian manifolds.

In [56] Jerison and Lee proved the following result:

Proposition 3.47 (Proposition 5.5) *With the notations above,* $\mathcal{S}_k^r(M) \subset L^s(M)$ *for* $\dfrac{1}{s} = \dfrac{1}{r} - \dfrac{k}{2n+2}$ *and* $1 < r < s < \infty$.

If we take $k = 1$ and $r = 2$, one shows that the Folland–Stein space $\mathcal{S}_1^2(M)$ is compactly embedded in $L^s(M)$ for $s \leq 1 + \dfrac{2}{n}$ and the inclusion is only continuous if $s = 2 + \dfrac{2}{n}$.

4 Prescribing the Webster scalar curvature on CR manifolds

4.1 Preliminaries

Let (M, θ) be a real compact orientable and integrable pseudo-Hermitian manifold of dimension $2n + 1$. We denote by $-L = -L_\theta = -(2 + \frac{2}{n})\Delta_b + R_\theta$ the conformal CR Laplacian of M, where Δ_b is the sub Laplacian operator and R_θ the Webster scalar curvature associated to θ. Let $K : M \to \mathbb{R}$ be a C^2 positive function. Our aim is to find suitable conditions on K which enable to prove the existence on M of a contact form $\tilde{\theta}$ CR conformal to θ, having the function K as Webster scalar curvature, $R_{\tilde{\theta}} = K$. We write $\tilde{\theta} = u^{\frac{2}{n}}\theta$, where u is a positive function defined on M. We obtain the following transformation law for the conformal Laplacians $-L_\theta$ and $-L_{\tilde{\theta}}$, see [56].

$$\left(-\left(2 + \frac{2}{n}\right)\tilde{\Delta}_b + R_{\tilde{\theta}}\right)\tilde{u} = r^{1-p}\left(-\left(2 + \frac{2}{n}\right)\Delta_b + R_\theta\right)\tilde{u} \tag{4.1}$$

with $p = \frac{2}{2+n}$ and $\tilde{u} = r^{-1}u$. If we substitute $r = u$ in (4.1), we obtain the following transformation law for the Webster scalar curvature R_θ of the contact form θ and the Webster scalar curvature $R_{\widetilde{\theta}}$ of the contact form $\widetilde{\theta}$:

$$R_{\widetilde{\theta}} = u^{1-p}\left(-\left(2 + \frac{2}{n}\right)\Delta_b + R_\theta\right)\tilde{u} \tag{4.2}$$

Hence, the fact of finding a solution for the prescribed scalar curvature problem is equivalent to solving the following partial differential equation:

$$\begin{cases} -\left(2 + \frac{2}{n}\right)\Delta_b u + R_\theta u = K u^{1+\frac{2}{n}} \\ u > 0 \end{cases}$$

This equation can be rewritten as

$$(P_K)\begin{cases} -Lu = K u^{1+\frac{2}{n}} \text{ on } M \\ \quad u > 0 \end{cases} \tag{4.3}$$

There is a big number of papers devoted to the prescription of a scalar curvature as well as for the multiplicity of solutions for the related differential equation. We can mention, for example, the papers [38, 40,63]. Concerning the most directly related literature using the method based on the theory of critical points at infinity, we refer first to the pioneer work in that direction [42] and for more recent ones, we can mention [45–48,53,54].

In case the function K to prescribe is constant, $K = \lambda$, $\lambda in\mathbb{R}$, equation (P_λ) is the so-called CR Yamabe Equation. If u_0 is a solution of (P_λ), then the contact form $\widetilde{\theta} = u^{\frac{2}{n}}\theta$, has a constant scalar curvature, $R_{\widetilde{\theta}} = \lambda$.

4.2 CR functional and compactness arguments

Equation (P_K) is the Euler–Lagrange equation of the functional

$$I(u) = \frac{1}{2}\int_M (|\nabla u|^2 \ + R_\theta u^2\ \theta \wedge \mathrm{d}\theta^n - \left(2+\frac{2}{n}\right)^{-1} \int_M K u^{2+\frac{2}{n}} \theta \wedge \mathrm{d}\theta^n$$

$u \in S_1^2(M)$.

$$I'(u) = 0 \Leftrightarrow -Lu = K u^{1+\frac{2}{n}}$$

Does this equation have a solution? Can we find u such that $u = -L^{-1}\ f(u)$, with $f(u) = K|u|^{p-1}\ u,\ \ p = 1+\frac{2}{n}$ by applying compactness arguments? The answer is negative. More precisely, for $s \le 1+\dfrac{2}{n}$, let F_s be the following function:

$$\begin{array}{rcl} F_s : S_1^2(M) & \longrightarrow & S_1^2(M) \\ u & \longmapsto & -L^{-1}\ K|u|^{s-1}\ u \end{array}$$

If $F_{1+\frac{2}{n}}$ has a fixed point, then it is a solution of our problem (P_K). First of all let us prove that F_s is well defined. We know that $\mathcal{S}_1^2(M)$ is embedded continuously in $L^r(M)$ for $r \le 2+\dfrac{2}{n}$ $(\mathcal{S}_1^2(M) \subset L^r(M)$. That means, for $r \le 2+\dfrac{2}{n}$, there exists a constant $C(r, M)$ such that

$$\|u\|_{L^r(M)} \le C(r, M)\ \|u\|_{S_1^2(M)}$$

If $u \in S_1^2(M) \hookrightarrow_{\text{continuously}}\ K|u|^{s-1}\ u \in L^{\left(2+\frac{2}{n}\right)\frac{1}{s}}(M)$; in other words

$$\int_M (K|u|^{s-1}\ u - K|v|^{s-1}\ v)^{\left(2+\frac{2}{n}\right)\frac{1}{s}} \to 0 \ \text{ if }\ u \to v\ \text{ in }\ S_1^2(M).$$

We know that

$$-L^{-1} : (S_1^2(M))^{-1} \longrightarrow S_1^2(M),$$

where $(S_1^2(M))^{-1}$ is the dual space of $S_1^2(M)$. Therefore, we have to prove that for

$$s \le 1+\frac{2}{n},\ L^{(2+\frac{2}{n})\frac{1}{s}}(M)\ \subset (S_1^2(M))^{-1}$$

which is equivalent to $S_1^2(M) \subset\ (L^{(2+\frac{2}{n})\frac{1}{s}}(M))^{-1}$, the dual of $L^{(2+\frac{2}{n})\frac{1}{s}}(M)$. We have $(L^{(2+\frac{2}{n})\frac{1}{s}}(M))^{-1} = L^q(M)$, with

$$q \le 2+\frac{2}{n} \text{ and } q + \frac{s}{2+\frac{2}{n}} = 1 \tag{4.4}$$

The above properties imply $q = \dfrac{1}{1-\frac{ns}{2n+2}} \le 2+\dfrac{2}{n}$; hence $s \le 1+\frac{2}{n}$, which is true.

So, the inclusion holds and the function F_s is well defined. The second step is to verify the compactness of F_s which means the following: does F_s send bounded subsets of $S_1^2(M)$ to relatively compact subsets of $S_1^2(M)$? We have two possible cases:

1) If $s < 1 + \frac{2}{n}$, then there exist $q < 2 + \frac{2}{n}$, and $p_1 < 1 + \frac{2}{n}$ such that

$$\left(2 + \frac{2}{n}\right)\frac{1}{p_1} = \frac{q}{s}. \tag{4.5}$$

Hence,

$$\begin{aligned} u \in S_1^2(M) \longrightarrow_{compat} u \in L^q \longrightarrow_{\text{continuous}} \; K|u|^{s-1}u \in L^{\frac{q}{s}}(M) \\ = L^{\left(2+\frac{2}{n}\right)\frac{1}{p_1}}(M) \longrightarrow -L^{-1}(K|u|^{s-1}\; u) \in S_1^2(M) \end{aligned} \tag{4.6}$$

So, in this case F_s is compact.

2) Let $s = 1 + \frac{2}{n}$, to ensure the embedding of $L^{(2+\frac{2}{n})\frac{1}{p_1}}(M)$ in $S_1^2(M)$; we suppose the existence of such real $p_1 \leq 1 + \frac{2}{n}$. It yields that equality (4.5) gives $q = 2 + \frac{2}{n}$. Hence, in this case, the inclusion of $S_1^2(M)$ in $L^{(2+\frac{2}{n})}$ is only continuous but not compact.

Therefore, the function $F_{1+\frac{2}{n}}$ is not compact and the methods based on compactness arguments do not apply to solve the scalar curvature equation (P_K).

4.3 Natural change of the CR functional

We write $u \in S_1^2(M)$ as $u = (|u|_{-L}.\frac{u}{|u|_{-L}}) = (\lambda.v)$. Hence,

$$I(u) = I(\lambda.v) = \frac{1}{2}\lambda^2 - \left(2 + \frac{2}{n}\right)^{-1}\lambda^{2+\frac{2}{n}}\int_M K v^{2+\frac{2}{n}}\theta \wedge \mathrm{d}\theta^n$$

$$I'_\lambda(\lambda.v) = \lambda - \lambda^{2+\frac{2}{n}}\int_M K v^{2+\frac{2}{n}}\theta \wedge \mathrm{d}\theta^n$$

$$I'_\lambda(\lambda.v) = 0 \Leftrightarrow \lambda(v) = \left(\frac{1}{\displaystyle\int_M K v^{2+\frac{2}{n}}\theta \wedge \mathrm{d}\theta^n}\right)^{\frac{n}{2}}. \tag{4.7}$$

The second derivative in $\lambda(v)$ is

$$I''_\lambda(\lambda.v) = 1 - \left(1 + \frac{2}{n}\right) < 0.$$

Hence, the second derivative at the critical point $\lambda(v) = (\frac{1}{\int_M K v^{2+\frac{2}{n}}\theta \wedge \mathrm{d}\theta^n})^{\frac{n}{2}}$ is negative, so $\lambda(v)$ is a maximum for $I(\lambda.v)$. A critical point of I has to satisfy

$$I'(\lambda.v) = 0 \Leftrightarrow \begin{cases} (1)\ I'_\lambda(\lambda.v) = 0 \\ (2)\ I'_v(\lambda.v) = 0; \end{cases} \tag{4.8}$$

since (1) is realized, if $\lambda.v = \lambda(v).v$, (2) is equivalent to $I'_v(\lambda(v).v) = 0$.
We consider the following function:

$$\begin{array}{rcccc} \Phi : S \subset S_1^2(M) & \longrightarrow_{\Phi_1} & S_1^2(M) & \longrightarrow_{\Phi_2} & \mathbb{R} \\ v & \longmapsto & \lambda(v).v & \longmapsto & I(\lambda(v), v) \end{array}$$

Where S is the unit sphere of the space $S_1^2(M)$ for the norm $||_{-L}$.
Denote by $T_v S$ the tangent space of S at v; we have

$$\begin{array}{rl} \Phi'(v) : T_v S & \longrightarrow \mathbb{R} \\ h & \longmapsto \Phi'(v)(h) = (I(\lambda(v).v))'(h) \end{array}$$

and

$$(I(\lambda(v).v))'(h) = I'(\lambda(v).v))[\lambda'(v).v + \lambda(v)](h).$$

This equation is reduced to

$$(I(\lambda(v).v))'(h) = \lambda(v)\ I'(\lambda(v).v))(h)$$

since

$$I'(\lambda(v).v))(v) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\ I(\lambda v)\ |_{\lambda(v)} = 0.$$

Hence, we have a new choice for the functional; we set

$$J(v) = I(\lambda(v).v))$$

for

$$\begin{aligned} v \in S = \{v \in S_1^2(M),\ |v|_{-L} = 1\} \\ J(v) = \max_{\lambda>0} I(\lambda(v).v)) \end{aligned}$$

The maximum is attained in a unique point $\lambda = \lambda(v)$ and

$$J(v) = \frac{1}{(2n+2)\left(\int_M K v^{2+\frac{2}{n}}\theta \wedge \mathrm{d}\theta^n\right)^{\frac{n}{2}}} \tag{4.9}$$

The functionals I and J are both of class C^2 and there is a one-to-one correspondence between the non zero critical points of I and J.

4.4 Hessian and the Morse Lemma [64]

Definition 4.1 Let M be a compact C^∞ manifold; a point x_0 is said to be a non degenerate critical point of a smooth function $f : M \to \mathbb{R}$ if

1) the derivative of f at x_0, $\mathrm{d}f_{x_0} \equiv 0$
2) the Hessian of f at x_0, $Hessf(x_0)$ is a non degenerate quadratic form.

Definition 4.2 If M is of finite dimension, the index of the non degenerate critical point x_0 of f, denoted by $ind(x_0)$, is the number of negative eigenvalues of $Hessf(x_0)$.

Theorem 4.3 *Morse Lemma Let M be a C^∞ manifold of dimension n and $f : M \to \mathbb{R}$ a smooth function having a non degenerate critical point x_0 of index λ. There exists an open neighborhood U of and a local chart $y = \phi(x)$, $\phi : U \longrightarrow \mathbb{R}^n$ such that $\phi(x_0) = 0$ and*

$$f \circ \phi^{-1}(y) = f \circ \phi^{-1}(0) - \sum_{i=1}^{i=\lambda} y_i^2 + \sum_{i=\lambda+1}^{i=n} y_i^2$$

4.5 Homotopy and homotopy type

Definition 4.4 Let X and Y be two topological spaces, $f_0 : X \to Y$ and $f_1 : X \to Y$ two continuous maps from X to Y. We say that f_0 is homotopic to f_1 if there exists a continuous map $F : X \times [0, 1] \longrightarrow Y$ such that

- $F(x, 0) = f_0(x), \ \forall x \in X.$
- $F(x, 1) = f_1(x) \in Y, \ \forall x \in X.$

It is an equivalence relation.

Definition 4.5 Two topological spaces X and Y are said to have the same homotopy type or homotopy equivalent, if there exist $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ two continuous maps such that $g \circ f$ is homotopic to the identity of X and $f \circ g$ is homotopic to the identity of Y.

Definition 4.6 A topological space X is said to be contractible if it has the same homotopy type of "a point".

Remark 4.7 A contractible space is simply connected.

The results and definitions given in this section are extracted from [64]

4.6 Deformation retract

Definition 4.8 Let $Y \subset X$ be two topological spaces and $r : X \to Y$ an onto continuous map from X to Y; r is called a retraction by deformation of X onto Y, if

- $r \circ i_Y = Id_Y, (i_Y : Y \hookrightarrow X$ is the inclusion map.)
- $i_Y \circ r$ is homotopic to Id_X.

Y is said to be a deformation retract of X.

Let M be a compact manifold and $f : M \longrightarrow \mathbb{R}$ a smooth function defined on M. Let $a, b \in \mathbb{R}, \ a < b$. Define

$$M^a := f^{-1}((-\infty, a]) = \{x \in M : \ f(x) \leq a\}$$
$$M^{ab} := f^{-1}([a, b]) = M^b \backslash \text{int}(M^a)$$

Theorem 4.9 *We suppose that M^{ab} is compact and contains no critical point of f. Then M^a is diffeomorphic to M^b. Furthermore, M^a is a deformation retract of M^b so that the inclusion map $i : M^a \hookrightarrow M^b$ is a homotopy equivalence.*

Theorem 4.10 *Let $f : M \longrightarrow \mathbb{R}$ be a smooth function and let x_0 be a non degenerate critical point with index λ. Setting $f(x_0) = c$, suppose that $f^{-1}([c - \epsilon, c + \epsilon])$ is compact, and contains no critical point of f other than x_0, for some $\epsilon > 0$. Then, for all sufficiently small ϵ, the set $M^{c+\epsilon}$ has the homotopy type of $M^{c-\epsilon}$ with a λ-cell attached.*

We give here some elements of the proof of the second theorem and for detailed proofs of the two theorems above one can see [64].

From the Morse Lemma, we have

$$f \circ \phi^{-1}(X, Y) = f(x_0) + |Y|^2 - |X|^2;$$

hence

$$f \circ \phi^{-1}(X, 0) = f(x_0) - |X|^2.$$

Denote by $B_r^-(\epsilon) = \{\phi^{-1}(X, 0), \ |X|^2 \leq \epsilon\}$ and suppose that the scalar product around x_0 is defined in the coordinates (X, Y) by $|X|^2 + |Y|^2$. The gradient of f is then given by

$$grad f_{(X,Y)} = 2 \Leftrightarrow \begin{cases} \dfrac{\partial f \circ \phi^{-1}}{\partial X}(X, Y) = -2X \\ \dfrac{\partial f \circ \phi^{-1}}{\partial Y}(X, Y) = 2Y \end{cases} \tag{4.10}$$

The differential equation, $\overbrace{(X, Y)}^{\cdot} = \text{grad} f_{(X,Y)} \Leftrightarrow$

$$\begin{cases} \dfrac{\partial X}{\partial s} = -2X \\ \dfrac{\partial Y}{\partial s} = 2Y, \end{cases} \tag{4.11}$$

is easy to integrate and gives

$$\begin{cases} X(s) = e^{-2s} X(0) \\ Y(s) = e^{2s} Y(0) \end{cases} \tag{4.12}$$

The trajectories of this differential equation "crash" on the Y axis when $s \to +\infty$ and " crash" on the X axis when $s \to -\infty$. So, the differential equation is defined on the hole manifold M and admits a solution $\eta(s, z)$, $\forall s$ and $\forall z \in M$ since M is compact.

Definition 4.11 The unstable manifold of x_0 for the vector field grad$f$ or ∇f denoted by $W_u(x_0)$ is the set of points $z \in M$ for which the solution $\eta(s, z)$ converges to x_0 as $s \to +\infty$.

Definition 4.12 The stable manifold of x_0 for the vector field grad f or ∇f denoted by $W_s(x_0)$ is the set of points $z \in M$ for which the solution $\eta(s, z)$ converges to x_0 as $s \to -\infty$.

The dimension of the unstable manifold $W_u(x_0)$ is equal to the index of the critical point x_0; in fact we have $W_u(x_0) = B_r^-(\epsilon)$. If $\epsilon > 0$ is sufficiently small then $M^{c+\epsilon}$ retracts by deformation on $M^{c-\epsilon} \bigcup B_r^-(\epsilon)$.

4.7 The Palais–Smale condition

The Palais–Smale condition, which relates both to the function and to the metric, imposes conditions at infinity which allow a control of the dynamics of the gradient on a manifold without boundary.

Suppose that I is a differentiable functional on a Hilbert space H; we say that I satisfies the Palais–Smale condition, (PS) in short, if for any sequence (u_k) of H satisfying

1) $I(u_k)$ is bounded.
2) $\partial I(u_k) \longrightarrow 0$.

We can extract from (u_k) a convergent subsequence.

Proposition 4.13 *If the Palais–Smale condition is not satisfied by a functional I in an interval $[a, b]$ and if I^b does not retract by deformation onto I^a, then I admits a critical value in $[a, b]$.*

Since the injection $S_1^2(M)$ in $L^{2+\frac{2}{n}}(M)$ is continuous but not compact, the functional J given by (4.9) fails to satisfy the (PS) condition. One can see that the standard solutions of the Yamabe problem on $\mathbb{H}^n$ after superposition are the good candidate sequences which violate the Palais–Smale condition.

4.8 The CR Yamabe problem

In [56], Jerison and Lee have extensively studied the CR Yamabe problem and showed that there is a deep analogy between the CR Yamabe problem and the Riemannian one. Their results can be formally compared to the partial completion of the proof of the Riemannian Yamabe conjecture by Aubin. As it was the case in the Riemannian settings, the CR Yamabe equation is the Euler–Lagrange equation for the constrained variational problem

$$\lambda(M) = \inf_{u \in S_1^2(M)} \{A_\theta(u)/B_\theta(u) = 1\} \tag{4.13}$$

$$A_\theta(u) = \int_M -Lu\, u\, \theta \wedge \mathrm{d}\theta^n; \quad B_\theta(u) = \int_M |\, u\,|^{2+\frac{2}{n}}\, \theta \wedge \mathrm{d}\theta^n.$$

In 1986, Jerison and Lee [56] gave a necessary condition on the conformal constant (4.13) to have existence of solutions for the CR Yamabe problem:

Theorem 4.14 *Let M be a compact, orientable strictly pseudo-convex and integrable CR manifold of dimension $2n+1$, and θ any contact form on M.*

1) *$\lambda(M)$ depends on the CR structure on M, not on the choice of θ.*
2) *$\lambda(M) \le \lambda(S^{2n+1})$, where $S^{2n+1} \subset \mathbb{C}^{n+1}$ is the unit sphere with its standard CR structure.*
3) *If $\lambda(M) < \lambda(S^{2n+1})$, then the infimum in (4.13) is attained by a positive C^∞ solution of (P_λ). If we denote this solution u, the contact form $\tilde{\theta} = u^{\frac{2}{n}}\theta$ has a constant Webster scalar curvature $R_{\tilde{\theta}} = \lambda(M)$.*

Theorem 4.14 (1) is an analogue of Aubin's Theorem 2.2 for the Riemannian Yamabe problem and its proof is also similar to the one given for Aubin's Theorem 2.2.

Since the asymptotic expansion of the Yamabe functional

$$Y(M,\theta) = \frac{\int_M \; R\,\theta \wedge \mathrm{d}\theta^n}{\left(\int_M \; \theta \wedge \mathrm{d}\theta^n\right)^{\frac{2}{p}}}$$

on M is expressed in terms of pseudoHermitian curvature and torsion invariants. In order to make the calculation as easy as possible Jerison and Lee refined their notion of normal coordinates, defined in [56], by constructing in [58] new intrinsic CR normal coordinates for an abstract CR manifold. These coordinates are called pseudoHermitian normal coordinates. Using these coordinates, Jerison and Lee have simplified the pseudo-Hermitian curvature and torsion invariants at a base point q and showed that the contact form can be chosen in a neighborhood of q so that the pseudo-Hermitian Ricci and torsion tensors and certain combination of their covariant derivatives vanish at this base point. The notions and results introduced and proved by Jerison and Lee are parallel, with drastically different techniques, to the one introduced by Lee and Parker for the Riemannian Yamabe conjecture [61]. For a family of contact forms θ^ϵ which concentrate more and more around the base point q. The asymptotic expression of the functional for θ^ϵ is given by

$$Y(M,\theta^\epsilon) = \begin{cases} \lambda(S^{2n+1})(1 - c(n)|S(q)|^2\epsilon^4) + O(\epsilon^5) & \text{for } n \ge 3 \\ \lambda(S^{2n+1})\left(1 - c(2)|S(q)|^2\epsilon^4 \log\frac{1}{\epsilon}\right) + O(\epsilon^4) & \text{for } n = 2. \end{cases} \tag{4.14}$$

Here, $S(q)$ is the Chern curvature tensor [30] of M evaluated at q and $c(n) > 0$ S is identically zero precisely when M is locally CR equivalent to the sphere.
In 1987, Jerison and Lee established the following result:

Theorem 4.15 [58] *Let M be a compact strictly pseudo-convex $2n+1$ dimensional CR manifold. If $n \ge 2$ and M is not locally CR equivalent to S^{2n+1}, then $\lambda(M) < \lambda(S^{2n+1})$. Hence, the CR Yamabe problem can be solved on M.*

The remaining cases left open by Jerison and Lee should by analogy be solved by using some CR positive mass theorem. Unfortunately, such a CR version of the positive mass theorem did not exist at that time. Besides the proof of Aubin and Schoen of the Riemannian Yamabe conjecture another proof by Bahri [6], Bahri and Brézis [17] of the same conjecture was available by techniques related to the theory of critical points at infinity. This proof is completely different in spirit as well as in techniques and details from the proof of Aubin and Schoen. It does not require the use of any theory of minimal surfaces neither the use of a CR positive mass theorem. It turns out that this proof can be carried to the CR framework.

We consider the subspace of $S_1^2(M)$, defined by

$$H = \left\{ u \in S_1^2(M) / \int_M |\mathrm{d}u|_\theta^2\, \theta \wedge \mathrm{d}\theta^n < \infty, \;\; \int_M |u|^{2+\frac{2}{n}}\, \theta \wedge \mathrm{d}\theta^n < \infty \right\}.$$

Let $\Sigma = \{u \in H, st\, \|u\|_H = 1\}$, $\;\|u\|_H = \left(\int_M ((2+\frac{2}{n})\,|\mathrm{d}u|_\theta^2 + R_\theta u^2)\theta \wedge \mathrm{d}\theta^n\right)^{\frac{1}{2}}$, and let

$$\Sigma_+ = \{u \in \Sigma, /u > 0\}.$$

For $u \in H$, we define the CR Yamabe functional:

$$J(u) = \frac{\int_M \left((2+\frac{2}{n})\,|\mathrm{d}u|_\theta^2 + R_\theta u^2\right)\theta \wedge \mathrm{d}\theta^n}{\left(\int_M |u|^{2+\frac{2}{n}}\, \theta \wedge \mathrm{d}\theta^n\right)^{\frac{n}{n+1}}}.$$

If u is a critical point of J on Σ_+, then $J(u)^{\frac{n}{2}}u$ is a solution of the Yamabe equation.

Let us recall that the standard solutions of the CR Yamabe equation on the Heisenberg group , $\mathbb{H}^n$, are obtained by left translations and dilations

$$(z,t) \to (\lambda z, \lambda^2 t), (\lambda \in \mathbb{R})$$

of the functions $u(z,t) = K\,|w+i|^{-n}$, $w = t + i\,|z|^2$ $(z,t) \in H^n$, $K \in \mathbb{C}$.
Since the injection $S_1^2(H^n) \to L^{2+\frac{2}{n}}(H^n)$ is continuous but not compact, the functional J does not satisfy the Palais–Smale condition denoted by (PS). More precisely, one can see that the standard solutions on H^n after superposition are the good candidate sequences which violate (PS). Therefore, the classical variational theory, based on compactness arguments, does not apply in this case.

The cases left open by Jerison and Lee of the CR Yamabe problem have been the purpose of two papers [43,44] published in 2001, in the first paper Yacoub and Gamara solved the CR Yamabe problem for spherical CR manifolds. The main result of [44] is

Theorem 4.16 *Let (M,θ) be an orientable compact $(2n+1)$-dimensional CR manifold, locally CR equivalent to $\mathbb{S}^{2n+1}$; then the CR Yamabe problem has a solution.*

In the second paper [43] Gamara completed the resolution of the CR Yamabe conjecture for all dimensions by solving the 3-dimensional non spherical case:

Theorem 4.17 *Let (M,θ) be a compact 3-dimensional CR manifold, not locally CR equivalent to the sphere $\mathbb{S}^3$; then the CR Yamabe problem has a solution.*

The proofs of Theorems 4.16 and 4.17 are both based on a contradiction argument. Before giving a sketch of the proofs of these theorems, we will introduce the general settings.

4.8.1 Spherical CR manifold: general settings

Let (M,θ) be a compact spherical CR manifold; we show the existence of a conformal factor $\tilde{u}_a^{\frac{2}{n}}$ depending differentiably on $a \in M$, such that if θ is replaced by $\tilde{u}_a^{\frac{2}{n}}\theta$ in a ball $B(a,\rho)$, then $(M, \tilde{u}_a^{\frac{2}{n}}\theta)$ is locally (H^n, θ_0). We may use in $B(a,\rho)$ the usual multiplication of H^n and the standard solutions of the CR Yamabe problem, which we denote by $\delta(b,\lambda)$, where $\lambda \in \mathbb{R}$. The function $\delta(b,\lambda)$ satisfies $\begin{cases} L_{\theta_0}\delta(b,\lambda) = \delta(b,\lambda)^{1+\frac{2}{n}} \text{ on } B(a,\rho) \\ b \in B(a,\rho) \end{cases}$
We then define on M a family of "almost solutions", which we denote by $\widehat{\delta}(a,\lambda)$. These functions are the solutions of

$$L\widehat{\delta}(a,\lambda) = \delta'(a,\lambda)^{1+\frac{2}{n}},$$

where

$$\begin{cases} \delta'(a,\lambda) = \omega_a \tilde{u}_a \delta(a,\lambda) \text{ on } & B(a,\rho) \\ \delta'(a,\lambda) = 0 & \text{on } B^c(a,\rho) \end{cases}$$

Here ω_a is a cut-off function used to localize our function near the base point a; as λ goes to infinity, we show that these "almost solution" closely approximate at infinity the Yamabe solutions of the Heisenberg group

$$\left|\hat{\delta}(a,\lambda) - \delta'(a,\lambda)\right| = O\left(\frac{1}{\lambda^n}\right),$$

$$\left|\hat{\delta}(a,\lambda) - \delta'(a,\lambda)\right|_{C^2} = O\left(\frac{1}{\lambda^n}\right), \text{ when } \lambda \to \infty.$$

Neighborhoods of critical points at infinity

Following Bahri [6,15], we set the following definitions and notations:

Definition 4.18 A critical point at infinity of J, on Σ_+ is a limit of a flow line $u(s)$ of the following equation:

$$\begin{cases} \frac{\partial u}{\partial s} = -\partial J(u) \\ u(0) = u_0 \end{cases}$$

We define neighborhoods of critical points at infinity of J as follows:

$$V(p,\varepsilon)=\left\{\begin{array}{c} u\in\Sigma_+ \text{ s.t, there exist } p \text{ concentration} \\ \text{points } a_1,\dots a_p \text{in } M \text{ and} \\ p \text{ concentrations } \lambda_1,\dots\lambda_p \text{ s.t} \\ \left\| u-\frac{1}{p^{\frac{1}{2}}S^{\frac{n}{2}}}\sum_{i=1}^{i=p}\widehat{\delta}(a_i,\lambda_i)\right\|_H<\varepsilon \\ \text{with } \lambda_i>\frac{1}{\varepsilon}, \text{ and for } i\neq j \\ \varepsilon_{ij}=\left(\frac{\lambda_i}{\lambda_j}+\frac{\lambda_j}{\lambda_i}+\lambda_i\lambda_j\tilde{d}^2(a_i,a_j)\right)^{-n}\geq\frac{1}{\varepsilon}, \end{array}\right\}$$

where $\tilde{d}(x,y)$, if x and y are in a small ball of M of radius ρ, is $\|\exp_x^{-1}(y)\|_{H^n}$ $\exp_x$ is the CR exponential map for the point x and $\tilde{d}(x,y)$ is equal to $\frac{\rho}{2}$ otherwise. (S is the Sobolev constant for the inclusion $S_1^2(H^n)\to L^{2+\frac{2}{n}}(H^n)$).
Because of the estimates of the "almost solution" given above, we can replace in the analysis of the (PS) condition, the functions δ' or δ by the functions $\widehat{\delta}$. Hence, we will be able to characterize the sequences of functions which violate the Palais–Smale condition.

In fact, we prove that, if (u_k) is a sequence of H satisfying $\partial J(u_k)\to 0$ and $J(u_k)$ bounded, then (u_k) has a weak limit $\overline{u}$ in H. Hence, If $\overline{u}$ is non-zero, we prove that $\overline{u}$ is a critical point of J. Since we will prove the CR Yamabe problem using a contradiction argument, we suppose that (P) has no solutions. Then we have the following characterization of the sequences failing (PS) condition

Proposition 4.19 *Let $\{u_k\}$ be a sequence such that $\partial J(u_k)\to 0$ and $J(u_k)$ is bounded. Then there exists an integer $p\in\mathbb{N}^+$, a sequence $\varepsilon_k\to 0$ $(\varepsilon_k>0)$ and an extracted subsequence of (u_k), such that $\frac{u_k}{\|u_k\|}\in V(p,\varepsilon_k)$.*

This proposition was first introduced in the Riemannian settings in [15,17]; the proof follows from iterated blow-up around the concentration points. For the CR settings, a complete proof is given in [44].

4.8.2 CR Yamabe problem: ideas of the Proof of theorem 4.16

Considering for $p\in\mathbb{N}$, the formal barycentric sets

$$B_0(M)=\emptyset$$
$$B_p(M)=\left\{\sum_1^p\alpha_i\delta_{x_i},\ \sum_{i=1}^p\alpha_i=1,\alpha_i>0,\ x_i\in M\right\},$$

where δ_{x_i} is the Dirac mass at the point x_i, and the following level sets of the functional J

$$W_p=\left\{u\in\Sigma_+\ /\ J(u)<(p+1)^{\frac{1}{n}}S\right\}.$$

We define a map $f_p(\lambda)$ from $B_p(M)$ to Σ_+ by

$$f_p(\lambda)\left(\sum_{i=1}^{i=p}\alpha_i\delta_{x_i}\right)=\frac{\sum_{i=1}^{i=p}\alpha_i\widehat{\delta}(x_i,\lambda_i)}{\|\sum_{i=1}^{i=p}\alpha_i\widehat{\delta}(x_i,\lambda_i)\|}.$$

It is proved in [44] that

Theorem 4.20 *1) For any integer $p\geq 1$, there exists a real $\lambda_p>0$, such that $f_p(\lambda)$ sends $B_p(M)$ in W_p, for any $\lambda>\lambda_p$.*
2) There exists an integer $p_0\geq 1$, such that for any integer $p\geq p_0$ and for any $\lambda>\lambda_{p_0}$, the map of pairs $f_p(\lambda):(B_p(M),B_{p-1}(M))\to(W_p,W_{p-1})$, is homologically trivial i.e.,

$$f_{p^*}(\lambda)=0,$$

where

$$f_{p^*}(\lambda):H_*(B_p(M),B_{p-1}(M))\to H_*(W_p,W_{p-1}).$$

and $H_*(\bullet)$ is the homology group with $\mathbb{Z}/2\mathbb{Z}$ coefficients of $\bullet$.

On the other hand, arguing by contradiction, we will assume that the weak limit $\overline{u}$ of any (PS) sequences (u_k) of H satisfying $\partial J(u_k) \to 0$ and $J(u_k)$ bounded, is zero; otherwise, our problem would be solved, since we would have found a solution. Then assuming that (u_k) is non- negative, we prove that we can extract from (u_k) a subsequence denoted again by (u_k), such that $\frac{u_k}{\|u_k\|_H} \in V(p, \varepsilon_k)$ with $\varepsilon_k > 0$ and $\lim_{k\to\infty} \varepsilon_k = 0$.

In this case, we proved that the pair (W_p, W_{p-1}) retracts by deformation on the pair $(W_{p-1} \cup A_p, W_{p-1})$, where $A_p \subset V(p, \varepsilon)$. More precisely, we prove that the elements of A_p are of the form $\sum_{i=1}^{i=p} \alpha_i \widehat{\delta}_{x_i,\lambda_i} + v$, v small in the norm $\|\|_H$. Therefore, the model $\ldots \subset B_{p-1}(M) \subset B_p(M) \subset \cdots$ can be compared via f_p to.... $\subset W_{p-1} \subset W_p$, and we proved that $f_{p^*}(\lambda) \neq 0$, for every $p \in \mathbb{N}^*$, which is a contradiction with the result of Theorem 4.20 and, therefore, achieves the proof of the CR Yamabe problem in this case.

4.8.3 CR Yamabe problem: ideas of the proof of theorem 4.17

In the paper [43], it is shown how the techniques of critical points at infinity can settle the case of a strictly pseudo-convex CR manifold (M, θ) of dimension $2n + 1$, without assuming that M is locally conformally flat. In fact in [43], we focus on the case $n = 1$, but the techniques apply to higher CR dimensions with no more assumptions. We have just to follow the sketch of the proof given for the case $n = 1$, with introducing where it is needed some required modifications due to the dimension of the CR manifold.
Here, we will give some ideas about the proof for the CR Yamabe problem in the case of a generic 3-dimensional non spherical CR manifold.
The proof of the result in this case is similar to the one given for the CR spherical case; it is obtained by using a contradiction argument.
We use the same techniques given by Bahri and Brézis in [17]. However, in this case the study of "the almost solutions" $\hat{\delta}_a$ is not straightforward as in the CR spherical case, where we have locally a relation between the conformal Laplacians of M and H^n. Here, we have to use the Green's function associated with L to derive a good asymptotic expansion of the Yamabe funtional J near the sets of its critical points at infinity. Finally, to compute the numerator and the denominator of J, we used the approach of Jerison and Lee who refined in [58] the notion of normal coordinates by constructing the so-called pseudo-Hermitian normal coordinates. In pseudo-Hermitian normal coordinates, Jerison and Lee gave the Taylor series of θ and $\{\theta^\alpha\}$ to high order at a base point $q \in M$, in terms of the pseudo-Hermitian curvature and torsion. Since the problem is CR invariant they had to choose θ so as to simplify the curvature and the torsion at a base point q as much as possible. Using the results of [58], we proved the following estimates in pseudo-Hermitian normal coordinates near a base point $q \in M$

$$R = O(2),\ W = Z + O(3),\ \overline{W} = \overline{Z} + O(3),\ L = -2(Z\overline{Z} + \overline{Z}Z) + O(2)$$
$$G_q(z, t) = C(\rho^{-2}(z, t)) + A + O(\rho(z, t)),\quad G_q > 0,$$

where $O(m)$ is a homogenous polynomial in ρ of degree a least m. Using these estimates and the topological method based on the theory of critical points at infinity explained earlier, we derive the result in this case.

4.9 The case of a CR spherical pseudo-Hermitian manifold of dimension 3

4.9.1 Introduction and main results

Let (M, θ) be a compact spherical pseudo-Hermitian 3-dimensional manifold and K a $\mathcal{C}^2$ positive function defined on M. Our aim was to find suitable conditions on K such that we can find a contact form $\tilde{\theta}$ conformal to θ having K as Webster scalar curvature. The new contact form reads $\tilde{\theta} = u^2\theta$, where u is a positive function on M.
The problem of prescribing the Webster scalar curvature is equivalent to the resolution of following partial differential equation:

$$(P_K) \begin{cases} -Lu = Ku^3 \text{ on } M, \\ \quad u > 0 \end{cases} \tag{4.15}$$

where $-L$ is the conformal Laplacian of M, $-L = -4\Delta_\theta + R_\theta$ where Δ_θ is the sub-Laplacian operator on (M, θ) and R_θ is the Webster scalar curvature of (M, θ).

Problem (P_K) has a variational structure, with associated Euler functional:

$$J(u) = \frac{\int_M -Lu\ u\ \theta \wedge \mathrm{d}\theta}{\left(\int K\ u^4\ \theta \wedge \mathrm{d}\theta\right)^{\frac{1}{2}}}, \quad u \in S_1^2(M).$$

A solution u of (P_K) is a critical point of J subject to the constraint $u \in \Sigma^+$, where

$$\Sigma = \left\{ u \in S_1^2(M);\ \|u\|_{S_1^2(M)} = \int_M -Lu\ u\ \theta \wedge \mathrm{d}\theta = 1 \right\} \text{ and } \Sigma^+ = \{u \in \Sigma, u \geq 0\}.$$

As for the Yamabe problem, the functional J fails to satisfy the Palais–Smale condition on Σ^+, which means that there exist noncompact sequences along which the functional J is bounded and its gradient goes to zero. The failure of the (PS) condition has been analyzed for the Riemannian case throughout the works of [5,15,18,27,28,60,62,65,67]. For the CR case, a complete description of sequences failing to satisfy (PS) is given in [44].

Since this problem has been formulated, obstructions have to be pointed out. The main encountered difficulty, when one tries to solve equation of type (P_K) consists of the failure of the Palais–Smale condition, which leads to the failure of classical existence mechanisms. We will use a gradient flow to overcome the noncompactness. Thinking of the sequences failing to satisfy the Palais–Smale condition as "critical points", our objective was to try to find suitable parameters, in order to complete a Morse Lemma at infinity analogous to the one given for the Riemannian case. The Morse Lemma is crucial to prove the existence of solution for equation (P_K); more precisely, the method, we used to prove the existence of solutions for problem (P_K) is based on the work of Bahri [5,18,24]. This method involves a Morse lemma at infinity, which establishes near the set of critical points at infinity of the functional J a change of variables in the space $(a_i, \alpha_i, \lambda_i, v)$, $1 \leq i \leq p$ to $(\widetilde{a}_i, \widetilde{\alpha}_i, \widetilde{\lambda}_i, V)$, $(\widetilde{\alpha}_i = \alpha_i)$, where V is a variable completely independent of $\widetilde{a}_i$ and $\widetilde{\lambda}_i$ such that $J(\sum \alpha_i \hat{\delta}_{a_i,\lambda_i})$ behaves like $J(\sum \alpha_i \hat{\delta}_{\widetilde{a}_i,\widetilde{\lambda}_i}) + \|V\|^2_{-L}$. The Morse lemma relies on the construction of a suitable pseudo-gradient for the associated variational problem, which is based on the expansion of J and its gradient ∂J near infinity. we define also a pseudo-gradient for the V-variable with the aim to make this variable disappear by setting $\frac{\partial V}{\partial s} = -\nu V$, where ν is taken to be a very large constant. Then, at $s = 1$, $V(s) = \exp(-\nu s)V(0)$ will be as small as we wish. This shows that in order to define our deformation, we can work as if V was zero. The deformation will be extended immediately with the same properties to a neighborhood of zero in the V-variable.

We prove that the Palais–Smale condition is satisfied along the decreasing flow lines of this pseudo-gradient, as long as these flow lines do not enter the neighborhood of a finite number of critical points of K. This method allows to study the critical points at infinity of the variational problem, by computing their total index and comparing this total index to the Euler-Poincaré characteristic of the space of variations. This procedure was extensively used in earlier Riemannian works and has displayed the role of the Green's function in equation of type (P_K).

It is important to recall that for the case we review, we have a balance phenomenon between the self interactions and interactions between the functions failing to satisfy the Palais–Smale condition.

To state our results we set up the following conditions and notations:

Let $G(a,)$ be a Green's function for L at $a \in M$ and A_a the value of the regular part of G evaluated at a. We assume that K has only nondegenerate critical points $\xi_1, \xi_2 \ldots \xi_r$ such that

$$-\frac{\Delta_\theta K(\xi_i)}{3K(\xi_i)} - 2A_{\xi_i} \neq 0, \quad i = 1, \ldots, r$$

Assume that ξ_i, $\ i = 1, \ldots, r_1$ are the critical points of K with $-\frac{\Delta_\theta K(\xi_i)}{3K(\xi_i)} - 2A_{\xi_i} > 0$. Let $\tau_l = (i_1, \ldots, i_l)$ denote any l-tuple of $(1, \ldots, r_1)$, $1 \leq l \leq r_1$, we define the following matrix $M(\tau_l) = (M_{st})$ with

$$M_{ss} = -\frac{\Delta_\theta K(\xi_s)}{3K^2(\xi_s)} - 2\frac{A_{\xi_s}}{K(\xi_s)}$$
$$M_{st} = -2\frac{G(\xi_s, \xi_t)}{\sqrt{K(\xi_s)K(\xi_t)}}, \quad \text{for } 1 \leq s \neq t \leq l$$

We say that K satisfies condition (C) if for any τ_l, $\ 1 \leq l \leq r_1$, $M(\tau_l)$ is nondegenerate. If we denote by k_{i_j} the index of the critical point ξ_{i_j} with respect to K, then $i(\tau_l) = 4l - 1 - \sum_{j=1}^{l} k_{i_j}$ is the index of the critical point at infinity τ_l. We obtain the following result:

Theorem 4.21 *Suppose the function K satisfies (4.16) and condition (C).*

$$\text{If} \quad \sum_{l=1}^{r_1} \sum_{\tau_l,\ M(\tau_l)>0} (-1)^{i(\tau_l)} \neq 1.$$

Then (P_K) has a solution.

This result means that if the total contribution of the critical points at infinity to the topology of the level sets of the associated functional J is not trivial, then we have a solution for (P_K).

4.9.2 Preliminaries

Let (M, θ) be a compact spherical pseudo-Hermitian manifold of dimension 3. Any point a in M has a neighborhood $V_a \supset B(a, \rho)$, ρ independent of a, such that the contact form of M is conformal to the contact form θ_0 of the Heisenberg group $\mathbb{H}^1$, so if there exists a conformal factor $\tilde{v}_a$ depending smoothly on a such that

$$\theta_0 = \tilde{v}_a^2 \theta$$

in the ball $B(a, \rho)$, then $(M, \tilde{v}_a^2 \theta)$ is locally $(\mathbb{H}^1, \theta_0)$. Therefore, we may use the usual multiplication of $\mathbb{H}^1$ in $B(a, \rho)$; we also may use the standard solutions of the CR Yamabe equation which we denote by $\delta(a, \lambda)$, where λ is a large positive parameter; we have

$$\delta(a, \lambda)(\xi) = c_0 \frac{\lambda}{|1 + \lambda^2(|z|^2 - it)|},$$

where $(z, t) = \exp_a^{-1}(\xi)$ and the constant c_0 is such that the following equation is satisfied:

$$-L_{\theta_0} \delta(a, \lambda) = \delta^3(a, \lambda) \quad \text{on } B(a, \rho).$$

Let $v_a(\xi) = \omega_a(\xi)\tilde{v}_a(\xi)$, where $\omega_a(\xi) = \chi(\|\xi\|)$, χ is a cut-off function which is used to localize the function $\delta(a, \lambda)$ near the base point a when $\lambda \to \infty$,

$$\begin{aligned} \chi : \mathbb{R} &\longrightarrow [0, 1] \\ t &\longmapsto \chi(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq \frac{\rho}{2} \\ 0 & \text{if } t \geq \rho. \end{cases} \end{aligned}$$

We define a family of "almost solutions" $\hat{\delta}(a, \lambda)$ to be the unique solutions on M of

$$-L\hat{\delta}(a, \lambda)(\xi) = {\delta'}^3(a, \lambda)(\xi)$$

with

$$\delta'(a, \lambda)(\xi) = \begin{cases} v_a \delta(a, \lambda)(\xi) & \text{on } B(a, \rho) \\ \quad 0 & \text{on } B^c(a, \rho). \end{cases}$$

□

The following result of Jerison and Lee will be very useful later:

Lemma 4.22 *Let Φ be in $C^2(B(a, \rho), \mathbb{R})$; we have the following relation between the conformal Laplacian of M and the one of $\mathbb{H}^1$:*

$$L(\tilde{v}_a \Phi) = \tilde{v}_a^3 L_{\theta_0}(\Phi).$$

For a proof one can see [56].

Let $H_{a,\lambda}(x) = \lambda(\hat{\delta}_{a,\lambda} - \delta'_{a,\lambda})(x)$; we have

Lemma 4.23 [42] *For λ large enough, there exists a constant $C = C(\rho)$ such that*

$$|H_{a,\lambda}(x)|_{L^\infty} \le C, \quad |\lambda \frac{\partial H_{a,\lambda}}{\partial \lambda}|_{L^\infty} \le C, \quad |\lambda^{-1} \frac{\partial H_{a,\lambda}}{\partial a}|_{L^\infty} \le C.$$

Moreover, for ρ small and λ large, we obtain

$$\lim_{\lambda \longrightarrow \infty} H_{a,\lambda}(a) = A_a,$$

and outside $B(a, 2\rho)$

$$\lim_{\lambda \longrightarrow \infty} H_{a,\lambda}(\xi) = G(a, \xi).$$

Now, we define the set of potential critical points at infinity of the functional J.
For any $\varepsilon > 0$ and $p \in \mathbb{N}^+$, let

$$V(p,\varepsilon) = \begin{cases} u \in \Sigma^+;\ \exists\, (a_1, \dots, a_p) \in M,\ \alpha_1, \dots, \alpha_p > 0 \text{ and } (\lambda_1, \dots, \lambda_p) \in (\varepsilon^{-1}, \infty)^p \text{ s.t :} \\ \left\| u - \sum_{i=1}^{p} \frac{\alpha_i \hat{\delta}_{a_i,\lambda_i}}{K(a_i)^{\frac{1}{2}}} \right\|_{S_1^2(M)} < \varepsilon, \\ \varepsilon_{ij} < \varepsilon,\ |\frac{\alpha_i^2 K(a_i)}{\alpha_j^2 K(a_j)} - 1| < \varepsilon, \quad \forall\ 1 \le i \ne j \le p. \end{cases}$$

Box where $\varepsilon_{ij} = \left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j (d(a_i, a_j)^2)\right)^{-1}$ and $d(x, y) = \| \exp_x^{-1}(y) \|_{\mathbb{H}^1}$ if x and y are in a small ball of M of radius r, and $d(x, y)$ is equal to $\frac{r}{2}$ otherwise.

Let (u_k) be a sequence of Σ^+ satisfying $J(u_k)$ bounded and $\partial J(u_k) \to 0$; then (u_k) is a bounded sequence in $S_1^2(M)$; hence (u_k) has a weak limit $\bar{u}$ in $S_1^2(M)$. If $\bar{u} \ne 0$, we prove that $\bar{u} > 0$, and it is a critical point of J. The proof is similar to the one given for the Yamabe case (Proposition 5 of [44]).
Since we are going to prove the existence of solution for problem P_K by contradiction, we assume that the weak limit $\bar{u}$ of any sequence (u_k) of Σ^+ satisfying $J(u_k)$ bounded and $\partial J(u_k) \to 0$ is zero.

Using the estimates of Lemma 4.22, one can replace in the analysis of the Palais–Smale condition the functions δ' or δ by the function $\hat{\delta}$; we then proceed as in [44], Proposition 8 to characterize the sequences which violate the (PS) condition as follows:

Proposition 4.24 *Let $\{u_k\}$ be a sequence such that $\partial J(u_k) \to 0$ and $J(u_k)$ is bounded. Then there exist an integer $p \in \mathbb{N}^*$, a sequence $\varepsilon_k \to 0$ $(\varepsilon_k > 0)$ and an extracted subsequence of $\{u_k\}$, again denoted by $\{u_k\}$, such that $u_k \in V(p, \varepsilon_k)$.*

While the final characterization of the sequences which violate the Palais–Smale condition is basically identical to the Riemannian case, the proof is different here from the one given by Struwe in [67]. M. Struwe used the H_0^1- spaces and the projections on them in order to give a characterization of the sequences violating the Palais–Smale condition in the Riemannian framework.

We consider the following minimization problem for a function $u \in V(p, \varepsilon)$, with ε small

$$\min_{\alpha_i > 0, \lambda_i > 0, a_i \in M} \left\| u - \sum_{i=1}^{p} \alpha_i \hat{\delta}_{a_i,\lambda_i} \right\|_{S_1^2(M)}. \tag{4.16}$$

We obtain as showed in [5,44] the following parametrization of $V(p, \varepsilon)$:

Proposition 4.25 *For any $p \in \mathbb{N}^*$, there exists $\varepsilon_p > 0$ such that, for any $0 < \varepsilon < \varepsilon_p$, $u \in V(p, \varepsilon)$, the minimization problem (4.16) has a unique solution $(\bar{\alpha}_1, \dots, \bar{\alpha}_p, \bar{\lambda}_1, \dots, \bar{\lambda}_p, \bar{a}_1, \dots, \bar{a}_p)$ (up to permutation on the set of indices $\{1, \dots, p\}$).*

In particular, we can write $u \in V(p, \varepsilon)$ as follows: $u = \sum_{i=1}^{p} \overline{\alpha}_i \hat{\delta}_{\overline{a}_i, \overline{\lambda}_i} + v$, where $v \in S_1^2(M)$ satisfies

$$(V_0) \quad < v, \psi >_{-L} = 0 \quad \text{for all} \quad \psi \in \left\{ \hat{\delta}_{a_i,\lambda_i}, \frac{\partial \hat{\delta}_{a_i,\lambda_i}}{\partial a_i}, \frac{\partial \hat{\delta}_{a_i,\lambda_i}}{\partial \lambda_i} \right\}_{1 \le i \le p}$$

Here $<,>_{-L}$ denotes the $-L$-scalar product defined on $S_1^2(M)$ by

$$< u, v >_{-L}= \int_M -Luv\ \theta \wedge \mathrm{d}\theta. \tag{4.17}$$

One of the basic phenomena that it displays is the behavior of the functional J with respect to v. We will prove the existence of a unique $\bar{v}$ which minimizes $J(\sum_{i=1}^p \alpha_i \hat{\delta}_{a_i,\lambda_i} + v)$ with respect to $v \in H_\varepsilon^p(a,\lambda)$, where

$$H_\varepsilon^p(a,\lambda) = H_\varepsilon^p(\hat{\delta}_{a_1,\lambda_1}, \ldots, \hat{\delta}_{a_p,\lambda_p}) = \left\{ v \in \mathcal{S}_1^2(M); \quad v \text{ satisfies } (V_0) \text{ and } \|v\|_{-L} < \frac{\varepsilon}{p} \right\}.$$

4.10 Expansion of the functional near the sets of potential critical points at infinity

For any $u = \sum \alpha_i \hat{\delta}_{a_i,\lambda_i} + v \in V(p,\varepsilon)$, $\varepsilon > 0$, we have

$$J(u) = \frac{\int_M -Lu\ u\ \theta \wedge \mathrm{d}\theta}{\left(\int_M K v^4 \theta \wedge \mathrm{d}\theta\right)^{\frac{1}{2}}} = \frac{N}{D}, \tag{4.18}$$

where

$$\begin{aligned} N &= \int_M -Lu\ u\theta \wedge \mathrm{d}\theta = \int_M -L\left(\sum \alpha_i \hat{\delta}_{a_i,\lambda_i} + v\right)\left(\sum \alpha_i \hat{\delta}_{a_i,\lambda_i} + v\right)\theta \wedge \mathrm{d}\theta \\ &= \sum \alpha_i^2 \int_M -L\hat{\delta}_{a_i,\lambda_i}\ \hat{\delta}_{a_i,\lambda_i}\theta \wedge \mathrm{d}\theta + 2\sum_{i<j} \alpha_i\alpha_j \int_M -L\hat{\delta}_{a_i,\lambda_i}\ \hat{\delta}_{a_j,\lambda_j}\theta \wedge \mathrm{d}\theta \\ &\quad + \int_M -Lv\ v\theta \wedge \mathrm{d}\theta. \end{aligned}$$

All the other terms are zero since u satisfies conditions (V_0).

We obtain the following expansion of the functional J :

Proposition 4.26 *There exists $\varepsilon_0 > 0$ such that, for any $u = \sum \alpha_i \hat{\delta}_{a_i,\lambda_i} + v \in V(p,\varepsilon)$, $\varepsilon < \varepsilon_0$, v satisfying (V_0), we have*

$$\begin{aligned} J(u) = \frac{\sum \alpha_i^2}{\left[\sum_{i=1}^p \alpha_i^4 K(a_i)\right]^{1/2}} S\Bigg[1 &- \frac{c}{4S^2} \sum_{i=1}^p \frac{\alpha_i^4}{\sum_k \alpha_k^4 K(a_k)} \frac{\Delta K(a_i)}{\lambda_i^2} \\ &+ S^{-2} \sum_{i \neq j} \left[c_{ij}\varepsilon_{ij} + c' \frac{H_j(a_i)}{\lambda_i \lambda_j} \right] \left(\frac{\alpha_i \alpha_j}{\sum_{k=1}^p \alpha_k^2} - \frac{2\alpha_i^3 \alpha_j K(a_i)}{\sum_{k=1}^p \alpha_k^4 K(a_k)} \right) \\ &+ \sum_i \frac{c'}{S^2} \frac{H_i(a_i)}{\lambda_i^2} \left[\left(\frac{\alpha_i^2}{\sum_{k=1}^p \alpha_k^2} - \frac{2\alpha_i^4 K(a_i)}{\sum_{k=1}^p \alpha_k^4 K(a_k)} \right) \right] \\ &+ f(v) + Q(v,v) + o\left(\sum_{i \neq j} \varepsilon_{ij} \right) + o(\|v\|^2) \Bigg], \end{aligned}$$

where f is a linear form in v and Q is a bilinear form in v given by

$$f(v) = -2 \int_M K(x) \sum_{i=1}^p (\alpha_i \hat{\delta}_{a_i,\lambda_i})^3 v\ \theta \wedge \mathrm{d}\theta$$

$$Q(v,v) = \frac{\|v\|_{-L}^2}{S \sum_{i=1}^p \alpha_i^2} - \frac{3}{S \sum_{i=1}^p \alpha_i^4 K(a_i)} \int_M K \left(\sum_{i=1}^p (\alpha_i \hat{\delta}_{a_i,\lambda_i})^2 \right) v^2.$$

and

$$S = c_0^4 \int_{\mathbb{H}^1} \frac{1}{\left|1+|z|^2 - it\right|^4} \theta_0 \wedge \mathrm{d}\theta_0.$$

Furthermore,

$$\|f\| = O\left(\sum_{=1}^{p}\left(\frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2}\right) + \sum_{i \neq j} \varepsilon_{ij}(Log\varepsilon_{ij}^{-1})^{\frac{1}{2}}\right)$$

Proof Let $u = \sum_{i=1}^{p} \alpha_i \hat{\delta}_{a_i,\lambda_i} + v$, v satisfies conditions (V_0). We derive from the expansions of N and D given in Appendix A of [42] the following:

$$\begin{aligned} J(u) = \frac{\sum_{i=1}^{p}\alpha_i^2 S}{\left[\sum_{i=1}^{p}\alpha_i^4 K(a_i)S\right]^{\frac{1}{2}}} & \Bigg[1 + \sum_{i\neq j} \frac{\alpha_i\alpha_j}{\sum_{k=1}^{p}\alpha_k S}\left(c_{ij}\varepsilon_{ij} + c'\frac{H_j(a_i)}{\lambda_i\lambda_j} + o(\varepsilon_{ij})\right) \\ & + \sum_{i} \frac{\alpha_i^2}{\sum_{k=1}^{p}\alpha_k S}\left(c'\frac{H_i(a_i)}{\lambda_i^2} + o\left(\sum_{i=1}^{p}\frac{1}{\lambda_i^2}\right)\right) + \frac{\|v\|_{-L}^2}{\sum_{k=1}^{p}\alpha_k S}\Bigg] \\ & \times\Bigg[1 + \frac{c}{2}\sum_{i=1}^{p}\frac{\alpha_i^4}{\sum_{k=1}^{p}\alpha_k^4 K(a_k)S}\frac{\Delta K(a_i)}{\lambda_i^2} \\ & + 4\sum_{i\neq j} c' \frac{\alpha_i^3\alpha_j K(a_i)}{\sum_{k=1}^{p}\alpha_k^4 K(a_k)S}\left(c_{ij}\varepsilon_{ij} + o(\varepsilon_{ij}) + \frac{H_j(a_i)}{\lambda_i\lambda_j}\right) + f(v) \\ & + O\left(\int_M |\nabla v|^2)^{\frac{1}{2}}\right)\left(\sum_{i=1}^{p}\left(\frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2}\right) + \sum_{i\neq j}\varepsilon_{ij}\left(Log\varepsilon_{ij}^{-1}\right)^{\frac{1}{2}}\right) \\ & + O\left(\int_M |\nabla v|^2)^{\frac{3}{2}}\right) + 6\sum_{i=1}^{p}\frac{\alpha_i^2 K(a_i)}{\sum_{k=1}^{p}\alpha_k^4 K(a_k)S}\int_{B(a_i,\rho)}(\delta'_{(a_i,\lambda_i)})^2 v^2\Bigg]^{\frac{-1}{2}} \end{aligned}$$

□

Lemma 4.27 ([42]) *For $\varepsilon > 0$ very small, there is $\alpha_0 > 0$ such that, for all $v \in H_\varepsilon^p(a, \lambda)$,*

$$Q(v, v) \geq \alpha_0 \|v\|_{-L}^2.$$

Lemma 4.28 *There exists a $\mathcal{C}^1$ map which to each $(\alpha_1, \ldots, \alpha_p, a_1, \ldots, a_p, \lambda_1, \ldots, \lambda_p)$ such that $\sum_{i=1}^{p} \alpha_i \hat{\delta}_{a_i,\lambda_i} \in V(p, \varepsilon)$, with small enough ε, associates $\bar{v} = \bar{v}(\alpha_i, a_i, \lambda_i)$ satisfying*

$$J\left(\sum_{i=1}^{p}\alpha_i\hat{\delta}_{a_i,\lambda_i} + \bar{v}\right) = \min_{v \text{ satisfies } (V_0)} J\left(\sum_{i=1}^{p}\alpha_i\hat{\delta}_{a_i,\lambda_i} + v\right).$$

Moreover, there exists $c > 0$ such that the following holds:

$$\|\bar{v}\|_{-L} \leq c\left(\sum_{i=1}^{p}\left(\frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2}\right) + \sum_{i\neq j}\varepsilon_{ij}(\mathrm{Log}(\varepsilon_{ij})^{-1})^{\frac{1}{2}}\right).$$

Proof We expand ∂J along a variation of h in the v-space $H_\varepsilon^p(a, \lambda)$ (that is h is a variation with respect to v with (α, a, λ) fixed). Since Q is positive definite, we write $Q(v, v) = Av$, then A is invertible and there exists a unique $\bar{v}$, which minimizes $J\left(\sum_{i=1}^{p}\alpha_i\hat{\delta}_{a_i,\lambda_i} + v\right)$ i.e

$$f + A\bar{v} + o(\|\bar{v}\|_{-L}) = 0.$$

Set

$$\bar{v} = A^{-1}(f) + o(1)$$

it yields

$$\|\bar{v}\|_{-L} \leq c\|A^{-1}f\| \leq c\|f\|$$

and

$$\|\bar{v}\|_{-L} = O(\|f\|,$$

where

$$\|f\| = O\left(\sum_{=1}^{p}\left(\frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2}\right) + \sum_{i\neq j}\varepsilon_{ij}(\text{Log}\varepsilon_{ij}^{-1})^{\frac{1}{2}}\right).$$

□

We have

$$f(\bar{v}) + Q(\bar{v}, \bar{v}) + o(\|\bar{v}\|_{-L}^2) = 0;$$

since $\bar{v}$ is a minimizer, it yields

$$f(v) + Q(v, v) + o\|v\|_{-L}^2 = Q(v - \bar{v}, v - \bar{v}) + o(\|\bar{v}\|_{-L}^2).$$

We derive

Proposition 4.29 *There exists $\varepsilon_0 > 0$ ($\varepsilon_0 < \varepsilon$) such that, for any*

$$u = \sum_{i=1}^{p}\alpha_i\hat{\delta}_{a_i,\lambda_i} + v,\ \ v \in H_\varepsilon^p(a, \lambda),$$

we have

$$\begin{aligned} J\left(\sum_{i=1}^{p}\alpha_i\hat{\delta}_{a_i,\lambda_i} + v\right) = \frac{\sum_{i=1}^{p}\alpha_i^2}{\left[\sum_{i=1}^{p}\alpha_i^4 K(a_i)\right]^{1/2}} S^{\frac{1}{2}}\Bigg[& 1 - \frac{c}{4S}\sum_i \frac{\alpha_i^4}{\sum_k \alpha_k^4 K(a_k)}\frac{\Delta K(a_i)}{\lambda_i^2} \\ & + \frac{1}{S}\sum_{i\neq j}\left(c_{ij}\varepsilon_{ij} + c'\frac{H_j(a_i)}{\lambda_i\lambda_j}\right)\left(\frac{\alpha_i\alpha_j}{\sum_{k=1}^{p}\alpha_k^2} - \frac{2\alpha_i^3\alpha_j K(a_i)}{\sum_{k=1}^{p}\alpha_k^4 K(a_k)}\right) \\ & + \sum_{i}^{p}\frac{c'}{S}\frac{H_i(a_i)}{\lambda_i^2}\left(\frac{\alpha_i^2}{\sum_{k=1}^{p}\alpha_k^2} - \frac{2\alpha_i^4 K(a_i)}{\sum_{k=1}^{p}\alpha_k^4 K(a_k)}\right) \\ & + Q(v - \bar{v}, v - \bar{v}) + o(\|\bar{v}\|_{-L}^2) + o\left(\sum_{i\neq j}\varepsilon_{ij}\right)\Bigg] \end{aligned}$$

4.11 Morse Lemma at infinity

We begin by characterizing the critical points at infinity of J in the sets $V(p,\varepsilon)$. This characterization is obtained through the construction of a suitable pseudo-gradient at infinity for the functional J for which the Palais–Smale condition is satisfied along its decreasing flow lines as long as these flow lines do not enter in the neighborhood of a finite number of critical points ξ_i; $1 \leq i \leq p$ satisfying condition (C). Notice that the deformation lemmas in Morse theory are realized by using the gradient flow lines or the flow lines of any decreasing pseudo-gradient vector field.

We first introduce some definitions and notations due to Bahri [5,15]. Let ∂J denote the gradient of the functional J.

Definition 4.30 A critical point at infinity of J on Σ^+ is a limit of a flow line $u(s)$ of the equation:

$$\begin{cases} \dfrac{\partial u}{\partial s} & = -\partial J(u) \\ u(0) = u_0 \end{cases}$$

such that $u(s)$ remains in $V(p,\varepsilon(s)$ for $s \geq s_0$, and $\varepsilon(s)$ satisfies $\lim_{s\longrightarrow\infty}\varepsilon(s) = 0$.

One can write $u(s) = \sum_{i=1}^{p} \alpha_i(s)\delta_{(a_i(s),\lambda_i(s))} + v(s)$; let $a_i := \lim_{s\longrightarrow\infty} a_i(s)$ and $\alpha_i := \lim_{s\longrightarrow\infty}\alpha_i(s)$; we denote such a critical point at infinity by

$$\xi_\infty \text{ or } (a_1,\ldots,a_p)_\infty \text{ or } \sum_{i=1}^{p}\alpha_i\hat{\delta}_{(a_i,\infty)}$$

To a critical point at infinity ξ_∞ are associated stable and unstable manifolds $W_s(\xi_\infty)$ and $W_u(\xi_\infty)$; those manifolds allow to compare critical points at infinity by what we call a "domination property", one can see [5,42], where a detailed description of theses manifolds is given.

Definition 4.31 A critical point at infinity ξ_∞ is said to be dominated by another critical point at infinity ξ'_∞, if

$$W_s(\xi_\infty) \cap W_u(\xi'_\infty) \neq \emptyset$$

and we write $\xi_{\infty'} > \xi_\infty$. □

If we assume that the intersection $W_s(\xi_\infty) \cap W_u(\xi'_\infty)$ is transverse, then we obtain
$\text{index}(\xi'_\infty) \geq \text{index}(\xi_\infty) + 1$.

4.11.1 Construction of the pseudo-gradient

In the set $V(p,\varepsilon)$, we obtain

Proposition 4.32 *Assume that K satisfies (4.16) and condition (C)*

For any p, there exists a pseudo-gradient W so that the following hold:

there is a positive constant c independent of $u = \sum_{i=1}^{p}\alpha_i\delta_{a_i,\lambda_i} \in V(p,\varepsilon)$, ε small enough such that, if we denote $\overline{u} = u + \overline{v}$, we have

1)

$$-J'(u)(W) \geq c\left(\sum_{i=1}^{p}\frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i=1}^{p}\frac{1}{\lambda_i^2} + \sum_{i\neq j}\varepsilon_{ij}\right)$$

2)

$$-J'(\bar{u})\left(W + \frac{\partial\bar{v}}{\partial(\alpha,a,\lambda)}(W)\right) \geq c\left(\sum_{i=1}^{p}\frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i=1}^{p}\frac{1}{\lambda_i^2} + \sum_{i\neq j}\varepsilon_{ij}\right)$$

3) $|W|$ is bounded and $d\lambda_{i_0} \leq c\lambda_{i_0}$ where λ_{i_0} is the highest of the concentration $\lambda_1,\lambda_2,\ldots,\lambda_p$.

We will now give the following result, which establishes our Morse Lemma at infinity:

Proposition 4.33 [42] *For any $u = \sum_{i=1}^{p} \alpha_i \hat{\delta}_{a_i,\lambda_i} \in V(p, \varepsilon_1)$, $(\varepsilon_1 < \frac{\varepsilon}{2})$, we find a change of variables in the space $(a_i, \alpha_i, \lambda_i, v)$, $1 \le i \le p$ to $(\widetilde{a}_i, \widetilde{\alpha}_i, \widetilde{\lambda}_i, V)$, $(\widetilde{\alpha}_i = \alpha_i)$, such that*

$$J\left(\sum_{i=1}^{p} \alpha_i \hat{\delta}_{a_i,\lambda_i} + \overline{v}(\alpha, a, \lambda)\right) = J\left(\sum_{i=1}^{p} \alpha_i \hat{\delta}_{\widetilde{a}_i,\widetilde{\lambda}_i},\right)$$

with

$$\sum_{i \neq j} \widetilde{\varepsilon}_{ij} + \sum_i \frac{1}{\widetilde{\lambda}_i^2} \longrightarrow 0 \Leftrightarrow \sum_{i \neq j} \varepsilon_{ij} + \sum_i \frac{1}{\lambda_i^2} \longrightarrow 0. \tag{4.19}$$

and

$$\|\widetilde{a}_i - a_i\| \longrightarrow 0 \ \ as \ \ \sum_{i \neq j} \varepsilon_{ij} + \sum_i \frac{1}{\lambda_i^2} \longrightarrow 0 \tag{4.20}$$

Proof Here, we give only the proof key idea. For the complete proof, see [42], (Lemma 4.4). Since the vector field W constructed in the next section is lipschitz, there is a one parameter group η_s generated by W solution of the equation

$$\frac{\partial}{\partial s}\eta_s\left(\sum_{i=1}^{p} \alpha_i \hat{\delta}_{a_i,\lambda_i}\right) = W\left(\eta_s\left(\sum_{i=1}^{p} \alpha_i \hat{\delta}_{a_i,\lambda_i}\right)\right)$$

with initial condition

$$\eta_0\left(\sum_{i=1}^{p} \alpha_i \hat{\delta}_{a_i,\lambda_i}\right) = \sum_{i=1}^{p} \alpha_i \hat{\delta}_{a_i,\lambda_i},$$

where $J(\eta_s(\sum_{i=1}^{p} \alpha_i \hat{\delta}_{a_i,\lambda_i}))$, and $J(\eta_s(\sum_{i=1}^{p} \alpha_i \hat{\delta}_{a_i,\lambda_i})) + \overline{v}(s)$ are decreasing functions of s. As $\overline{v}(s)$ is a minimizer, we have

$$J\left(\sum_{i=1}^{p} \alpha_i \hat{\delta}_{a_i,\lambda_i} + \overline{v}(s)\right) \le J\left(\eta_0\left(\sum_{i=1}^{p} \alpha_i \hat{\delta}_{a_i,\lambda_i}\right)\right).$$

Regarding the construction of the vector field W the flow line $\eta_s(\sum_{i=1}^{p} \alpha_i \hat{\delta}_{a_i,\lambda_i})$ satisfies the Palais–Smale condition if it does not approach the critical points at infinity. Since the maximum of the $\lambda_i(s)'s$ is a decreasing function, and the flow line started far away from these critical points at infinity, it will take an infinite time to this later to go to infinity. During this trip, we would be down the level $J(\sum_{i=1}^{p} \alpha_i \hat{\delta}_{a_i,\lambda_i}) + \overline{v}(s)$. In any case, as long as we do not cut the lower bound level, the speed of decay is at least $-c$. Hence, we are forced to cut the level $J(\sum_{i=1}^{p} \alpha_i \hat{\delta}_{a_i,\lambda_i}) + \overline{v}(s)$ unless the flow line exits $V(p, \epsilon)$ which means there is at most one solution of the equation:

$$J\left(\eta_s\left(\sum_{i=1}^{p} \alpha_i \hat{\delta}_{a_i,\lambda_i}\right)\right) = J\left(\sum_{i=1}^{p} \alpha_i \hat{\delta}_{a_i,\lambda_i}\right) + \overline{v}(s) \tag{4.21}$$

Does the flow line η_s exit from $V(p, \epsilon)$? We assume that $\sum_{i=1}^{p} \alpha_i \hat{\delta}_{a_i,\lambda_i} \in V(p, \epsilon_1)$, $\epsilon_1 < \frac{\epsilon}{2}$; then we have

$$-\partial J\left(\eta_s \sum_{i=1}^{p} \alpha_i \hat{\delta}_{a_i,\lambda_i}\right) W\left(\eta_s \sum_{i=1}^{p} \alpha_i \hat{\delta}_{a_i,\lambda_i}\right) \ge C\left(\sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} + \sum_{i \neq j} \varepsilon_{ij}\right) \ge c(\varepsilon) > 0 \tag{4.22}$$

during the trip between the boundaries of $V(p, \epsilon_1)$ and $V(p, \epsilon)$, which we suppose of length $l(\varepsilon)$. If we denote Δs the corresponding time to travel on this portion of the flow trajectory, we have $l(\varepsilon) \le c\Delta s$. Let

$\delta(\varepsilon) = -\frac{c(\varepsilon)l(\varepsilon)}{c}$; then $J(\eta_s \sum_{i=1}^p \alpha_i \hat{\delta}_{a_i,\lambda_i})$ decreases at least $-\delta(\varepsilon)$ during the trip from the boundary of $V(p,\epsilon_1)$ to the boundary of $V(p,\epsilon)$. To prove the result, we have to show that

$$J\left(\eta_s \sum_{i=1}^p \alpha_i \hat{\delta}_{a_i,\lambda_i}\right) W\left(\eta_s \sum_{i=1}^p \alpha_i \hat{\delta}_{a_i,\lambda_i} + \bar{v}\right) > J\left(\eta_s \sum_{i=1}^p \alpha_i \hat{\delta}_{a_i,\lambda_i}\right) - \delta(\varepsilon) \tag{4.23}$$

To this end, we know from [42] that $J(\eta_s \sum_{i=1}^p \alpha_i \hat{\delta}_{a_i,\lambda_i}) - J(\eta_s \sum_{i=1}^p \alpha_i \hat{\delta}_{a_i,\lambda_i} + \bar{v})$ goes to 0 as ε_1 tends to 0. Hence, by choosing ε_1 small enough, we have (4.23) and, therefore, Eq. (4.21) has a unique solution that we denote by $\eta_{s_0}(\sum_{i=1}^p \alpha_i \delta_{a_i,\lambda_i})$.

Next, we are going to prove (4.19), set $\sum_{i=1}^p \alpha_i(s)\hat{\delta}_{a_i(s),\lambda_i(s)}) = \eta_s(\sum_{i=1}^p \alpha_i \hat{\delta}_{a_i,\lambda_i})$. Since the vector field W has no action on the variables α_i, we have

$$W = \sum_{i=1}^p \alpha_i \frac{1}{\lambda_i(s)} \frac{\partial \hat{\delta}_{a_i(s),\lambda_i(s)}}{\partial a_i(s)} (\lambda_i(s)\dot{a}_i(s)) + \sum_{i=1}^p \alpha_i \lambda_i(s) \frac{\partial \hat{\delta}_{a_i(s),\lambda_i(s)}}{\partial \lambda_i(s)} \left(\frac{\dot{\lambda}_i(s)}{\lambda_i(s)}\right)$$

where $\dot{a}_i(s)$ and $\dot{\lambda}_i(s)$ denote the actions of W on the variables a_i and λ_i. Since W is bounded, $\frac{1}{\lambda_i(s)} \frac{\partial \hat{\delta}_{a_i(s),\lambda_i(s)}}{\partial a_i(s)}$ and $\lambda_i(s) \frac{\partial \hat{\delta}_{a_i(s),\lambda_i(s)}}{\partial \lambda_i(s)}$ are nearly orthogonal and bounded (both are $O(\delta_{a_i(s),\lambda_i(s)})$, it yields that $|\lambda_i(s)\dot{a}_i(s)| + |\frac{\dot{\lambda}_i(s)}{\lambda_i(s)}| \leq C'$, $i = 1, \ldots, p$. Regarding $\frac{1}{\lambda_i} \frac{\partial \varepsilon_{ij}}{\partial a_i}$ and $\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i}$ both are $O(\varepsilon_{ij})$ since $\varepsilon_{ij} = o(1)$ and we obtain $\frac{\partial \varepsilon_{ij}}{\partial s} \leq C\varepsilon_{ij}$. Therefore,

$$\exp -cs \leq \frac{\varepsilon_{ij}(s)}{\varepsilon_{ij}(0)} \exp cs,$$
$$\text{and} \quad \exp -cs \leq \frac{\lambda_i(s)}{\lambda_i(0)} \exp cs,$$

which establishes (4.19). □

Now, we will prove (4.20): we have $|\dot{a}_i(s)| \leq \frac{C'}{\lambda_i(s)} \leq C' \frac{\exp cs}{\lambda_i(0)}$; thus

$$|a_i(s) - a_i| \leq C's \frac{\exp cs}{\lambda_i(0)}$$

and since s_0 satisfies Eq. (4.21), $|a_i(s) - a_i|$ is bounded; hence we have (4.20).

Here and for the sake of simplicity, we will use the notation $\hat{\delta}_j$ instead of $\hat{\delta}_{a_j,\lambda_j}$.

To construct a vector field W satisfying Proposition 4.32, we have first to find the expansions of $< \partial J(u), \lambda_j \frac{\partial \hat{\delta}_j}{\partial \lambda_j} >$, and $< \partial J(u), \frac{1}{\lambda_j} \frac{\partial \hat{\delta}_j}{\partial a_j} >$.

We have the following result:

Lemma 4.34 [42] *For $u = \sum_{i=1}^p \alpha_i \hat{\delta}_i \in V(p,\varepsilon)$, we obtain*

1)

$$\left\langle -\partial J(u), \lambda_j \frac{\partial \hat{\delta}_j}{\partial \lambda_j} \right\rangle = 2J(u)\left[\frac{\omega_3}{4} \sum_i \alpha_i \frac{H_i(a_j)}{\lambda_i \lambda_j}(1+o(1)) - \frac{\omega_3}{24} \alpha_j \frac{\Delta K(a_j)}{K(a_j)\lambda_j^2}(1+o(1)) + \sum_{i \neq j} c_{ij} \alpha_i \lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j}(1+o(1)) + o\left(\sum_{i \neq j} \varepsilon_{ij}\right)\right]$$

2)

$$\left\langle -\partial J(u), \frac{1}{\lambda_j} \frac{\partial \hat{\delta}_j}{\partial a_j} \right\rangle = 2J(u)\left[\frac{\alpha_j}{K(a_j)} \frac{\omega_3}{48} \frac{\nabla K(a_j)}{\lambda_j}(1+o(1)) + O\left(\sum_{i \neq j} \varepsilon_{ij} + \frac{1}{\lambda_j^2}\right)\right]$$

□

For the proof, one can see the Appendix of [42]. The second step for the construction of the vector field W is to divide the domains $V(p, \varepsilon)$ in subdomains and to construct partial vector field on such domains satisfying Proposition 4.32. Thus, the final vector field will be defined as a convex combination of the vector fields constructed in the subdomains of $V(p, \varepsilon)$. For details of this construction, we refer to [42].

For technical reasons and for $\varepsilon_0 > 0$ small enough, we introduce the following neighborhood of Σ^+ :

$$V_{\varepsilon_0}(\Sigma^+) = \{u \in \Sigma;\ \|u^-\|_{L^4} \leq \varepsilon_0\},$$

where $u^- = \max(0, -u)$ is the negative part of u and $\|u^-\|_{L^4} = \left(\int_M |u^-|^4 \theta \wedge \mathrm{d}\theta\right)^{\frac{1}{4}}$.
Once the vector field W is constructed in the new variables, we build a global vector field Z on $V_{\varepsilon_0}(\Sigma^+)$ such that Proposition 4.32 is satisfied. For the V-part, we construct a pseudo-gradient T by setting $\frac{\partial V}{\partial s} = -\nu V$, locally on the base space of the bundle $V(p, \varepsilon)$, where ν is taken to be a very large constant. Define Z on $V_{\varepsilon_0}(\Sigma^+)$ to be $Z = W + T$. Thus, the defined vector field Z is a pseudo-gradient vector field for the functional $-J$ on $V_{\varepsilon_0}(\Sigma^+)$ which is invariant under the flow generated by Z (the proof of this claim is similar to the one given in [27]).

4.11.2 Critical points at infinity

In the sequel, we have to check the critical points at infinity of the functional J, which lead us to the study of the concentration phenomenon of J. First we claim that if $u_0 \in V_{\varepsilon_0}(\Sigma^+)$ there is $p \in \mathbb{N}*$ and $s_0 \geq 0$ such that if $\eta(s, u_0)$ denotes the flow line of the vector field Z with initial condition u_0, that is $\eta(s, u_0)$ satisfies

$$\begin{cases} \frac{\partial}{\partial s}\eta(s, u_0) = Z(\eta(s, u_0)) \\ \eta(0, u_0) = u_0 \end{cases} \tag{4.24}$$

$\eta(s, u_0)$ is in $V(p, \frac{3\varepsilon}{4})$ for $s \geq s_0$. Indeed outside $\bigcup_{p=1}^{r_1} V(p, \frac{3\varepsilon}{4})$, $-\partial J(Z(u)) \geq c > 0$.

We come back to the subdivision of the neighborhood $V(p, \varepsilon)$. We denote by $\bar{V}(p, \varrho, \varepsilon)$ the subset of $V(p, \varepsilon)$ containing $u = \sum_{i=1}^{p} \alpha_i \hat{\delta}_{a_i,\lambda_i} + v$, v satisfying conditions (V_0) and for which there is a subcollection of the critical points $\xi_1, \ldots, \xi_r$ of K such that any point is very close to all the concentration points $a_1 \ldots a_p$ and where $\varrho < \frac{1}{3} \min_{i \neq j} d(\xi_i, \xi_j)$. Then, we consider the subset of $\bar{V}(p, \varrho, \varepsilon)$ which we denote by $\bar{V}_2(p, \varrho, \varepsilon)$ and which contains $u = \sum_{i=1}^{p} \alpha_i \hat{\delta}_{a_i,\lambda_i} + v$ such that

1) two different concentration points are close to different critical points of K.
2) for which the matrix $M(\tau)$ for $\tau = (\xi_{\eta_1}, \ldots, \xi_{\eta_p})$ defined in (4.16) is positive definite.
3)

$$\frac{\Delta_\theta K(a_i)}{3K(a_i)} + 2A_{a_i} < 0, \quad i = 1, \ldots, p.$$

We have the following result:

Lemma 4.35 *The critical points at infinity of the functional J lie in $\bigcup_{p=1}^{r_1} \bar{V}_2(p, \varrho, \varepsilon)$ for any ε, $\varrho > 0$ small.*

Proof By using the argument above $\eta(s, u_0) = \sum_{i=1}^{p} \alpha_i \hat{\delta}_{a_i(s),\lambda_i(s)} + v(s)$ is in $V(p, \frac{3\varepsilon}{4})$. Suppose that in the new variables $\sum_{i=1}^{p} \widetilde{\alpha}_i(s) \hat{\delta}_{\widetilde{a}_i(s),\widetilde{\lambda}_i(s)}$ is outside $\bar{V}_2(p, \varrho, \frac{\varepsilon}{2})$, then we derive from the construction of Z that the maximum of the $\lambda_i(s)$ and the $\widetilde{\lambda}_i(s)$ are bounded by a positive constant c (we refer to [42] for all the details). Since $-J'(u)Z(u) > 0$ and $\sum_{i=1}^{p} \alpha_i \hat{\delta}_{a_i,\lambda_i}$ is in the compact set $\{\alpha_i \leq 1, \lambda_i \leq c, a_i \in M\}$, the minimum is achieved; hence $-J'(u)Z(u) > C > 0$. Therefore,

$$\begin{aligned} J(\eta(s, u_0)) &= J(\eta(0, u_0)) + \int_0^s J'(u)(t)Z(u)(t)\mathrm{d}t \\ &\leq J(\eta(0, u_0)) - C(s - s_0), \end{aligned}$$

which gives that J is not bounded; hence a contradiction. □

Lemma 4.36 *For any* $u=\sum_{i=1}^p \alpha_i \hat{\delta}_{a_i,\lambda_i}$ *in* $\bar{V}_2(p,\varrho,\varepsilon)$ $(\varepsilon,\ \varrho>0$ *small) close to a critical point at infinity of* J, *we obtain the following expansion of* J *in the new variables:*

$$J(u)=S\Big(\sum_{i=1}^{p}\frac{1}{K(\xi_i)}\Big)^{\frac{1}{2}}\Big(1-|Q|^2+\sum_{i=1}^{p}(|a_i^s|^2-|a_i^u|^2)+c\sum_{i=1}^{p}\frac{1}{\lambda_i^2}\Big), \tag{4.25}$$

where (a_i^s,a_i^u) *are the coordinates of* a_i *near* ξ_i *along the manifolds* $W_s(\xi_i)$ *and* $W_u(\xi_i)$ *and* $Q\in\mathbb{R}^{p-1}$ *is the coordinate of* $(\alpha_1,\ldots,\alpha_p)$.

Proof Using Proposition 4.29, we obtain the following expansion of the functional J in $\bar{V}_2(p,\varrho,\varepsilon)$ in the new variables $(v=0)$:

$$\begin{aligned}J\left(\sum_{i=1}^{p}\alpha_i\hat{\delta}_{a_i,\lambda_i}\right)=&\frac{\sum_{i=1}^p\alpha_i^2 S}{[\sum_{i=1}^p\alpha_i^4K(a_i)]^{1/2}}\Bigg[1-\frac{c}{4S^2}\sum_{i=1}^{p}\frac{\alpha_i^4}{\sum_{l=1}^p\alpha_l^4K(a_l)}\frac{\triangle K(a_i)}{\lambda_i^2}\\&+\frac{c'}{S^2}\sum_{i\neq j}\frac{G(a_i,a_j)}{\lambda_i\lambda_j}\Big(\frac{\alpha_i\alpha_j}{\sum_{l=1}^p\alpha_l^2}-\frac{2\alpha_i^3\alpha_jK(a_i)}{\sum_{l=1}^p\alpha_l^4K(a_l)}\Big)\\&+\frac{c'}{S^2}\sum_{i=1}^{p}\frac{A_{a_i}}{\lambda_i^2}\left(\frac{\alpha_i^2}{\sum_{l=1}^p\alpha_l^4}-2\frac{\alpha_i^4K(a_i)}{\sum_{l=1}^p\alpha_l^4K(a_l)}\right)+o\Big(\sum_{i\neq j}\varepsilon_{ij}\Big)+o\left(\frac{1}{\lambda_i^2}\right)\Bigg].\end{aligned}$$

Under the assumption that $u=\sum_{i=1}^p\alpha_i\hat{\delta}_{a_i,\lambda_i}$ belongs to $\bar{V}_2(p,\varrho,\varepsilon)$, the expansion of the functional can be rewritten as follows:

$$\begin{aligned}J(u)=&\frac{\sum_{i=1}^p\alpha_i^2S}{\big[\sum_{i=1}^p\alpha_i^4K(a_i)\big]^{1/2}}\Bigg[1-\frac{\omega_3}{8S^2\sum_{k=1}^p\frac{1}{K(a_k)}}\sum_{i=1}^{p}\left[\frac{\triangle K(a_i)}{3K^2(a_i)}+\frac{2A_{a_i}}{K(a_i)}\right]\frac{1}{\lambda_i^2}\\&+\frac{\omega_3}{8S^2}\sum_{i\neq j}\frac{2G(a_i,a_j)}{\lambda_i\lambda_j}\frac{(K(a_i)K(a_j))^{\frac{1}{2}}}{\sum_{k=1}^p\frac{1}{K(a_k)}}+o\left(\frac{1}{\lambda_i^2}\right)\Bigg].\end{aligned}$$

We can refine the expansion of J, since in this set, we have $\alpha_i^2K(a_i)\simeq\alpha_j^2K(a_j)$ and $\varepsilon_{ij}=(\lambda_i\lambda_jd^2(a_i,a_i))^{-1}$. Hence, we obtain

$$J\left(\sum_{i=1}^{p}\alpha_i\hat{\delta}_{a_i,\lambda_i}\right)=\frac{\sum_{i=1}^p\alpha_i^2}{\big[\sum_{i=1}^p\alpha_i^4K(a_i)\big]^{1/2}}S\Bigg[1+\frac{\omega_3}{8S^2\sum_{k=1}^p\frac{1}{K(a_k)}}\Lambda\big(M+o(1),\big)\Lambda^t\Bigg]$$

where $\Lambda=(\frac{1}{\lambda_1},\ldots,\frac{1}{\lambda_p})$.

Let us turn now to the term

$$\mathcal{G}(\alpha_1,\ldots,\alpha_p)=\frac{\sum_{i=1}^p\alpha_i^2}{\big(\sum_{i=1}^p\alpha_i^4K(a_i)\big)^{1/2}},$$

where $\mathcal{G}$ is a homogeneous function and $(\frac{1}{K(a_1)},\ldots,\frac{1}{K(a_p)})$ is a critical point (a maximum) with critical value $\sum_{j=1}^p\frac{1}{K(a_i)}$.
By performing a Morse lemma for $\mathcal{G}$, we obtain in the new variables

$$J(u)=S\left(\sum_{i=1}^{p}\frac{1}{K(\xi_i)}\right)\left[1-|Q|^2+\sum_{i=2}^{p}\big(|a_i^s|^2-|a_i^u|^2\big)+c\sum_{i=1}^{p}\frac{1}{\lambda_i^2}\right]$$

since $\Lambda M\Lambda^t\geq c|\Lambda|^2=\sum_{j=1}^p\frac{c}{\lambda_i^2}$, and the lemma follows. □

4.12 Topological argument

For any l-tuple $\tau_l = (i_1, \dots, i_l)$, $1 \le i_j \le r_1$, $j = 1, \dots, l$ such that $M(\tau_l)$ is positive definite, let $c(\tau_l) = \frac{S}{\sum_{j=1}^{l} K(\xi_{i_j})}$ denote the associated critical value. Here, we choose to consider a simplified situation, where for $\tau \ne \tau'$ we have $c(\tau) \ne c(\tau')$ and thus order the $c(\tau)$'s as $c(\tau_1) < c(\tau_2) < \cdots < c(\tau_k)$.

By using a deformation lemma (one can see [15]), we derive the existence of a positive constant $\sigma_0(\varepsilon, \varrho)$ such that for any $0 < \sigma < \sigma_0$, the set $J^{c(\tau_l)-\sigma} \cup W_u^{\infty}(\xi_l)_\infty$ is a retract by deformation of $J^{c(\tau_l)+\sigma}$, where J^a denotes the level set for the functional, $J^a = \{u \in \sum^+ / J(u) \le a\}$ and $W_u^{\infty}(\xi_l)_\infty$ is the unstable manifold of the critical point at infinity $(\xi_l)_\infty$.

Lemma 4.37 *If $c(\tau_{l-1}) < a < c(\tau_l) < b < c(\tau_{l+1})$, then for any coefficient group G, we have*

$$H_q(J^b, J^a) = \begin{cases} 0 & \text{if} \quad q \ne i(\tau_l), \\ G & \text{if} \quad q = i(\tau_l). \end{cases}$$

where $i(\tau_l) = 4l - 1 - \sum_{j=1}^{l} k_{i_j}$ with $k_{i_j} = ind(K, \xi_j)$. We are now ready to state the proof of our result.

4.12.1 Proof of Theorem 4.21

Let $b_1 < c(\tau_1) = \min_{u \in \Sigma^+} J(u) < b_2 < c(\tau_2) < \cdots < b_k < c(\tau_k) < b_{k+1}$. Since we assume that problem (P_K) has no solution, $J^{b_{k+1}}$ is a retract by deformation of the set Σ^+, which is a retract by deformation of $V_{\varepsilon_0}(\Sigma^+)$ and hence they have the same Euler-Poincaré characteristic,

$$\chi(V_{\varepsilon_0}(\Sigma^+)) = \chi(J^{b_{k+1}}).$$

By Lemma 4.37, we obtain

$$\chi(J^{b_{k+1}}) = \chi(J^{b_k}) + (-1)^{i(\tau_k)}.$$

We derive after recalling that $\chi(J^{b_1}) = \chi(\emptyset) = 0$, that

$$\sum_{l=1}^{r_1} \sum_{\tau_l = (i_1, \dots, i_l), M(\tau_l) > 0} (-1)^{i(\tau_l)} = 1.$$

Therefore, (P_K) has a solution u_0 in $V_{\varepsilon_0}(\Sigma^+)$ if the equality above is not true.

We claim that $u_0 > 0$, when ε_0 is small enough. Otherwise, by multiplying (P_K) by u_0^- and integrating, using the fact that u_0 is in $V_{\varepsilon_0}(\Sigma^+)$, we obtain

$$\|u_0^-\|^2 \le C\|u_0^-\|_{L^4}^4 \le C'\|u_0^-\|^2.$$

Hence, either $u_0^- = 0$ or $\|u_0^-\| \ge C_0$, where $C_0 > 0$. Thus, we have a contradiction if ε_0 is small enough. Therefore, $u_0^- = 0$ and $u_0 > 0$.

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