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Liouville-type theorems for a system governed by degenerate elliptic operators of fractional orders

This work is dedicated to the memory of Abbas Bahri

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Abstract We study the nonexistence of nontrivial solutions for the nonlinear elliptic system

$$\begin{cases} (-\Delta_x)^{\alpha/2}u + |x|^{2\delta}(-\Delta_y)^{\beta/2}u + |x|^{2\eta}|y|^{2\theta}(-\Delta_z)^{\gamma/2}u = v^p, \\ (-\Delta_x)^{\mu/2}v + |x|^{2\delta}(-\Delta_y)^{\nu/2}v + |x|^{2\eta}|y|^{2\theta}(-\Delta_z)^{\sigma/2}v = u^q, \end{cases}$$

where $(x, y, z) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3}$, $0 < \alpha, \beta, \gamma, \mu, \nu, \sigma \leq 2$, $\delta, \eta, \theta \geq 0$, and $p, q > 1$. Here, $(-\Delta_x)^{\alpha/2}$, $0 < \alpha < 2$, is the fractional Laplacian operator of order $\alpha/2$ with respect to the variable $x \in \mathbb{R}^{N_1}$, $(-\Delta_y)^{\beta/2}$, $0 < \beta < 2$, is the fractional Laplacian operator of order $\beta/2$ with respect to the variable $y \in \mathbb{R}^{N_2}$, and $(-\Delta_z)^{\gamma/2}$, $0 < \gamma < 2$, is the fractional Laplacian operator of order $\gamma/2$ with respect to the variable $z \in \mathbb{R}^{N_3}$. Using a weak formulation approach, sufficient conditions are provided in terms of space dimension and system parameters.

Mathematics Subject Classification 35B53 · 35J70 · 35R11

الملخص

ندرس عدم وجدود حلول غير تافهة للنظام الإهليلجي غير الخطي:

$$\begin{cases} (-\Delta_x)^{\alpha/2}u + |x|^{2\delta}(-\Delta_y)^{\beta/2}u + |x|^{2\eta}|y|^{2\theta}(-\Delta_z)^{\gamma/2}u = v^p, \\ (-\Delta_x)^{\mu/2}v + |x|^{2\delta}(-\Delta_y)^{\nu/2}v + |x|^{2\eta}|y|^{2\theta}(-\Delta_z)^{\sigma/2}v = u^q, \end{cases}$$

حيث $(x,y,z) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3}, 0 < \alpha,\beta,\gamma,\mu,\nu,\sigma \leq 2. \delta,\eta,\theta \geq 0$، وَ $p,q > 1$. هنا $(-\Delta_x)^{\alpha/2}, 0 < \alpha < 2$ مؤثر لابلاس الكسري من الرتبة $\alpha/2$ بالنسبة للمتغير $x \in \mathbb{R}^{N_1}$، وَ $(-\Delta_y)^{\beta/2}, 0 < \beta < 2$ مؤثر لابلاس الكسري من الرتبة $\beta/2$ بالنسبة للمتغير $y \in \mathbb{R}^{N_2}$ وحيث وَ $(-\Delta_z)^{\gamma/2}, 0 < \gamma < 2$ مؤثر لابلاس الكسري من الرتبة $\gamma/2$ بالنسبة للمتغير $z \in \mathbb{R}^{N_3}$. باستخدام مقاربة تركيب، أُعطيت شروط كافية بدلالة بعد الفضاء ومَعْلمات النظام. بحسب علمنا، لم يتم التطرق لهذه المسألة من قبل.

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1 Introduction

Any bounded complex function which is harmonic (or holomorphic) on the entire space is constant. This result is known as Liouville theorem [17], and was first proved by Cauchy in [2]. Gidas and Spruck [10] extended Liouville theorem to the case of non-negative solutions of semilinear elliptic equations in the whole space $\mathbb{R}^N$ or in half-spaces. In the case of $\mathbb{R}^N$, they proved that the unique non-negative solution of

$$-\Delta u = u^p, \quad \text{in } \mathbb{R}^N$$

is the trivial solution, provided that $1 \leq p < \frac{N+2}{N-2}$. Chen and Li [3] presented a simple proof based on the moving planes method for $0 < p < \frac{N+2}{N-2}$. Such result is optimal, i.e., if $p \geq \frac{N+2}{N-2}$, we have infinitely many positive solutions.

There are several works in the literature dealing with Liouville-type properties for different classes of degenerate elliptic equations and systems. Serrin and Zou [26] studied p-harmonic functions on the whole space and exterior domains. In [14], Liouville-type results in halfspaces for a class of evolution hypoelliptic equations are derived. In [15], a Liouville-type theorem was proved for X-elliptic operators. Dolcetta and Cutri [6] considered an elliptic inequality involving the Grushin operator. More precisely, they studied the problem

$$(-\Delta_x)u + |x|^{2\theta}(-\Delta_y)u \geq u^p, \quad \text{in } \mathbb{R}^{N_1} \times \mathbb{R}^{N_2},$$

where Δ_x is the Laplacian operator with respect to the variable $x \in \mathbb{R}^{N_1}$, and Δ_y is the Laplacian operator with respect to the variable $y \in \mathbb{R}^{N_2}$. They proved that if $1 < p < \frac{Q}{Q-2}$, then the only solution of the above inequality is the trivial solution. Here, Q is the homogeneous dimension of the space, given by $Q = N_1 + (\theta + 1)N_2$. In [1], Anh and My Considered an elliptic system of inequalities involving the Δ_λ Laplace operator. Some Liouville-type theorems are obtained for such system. For other related results, we refer to [5,22,23,28].

Recently, a lot of attention has been paid to the study of Liouville-type properties for elliptic equations and systems governed by fractional operators. Ma and Chen [18] considered the system of equations

$$\begin{cases} (-\Delta)^{\mu/2}u = v^q, \\ (-\Delta)^{\mu/2}v = u^p, \end{cases}$$

where $\mu \in (0, 2)$, $1 < p, q \leq \frac{N+\mu}{N-\mu}$, $N \geq 2$, and $(-\Delta)^{\mu/2}$ is the fractional Laplacian operator of order $\mu/2$. Using the moving plane method, they obtained a Liouville-type result for the above system. Dahmani et al. [4] extended the result in [18] to various classes of systems involving fractional Laplacian operators with different orders, using the test function method [20]. Some Liouville-type results were established recently by Quaas and Xia in [25] for a class of fractional elliptic equations and systems in the half space. For other related works, we refer to [7–9,11,13], and the references therein.

In this work, we establish Liouville-type results for the system

$$\begin{cases} (-\Delta_x)^{\alpha/2}u + |x|^{2\delta}(-\Delta_y)^{\beta/2}u + |x|^{2\eta}|y|^{2\theta}(-\Delta_z)^{\gamma/2}u = v^p, \\ (-\Delta_x)^{\mu/2}v + |x|^{2\delta}(-\Delta_y)^{\nu/2}v + |x|^{2\eta}|y|^{2\theta}(-\Delta_z)^{\sigma/2}v = u^q, \end{cases} \tag{1.1}$$

where $(x, y, z) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3}$, $0 < \alpha, \beta, \gamma, \mu, \nu, \sigma \leq 2$, $\delta, \eta, \theta \geq 0$, and $p, q > 1$. Here, $(-\Delta_x)^{\alpha/2}$, $0 < \alpha < 2$, is the fractional Laplacian operator of order $\alpha/2$ with respect to the variable $x \in \mathbb{R}^{N_1}$, $(-\Delta_y)^{\beta/2}$, $0 < \beta < 2$, is the fractional Laplacian operator of order $\beta/2$ with respect to the variable $y \in \mathbb{R}^{N_2}$, and $(-\Delta_z)^{\gamma/2}$, $0 < \gamma < 2$, is the fractional Laplacian operator of order $\gamma/2$ with respect to the variable $z \in \mathbb{R}^{N_3}$. We provide sufficient conditions for the nonexistence of nontrivial solutions to System (1.1) in terms of space dimension and system parameters.

Our approach is based on the test function method, which is based on the scaling invariance property of the operator. The corresponding literature is very extensive. We only quote the papers in which Mitidieri and Pohozaev explain how a suitable choice of the test function gives a nonexistence result. A deep description of this technique can be found in [20], see also [19,21,24]. Note that in our case, the moving plane approach used by Ma and Chen [18] cannot be applied. Indeed, in such approach an integral representation of the solution is required, which is not possible in our situation.

Recall that the nonlocal operator $(-\Delta)^s$, $0 < s < 1$, is defined for any function h in the Schwartz class through the Fourier transform

$$(-\Delta)^s h(x) = \mathcal{F}^{-1}\left(|\xi|^{2s}\mathcal{F}(h)(\xi)\right)(x),$$

where $\mathcal{F}$ stands for the Fourier transform and $\mathcal{F}^{-1}$ for its inverse. It can be also defined via the Riesz potential

$$(-\Delta)^s h(x) = c_{N,s}\,\mathrm{PV}\int_{\mathbb{R}^N}\frac{h(x)-h(\overline{x})}{|x-\overline{x}|^{N+2s}}\,\mathrm{d}\overline{x},$$

where $c_{N,s}$ is a normalisation constant and PV is the Cauchy principal value (see [16,27]).

The following inequality, known as Ju's inequality, will be useful for the proof of our main result (see [12]).

Lemma 1.1 *Suppose that $\delta \in (0,2]$, $\beta + 1 \geq 0$, and $\psi \in C_0^\infty(\mathbb{R}^N)$, $\psi \geq 0$. Then the following point-wise inequality holds:*

$$(-\Delta)^{\delta/2}\psi^{\beta+2}(x) \leq (\beta+2)\psi^{\beta+1}(x)(-\Delta)^{\delta/2}\psi(x).$$

2 Main results

In this section, we state and prove the main results in this paper.

We consider the system (1.1) under the assumptions

$$0 < \alpha, \beta, \gamma, \mu, \nu, \sigma \leq 2, \;\; \delta, \eta, \theta \geq 0, \;\; p > 1, \; q > 1.$$

Let

$$N = N_1 + N_2 + N_3 \quad \text{and} \quad Q = N_1 + (\delta+1)N_2 + (\eta + (\delta+1)\theta + 1)N_3.$$

The definition of solutions we adopt for (1.1) is the following.

Definition 2.1 We say that the pair (u, v) is a weak solution of (1.1) if, $u \geq 0$, $v \geq 0$, $(u, v) \in L^q_{\mathrm{loc}}(\mathbb{R}^{\mathbb{N}}) \times L^p_{\mathrm{loc}}(\mathbb{R}^{\mathbb{N}})$, and

$$\begin{aligned}\int_{\mathbb{R}^N} v^p\varphi\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_{\mathbb{R}^N} u(-\Delta_x)^{\alpha/2}\varphi\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \int_{\mathbb{R}^N}|x|^{2\delta}u(-\Delta_y)^{\beta/2}\varphi\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\ &\quad + \int_{\mathbb{R}^N}|x|^{2\eta}|y|^{2\theta}u(-\Delta_z)^{\gamma/2}\varphi\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,\end{aligned}$$

$$\begin{aligned}\int_{\mathbb{R}^N} u^q\varphi\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_{\mathbb{R}^N} v(-\Delta_x)^{\mu/2}\varphi\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \int_{\mathbb{R}^N}|x|^{2\delta}v(-\Delta_y)^{\nu/2}\varphi\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\ &\quad + \int_{\mathbb{R}^N}|x|^{2\eta}|y|^{2\theta}v(-\Delta_z)^{\sigma/2}\varphi\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,\end{aligned}$$

for every $\varphi \in C_0^\infty(\mathbb{R}^N)$, $\varphi \geq 0$.

Let us introduce the following parameters:

$$\begin{aligned}L_1 &= \min\{\alpha, -2\delta + \beta(\delta+1), -2\eta - \theta(\delta+1)(2-\gamma) + \gamma(\eta+1)\},\\ L_2 &= \min\{\mu, -2\delta + \nu(\delta+1), -2\eta - \theta(\delta+1)(2-\sigma) + \sigma(\eta+1)\},\\ Q_1 &= \frac{pq}{pq-1}\left(L_2 + \frac{L_1}{p}\right),\\ Q_2 &= \frac{pq}{pq-1}\left(L_1 + \frac{L_2}{q}\right).\end{aligned}$$

Our main result in this paper is the following Liouville-type theorem.

Theorem 2.2 *Let (u, v) be a weak solution of System (1.1). If*

$$Q < \max\{Q_1, Q_2\}, \tag{2.1}$$

then (u, v) is trivial, i.e., $(u, v) \equiv (0, 0)$.

Proof Suppose that (u, v) is a weak solution of (1.1) such that $(u, v) \not\equiv (0, 0)$. Let ω be a real number such that

$$\omega > \max\left\{\frac{q}{q-1}, \frac{p}{p-1}\right\}. \tag{2.2}$$

By the weak formulation of (1.1), for all $\varphi \in C_0^\infty(\mathbb{R}^N)$, $\varphi \geq 0$, we have

$$\begin{aligned}\int_{\mathbb{R}^N} v^p \varphi^\omega \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_{\mathbb{R}^N} u(-\Delta_x)^{\alpha/2}\varphi^\omega \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \int_{\mathbb{R}^N} |x|^{2\delta} u(-\Delta_y)^{\beta/2}\varphi^\omega \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\ &\quad + \int_{\mathbb{R}^N} |x|^{2\eta}|y|^{2\theta} u(-\Delta_z)^{\gamma/2}\varphi^\omega \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}\int_{\mathbb{R}^N} u^q \varphi^\omega \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_{\mathbb{R}^N} v(-\Delta_x)^{\mu/2}\varphi^\omega \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \int_{\mathbb{R}^N} |x|^{2\delta} v(-\Delta_y)^{\nu/2}\varphi^\omega \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\ &\quad + \int_{\mathbb{R}^N} |x|^{2\eta}|y|^{2\theta} v(-\Delta_z)^{\sigma/2}\varphi^\omega \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.\end{aligned} \tag{2.4}$$

Using Lemma 1.1 and Hölder's inequality with parameters q and $\frac{q}{q-1}$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} u(-\Delta_x)^{\alpha/2}\varphi^\omega \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &\leq \omega \int_{\mathbb{R}^N} u\varphi^{\omega-1}|(-\Delta_x)^{\alpha/2}\varphi| \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\ &= \omega \int_{\mathbb{R}^N} u\varphi^{\frac{\omega}{q}}\varphi^{\left(\omega-1-\frac{\omega}{q}\right)}|(-\Delta_x)^{\alpha/2}\varphi| \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\ &\leq \omega\left(\int_{\mathbb{R}^N} u^q\varphi^\omega \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{\frac{1}{q}}\left(\int_{\mathbb{R}^N} \varphi^{\left(\omega-1-\frac{\omega}{q}\right)\frac{q}{q-1}}|(-\Delta_x)^{\alpha/2}\varphi|^{\frac{q}{q-1}} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{\frac{q-1}{q}}\\ &= \omega\left(\int_{\mathbb{R}^N} u^q\varphi^\omega \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{\frac{1}{q}}\left(\int_{\mathbb{R}^N} \varphi^{\omega-\frac{q}{q-1}}|(-\Delta_x)^{\alpha/2}\varphi|^{\frac{q}{q-1}} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{\frac{q-1}{q}}.\end{aligned}$$

Thanks to the choice (2.2) of the parameter ω, we have

$$\int_{\mathbb{R}^N} \varphi^{\omega-\frac{q}{q-1}}|(-\Delta_x)^{\alpha/2}\varphi|^{\frac{q}{q-1}} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z < \infty.$$

Therefore, we have the estimate

$$\int_{\mathbb{R}^N} u(-\Delta_x)^{\alpha/2}\varphi^\omega \, dx\, dy\, dz \leq \omega\left(\int_{\mathbb{R}^N} u^q\varphi^\omega \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{\frac{1}{q}}\left(\int_{\mathbb{R}^N} \varphi^{\omega-\frac{q}{q-1}}|(-\Delta_x)^{\alpha/2}\varphi|^{\frac{q}{q-1}} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{\frac{q-1}{q}}. \tag{2.5}$$

Again, using Lemma 1.1 and Hölder's inequality with parameters q and $\frac{q}{q-1}$, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N} |x|^{2\delta} u(-\Delta_y)^{\beta/2}\varphi^\omega \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\ &\leq \omega \int_{\mathbb{R}^N} u|x|^{2\delta}\varphi^{\omega-1}|(-\Delta_y)^{\beta/2}\varphi| \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\ &= \omega \int_{\mathbb{R}^N} u\varphi^{\frac{\omega}{q}}|x|^{2\delta}\varphi^{\left(\omega-1-\frac{\omega}{q}\right)}|(-\Delta_y)^{\beta/2}\varphi| \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\ &\leq \omega\left(\int_{\mathbb{R}^N} u^q\varphi^\omega \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{\frac{1}{q}}\left(\int_{\mathbb{R}^N} |x|^{\frac{2\delta q}{q-1}}\varphi^{\left(\omega-1-\frac{\omega}{q}\right)\frac{q}{q-1}}|(-\Delta_y)^{\beta/2}\varphi|^{\frac{q}{q-1}} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{\frac{q-1}{q}}\\ &= \omega\left(\int_{\mathbb{R}^N} u^q\varphi^\omega \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{\frac{1}{q}}\left(\int_{\mathbb{R}^N} |x|^{\frac{2\delta q}{q-1}}\varphi^{\omega-\frac{q}{q-1}}|(-\Delta_y)^{\beta/2}\varphi|^{\frac{q}{q-1}} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{\frac{q-1}{q}}.\end{aligned}$$

Thanks to (2.2), we have

$$\int_{\mathbb{R}^N} |x|^{\frac{2\delta q}{q-1}} \varphi^{\omega-\frac{q}{q-1}} |(-\Delta_y)^{\beta/2}\varphi|^{\frac{q}{q-1}} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z < \infty.$$

Therefore, we obtain the estimate

$$\int_{\mathbb{R}^N} |x|^{2\delta} u(-\Delta_y)^{\beta/2}\varphi^\omega \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \le \omega \left(\int_{\mathbb{R}^N} u^q\varphi^\omega \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^N} |x|^{\frac{2\delta q}{q-1}} \varphi^{\omega-\frac{q}{q-1}} |(-\Delta_y)^{\beta/2}\varphi|^{\frac{q}{q-1}} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{\frac{q-1}{q}}. \tag{2.6}$$

Similarly, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} |x|^{2\eta}|y|^{2\theta} u(-\Delta_z)^{\gamma/2}\varphi^\omega \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&\le \omega \int_{\mathbb{R}^N} u|x|^{2\eta}|y|^{2\theta}\varphi^{\omega-1}|(-\Delta_z)^{\gamma/2}\varphi| \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&= \omega \int_{\mathbb{R}^N} u\varphi^{\frac{\omega}{q}}|x|^{2\eta}|y|^{2\theta}\varphi^{\left(\omega-1-\frac{\omega}{q}\right)}|(-\Delta_z)^{\gamma/2}\varphi| \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&\le \omega \left(\int_{\mathbb{R}^N} u^q\varphi^\omega \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^N} |x|^{\frac{2\eta q}{q-1}}|y|^{\frac{2\theta q}{q-1}}\varphi^{\left(\omega-1-\frac{\omega}{q}\right)\frac{q}{q-1}}|(-\Delta_z)^{\gamma/2}\varphi|^{\frac{q}{q-1}} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{\frac{q-1}{q}} \\
&= \omega \left(\int_{\mathbb{R}^N} u^q\varphi^\omega \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^N} |x|^{\frac{2\eta q}{q-1}}|y|^{\frac{2\theta q}{q-1}}\varphi^{\omega-\frac{q}{q-1}}|(-\Delta_z)^{\gamma/2}\varphi|^{\frac{q}{q-1}} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{\frac{q-1}{q}},
\end{aligned}$$

which yields the estimate

$$\begin{aligned}
&\int_{\mathbb{R}^N} |x|^{2\eta}|y|^{2\theta} u(-\Delta_z)^{\gamma/2}\varphi^\omega \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&\le \omega \left(\int_{\mathbb{R}^N} u^q\varphi^\omega \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^N} |x|^{\frac{2\eta q}{q-1}}|y|^{\frac{2\theta q}{q-1}}\varphi^{\omega-\frac{q}{q-1}}|(-\Delta_z)^{\gamma/2}\varphi|^{\frac{q}{q-1}} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{\frac{q-1}{q}}.
\end{aligned} \tag{2.7}$$

Now, combining (2.3) with the estimates (2.5), (2.6) and (2.7), we obtain

$$\int_{\mathbb{R}^N} v^p\varphi^\omega \, dx\,dy\,dz \le (A_1(\varphi)+B_1(\varphi)+C_1(\varphi)) \left(\int_{\mathbb{R}^N} u^q\varphi^\omega \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{\frac{1}{q}}, \tag{2.8}$$

where

$$\begin{aligned}
A_1(\varphi) &= \omega \left(\int_{\mathbb{R}^N} \varphi^{\omega-\frac{q}{q-1}}|(-\Delta_x)^{\alpha/2}\varphi|^{\frac{q}{q-1}} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{\frac{q-1}{q}}, \\
B_1(\varphi) &= \omega \left(\int_{\mathbb{R}^N} |x|^{\frac{2\delta q}{q-1}}\varphi^{\omega-\frac{q}{q-1}}|(-\Delta_y)^{\beta/2}\varphi|^{\frac{q}{q-1}} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{\frac{q-1}{q}}, \\
C_1(\varphi) &= \omega \left(\int_{\mathbb{R}^N} |x|^{\frac{2\eta q}{q-1}}|y|^{\frac{2\theta q}{q-1}}\varphi^{\omega-\frac{q}{q-1}}|(-\Delta_z)^{\gamma/2}\varphi|^{\frac{q}{q-1}} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{\frac{q-1}{q}}.
\end{aligned}$$

Similarly, using Hölder's inequality with parameters p and $\frac{p}{p-1}$, we obtain the estimates

$$\int_{\mathbb{R}^N} v(-\Delta_x)^{\mu/2}\varphi^\omega \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \le \omega \left(\int_{\mathbb{R}^N} v^p\varphi^\omega \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^N} \varphi^{\omega-\frac{p}{p-1}}|(-\Delta_x)^{\mu/2}\varphi|^{\frac{p}{p-1}} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{\frac{p-1}{p}}, \tag{2.9}$$

$$\int_{\mathbb{R}^N} |x|^{2\delta} v(-\Delta_y)^{\nu/2}\varphi^\omega \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \le \omega \left(\int_{\mathbb{R}^N} v^p\varphi^\omega \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^N} |x|^{\frac{2\delta p}{p-1}}\varphi^{\omega-\frac{p}{p-1}}|(-\Delta_y)^{\nu/2}\varphi|^{\frac{p}{p-1}} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{\frac{p-1}{p}} \tag{2.10}$$

and

$$\int_{\mathbb{R}^N} |x|^{2\eta}|y|^{2\theta} v(-\Delta_z)^{\sigma/2}\varphi^\omega \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$
$$\leq \omega\left(\int_{\mathbb{R}^N} v^p\varphi^\omega \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}^N} |x|^{\frac{2\eta p}{p-1}}|y|^{\frac{2\theta p}{p-1}}\varphi^{\omega-\frac{p}{p-1}}|(-\Delta_z)^{\sigma/2}\varphi|^{\frac{p}{p-1}} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{\frac{p-1}{p}}. \quad (2.11)$$

Combining (2.4) with the estimates (2.9), (2.10) and (2.11), we obtain

$$\int_{\mathbb{R}^N} u^q\varphi^\omega \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \leq (A_2(\varphi)+B_2(\varphi)+C_2(\varphi))\left(\int_{\mathbb{R}^N} v^p\varphi^\omega \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{\frac{1}{p}}, \quad (2.12)$$

where

$$A_2(\varphi) = \omega\left(\int_{\mathbb{R}^N} \varphi^{\omega-\frac{p}{p-1}}|(-\Delta_x)^{\mu/2}\varphi|^{\frac{p}{p-1}} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{\frac{p-1}{p}},$$
$$B_2(\varphi) = \omega\left(\int_{\mathbb{R}^N} |x|^{\frac{2\delta p}{p-1}}\varphi^{\omega-\frac{p}{p-1}}|(-\Delta_y)^{\nu/2}\varphi|^{\frac{p}{p-1}} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{\frac{p-1}{p}},$$
$$C_2(\varphi) = \omega\left(\int_{\mathbb{R}^N} |x|^{\frac{2\eta p}{p-1}}|y|^{\frac{2\theta p}{p-1}}\varphi^{\omega-\frac{p}{p-1}}|(-\Delta_z)^{\sigma/2}\varphi|^{\frac{p}{p-1}} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{\frac{p-1}{p}}.$$

Using (2.8) and (2.12), we obtain

$$\left(\int_{\mathbb{R}^N} u^q\varphi^\omega \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{1-\frac{1}{pq}} \leq (A_2(\varphi)+B_2(\varphi)+C_2(\varphi))\,(A_1(\varphi)+B_1(\varphi)+C_1(\varphi))^{\frac{1}{p}} \quad (2.13)$$

and

$$\left(\int_{\mathbb{R}^N} v^p\varphi^\omega \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{1-\frac{1}{pq}} \leq (A_1(\varphi)+B_1(\varphi)+C_1(\varphi))\,(A_2(\varphi)+B_2(\varphi)+C_2(\varphi))^{\frac{1}{q}}. \quad (2.14)$$

Now, as a test function, we take

$$\varphi(x,y,z) = \varphi_0\left(\frac{|x|^2}{R^2}+\frac{|y|^2}{R^{2(\delta+1)}}+\frac{|z|^2}{R^{2(\eta+(\delta+1)\theta+1)}}\right), \quad (x,y,z)\in\mathbb{R}^{N_1}\times\mathbb{R}^{N_2}\times\mathbb{R}^{N_3},$$

where φ_0 is the classical cutoff function, that is, $\varphi_0 \in C_0^\infty(0,\infty)$ is a smooth decreasing function such that

$$0\leq\varphi_0\leq 1, \quad |\varphi_0'(\xi)|\leq C\xi^{-1}$$

and

$$\varphi_0(\xi) = \begin{cases} 1 & \text{if } 0<\xi\leq 1,\\ 0 & \text{if } \xi\geq 2.\end{cases}$$

We use the change of variables

$$x = R\rho, \quad y = R^{\delta+1}\tau, \quad z = R^{\eta+(\delta+1)\theta+1}\vartheta.$$

In this case, we have

$$\kappa := \frac{|x|^2}{R^2}+\frac{|y|^2}{R^{2(\delta+1)}}+\frac{|z|^2}{R^{2(\eta+(\delta+1)\theta+1)}} = |\rho|^2+|\tau|^2+|\vartheta|^2, \quad (\rho,\tau,\vartheta)\in\mathbb{R}^{N_1}\times\mathbb{R}^{N_2}\times\mathbb{R}^{N_3}.$$

Let Ω be the subset of $\mathbb{R}^{N_1}\times\mathbb{R}^{N_2}\times\mathbb{R}^{N_3}$ defined by

$$\Omega = \{(\rho,\tau,\vartheta)\in\mathbb{R}^{N_1}\times\mathbb{R}^{N_2}\times\mathbb{R}^{N_3} : 1\leq|\rho|^2+|\tau|^2+|\vartheta|^2\leq 2\}.$$

We have the following estimates.

• Estimates of $A_i(\varphi)$, $i = 1, 2$.
Using the above change of variables, we obtain

$$A_1(\varphi) = \omega R^{-\alpha+\frac{Q(q-1)}{q}} \left(\int_\Omega [\varphi_0(\kappa)]^{\omega-\frac{q}{q-1}} |(-\Delta_\rho)^{\alpha/2}\varphi_0(\kappa)|^{\frac{q}{q-1}} \,\mathrm{d}\rho\,\mathrm{d}\tau\,\mathrm{d}\vartheta \right)^{\frac{q-1}{q}}.$$

Therefore, we have

$$A_1(\varphi) = C R^{-\alpha+\frac{Q(q-1)}{q}}. \tag{2.15}$$

Similarly, we obtain

$$A_2(\varphi) = C R^{-\mu+\frac{Q(p-1)}{p}}. \tag{2.16}$$

• Estimates of $B_i(\varphi)$, $i = 1, 2$.
Under the same change of variables, we obtain

$$B_1(\varphi) = \omega R^{2\delta-\beta(\delta+1)+\frac{Q(q-1)}{q}} \left(\int_\Omega |\rho|^{\frac{2\delta q}{q-1}} [\varphi_0(\kappa)]^{\omega-\frac{q}{q-1}} |(-\Delta_\tau)^{\beta/2}\varphi_0(\kappa)|^{\frac{q}{q-1}} \,\mathrm{d}\rho\,\mathrm{d}\tau\,\mathrm{d}\vartheta \right)^{\frac{q-1}{q}}.$$

Therefore, we have

$$B_1(\varphi) = C R^{2\delta-\beta(\delta+1)+\frac{Q(q-1)}{q}}. \tag{2.17}$$

Similarly, we obtain

$$B_2(\varphi) = C R^{2\delta-\nu(\delta+1)+\frac{Q(p-1)}{p}}. \tag{2.18}$$

• Estimates of $C_i(\varphi)$, $i = 1, 2$.
A simple computation yields

$$C_1(\varphi) = \omega R^{2\eta+\theta(\delta+1)(2-\gamma)-\gamma(\eta+1)+\frac{Q(q-1)}{q}} \left(\int_\Omega |\rho|^{\frac{2\eta q}{q-1}} |\tau|^{\frac{2\theta q}{q-1}} [\varphi_0(\kappa)]^{\omega-\frac{q}{q-1}} |(-\Delta_\vartheta)^{\gamma/2}\varphi_0(\kappa)|^{\frac{q}{q-1}} \,\mathrm{d}\rho\,\mathrm{d}\tau\,\mathrm{d}\vartheta \right)^{\frac{q-1}{q}}.$$

Then

$$C_1(\varphi) = C R^{2\eta+\theta(\delta+1)(2-\gamma)-\gamma(\eta+1)+\frac{Q(q-1)}{q}}. \tag{2.19}$$

Similarly, we have

$$C_2(\varphi) = C R^{2\eta+\theta(\delta+1)(2-\sigma)-\sigma(\eta+1)+\frac{Q(p-1)}{p}}. \tag{2.20}$$

• Estimate of $A_1(\varphi) + B_1(\varphi) + C_1(\varphi)$.
Using the estimates (2.15), (2.17) and (2.19), for R large enough, we obtain

$$\begin{aligned} A_1(\varphi) + B_1(\varphi) + C_1(\varphi) &= C \left(R^{-\alpha+\frac{Q(q-1)}{q}} + R^{2\delta-\beta(\delta+1)+\frac{Q(q-1)}{q}} + R^{2\eta+\theta(\delta+1)(2-\gamma)-\gamma(\eta+1)+\frac{Q(q-1)}{q}} \right) \\ &= C R^{\frac{Q(q-1)}{q}} \left(R^{-\alpha} + R^{2\delta-\beta(\delta+1)} + R^{2\eta+\theta(\delta+1)(2-\gamma)-\gamma(\eta+1)} \right) \\ &\le C R^{\frac{Q(q-1)}{q}} R^{\max\{-\alpha, 2\delta-\beta(\delta+1), 2\eta+\theta(\delta+1)(2-\gamma)-\gamma(\eta+1)\}} \\ &= C R^{\frac{Q(q-1)}{q}-L_1}, \end{aligned}$$

i.e.,

$$A_1(\varphi) + B_1(\varphi) + C_1(\varphi) \le C R^{\frac{Q(q-1)}{q}-L_1}. \tag{2.21}$$

• Estimate of $A_2(\varphi) + B_2(\varphi) + C_2(\varphi)$.
Similarly, using the estimates (2.16), (2.18) and (2.20), for R large enough, we obtain

$$A_2(\varphi) + B_2(\varphi) + C_2(\varphi) \le C R^{\frac{Q(p-1)}{p}-L_2}. \tag{2.22}$$

The estimates (2.13), (2.21) and (2.22) yield

$$\left(\int_{\mathbb{R}^N} u^q \varphi_0^\omega\left(\frac{|x|^2}{R^2}+\frac{|y|^2}{R^{2(\delta+1)}}+\frac{|z|^2}{R^{2(\eta+(\delta+1)\theta+1)}}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{1-\frac{1}{pq}} \leq C R^{Q\left(\frac{pq-1}{pq}\right)-L_2-\frac{L_1}{p}}. \tag{2.23}$$

Similarly, the estimates (2.14), (2.21) and (2.22) yield

$$\left(\int_{\mathbb{R}^N} v^p \varphi_0^\omega\left(\frac{|x|^2}{R^2}+\frac{|y|^2}{R^{2(\delta+1)}}+\frac{|z|^2}{R^{2(\eta+(\delta+1)\theta+1)}}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{1-\frac{1}{pq}} \leq C R^{Q\left(\frac{pq-1}{pq}\right)-L_1-\frac{L_2}{q}}. \tag{2.24}$$

Observe that condition (2.1) is equivalent to

$$Q\left(\frac{pq-1}{pq}\right)-L_2-\frac{L_1}{p}<0$$

or

$$Q\left(\frac{pq-1}{pq}\right)-L_1-\frac{L_2}{q}<0.$$

Therefore, we have two cases.

• Case 1. If

$$Q\left(\frac{pq-1}{pq}\right)-L_2-\frac{L_1}{p}<0.$$

In this case, passing to the limit as $R\to\infty$ in (2.23), using the monotone convergence theorem, and (2.8), we obtain

$$\int_{\mathbb{R}^N} u^q\, dx\, dy\, dz=\int_{\mathbb{R}^N} v^p\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=0,$$

which is a contradiction with the fact that (u,v) is a nontrivial solution.

• Case 2. If

$$Q\left(\frac{pq-1}{pq}\right)-L_1-\frac{L_2}{q}<0.$$

As in the previous case, passing to the limit as $R\to\infty$ in (2.24), using the monotone convergence theorem, and (2.12), we obtain

$$\int_{\mathbb{R}^N} u^q\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=\int_{\mathbb{R}^N} v^p\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=0,$$

which is a contradiction.

Therefore, in both cases, we get a contradiction. As a consequence, we infer that the only weak solution of System (1.1) is the trivial solution, provided that (2.1) is satisfied. □

Different Liouville-type results can be deduced from Theorem 2.2 for equations and systems.

Taking $\alpha=\mu$, $\beta=\nu=2$, $\gamma=\sigma=2$, in Theorem 2.2, the following result follows.

Corollary 2.3 *Let (u,v) be a weak solution of the system*

$$\begin{cases}(-\Delta_x)^{\alpha/2}u+|x|^{2\delta}(-\Delta_y)u+|x|^{2\eta}|y|^{2\theta}(-\Delta_z)u=v^p,\\[2mm] (-\Delta_x)^{\alpha/2}v+|x|^{2\delta}(-\Delta_y)v+|x|^{2\eta}|y|^{2\theta}(-\Delta_z)v=u^q,\end{cases}$$

where $0<\alpha\leq 2$, $\delta,\eta,\theta\geq 0$, $p>1$, and $q>1$. If

$$Q<\frac{\alpha\,(pq+\max\{p,q\})}{pq-1},$$

then (u,v) is trivial.

Taking $\alpha = \mu = 2$, $\beta = \nu$, $\gamma = \sigma = 2$, in Theorem 2.2, the following result follows.

Corollary 2.4 *Let (u, v) be a weak solution of the system*

$$\begin{cases} (-\Delta_x)u + |x|^{2\delta}(-\Delta_y)^{\beta/2}u + |x|^{2\eta}|y|^{2\theta}(-\Delta_z)u = v^p, \\ (-\Delta_x)v + |x|^{2\delta}(-\Delta_y)^{\beta/2}v + |x|^{2\eta}|y|^{2\theta}(-\Delta_z)v = u^q, \end{cases}$$

where $0 < \beta \leq 2$, $\delta, \eta, \theta \geq 0$, $p > 1$, and $q > 1$. If

$$Q < \frac{(\delta(\beta - 2) + \beta)\,(pq + \max\{p, q\})}{pq - 1},$$

then (u, v) is trivial.

Taking $\alpha = \mu = 2$, $\beta = \nu = 2$, $\gamma = \sigma$, in Theorem 2.2, the following result follows.

Corollary 2.5 *Let (u, v) be a weak solution of the system*

$$\begin{cases} (-\Delta_x)u + |x|^{2\delta}(-\Delta_y)u + |x|^{2\eta}|y|^{2\theta}(-\Delta_z)^{\gamma/2}u = v^p, \\ (-\Delta_x)v + |x|^{2\delta}(-\Delta_y)v + |x|^{2\eta}|y|^{2\theta}(-\Delta_z)^{\gamma/2}v = u^q, \end{cases}$$

where $0 < \gamma \leq 2$, $\delta, \eta, \theta \geq 0$, $p > 1$, and $q > 1$. If

$$Q < \frac{(pq + \max\{p, q\})\,\big((\gamma - 2)(\eta + \theta(\delta + 1)) + \gamma\big)}{pq - 1},$$

then (u, v) is trivial.

Taking $\alpha = \beta = \gamma = \mu = \nu = \sigma = 2$ in Theorem 2.2, or $\alpha = 2$ in Corollary 2.3, or $\beta = 2$ in Corollary 2.4, or $\gamma = 2$ in Corollary 2.5, the following result follows.

Corollary 2.6 *Let (u, v) be a weak solution of the system*

$$\begin{cases} (-\Delta_x)u + |x|^{2\delta}(-\Delta_y)u + |x|^{2\eta}|y|^{2\theta}(-\Delta_z)u = v^p, \\ (-\Delta_x)v + |x|^{2\delta}(-\Delta_y)v + |x|^{2\eta}|y|^{2\theta}(-\Delta_z)v = u^q, \end{cases}$$

where $\delta, \eta, \theta \geq 0$, $p > 1$, and $q > 1$. If

$$Q < \frac{2\,(pq + \max\{p, q\})}{pq - 1},$$

then (u, v) is trivial.

Taking $\alpha = \mu$, $\beta = \nu$, $\gamma = \sigma$, $p = q$, and $u = v$ in Theorem 2.2, the following Liouville-type result follows.

Corollary 2.7 *Let u be a weak solution of*

$$(-\Delta_x)^{\alpha/2}u + |x|^{2\delta}(-\Delta_y)^{\beta/2}u + |x|^{2\eta}|y|^{2\theta}(-\Delta_z)^{\gamma/2}u = u^p,$$

where $0 < \alpha, \beta, \gamma \leq 2$, $\delta, \eta, \theta \geq 0$, and $p > 1$. If

$$Q < \frac{L_1 p}{p - 1},$$

then u is trivial.

Taking $\beta = \gamma = 2$ in Corollary 2.7, the following Liouville-type result follows.

Corollary 2.8 *Let u be a weak solution of*

$$(-\Delta_x)^{\alpha/2}u + |x|^{2\delta}(-\Delta_y)u + |x|^{2\eta}|y|^{2\theta}(-\Delta_z)u = u^p,$$

where $0 < \alpha \leq 2$ *and* $\delta, \eta, \theta \geq 0$. *If*

$$1 < p < \frac{Q}{Q-\alpha},$$

then u is trivial.

Taking $\alpha = \gamma = 2$ in Corollary 2.7, the following Liouville-type result follows.

Corollary 2.9 *Let u be a weak solution of*

$$(-\Delta_x)u + |x|^{2\delta}(-\Delta_y)^{\beta/2}u + |x|^{2\eta}|y|^{2\theta}(-\Delta_z)u = u^p,$$

where $0 < \beta \leq 2$, $\delta, \eta, \theta \geq 0$, *and* $p > 1$. *If*

$$Q < \frac{(\delta(\beta-2)+\beta)p}{p-1},$$

then u is trivial.

Taking $\alpha = \beta = 2$ in Corollary 2.7, the following Liouville-type result follows.

Corollary 2.10 *Let u be a weak solution of*

$$(-\Delta_x)u + |x|^{2\delta}(-\Delta_y)u + |x|^{2\eta}|y|^{2\theta}(-\Delta_z)^{\gamma/2}u = u^p,$$

where $0 < \gamma \leq 2$, $\delta, \eta, \theta \geq 0$, *and* $p > 1$. *If*

$$Q < \frac{p\big((\gamma-2)(\eta+\theta(\delta+1))+\gamma\big)}{p-1},$$

then u is trivial.

Taking $\gamma = 2$ in Corollary 2.10, we obtain the following result.

Corollary 2.11 *Let u be a weak solution of*

$$(-\Delta_x)u + |x|^{2\delta}(-\Delta_y)u + |x|^{2\eta}|y|^{2\theta}(-\Delta_z)u = u^p,$$

where $\delta, \eta, \theta \geq 0$, *If*

$$1 < p < \frac{Q}{Q-2},$$

then u is trivial.

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