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# (1,1)-Tensor sphere bundle of Cheeger–Gromoll type

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**Abstract** We construct a metrical framed $f(3, -1)$-structure on the (1, 1)-tensor bundle of a Riemannian manifold equipped with a Cheeger–Gromoll type metric and by restricting this structure to the (1, 1)-tensor sphere bundle, we obtain an almost metrical paracontact structure on the (1, 1)-tensor sphere bundle. Moreover, we show that the (1, 1)-tensor sphere bundles endowed with the induced metric are never space forms.

**Mathematics Subject Classification** 53C15 · 53C21

**الملخص.**

نقوم بإنشاء بنية $f$(3،-1) مترية ومؤطرة على حزمة (1،1)-موتّرمن متنوعة **ريمان** مجهزة بمتري (دالة مسافة) من نوع **شيجر- جرومول**. وبقصر هذا الهيكل على حزمة (1،1)-موتّركرات، نحصل على بنية شبه اتصال تقريبا على حزمة (1،1)-موتّركرات. وإضافة إلى ذلك، نبين أن حزمة (1،1)-موتّر الكرات، المرفقة بالمتري المحدث، لن تكون أبدا فضاء أشكال.

## 1 Introduction

Maybe, the best known Riemannian metric on the tangent bundle is introduced by Sasaki in 1958 [20]. However, in most cases, the study of some geometric properties of the tangent bundle equipped with this metric lead to the flatness of the base manifold. A few years later, some researchers became interested in finding other lifted structures on the tangent bundles, cotangent, and tangent sphere bundles with interesting properties (see [2,4–10,13,16,21]).

The tangent sphere bundle $T_r M$ consisting of spheres with constant radius $r$ seen as hypersurfaces of the tangent bundle $TM$ has significant applications in geometry [11,12]. Recently, some interesting results were obtained by endowing the tangent sphere bundles with Riemannian metrics induced by the natural lifted metrics from $TM$, which are different from Sasakian (see [1,8,15]).

Tensor bundles $T_q^p M$ of type $(p, q)$ over a differentiable manifold $M$ are prime examples of fiber bundles, which are studied by mathematicians such as Ledger, Yano, Cengiz, and Salimov [3,14,18]. The tangent bundle $TM$ and cotangent bundle $T^*M$ are the special cases of $T_q^p M$.

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Salimov and Gezer [19] introduced the Sasaki metric ${}^{S}g$ on the (1, 1)-tensor bundle $T_1^1M$ of a Riemannian manifold $M$ and studied some geometric properties of this metric. By the similar method used in the tangent bundle, the present authors defined in [17] the Cheeger–Gromoll type metric ${}^{CG}g$ on $T_1^1M$ which is an extension of Sasaki metric. Then, the authors studied some relations between the geometric properties of the base manifold $(M, g)$ and $(T_1^1M, {}^{CG}g)$. In the present paper, we consider Cheeger–Gromoll type metric ${}^{CG}g$ on $T_1^1M$, and by applying it, we introduce a metrical framed $f(3, -1)$-structure on $T_1^1M$. Then, by restricting this structure to the (1, 1)-tensor sphere bundle of constant radius $r$, $T_{1r}^1M$, we obtain a metrical almost paracontact structure on $T_{1r}^1M$. Finally, we show that the (1, 1)-tensor sphere bundles endowed with the induced metric are never space forms.

## 2 Preliminaries

Let $M$ be a smooth $n$-dimensional manifold. We define the bundle of (1, 1)-tenors on $M$ as $T_1^1M = \coprod_{p\in M} T_1^1(p)$, where $\coprod$ denotes the disjoint union, and we call it (1, 1)-tensor bundle. We also define the projection $\pi : T_1^1M \to M$ to $p$. If $(x^i)$ are any local coordinates on $U \subset M$, and $p \in U$, the coordinate vectors $\{\partial_i\}$, where $\partial_i := \frac{\partial}{\partial x^i}$, form a basis for $T_pM$ whose dual basis is $\mathrm{d}x^i$. Any tensor $t \in T_1^1M$ can be expressed in terms of this basis as $t = t_j^i \partial_i \otimes \mathrm{d}x^j$.

For any coordinate chart $(U, (x^i))$ on $M$, correspondence $t \in T_1^1(x) \to (x, (t_j^i)) \in U \times R^{n^2}$ determines local trivializations $\phi : \pi^{-1}(U) \subset T_1^1M \to U \times R^{n^2}$, which shows that $T_1^1M$ is a vector bundle on $M$. Therefore, each local coordinate neighborhood $\{(U, x^j)\}_{j=1}^n$ in $M$ induces on $T_1^1M$ a local coordinate neighborhood $\{\pi^{-1}(U);\ \ x^j,\ x^{\bar{j}} = t_j^i\}_{j=1}^n$, $\bar{j} = n + j$, i.e., $T_1^1M$ is a smooth manifold of dimension $n + n^2$.

We denote by $F(M)$ and $\Im_1^1(M)$, the ring of real-valued $C^\infty$ functions and the space of all $C^\infty$ tensor fields of type (1, 1) on $M$. If $\alpha \in \Im_1^1(M)$, then by contraction, it is regarded as a function on $T_1^1M$, which we denote by $\imath\alpha$. If $\alpha$ has the local expression $\alpha = \alpha_i^j \frac{\partial}{\partial x^j} \otimes \mathrm{d}x^i$ in a coordinate neighborhood $U(x^j) \subset M$, then $\imath(\alpha) = \alpha(t)$ has the local expression $\imath\alpha = \alpha_i^j t_j^i$ with respect to the coordinates $(x^j, x^{\bar{j}})$ in $\pi^{-1}(U)$.

Suppose that $A \in \Im_1^1(M)$. Then, the vertical lift ${}^{V}A \in \Im_0^1(T_1^1M)$ of $A$ has the following local expression with respect to the coordinates $(x^j, x^{\bar{j}})$ in $T_1^1M$:

$$ {}^{V}A = {}^{V}A^{\bar{j}}\partial_{\bar{j}}, \tag{2.1} $$

where ${}^{V}A^{\bar{j}} = A_j^i$ and $\partial_{\bar{j}} := \frac{\partial}{\partial x^{\bar{j}}} = \frac{\partial}{\partial t_j^i}$. Moreover, if $V \in \Im_0^1(M)$, then the complete lift ${}^{C}V$ and the horizontal lift ${}^{H}V \in \Im_0^1(T_1^1M)$ of $V$ to $T_1^1M$ have the following local expressions with respect to the coordinates $(x^j, x^{\bar{j}})$ in $T_1^1M$ (see [3] and [14]):

$$ {}^{C}V = V^j\partial_j + \left(t_j^m\left(\partial_m V^i\right) - t_m^i\left(\partial_j V^m\right)\right)\partial_{\bar{j}}, \tag{2.2} $$

$$ {}^{H}V = V^j\partial_j + V^s\left(\Gamma_{sj}^m t_m^i - \Gamma_{sm}^i t_j^m\right)\partial_{\bar{j}}, \tag{2.3} $$

where $\Gamma_{ij}^k$ are the local components of a symmetric affine connection $\nabla$ on $M$.

Let $U(x^h)$ be a local chart of $M$. Using (2.1) and (2.3), we obtain

$$ e_j := {}^{H}\partial_j = {}^{H}\left(\delta_j^h\partial_h\right) = \delta_j^h\partial_h + \left(\Gamma_{jh}^s t_s^k - \Gamma_{js}^k t_h^s\right)\partial_{\bar{h}}, \tag{2.4} $$

$$ e_{\bar{j}} := {}^{V}\left(\partial_i \otimes dx^j\right) = {}^{V}\left(\delta_i^k\delta_h^j\partial_k \otimes dx^h\right) = \delta_i^k\delta_h^j\partial_{\bar{h}}, \tag{2.5} $$

where $\delta_j^h$ is the Kronecker's symbol and $\bar{j} = n + 1, \ldots, n + n^2$. These $n + n^2$ vector fields are linearly independent and generate the horizontal distribution of $\nabla$ and vertical distribution of $T_1^1M$, respectively. Indeed, we have ${}^{H}X = X^j e_j$ and ${}^{V}A = A_j^i e_{\bar{j}}$ (see [19]). The set $\{e_\beta\} = \{e_j, e_{\bar{j}}\}$ is called the frame adapted to the affine connection $\nabla$ on $\pi^{-1}(U) \subset T_1^1M$.

**Lemma 2.1** *Let $\alpha_1$, $\alpha_2$, $\alpha_3$, and $\alpha_4$ be smooth functions on $T_1^1M$, such that*

$$\alpha_1 g_{ti} g^{lj} \delta_r^m \delta_n^v + \alpha_2 g_{ni} g^{mj} \delta_r^l \delta_t^v + \alpha_3 \bar{t}_n^m \bar{t}_i^j \delta_r^l \delta_t^v + \alpha_4 \bar{t}_t^l \bar{t}_i^j \delta_r^m \delta_n^v = 0. \tag{2.6}$$

*Then,* $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$.

*Proof* Contacting (2.6) with $\bar{t}_v^r$, then differentiating the obtained expression three times, it follows that, $\alpha_3 = -\alpha_4$. Also differentiating the remaining expression two times, we have

$$\alpha_1 g_{ti} g^{lj} \bar{t}_n^m - \alpha_2 g_{ni} g^{mj} \bar{t}_t^l = 0.$$

Contacting the above equation with $t_i^j$, yield $\alpha_1 = -\alpha_2$. Multiplying (2.6) by $g_{jh} g^{ik}$ and $\delta_m^h \delta_k^n$, we obtain $\alpha_3 = \alpha_4 = 0$. Finally contacting (2.6) with $t_i^j$, $t_n^m$, we conclude that $\alpha_1 = \alpha_2 = 0$. □

## 3 Cheeger–Gromoll type metric on $T_1^1M$

For each $p \in M$, the extension of the scalar product $g$, denoted by $G$, is defined on the tensor space $\pi^{-1}(p) = T_1^1(p)$ by

$$G(A, B) = g_{it} g^{jl} A_j^i B_l^t, \quad A, B \in \Im_1^1(p),$$

where $g_{ij}$ and $g^{ij}$ are the local covariant and contravariant tensors associated with the metric $g$ on $M$.

Now, we consider on $T_1^1M$ a Riemannian metric ${}^{CG}g$ of Cheeger–Gromoll type, as follows [17]:

$$\begin{cases} {}^{CG}g({}^VA, {}^VB) = {}^V\Big(aG(A, B) + bG(t, A)G(t, B)\Big), \\ {}^{CG}g({}^HX, {}^HY) = {}^V(g(X, Y)), \\ {}^{CG}g({}^VA, {}^HY) = 0, \end{cases} \tag{3.1}$$

for each $X, Y \in \Im_0^1(M)$ and $A, B \in \Im_1^1(M)$, where $a$ and $b$ are smooth functions of $\tau = ||t||^2 = t_j^i t_l^t g_{it}(x) g^{jl}(x)$ on $T_1^1M$ that satisfies the conditions $a > 0$ and $a + b\tau > 0$.

The symmetric matrix of type $2n \times 2n$

$$\begin{pmatrix} g_{jl} & 0 \\ 0 & a g^{jl} g_{it} + b \bar{t}_i^j \bar{t}_t^l \end{pmatrix}, \tag{3.2}$$

associated with the metric ${}^{CG}g$ in the adapted frame $\{e_\beta\}$, has the inverse

$$\begin{pmatrix} g^{jl} & 0 \\ 0 & \frac{1}{a} g_{jl} g^{it} - \frac{b}{a(a+b\tau)} t_j^i t_l^t \end{pmatrix}, \tag{3.3}$$

where $\bar{t}_i^j = g^{jh} g_{ik} t_h^k$. In the special case, if $a = 1$ and $b = 0$, we have the Sasaki metric ${}^Sg$ (see [19]).

Let $\varphi = \varphi_j^i \frac{\partial}{\partial x^i} \otimes dx^j$ be a tensor field on $M$. Then, $\gamma\varphi = (t_j^m \varphi_m^i) \frac{\partial}{\partial \bar{x}^j}$ and $\widetilde{\gamma}\varphi = (t_m^i \varphi_j^m) \frac{\partial}{\partial \bar{x}^j}$ are vector fields on $T_1^1M$. The bracket operation of vertical and horizontal vector fields is given by the formulas

$$[{}^VA, {}^VB] = 0, \quad [{}^HX, {}^VA] = {}^V(\nabla_X A), \tag{3.4}$$

$$[{}^HX, {}^HY] = {}^H[X, Y] + (\widetilde{\gamma} - \gamma)R(X, Y), \tag{3.5}$$

where $R$ denotes the curvature tensor field of the connection $\nabla$ and $\widetilde{\gamma} - \gamma : \varphi \to \Im_0^1(T_1^1M)$ is the operator defined by

$$(\widetilde{\gamma} - \gamma)\varphi = \begin{pmatrix} 0 \\ t_m^i \varphi_j^m - t_j^m \varphi_m^i \end{pmatrix}, \quad \forall \varphi \in \Im_1^1(M).$$

**Proposition 3.1** [17] *The Levi-Civita connection ${}^{CG}\nabla$ associated with the Riemannian metric ${}^{CG}g$ on the* $(1,1)$*-tensor bundle $T^1_1M$ has the form*

$$\begin{aligned}
{}^{CG}\nabla_{e_l}^{e_j} &= \Gamma^r_{lj}e_r + \frac{1}{2}\left(R_{ljr}{}^s t^v_s - R_{ljs}{}^v t^s_r\right)e_{\bar r},\\
{}^{CG}\nabla_{e_{\bar l}}^{e_j} &= \frac{a}{2}\left(g_{ta}R^{sl}{}_j{}^r t^a_s - g^{lb}R_{tsj}{}^r t^s_b\right)e_r,\\
{}^{CG}\nabla_{e_l}^{e_{\bar j}} &= \frac{a}{2}\left(g_{ia}R^{sj}{}_l{}^r t^a_s - g^{jb}R_{isl}{}^r t^s_b\right)e_r + \left(\Gamma^v_{li}\delta^j_r - \Gamma^j_{lr}\delta^v_i\right)e_{\bar r},\\
{}^{CG}\nabla_{e_{\bar l}}^{e_{\bar j}} &= \left(L(\bar t^l_t\delta^j_r\delta^v_i + \bar t^j_i\delta^l_r\delta^v_t) + Mg^{lj}g_{ti}t^v_r + N\bar t^l_t\bar t^j_i t^v_r\right)e_{\bar r},
\end{aligned}$$

*where $R_{ljr}{}^s$ are the components of the curvature tensor field of the Levi-Civita connection on the base manifold* $(M,g)$ *and* $L := \frac{a'}{a}$, $M := \frac{-a'+2b}{a+b\tau}$, *and* $N := \frac{b'a-2a'b}{a(a+b\tau)}$.

In the following sections, we consider the subset $T^1_{1r}M$ of $T^1_1M$ consisting of sphere of constant radius $r$. Now, we consider the $(1,1)$-tensor field P on $T^1_1M$ as follows: [17]

$$\begin{cases} P\,{}^HX = c_1\,{}^V(X\otimes\widetilde E) + d_1 g(X,E)\,{}^V(E\otimes\widetilde E),\\ P\,{}^V(X\otimes\widetilde E) = c_2\,{}^HX + d_2 g(X,E)\,{}^HE,\\ P({}^VA) = {}^VA,\end{cases}$$

where $c_1$, $c_2$, $d_1$, and $d_2$ are smooth functions of the energy density $t$ and $\widetilde E = g\circ E\in\Im^0_1(M)$. Using the adapted frame $\{e_i, E_je_{\bar j}, e_{\bar j}\}$ to $T^1_1M$, $P$ has the following locally expression:

$$\begin{cases} P(e_i) = c_1E_je_{\bar j} + d_1E_iE^vE_re_{\bar r},\\ P(E_je_{\bar j}) = c_2e_i + d_2E_iE^re_r,\\ P(e_{\bar r}) = e_{\bar r},\end{cases} \tag{3.6}$$

where $E_k = g_{rk}E^r$. We have

**Theorem 3.2** [17] *The natural tensor field $P$ of type $(1,1)$ on $T^1_1M$, defined by the relations (3.6), is an almost product structure on $T^1_1M$, if and only if its coefficients are related by*

$$c_1c_2 = 1,\quad (c_1 + d_1||E||^2)(c_2 + d_2||E||^2) = 1. \tag{3.7}$$

**Theorem 3.3** [17] *$({}^{CG}g, P)$ is a Riemannian almost product structure on $T^1_1M$ if and only if*

$$c_1 = \frac{1}{\sqrt{a}||E||},\ c_2 = ||E||\sqrt{a},\ d_1 = \frac{-2}{\sqrt{a}||E||^3},\ d_2 = \frac{-2\sqrt{a}}{|E||}, \tag{3.8}$$

*and (3.7) hold good.*

Now, we consider vector fields

$$\xi_1 := \alpha\,{}^HE,\quad \xi_2 := \beta\,{}^V(E\otimes\widetilde E),\quad \xi_3 := \kappa\,{}^VA, \tag{3.9}$$

and 1-forms

$$\eta^1 = \gamma E_v dx^v,\quad \eta^2 = \lambda E_vE^r\delta t^v_r,\quad \eta^3 = \rho\bar t^r_v\delta t^v_r, \tag{3.10}$$

on $T^1_1M$, where $\alpha$, $\beta$, $\kappa$, $\gamma$, $\lambda$, and $\rho$ are smooth functions of the energy density on $T^1_1M$ and $\delta t^v_r$ is a dual of $e_{\bar r}$. Using (3.6) and (3.9), we get

$$P(\xi_1) = \frac{\alpha}{\beta}(c_1 + d_1||E||^2)\xi_2,\quad P(\xi_2) = \frac{\beta}{\alpha}(c_2 + d_2||E||^2)\xi_1,\quad P(\xi_3) = \xi_3, \tag{3.11}$$

and

$$\eta^1(\xi_1) = \alpha\gamma||E||^2,\quad \eta^2(\xi_2) = \beta\lambda||E||^4,\quad \eta^3(\xi_3) = \kappa\rho\tau,\quad \eta^a(\xi_b) = 0, \tag{3.12}$$

where $a, b = 1, 2, 3$ with condition $a \neq b$. We have also the following equations using (3.6) and (3.10):

$$\eta^1 \circ P = \frac{\gamma}{\lambda||E||^2}(c_2 + d_2||E||^2)\eta^2, \quad \eta^2 \circ P = \frac{\lambda||E||^2}{\gamma}(c_1 + d_1||E||^2)\eta^1, \quad \eta^3 \circ P = \eta^3. \tag{3.13}$$

Now, we define a tensor field $p$ of type (1,1) on $T_1^1 M$ by

$$p(X) = P(X) - \eta^1(X)\xi_2 - \eta^2(X)\xi_1 - \eta^3(X)\xi_3. \tag{3.14}$$

This can be written in a more compact from as $p = P - \eta^1 \otimes \xi_2 - \eta^2 \otimes \xi_1 - \eta^3 \otimes \xi_3$. From (3.14), the following local expression of $p$ yields:

$$\begin{cases} p(e_i) = \Big(c_1 \delta_i^v + (d_1 - \beta\gamma) E_i E^v\Big) E_r e_{\bar{r}}, \\ p(E_j e_{\bar{j}}) = \Big(c_2 \delta_i^r + (d_2 - \alpha\lambda||E||^2) E_i E^r\Big) e_r, \\ p(e_{\bar{j}}) = \Big(\delta_r^j \delta_i^v - \kappa\rho \bar{t}_i^j t_r^v\Big) e_{\bar{r}}. \end{cases} \tag{3.15}$$

**Lemma 3.4** *We have*

$$\begin{cases} p(\xi_1) = \frac{\alpha}{\beta}\Big(c_1 + (d_1 - \beta\gamma)||E||^2\Big)\xi_2, \\ p(\xi_2) = \frac{\beta}{\alpha}\Big(c_2 + (d_2 - \alpha\lambda||E||^2)||E||^2\Big)\xi_1, \\ p(\xi_3) = (1 - \kappa\rho\tau)\xi_3, \end{cases} \tag{3.16}$$

$$\begin{cases} \eta^1 \circ p = \frac{\gamma}{\lambda||E||^2}\Big(c_2 + (d_2 - \alpha\lambda||E||^2)||E||^2\Big)\eta^2, \\ \eta^2 \circ p = \frac{\lambda||E||^2}{\gamma}\Big(c_1 + (d_1 - \beta\gamma)||E||^2\Big)\eta^1, \\ \eta^3 \circ p = \Big(1 - \kappa\rho\tau\Big)\eta^3, \end{cases} \tag{3.17}$$

$$\begin{aligned} p^2 = I &- \Big(\frac{\beta}{\alpha}(c_2 + d_2||E||^2) + \frac{\lambda||E||^2}{\gamma}(c_1 + d_1||E||^2) - \beta\lambda||E||^4\Big)\eta^1 \otimes \xi_1 \\ &-\Big(\frac{\alpha}{\beta}(c_1 + d_1||E||^2) + \frac{\gamma}{\lambda||E||^2}(c_2 + d_2||E||^2) - \alpha\gamma||E||^2\Big)\eta^2 \otimes \xi_2, \\ &+ (\kappa\rho\tau - 2)\eta^3 \otimes \xi_3. \end{aligned} \tag{3.18}$$

*Proof* We only prove (3.18). Using (3.11), (3.12), and (3.13), we have

$$\begin{aligned} p^2(X) = p(p(X)) = P&\left[P(X) - \eta^1(X)\xi_2 - \eta^2(X)\xi_1 - \eta^3(X)\xi_3\right] \\ &- \eta^1\left[P(X) - \eta^2(X)\xi_1\right]\xi_2 - \eta^2\left[P(X) - \eta^1(X)\xi_2\right]\xi_1 \\ &- \eta^3\left[P(X) - \eta^3(X)\xi_3\right]\xi_1 = X - \frac{\beta}{\alpha}\left(c_2 + d_2||E||^2\right)\eta^1(X)\xi_1 \\ &- \frac{\alpha}{\beta}\left(c_1 + d_1||E||^2\right)\eta^2(X)\xi_2 - \frac{\gamma}{\lambda||E||^2}\left(c_2 + d_2||E||^2\right)\eta^2(X)\xi_2 \\ &+ ||E||^2\alpha\gamma\eta^2(X)\xi_2 - 2\eta^3(X)\xi_3 - \frac{\lambda||E||^2}{\gamma}\left(c_1 + d_1|E||^2\right)\eta^1(X)\xi_1 \\ &+ ||E||^4\beta\lambda\eta^1(X)\xi_1 + \kappa\rho\tau\eta^3(X)\xi_3. \end{aligned}$$

The above equation gives us (3.18). □

**Lemma 3.5** *Let $P$ satisfy Theorem 3.2. If*

$$\alpha\gamma||E||^2 = 1, \quad \beta\lambda||E||^4 = 1, \quad \kappa\rho\tau = 1, \quad \lambda = \frac{\gamma}{||E||^2}(c_2 + d_2||E||^2), \tag{3.19}$$

*then $p^3 - p = 0$ and $p$ has the rank $n + n^2 - 3$ (or corank 3).*

*Proof* If (3.19) holds, then from the above lemma, we obtain

$$p^2 = I - \eta^1 \otimes \xi_1 - \eta^2 \otimes \xi_2 - \eta^3 \otimes \xi_3, \quad p(\xi_k) = 0, \quad \eta^k(\xi_l) = \delta_l^k, \quad \eta^k \circ p = 0, \tag{3.20}$$

where $k, l = 1, 2, 3$. Therefore, we have $p^3 = p$. To prove the second part of the lemma, it is sufficient to show that $\ker p = \text{span}\{\xi_1, \xi_2, \xi_3\}$. From the second relation in (3.20), we notice that $\text{span}\{\xi_1, \xi_2, \xi_3\} \subset \ker p$. Now, let $X = X^r e_r + X^v E_r e_{\bar{r}} + X^{\bar{r}} e_{\bar{r}} \in \ker p$. Then, $p(X) = 0$ implies that

$$P(X) - \eta^1(X)\xi_2 - \eta^2(X)\xi_1 - \eta^3 \otimes \xi_3 = 0.$$

Thus

$$P^2(X) = \eta^1(X)P(\xi_2) + \eta^2(X)P(\xi_1) + \eta^3(X)P(\xi_3).$$

Since $P^2 = I$, then using (3.11), we get

$$X = \frac{\beta}{\alpha}(c_2 + d_2||E||^2)\eta^1(X)\xi_1 + \frac{\alpha}{\beta}(c_1 + d_1||E||^2)\eta^2(X)\xi_2 + \eta^3(X)\xi_3,$$

that is $X \in \text{span}\{\xi_1, \xi_2, \xi_3\}$, i.e., $\ker p \subseteq \text{span}\{\xi_1, \xi_2, \xi_3\}$. □

**Theorem 3.6** *Let $P$ be the almost product structure characterized in Theorem 3.2 and $\xi_k$, $\eta^k$, $k = 1, 2, 3$, and $p$ be defined by (3.9), (3.10), and (3.14), respectively. Then, the triple $(p, (\xi_k), (\eta^k))$ provides a framed $f(3, -1)$- structure if and only if (3.19) holds.*

*Proof* Let $(p, (\xi_k), (\eta^k))$ be a framed $f(3, -1)$-structure on $T_1^1 M$. Then, by the definition of a framed $f(3, -1)$-structure, we have $\eta^k(\xi_l) = \delta_l^k$, where $k, l = 1, 2, 3$. Thus, (3.12) gives us

$$\alpha\gamma||E||^2 = \beta\lambda||E||^4 = \kappa\rho\tau = 1. \tag{3.21}$$

We have also $p(\xi_3) = 0$. The above equation and the second relation in (3.16) yield $\lambda = \frac{\gamma}{||E||^2}(c_2 + d_2||E||^2)$. Using Lemmas 3.4 and 3.5, the converse of the theorem is proved. □

**Lemma 3.7** *Let $({}^{CG}g, P)$ satisfy Theorem 3.3. Then, the Riemannian metric ${}^{CG}g$ satisfies*

$$\begin{aligned}{}^{CG}g(pX, pY) = {}^{CG}g(X, Y) &- a\beta\left(\frac{2(c_1 + d_1||E||^2)}{\gamma} - \beta||E||^2\right)||E||^2\eta^1(X)\eta^1(Y) \\ &- \alpha\left(\frac{2(c_2 + d_2||E||^2)}{\lambda||E||^2} - \alpha||E||^2\right)\eta^2(X)\eta^2(Y) \\ &- \kappa(a + b\tau)\left(\frac{2}{\rho} - \kappa\tau\right)\eta^3(X)\eta^3(Y),\end{aligned}$$

*for each $X, Y \in \Im_0^1(T_1^1 M)$.*

*Proof* Obviously, we have ${}^{CG}g(\xi_1, \xi_2) = 0$. Using (3.9), we deduce

$${}^{CG}g(\xi_1, \xi_1) = \alpha^2||E||^2, \; {}^{CG}g(\xi_2, \xi_2) = a\beta^2||E||^4, \; {}^{CG}g(\xi_3, \xi_3) = \kappa^2(a + b\tau)\tau.$$

We have also

$${}^{CG}g(X, \xi_1) = \frac{\alpha}{\gamma}\eta^1(X), \; {}^{CG}g(X, \xi_2) = \frac{a\beta}{\lambda}\eta^2(X), \; {}^{CG}g(X, \xi_3) = \frac{\kappa}{\rho}(a + b\tau)\eta^3(X).$$

Using (3.13) and the above equations, we deduce

$$\begin{aligned}{}^{CG}g(pX, pY) = {}^{CG}g(PX, PY) &- \frac{2a\beta}{\gamma}(c_1 + d_1||E||^2)||E||^2\eta^1(X)\eta^1(Y) \\ &+ \alpha^2||E||^2\eta^2(X)\eta^2(Y) + a\beta^2||E||^4\eta^1(X)\eta^1(Y) \\ &- \frac{2\alpha}{\lambda||E||^2}\left(c_2 + d_2||E||^2\right)\eta^2(X)\eta^2(Y)\end{aligned}$$

$$-\kappa(a+b\tau)\left(\frac{2}{\rho}-\kappa\tau\right)\eta^3(X)\eta^3(Y).$$

However, ${}^{CG}g(PX, PY) = {}^{CG}g(X, Y)$, since $({}^{CG}g, P)$ is a Riemannian almost product structure. Thus, the lemma is proved. □

**Theorem 3.8** *If* $({}^{CG}g, P)$ *is the Riemannian almost product structure characterized in Theorem 3.3, and* $\xi_k$, $\eta^k$, $k = 1, 2, 3$, $p$ *are defined by (3.9), (3.10), and (3.14), respectively, then* $({}^{CG}g, p, (\xi_k), (\eta^k))$ *provides a metrical framed* $f(3, -1)$*-structure if and only if (3.19) and*

$$\gamma = \alpha, \quad \lambda = a\beta, \quad \rho = \kappa(a + b\tau), \tag{3.22}$$

*hold good.*

*Proof* Using Lemma 3.7, it is easy to see that the metricity condition

$${}^{CG}g(pX, pY) = {}^{CG}g(X, Y) - \eta^1(X)\eta^1(Y) - \eta^2(X)\eta^2(Y) - \eta^3(X)\eta^3(Y),$$

of the framed $f(3, -1)$ structure characterized by (3.19) is satisfied if and only if (3.22) holds good. Thus, the proof is complete. □

## 4 On (1, 1)-tensor sphere bundle

Let $r$ be a positive number. Then, the (1, 1)-tensor sphere bundle of radius $r$ over a Riemannian $(M, g)$ is the hypersurface $T^1_{1r}(M) = \{(x, t) \in T^1_1 M | G_x(t, t) = r^2\}$. It is easy to check that the tensor field

$$N = t^i_j e_{\bar{j}},$$

is a tensor field on $TM^1_1$ which is normal to $T^1_{1r}M$.

In general for any tensor field $A \in \Im^1_1(M)$, the vertical lift ${}^VA$ is not tangent to $T^1_{1r}M$ at point $(x, t)$. We define the tangential lift ${}^TA$ of a tensor field $A$ to $(x, t) \in T^1_{1r}M$ by

$${}^TA_{(x,t)} = {}^VA_{(x,t)} - \frac{1}{r^2}G_x(A, t)N_{(x,t)}. \tag{4.1}$$

Now, the tangent space $TT^1_{1r}M$ is spanned by $e_j$ and $e^T_{\bar{j}} = \partial_{\bar{j}} - \frac{1}{r^2}\bar{t}^j_i t^v_r \partial_{\bar{r}}$. We notice that there is the relation $t^i_j e^T_{\bar{j}} = 0$, and hence, in any point of $T^1_{1r}M$, the vectors $e^T_{\bar{j}}$, $\bar{j} = n+1, \ldots, n+n^2$, span an $(n^2-1)$-dimensional subspace of $TT^1_{1r}(M)$. Using (4.1) and the computation starting with the formula (3.1), we see that the Riemannian metric $\widetilde{g}$ on $T^1_1M$, induced from ${}^{CG}g$, is completely determined by the identities

$$\begin{aligned}
&\widetilde{g}({}^TA, {}^TB) = a^V\left(G(A, B) - \frac{1}{r^2}G(t, A)G(t, B)\right),\\
&\widetilde{g}({}^TA, {}^HY) = 0,\\
&\widetilde{g}({}^HX, {}^HY) = {}^V(g(X, Y)),
\end{aligned} \tag{4.2}$$

for all $X, Y \in \Im^1_0(M)$ and $A, B \in \Im^1_1(M)$, where $a$ is constant that satisfy $a > 0$.

The bracket operation of tangential and horizontal vector fields is given by the formulas

$$\begin{aligned}
\left[e^T_{\bar{l}}, e^T_{\bar{j}}\right] &= \frac{1}{r^2}\left(\bar{t}^l_t\delta^v_i\delta^j_r - \bar{t}^j_i\delta^v_t\delta^l_r\right)e^T_{\bar{r}},\\
\left[e_l, e^T_{\bar{j}}\right] &= \left(\Gamma^v_{li}\delta^j_r - \Gamma^j_{lr}\delta^v_i\right)e^T_{\bar{r}},\\
\left[e_l, e_j\right] &= \left(R_{ljr}{}^s t^v_s - R_{ljs}{}^v t^s_r\right)e^T_{\bar{r}}.
\end{aligned}$$

Using the Levi-Civita connection of the Cheeger–Gromoll type metric introduced by the authors in [17], we can conclude the following:

**Proposition 4.1** *The Levi-Civita connection $\widetilde{\nabla}$, associated with the Riemannian metric $\widetilde{g}$ on the tensor bundle $T^1_{1r}M$, has the form*

$$\begin{aligned}
\widetilde{\nabla}_{e_l}^{e_j} &= \Gamma^r_{lj} e_r + \frac{1}{2}\left(R_{ljr}{}^s t^v_s - R_{ljs}{}^v t^s_r\right) e^T_{\bar r},\\
\widetilde{\nabla}_{e^T_{\bar l}}^{e_j} &= \frac{a}{2}\left(g_{ta} R^{sl}{}_j{}^r t^a_s - g^{lb} R_{tsj}{}^r t^s_b\right) e_r,\\
\widetilde{\nabla}_{e_l}^{e^T_{\bar j}} &= \frac{a}{2}\left(g_{ia} R^{sj}{}_l{}^r t^a_s - g^{jb} R_{isl}{}^r t^s_b\right) e_r + \left(\Gamma^v_{li}\delta^j_r - \Gamma^j_{lr}\delta^v_i\right) e^T_{\bar r},\\
\widetilde{\nabla}_{e^T_{\bar l}}^{e^T_{\bar j}} &= -\frac{1}{r^2}\bar t^j_i \delta^l_r \delta^v_t e^T_{\bar r}.
\end{aligned}$$

## 4.1 An almost paracontact structure on $T^1_{1r}M$

In this section, we show that the framed $f(3,-1)$-structure on $T^1_1 M$, given by Theorem 3.6, induces an almost paracontact structure on $T^1_{1r}M$.

First, we show that $\xi_2$ and $\xi_3$ are unit normal vector fields with respect to the metric ${}^{CG}g$. Let

$$x^i = x^i(u^\alpha), \quad t^i_j = t^i_j(u^\alpha), \quad \alpha \in \{1, ..., n\}, \tag{4.3}$$

be the local equations of $T^1_{1r}M$ in $T^1_1M$. Since $\tau = t^i_j t^t_l g^{jl} g_{it} = r^2$, we have

$$\frac{\partial \tau}{\partial x^j}\frac{\partial x^j}{\partial u^\alpha} + \frac{\partial \tau}{\partial t^k_h}\frac{\partial t^k_h}{\partial u^\alpha} = 0. \tag{4.4}$$

However, we have

$$\frac{\partial \tau}{\partial x^j} = 2\left(\Gamma^k_{js} t^s_h - \Gamma^s_{jh} t^k_s\right)\bar t^h_k, \qquad \frac{\partial \tau}{\partial t^k_h} = 2\bar t^h_k. \tag{4.5}$$

By replacing (4.5) into (4.4), we get

$$\left(\left(\Gamma^k_{js} t^s_h - \Gamma^s_{jh} t^k_s\right)\frac{\partial x^j}{\partial u^\alpha} + \frac{\partial t^k_h}{\partial u^\alpha}\right)\bar t^h_k = 0. \tag{4.6}$$

The natural frame field on $T^1_{1r}M$ is represented by

$$\frac{\partial}{\partial u^\alpha} = \frac{\partial x^j}{\partial u^\alpha}\frac{\partial}{\partial x^j} + \frac{\partial t^k_h}{\partial u^\alpha}\frac{\partial}{\partial t^k_h} = \frac{\partial x^j}{\partial u^\alpha} e_j + \left(\left(\Gamma^k_{js} t^s_h - \Gamma^s_{jh} t^k_s\right)\frac{\partial x^j}{\partial u^\alpha} + \frac{\partial t^k_h}{\partial u^\alpha}\right) e_{\bar h}. \tag{4.7}$$

Then, by (4.6), we deduce that

$${}^{CG}g\left(\frac{\partial}{\partial u^\alpha}, \xi_3\right) = \kappa(a + b\tau)\left(\left(\Gamma^k_{js} t^s_h - \Gamma^s_{jh} t^k_s\right)\frac{\partial x^j}{\partial u^\alpha} + \frac{\partial t^k_h}{\partial u^\alpha}\right)\bar t^h_k = 0. \tag{4.8}$$

Similarly, we obtain ${}^{CG}g(\frac{\partial}{\partial u^\alpha}, \xi_2) = 0$. Thus, $\xi_2$ and $\xi_3$ are orthogonal to any vector tangent to $T^1_{1r}M$. The vector field $\xi_1$ is tangent to $T^1_{1r}M$, since ${}^{CG}g(\xi_1, \xi_2) = 0$.

**Lemma 4.2** *On $T^1_{1r}M$, we have*

$$\eta^2 = \eta^3 = 0, \quad p(X) = P(X) - \eta^1(X)\xi_1, \quad \forall X \in \chi(T^1_{1r}M).$$

*Proof* Using $\eta^i|_{T^1_{1r}M}(X) = {}^{CG}g(X, \xi_i) = 0$, $i = 2, 3$, the proof is obvious. □

We put $\xi_1|_{T^1_{1r}M} = \xi$, $\eta^1|_{T^1_{1r}M} = \eta$ and $p|_{T^1_{1r}M} = p$. Then, Theorem 3.6 and Lemma 4.2 imply the following.

**Theorem 4.3** *If* (3.19) *holds, then the triple* $(p, \xi, \eta)$ *defines an almost paracontact structure on* $T^1_{1r}M$*, that is,*

(i) $\eta(\xi) = 1, \quad p(\xi) = 0, \quad \eta \circ p = 0.$
(ii) $p^2(X) = X - \eta(X)\xi, \quad X \in \chi(T^1_{1r}M).$

It is easy to show that if (3.19) and (3.22) hold, then the Riemannian metric $\widetilde{g}$ satisfies

$$\widetilde{g}(pX, pY) = \widetilde{g}(X, Y) - \eta(X)\eta(Y), \quad X, Y \in \chi(T^1_{1r}M). \tag{4.9}$$

Using the equation (4.9) and Theorem 4.3, we conclude the following:

**Theorem 4.4** *If* (3.19) *and* (3.22) *hold, then the ensemble* $(p, \xi, \eta, \widetilde{g})$ *defines an almost metrical paracontact structure on the tangent sphere bundle* $T^1_{1r}M$.

### 4.2 Non-existence (1, 1)-tensor sphere bundles space form

The curvature tensor field $\widetilde{R}$ of the connection $\widetilde{\nabla}$ is defined by the well-known formula

$$\widetilde{R}(\widetilde{X}, \widetilde{Y})\widetilde{Z} = \widetilde{\nabla}_{\widetilde{X}}\widetilde{\nabla}_{\widetilde{Y}}\widetilde{Z} - \widetilde{\nabla}_{\widetilde{Y}}\widetilde{\nabla}_{\widetilde{X}}\widetilde{Z} - \widetilde{\nabla}_{[\widetilde{X},\widetilde{Y}]}\widetilde{Z},$$

where $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \Im^1_0(T^1_{1r}M)$. Using the above equation, Proposition 4.1, and the local frame $\{e_j, e^T_{\bar{j}}\}$, we obtain

$$\widetilde{R}(e_m, e_l)e_j = HHHH^r_{mlj}e_r + HHHT^{\bar{r}}_{mlj}e^T_{\bar{r}}, \tag{4.10}$$
$$\widetilde{R}(e_m, e_l)e^T_{\bar{j}} = HHTH^r_{ml\bar{j}}e_r + HHTT^{\bar{r}}_{ml\bar{j}}e^T_{\bar{r}}, \tag{4.11}$$
$$\widetilde{R}(e_m, e^T_{\bar{l}})e_j = HTHH^r_{m\bar{l}j}e_r + HTHT^{\bar{r}}_{m\bar{l}j}e^T_{\bar{r}}, \tag{4.12}$$
$$\widetilde{R}(e_m, e^T_{\bar{l}})e^T_{\bar{j}} = HTTH^r_{m\bar{l}\bar{j}}e_r, \tag{4.13}$$
$$\widetilde{R}(e^T_{\bar{m}}, e^T_{\bar{l}})e_j = TTHH^r_{\bar{m}\bar{l}j}e_r, \tag{4.14}$$
$$\widetilde{R}(e^T_{\bar{m}}, e^T_{\bar{l}})e^T_{\bar{j}} = TTTT^{\bar{r}}_{\bar{m}\bar{l}\bar{j}}e^T_{\bar{r}}, \tag{4.15}$$

where

$$\begin{aligned}
HHHH_{mlj}{}^{r} = {} & R_{mlj}{}^{r} + \frac{a}{4}\Big\{g_{ka}\left(R^{sh}{}_{m}{}^{r}R_{ljh}{}^{p} - R^{sh}{}_{l}{}^{r}R_{mjh}{}^{p} - 2R^{sh}{}_{j}{}^{r}R_{mlh}{}^{p}\right)t^a_s t^k_p \\
& + g_{ka}\left(R^{sh}{}_{l}{}^{r}R_{mjp}{}^{k} - R^{sh}{}_{m}{}^{r}R_{ljp}{}^{k} + 2R^{sh}{}_{j}{}^{r}R_{mlp}{}^{k}\right)t^a_s t^p_h \\
& + g^{hb}\left(R_{kpl}{}^{r}R_{mjh}{}^{s} - R_{kpm}{}^{r}R_{ljh}{}^{s} + 2R_{kpj}{}^{r}R_{mlh}{}^{s}\right)t^p_b t^k_s \\
& + g^{hb}\left(R_{ksm}{}^{r}R_{ljp}{}^{k} - R_{ksl}{}^{r}R_{mjp}{}^{k} - 2R_{ksj}{}^{r}R_{mlp}{}^{k}\right)t^s_b t^p_h\Big\}, \\
HHHT_{mlj}{}^{\bar{r}} = {} & \frac{1}{2}\left\{\nabla_m R_{ljr}{}^{s}t^v_s - \nabla_l R_{mjr}{}^{s}t^v_s + \nabla_l R_{mjs}{}^{v}t^s_r - \nabla_m R_{ljs}{}^{v}t^s_r\right\}, \\
HHTH_{ml\bar{j}}{}^{r} = {} & \frac{a}{2}\left\{g_{ia}\nabla_m R^{sj}{}_{l}{}^{r}t^a_s - g_{ia}\nabla_l R^{sj}{}_{m}{}^{r}t^a_s + g^{jb}\nabla_l R_{ism}{}^{r}t^s_b - g^{jb}\nabla_m R_{isl}{}^{r}t^s_b\right\}, \\
HHTT_{ml\bar{j}}{}^{\bar{r}} = {} & R_{mli}{}^{v}\delta^j_r - R_{mlr}{}^{j}\delta^v_i + \frac{a}{4}\Big\{g_{ia}\left(R_{mhr}{}^{s}R^{pj}{}_{l}{}^{h} - R_{lhr}{}^{s}R^{pj}{}_{m}{}^{h}\right)t^v_s t^a_p \\
& + g_{ia}\left(R_{lhp}{}^{v}R^{sj}{}_{m}{}^{h} - R_{mhp}{}^{v}R^{sj}{}_{l}{}^{h}\right)t^a_s t^p_r + g^{jb}\left(R_{lhr}{}^{s}R_{ipm}{}^{h} - R_{mhr}{}^{s}R_{ipl}{}^{h}\right)t^p_b t^v_s \\
& + g^{jb}\left(R_{mhs}{}^{v}R_{ipl}{}^{h} - R_{lhs}{}^{v}R_{ipm}{}^{h}\right)t^s_r t^p_b\Big\} + \frac{1}{r^2}\left(R_{mlr}{}^{s}t^v_s - R_{mls}{}^{v}t^s_r\right)\bar{t}^j_i, \\
HTHH_{m\bar{l}j}{}^{r} = {} & \frac{a}{2}\left\{g_{ta}\nabla_m R^{sl}{}_{j}{}^{r}t^a_s - g^{lb}\nabla_m R_{tsj}{}^{r}t^s_b\right\}, \\
HTHT_{m\bar{l}j}{}^{\bar{r}} = {} & -\frac{1}{2}\left(R_{mjr}{}^{l}\delta^v_t - R_{mjt}{}^{v}\delta^l_r\right) + \frac{a}{4}\Big\{g_{ta}R^{pl}{}_{j}{}^{h}R_{mhr}{}^{s}t^v_s t^a_p
\end{aligned}$$

$$- g^{lb} R_{tpj}{}^{h} R_{mhr}{}^{s} t_s^v t_b^p - g_{ta} R^{sl}{}_{j}{}^{h} R_{mhp}{}^{v} t_r^p t_s^a$$
$$+ g^{lb} R_{tpj}{}^{h} R_{mhs}{}^{v} t_r^s t_b^p \Big\},$$
$$HTTH_{m\bar{l}\bar{j}}{}^{r} = \frac{a}{2}\left(g^{jl} R_{itm}{}^{r} - g_{it} R^{lj}{}_{m}{}^{r}\right) + \frac{a^2}{4}\Big\{ g_{ta} R^{sl}{}_{h}{}^{r} g^{jb} R_{ipm}{}^{h} t_s^a t_b^p$$
$$- g_{ta} R^{sl}{}_{h}{}^{r} g_{ib} R^{pj}{}_{m}{}^{h} t_s^a t_p^b + g^{lb} R_{tph}{}^{r} g_{ia} R^{sj}{}_{m}{}^{h} t_b^p t_s^a$$
$$- g^{la} R_{tsh}{}^{r} g^{jb} R_{ipm}{}^{h} t_a^s t_b^p \Big\} - \frac{a}{2r^2}\Big( g_{ta} R^{sl}{}_{m}{}^{r} t_s^a$$
$$- g^{lb} R_{tsm}{}^{r} t_b^s \Big) \bar{t}_i^j,$$
$$TTHH_{\overline{m}\bar{l}j}{}^{r} = a\left(g_{tn} R^{ml}{}_{j}{}^{r} - g^{lm} R_{tnj}{}^{r}\right) + \frac{a^2}{4}\Big\{ g_{na} R^{sm}{}_{h}{}^{r} g_{tb} R^{pl}{}_{j}{}^{h} t_s^a t_p^b$$
$$- g_{ta} R^{sl}{}_{h}{}^{r} g_{nb} R^{pm}{}_{j}{}^{h} t_s^a t_p^b + g_{ta} R^{sl}{}_{h}{}^{r} g^{mb} R_{npj}{}^{h} t_s^a t_b^p$$
$$- g_{na} R^{sm}{}_{h}{}^{r} g^{lb} R_{tpj}{}^{h} t_s^a t_b^p + g^{lb} R_{tph}{}^{r} g_{na} R^{sm}{}_{j}{}^{h} t_b^p t_s^a$$
$$- g^{mb} R_{nph}{}^{r} g_{ta} R^{sl}{}_{j}{}^{h} t_b^p t_s^a + g^{ma} R_{nsh}{}^{r} g^{lb} R_{tsj}{}^{h} t_b^p t_a^s$$
$$- g^{la} R_{tsh}{}^{r} g^{mb} R_{npj}{}^{h} t_b^p t_a^s \Big\},$$
$$TTTT_{\overline{m}\bar{l}\bar{j}}{}^{\bar{r}} = \frac{1}{r^4}\left(\bar{t}_n^m \bar{t}_i^j \delta_r^l \delta_t^v - \bar{t}_t^l \bar{t}_i^j \delta_r^m \delta_n^v\right) + \frac{1}{r^2}\Big( g^{lj} g_{ti} \delta_r^m \delta_n^v$$
$$- g^{mj} g_{ni} \delta_r^m \delta_n^v \Big).$$

In the following, we calculate the Ricci tensor $\widetilde{\mathrm{Ric}}$ of $(T^1_{1r}(M), \widetilde{g})$ using the well-known formula:

$$\widetilde{\mathrm{Ric}} = \mathrm{trace}(X \to \widetilde{R}(\widetilde{X}, \widetilde{Y})\widetilde{Z}), \quad \forall \widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \Im^1_0(T^1_{1r}M).$$

Let $(E_1, \ldots, E_{n^2+n})$ be the orthonormal frame, such that the first $n$ vectors $E_1, \ldots, E_n$ are vectors of a frame in $HTM$ and the last $n^2$ vectors $E_{n+1}, \ldots, E_{n^2+n}$ are vectors of a frame in $VTM$ [8]. We consider the last vector $E_{n^2+n}$ as the unitary vector of the normal vector $N = t^i_j e_{\bar{j}}$ to $T^1_{1r}(M)$. It is easy to see that the vector fields $e^T_1, \ldots, e^T_{n^2}$ are not independent. Considering the basis $e_1, \ldots, e_n, e^T_1, \ldots, e^T_{n^2-1}$ for $TT^1_{1r}(M)$, on an open set of $T^1_{1r}(M)$ where $t^i_j \neq 0$, we can write the last vector $e^T_{n^2}$ as follows:

$$e^T_{n^2} = e^T_{\bar{n}} = -\frac{1}{t^n_n} \sum_{\substack{i,j=1\\ i\neq j\neq n}}^{n} t^i_j e^T_{\bar{j}}.$$

Using the definition of the Ricci tensor, we have

$$\widetilde{\mathrm{Ric}}(e^T_{\bar{l}}, e^T_{\bar{j}}) = TTTT_{\bar{r}\bar{l}\bar{j}}{}^{\bar{r}} + HTTH_{r\bar{l}\bar{j}}{}^{r}.$$

Direct calculations give us

$$TTTT_{\bar{s}\bar{l}\bar{j}}{}^{\bar{r}} e^T_{\bar{r}} = \sum_{\substack{k,h=1\\ k\neq h\neq n}}^{n} TTTT_{\bar{s}\bar{l}\bar{j}}{}^{\bar{k}} e^T_{\bar{k}} + TTTT_{\bar{s}\bar{l}\bar{j}}{}^{\bar{n}} e^T_{n^2}$$
$$= \sum_{\substack{k,h=1\\ k\neq h\neq n}}^{n} TTTT_{\bar{s}\bar{l}\bar{j}}{}^{\bar{k}} e^T_{\bar{k}} - TTTT_{\bar{s}\bar{l}\bar{j}}{}^{\bar{n}} \frac{1}{t^n_n} \sum_{\substack{k,h=1\\ k\neq h\neq n}}^{n} t^h_k e^T_{\bar{k}}$$
$$= TTTT_{\bar{s}\bar{l}\bar{j}}{}^{\bar{r}} e^T_{\bar{r}} - TTTT_{\bar{s}\bar{l}\bar{j}}{}^{\bar{n}} \frac{1}{t^n_n} t^v_r e^T_{\bar{r}}.$$

Setting $\bar{s} = \bar{r}$ in the above equation, we have

$$TTTT_{\bar{r}l\bar{j}}{}^{\bar{r}} = TTTT_{\bar{r}l\bar{j}}{}^{\bar{r}} - \frac{1}{t_n^n} t_r^v TTTT_{\bar{r}l\bar{j}}{}^{\bar{n}}.$$

Note that in the left side of the above equation, summation index $r$ is different from the summation index $r$ in the right side. Using the above expression of $TTTT_{\bar{m}l\bar{j}}{}^{\bar{r}}$ and (4.15), we get

$$t_r^v TTTT_{\bar{r}l\bar{j}}{}^{\bar{n}} = \frac{1}{r^2} g^{lj} g_{ti} t_n^n - \frac{1}{r^4} \bar{t}_t^l \bar{t}_i^j t_n^n.$$

Hence

$$\frac{1}{t_n^n} t_r^v TTTT_{\bar{r}l\bar{j}}{}^{\bar{n}} = \frac{1}{r^2} g^{lj} g_{ti} - \frac{1}{r^4} \bar{t}_t^l \bar{t}_i^j.$$

It follows that:

$$\begin{aligned}\widetilde{Ric}\left(e_{\bar{l}}^T, e_{\bar{j}}^T\right) &= TTTT_{\bar{r}l\bar{j}}{}^{\bar{r}} + HTTH_{rl\bar{j}}{}^{r} - \frac{1}{r^2} g^{lj} g_{ti} + \frac{1}{r^4} \bar{t}_t^l \bar{t}_i^j \\ &= \frac{1}{r^2}(n^2-2)\left(g_{ti} g^{lj} - \frac{1}{r^2} \bar{t}_i^j \bar{t}_t^l\right) + \frac{a^2}{4}\left\{g^{lb} R_{tph}{}^{r} g_{ia} R^{sj}{}_{r}{}^{h} t_b^p t_s^a\right. \\ &\quad - g_{ta} R^{sl}{}_{h}{}^{r} g_{ib} R^{pj}{}_{r}{}^{h} t_s^a t_p^b - g^{la} R_{tsh}{}^{r} g^{jb} R_{ipr}{}^{h} t_a^s t_b^p \\ &\quad \left. + g_{ta} R^{sl}{}_{h}{}^{r} g^{jb} R_{ipr}{}^{h} t_s^a t_b^p\right\}.\end{aligned}$$

In a similar way, we get other components of the Ricci tensor on $T_{1r}^1(M)$ as follows:

$$\begin{aligned}\widetilde{\mathrm{Ric}}(e_{\bar{l}}^T, e_j) &= HTHH_{r\bar{l}j}{}^{r} = \frac{a}{2}\left\{g_{ta} \nabla_r R^{sl}{}_{j}{}^{r} t_s^a - g^{lb} \nabla_r R_{tsj}{}^{r} t_b^s\right\}, \\ \widetilde{\mathrm{Ric}}(e_l, e_{\bar{j}}^T) &= HHTH_{rl\bar{j}}{}^{r} = \frac{a}{2}\left\{g_{ia} \nabla_r R^{sj}{}_{l}{}^{r} t_s^a - g^{jb} \nabla_r R_{isl}{}^{r} t_b^s\right\}, \\ \widetilde{\mathrm{Ric}}(e_l, e_j) &= HHHH_{rlj}{}^{r} + THHT_{\bar{r}lj}{}^{\bar{r}} - \frac{1}{t_n^n} t_r^v THHT_{\bar{r}lj}{}^{\bar{n}} \\ &= R_{lj} + \frac{a}{2}\left\{g^{hb} R_{kpj}{}^{r} R_{rlh}{}^{s} t_b^p t_s^k - g_{ka} R^{sh}{}_{j}{}^{r} R_{rlh}{}^{p} t_s^a t_p^k\right. \\ &\quad \left. - g^{hb} R_{ksj}{}^{r} R_{rlp}{}^{k} t_b^s t_h^p + g_{ka} R^{sh}{}_{j}{}^{r} R_{rlp}{}^{k} t_s^a t_h^p\right\} \\ &\quad - \frac{a}{4}\left\{g_{ka} R^{sh}{}_{l}{}^{r} R_{rjh}{}^{p} t_s^a t_p^k + g_{va} R^{pr}{}_{j}{}^{h} R_{lhr}{}^{s} t_s^v t_p^a\right. \\ &\quad \left. + g^{hb} R_{ksl}{}^{r} R_{rjp}{}^{k} t_b^s t_h^p + g^{rb} R_{vpj}{}^{h} R_{lhs}{}^{v} t_r^s t_b^p\right\}.\end{aligned}$$

**Theorem 4.5** *(1, 1)-tensor sphere bundle $T_{1r}^1 M$, with the Riemannian metric $\tilde{g}$ induced from the metric ${}^{CG}g$ on $T_1^1 M$, has never constant sectional curvature.*

*Proof* It is known that the curvature tensor field of the Riemannian manifold $(T_{1r}^1 M, \tilde{g})$ with constant section curvature $k$ satisfies the relation

$$\widetilde{R}(\widetilde{X}, \widetilde{Y})\widetilde{Z} = k\left\{\tilde{g}(\widetilde{Y}, \widetilde{Z})\widetilde{X} - \tilde{g}(\widetilde{X}, \widetilde{Z})\widetilde{Y}\right\}, \tag{4.16}$$

where $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \Im_0^1(T_{1r}^1 M)$. If $(T_{1r}^1 M, \tilde{g})$ has constant sectional curvature $k$, then we have

$$\widetilde{R}\left(e_{\bar{m}}^T, e_{\bar{l}}^T\right) e_{\bar{j}}^T - k\left\{\tilde{g}(e_{\bar{l}}^T, e_{\bar{j}}^T) e_{\bar{m}}^T - \tilde{g}\left(e_{\bar{m}}^T, e_{\bar{j}}^T\right) e_{\bar{l}}^T\right\} = 0. \tag{4.17}$$

Using (4.17) and (4.15), we get

$$\frac{1-kr^2 a}{r^2}\left[g_{ti} g^{lj} \delta_r^m \delta_n^v - g_{ni} g^{mj} \delta_r^l \delta_t^v + \frac{1}{r^2}\left(\bar{t}_n^m \bar{t}_i^j \delta_r^l \delta_t^v - \bar{t}_t^l \bar{t}_i^j \delta_r^m \delta_n^v\right)\right] = 0. \tag{4.18}$$

Using the above equation and Lemma 2.1, we deduce $k \neq 0$ and $a = \frac{1}{kr^2}$. Since $(T^1_{1r}M, \widetilde{g})$ has constant sectional curvature $k$, we have

$$\widetilde{R}(e_m, e_l)e_j - k\left\{\widetilde{g}(e_l, e_j)e_m - \widetilde{g}(e_m, e_j)e_l\right\} = 0. \tag{4.19}$$

(4.10) and (4.19) give us

$$\begin{aligned}
&R_{mlj}{}^r - k\left(g_{lj}\delta^r_m - g_{mj}\delta^r_l\right) + \frac{a}{4}\Big\{g_{ka}\Big(R^{sh}{}_m{}^r R_{ljh}{}^p \\
&-R^{sh}{}_l{}^r R_{mjh}{}^p - 2R^{sh}{}_j{}^r R_{mlh}{}^p\Big)t^a_s t^k_p + g_{ka}\Big(R^{sh}{}_l{}^r R_{mjp}{}^k \\
&-R^{sh}{}_m{}^r R_{ljp}{}^k + 2R^{sh}{}_j{}^r R_{mlp}{}^k\Big)t^a_s t^p_h + g^{hb}\Big(R_{ksm}{}^r R_{ljp}{}^k \\
&-R_{ksl}{}^r R_{mjp}{}^k - 2R_{ksj}{}^r R_{mlp}{}^k\Big)t^s_b t^p_h + g^{hb}\Big(R_{kpl}{}^r R_{mjh}{}^s \\
&-R_{kpm}{}^r R_{ljh}{}^s + 2R_{kpj}{}^r R_{mlh}{}^s\Big)t^p_b t^k_s\Big\} = 0.
\end{aligned} \tag{4.20}$$

Differentiating the expression (4.20) two times, in the tangential coordinates $x^{\bar{j}}$; $\bar{j} = 1, \ldots, n + n^2$, we conclude

$$R_{mlj}{}^r = k\left(g_{lj}\delta^r_m - g_{mj}\delta^r_l\right). \tag{4.21}$$

In addition, we have

$$\widetilde{R}\left(e^T_{\bar{m}}, e_l\right)e^T_{\bar{j}} - k\left\{\widetilde{g}\left(e_l, e^T_{\bar{j}}\right)e^T_{\bar{m}} - \widetilde{g}\left(e^T_{\bar{m}}, e^T_{\bar{j}}\right)e_l\right\} = 0. \tag{4.22}$$

Setting $a = \frac{1}{kr^2}$ and (4.21) in (4.13) and then using (4.22), we obtain

$$\begin{aligned}
&-\frac{1}{2r^2}\left[g^{jl}\left(g_{tm}\delta^r_i - g_{im}\delta^r_t + 2g_{it}\delta^r_m\right) + g_{it}\left(g^{jr}\delta^l_m - g^{lr}\delta^j_m\right)\right] \\
&-\frac{1}{4r^4}\Big[g_{ta}g^{jb}\left(g_{pm}g^{sr}\delta^l_i t^a_s t^p_b - g_{im}g^{sr}t^a_s t^l_b - g_{pm}g^{lr}t^a_i t^p_b + g_{im}g^{lr}t^a_p t^p_b\right) \\
&+g_{ta}g_{ib}\left(g^{sr}g^{jl}t^a_s t^b_m - g^{sr}g^{lp}\delta^j_m t^a_s t^b_p + g^{lr}g^{sp}\delta^j_m t^a_s t^b_p - g^{lr}g^{js}t^a_s t^b_m\right) \\
&+g^{la}g^{jb}\left(g_{sp}g_{im}\delta^r_t t^s_a t^p_b - g_{si}g_{pm}\delta^r_t t^s_a t^p_b + g_{ti}g_{pm}t^r_a t^p_b - g_{tp}g_{im}t^r_a t^p_b\right) \\
&+g_{ia}g^{lb}\left(\delta^j_m\delta^r_t t^p_b t^a_p - \delta^r_t t^j_b t^a_m - \delta^j_m t^r_b t^a_t + \delta^j_t t^r_b t^a_m\right)\Big] \\
&+\frac{1}{2r^4}\left[\left(g_{ta}g^{sr}\delta^l_m t^a_s - g_{ta}g^{lr}t^a_m - g_{sm}g^{lb}\delta^r_t t^s_b + g_{tm}g^{lb}t^r_b + 2\delta^r_m\bar{t}^l_t\right)\bar{t}^j_i\right] = 0.
\end{aligned}$$

From the above equation in the point $(x^i, t^j_i) = (x^i, \delta^j_i) \in T^1_1M$, we get

$$-\frac{1}{2r^2}\left[g^{jl}\left(g_{tm}\delta^r_i - g_{im}\delta^r_t + 2g_{it}\delta^r_m\right) + g_{it}\left(g^{jr}\delta^l_m - g^{lr}\delta^j_m\right)\right] + \frac{1}{r^4}\delta^r_m\delta^l_t\delta^j_i = 0,$$

which is a contradiction. Thus, we conclude that the manifold $(T^1_{1r}M, \widetilde{g})$ may never be a space form. □

For Sasaki metric $S_g$ we have $a = 1$. Then using Theorem 4.5, we have

**Corollary 4.6** *The* $(1, 1)$*-tensor sphere bundle* $T^1_{1r}M$*, endowed with the metric induced by the Sasaki metric* ${}^Sg$ *from* $T^1_1M$*, is never a space form.*

In this paper, we show that considering Cheeger–Gromoll type metric ${}^{CG}g$ on $T^1_1M$, we can construct a metrical framed $f(3, -1)$-structure on $T^1_1M$. In addition, by restricting this structure to the $(1, 1)$-tensor sphere bundle with constant radius $r$, $T^1_{1r}M$, we obtain a metrical almost paracontact structure on $T^1_{1r}M$. Moreover, we deduce that $(1, 1)$-tensor sphere bundles endowed with the induced metric are never space forms.

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