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(k, s)-Riemann–Liouville fractional integral inequalities for continuous random variables

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Abstract In this paper, we introduce some new concepts to the field of probability theory: (k, s)-Riemann–Liouville fractional expectation and variance functions. Some generalized integral inequalities are established for (k, s)-Riemann–Liouville expectation and variance functions.

Mathematics Subject Classification 26D15 · 26A33 · 60E15

الملخص

في هذه الورقة نقدم بعض المفاهيم الجديدة في نظرية الاحتمالات: دالتي توقع ريمان – ليوفيل – (k, s) الكسري والتباين. نثبت بعض متباينات التكاملية المعممة لدالتي توقع ريمان – ليوفيل – (k, s) الكسري والتباين.

1 Introduction

The integral inequalities play a very important role in the theory of differential equations and applied sciences. Many researchers have carried out important studies on the theory of integral inequalities. Also, integral inequalities have been used in probability theory for a long time and continue to attract the attention of researchers. In [5], Barnett et al. have obtained some recent inequalities for cumulative distribution functions, expectation, variance, and applications. In [14], Kumar has obtained some results based on the Korkine's identity and integral inequalities of Hölder and Grüss for moments of a continuous random variable whose probability distribution is a convex function on the interval of real numbers. In addition, applications of these results are considered by deriving the inequalities involving higher moments and also by special means and also in evaluating moments of a beta random variable. In [15], Ostrowski type integral inequalities involving moments of a continuous random variable defined on a finite interval, is established. Also, the author has derived bounds for moments from these inequalities. Moreover, in [11], several inequalities for differentiable convex, wright-convex and quasi-convex mappings have been obtained, respectively, that are connected with the celebrated Hermite–Hadamard integral inequality. In the same paper, some error estimates for weighted Trapezoid formula and higher moments of random variables have been given. More details and information can

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be seen in the papers [1–4,6,9,12,18]. Very recently, new concepts on fractional random variables have been introduced by Dahmani [7]. In this study, Dahmani establishes several integral inequalities for the fractional expectation and the fractional variance functions of a continuous random variable.

First, the Riemann–Liouville fractional integral of order $\alpha \geq 0$ for a continuous function f on $[a, b]$ is defined by

$$J_a^\alpha [f(t)] = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) \mathrm{d}\tau; \quad \alpha \geq 0, \quad a \leq t \leq b.$$

This integral is motivated by the well-known Cauchy formula:

$$\int_a^x \mathrm{d}t_1 \int_a^{t_1} \mathrm{d}t_2 \ldots \int_a^{t_{n-1}} f(t_n) \mathrm{d}t_n = \frac{1}{\Gamma(n)} \int_a^x (x-t)^{n-1} f(t) \mathrm{d}t; \quad n \in \mathbb{N}^*.$$

The second is the Hadamard fractional integral introduced by Hadamard [10]. It is given by:

$$J_a^\alpha [f(t)] = \frac{1}{\Gamma(n)} \int_a^x \left(\log \frac{x}{t}\right)^{\alpha-1} f(t) \frac{\mathrm{d}t}{t}, \quad \alpha > 0, \quad x > a.$$

The Hadamard integral is based on the generalization of the integral

$$\int_a^x \frac{\mathrm{d}t_1}{t_1} \int_a^{t_1} \frac{\mathrm{d}t_2}{t_2} \ldots \int_a^{t_{n-1}} \frac{f(t_n)}{t_n} \mathrm{d}t_n = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\log \frac{x}{t}\right)^{n-1} f(t) \frac{\mathrm{d}t}{t}$$

for $n \in \mathbb{N}^*$.

In [13], Katugampola gave a new fractional integration which generalizes both the Riemann–Liouville and Hadamard fractional integrals into a single form. This generalization is based on the observation that, for $n \in \mathbb{N}^*$

$$\int_a^x t_1^s \mathrm{d}t_1 \int_a^{t_1} t_2^s \mathrm{d}t_2 \ldots \int_a^{t_{n-1}} t_n^s f(t_n) \mathrm{d}t_n = \frac{(s+1)^{1-n}}{\Gamma(\alpha)} \int_a^x (x^{s+1} - t^{s+1})^{n-1} t^s f(t) \mathrm{d}t$$

which gives the following fractional version

$$^s J_a^\alpha [f(t)] := \frac{(s+1)^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \left(t^{s+1} - \tau^{s+1}\right)^{\alpha-1} \tau^s f(\tau) \mathrm{d}\tau, \tag{1}$$

where α and $s \neq -1$ are real numbers.

In [8], Diaz and Pariguan have defined new functions called k-gamma and k-beta functions and the Pochhammer k-symbol that is, respectively, generalization of the classical gamma and beta functions and the classical Pochhammer symbol:

$$\Gamma_k(x) = \lim_{n \longrightarrow \infty} \frac{n! k^n (nk)^{\frac{x}{k}-1}}{(x)_{n,k}}, \quad (k > 0),$$

where $(x)_{n,k}$ is the Pochhammer k-symbol for factorial function. It has been shown that the Mellin transform of the exponential function $e^{-\frac{t^k}{k}}$ is the k-gamma function, explicitly given by

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}} \mathrm{d}t, \quad x > 0.$$

Clearly, $\Gamma(x) = \lim_{k \to 1} \Gamma_k(x)$ and $\Gamma_k(x+k) = x\Gamma_k(x)$.

Later, under the above definitions, in [17], Mubeen and Habibullah have introduced k-fractional integral of Riemann–Liouville as follows:

$$_k J_a^\alpha [f(t)] = \frac{1}{\Gamma_k(x)} \int_a^t (t-\tau)^{\frac{\alpha}{k}-1} f(\tau) \mathrm{d}\tau. \tag{2}$$

Note that when $k = 1$ in the above integral, then it reduces to the classical Riemann–Liouville fractional integral.

2 Preliminaries and definitions

In a recent paper [20], Sarikaya et al. have introduced a new fractional integration which generalizes both the k- Riemann–Liouville and Katugampola's fractional integrals into a single form as the following.

Definition 2.1 Let f be a continuous function on a the finite real interval $[a, b]$. Then the (k, s)-Riemann–Liouville fractional integral of f of order $\alpha > 0$ is defined by:

$$ {}_k^sJ_a^\alpha \left[f(t)\right] := \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^t \left(t^{s+1} - \tau^{s+1}\right)^{\frac{\alpha}{k}-1} \tau^s f(\tau)\, \mathrm{d}\tau, \quad \tau \in [a, t], \tag{3} $$

where $k > 0$, $s \in \mathbb{R}\backslash\{-1\}$.

Also, in [20], the following results can be seen.

Theorem 2.2 *Let f be continuous on $[a; b]$, $k > 0$ and $s \in \mathbb{R}\backslash\{-1\}$. Then,*

$$ {}_k^sJ_a^\alpha \left[{}_k^sJ_a^\beta f(t)\right] = {}_k^sJ_a^{\alpha+\beta} f(t) = {}_k^sJ_a^\beta \left[{}_k^sJ_a^\alpha f(t)\right] \tag{4} $$

for all $\alpha, \beta > 0$, $t \in [a, b]$.

Theorem 2.3 *Let f be continuous on $[a, b]$, $k > 0$ and $s \in \mathbb{R}\backslash\{-1\}$. Then,*

$$ {}_k^sJ_a^\alpha \left[\left(x^{s+1} - a^{s+1}\right)^{\frac{\beta}{k}-1}\right] = \frac{\Gamma_k(\beta)}{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+\beta)} \left(x^{s+1} - a^{s+1}\right)^{\frac{\alpha+\beta}{k}-1}, $$

where Γ_k denotes the k-gamma function.

Next two theorems, Grüss type inequalities for (k, s)-Riemann–Liouville fractional integrals have been obtained by Set et al. in [21].

Theorem 2.4 *Let f and g be two integrable functions on $[0, \infty)$ with $\varphi < f(t) < \Phi$, $\psi < g < \Psi$ and let p be a positive function on $[0, \infty)$. Then for all $t > 0$, $k > 0$, $\alpha > 0$, $s \in \mathbb{R}\backslash\{-1\}$, we have:*

$$ \left|{}_k^sJ_a^\alpha\left[p(t)\right]{}_k^sJ_a^\alpha\left[pfg(t)\right] - {}_k^sJ_a^\alpha\left[pf(t)\right]{}_k^sJ_a^\alpha\left[pg(t)\right]\right| \le \left(\frac{{}_k^sJ_a^\alpha\left[p(t)\right]}{2}\right)^2 (\Phi-\varphi)(\Psi-\psi). \tag{5} $$

Theorem 2.5 *Let f and g be two integrable function on $[0, \infty)$ with $\varphi < f(t) < \Phi$, $\psi < g < \Psi$ and let p be a positive function on $[0, \infty)$. Then for all $t > 0$, $k > 0$, $s \in \mathbb{R}\backslash\{-1\}$, $\alpha > 0$, $\beta > 0$, we have:*

$$ \begin{aligned} &\Big\{{}_k^sJ_a^\alpha\left[p(t)\right]{}_k^sJ_a^\beta\left[pfg(t)\right] + {}_k^sJ_a^\beta\left[p(t)\right]{}_k^sJ_a^\alpha\left[pfg(t)\right] \\ &\quad - {}_k^sJ_a^\alpha\left[pf(t)\right]{}_k^sJ_a^\beta\left[pg(t)\right] - {}_k^sJ_a^\beta\left[pf(t)\right]{}_k^sJ_a^\alpha\left[pg(t)\right]\Big\}^2 \\ &\le \Big\{\left(\Phi\,{}_k^sJ_a^\alpha\left[p(t)\right] - {}_k^sJ_a^\alpha\left[pf(t)\right]\right)\left({}_k^sJ_a^\beta\left[pf(t)\right] - \varphi\,{}_k^sJ_a^\beta\left[p(t)\right]\right) \\ &\quad + \left({}_k^sJ_a^\alpha\left[pf(t)\right] - \varphi\,{}_k^sJ_a^\alpha\left[p(t)\right]\right)\left(\Phi\,{}_k^sJ_a^\beta\left[p(t)\right] - {}_k^sJ_a^\beta\left[pf(t)\right]\right)\Big\} \\ &\quad \times \Big\{\left(\Psi\,{}_k^sJ_a^\alpha\left[p(t)\right] - {}_k^sJ_a^\alpha\left[pg(t)\right]\right)\left({}_k^sJ_a^\beta\left[pg(t)\right] - \psi\,{}_k^sJ_a^\beta\left[p(t)\right]\right) \\ &\quad + \left({}_k^sJ_a^\alpha\left[pg(t)\right] - \psi\,{}_k^sJ_a^\alpha\left[p(t)\right]\right)\left(\Psi\,{}_k^sJ_a^\beta\left[p(t)\right] - {}_k^sJ_a^\beta\left[pg(t)\right]\right)\Big\}. \end{aligned} \tag{6} $$

In the following section, we introduce some new concepts for the Riemann–Liouville (k, s)-fractional integral. Also, we give new integral inequalities for the (k, s)-fractional expectation and variance functions of a continuous random variable X having the probability density function f.

3 (k, s)-Riemann–Liouville fractional integral inequalities

Now, we give the following new definitions for (k, s)-fractional integral operators:

Definition 3.1 The (k, s)-fractional expectation function of order $\alpha \geq 0$, for a random variable X with a positive probability density function f defined on $[a, b]$ is defined as

$$\begin{aligned} {}_k^s E_{X,\alpha}(t) &:= {}_k^s J_a^\alpha \left[t f(t)\right] \\ &= \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^t \left(t^{s+1} - \tau^{s+1}\right)^{\frac{\alpha}{k}-1} \tau^{s+1} f(\tau)\, \mathrm{d}\tau; \end{aligned}$$

$\alpha > 0,\ k > 0,\ s \in \mathbb{R}\backslash\{-1\}$ and $a < t \leq b$.

Likewise, we will define the (k, s)-fractional expectation function of $X - E(X)$:

Definition 3.2 The (k, s)-fractional expectation function of order $\alpha > 0$ for a random variable $X - E(X)$ with a positive probability density function f defined on $[a, b]$ is defined as

$$\begin{aligned} {}_k^s E_{X-E(X),\alpha}(t) &:= {}_k^s J_a^\alpha \left[(t - E(X))\, f(t)\right] \\ &= \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^t \left(t^{s+1} - \tau^{s+1}\right)^{\frac{\alpha}{k}-1} \tau^{s}(\tau - E(X)) f(\tau)\mathrm{d}\tau; \end{aligned}$$

$\alpha > 0,\ k > 0,\ s \in \mathbb{R}\backslash\{-1\}$ and $a < t \leq b$.

Definition 3.3 The (k, s)-fractional expectation of order $\alpha > 0$ for a random variable X with a positive probability density function f defined on $[a, b]$ is defined as

$${}_k^s E_{X,\alpha} = \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^b \left(t^{s+1} - \tau^{s+1}\right)^{\frac{\alpha}{k}-1} \tau^{s+1} f(\tau)\mathrm{d}\tau;$$

$\alpha > 0$ and $k > 0,\ s \in \mathbb{R}\backslash\{-1\}$.

With a similar logic, we introduce (k, s)-fractional variance function and variance as follows:

Definition 3.4 The (k, s)-fractional variance function of order $\alpha \geq 0$ for a random variable X with a positive probability density function f defined on $[a, b]$ is defined as

$$\begin{aligned} {}_k^s \sigma^2_{X,\alpha}(t) &:= {}_k^s J_a^\alpha \left[(t - E(X))^2 f(t)\right] \\ &= \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^t \left(t^{s+1} - \tau^{s+1}\right)^{\frac{\alpha}{k}-1} \tau^{s} (\tau - E(X))^2 f(\tau)\mathrm{d}\tau; \end{aligned}$$

$\alpha > 0$ and $k > 0,\ s \in \mathbb{R}\backslash\{-1\}$.

Definition 3.5 The (k, s) fractional variance of order $\alpha \geq 0$, for a random variable X with a positive probability density function f defined on $[a, b]$ is defined as

$${}_k^s \sigma^2_{X,\alpha} = \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^b \left(t^{s+1} - \tau^{s+1}\right)^{\frac{\alpha}{k}-1} \tau^{s} (\tau - E(X))^2 f(\tau)\mathrm{d}\tau;$$

$\alpha > 0$ and $k > 0,\ s \in \mathbb{R}\backslash\{-1\}$.

Let us give the following important properties:

(i) If we take $s = 0$ and $k = 1$ in Definitions 3.1 and 3.4, we obtain the functions of fractional expectation and variance in [7], respectively.
(ii) If we take $s = 0$ and $k = 1$ in Definitions 3.3 and 3.5, we obtain Definitions 2.4 and 2.6 in [7], respectively.
3. If we take $s = 0$ and $\alpha = k = 1$ in Definition 3.3, we obtain the classical expectation: ${}_k^s E_{X,\alpha} = E(X)$.
4. If we take $s = 0$ and $\alpha = k = 1$ in Definition 3.5, we obtain the classical variance: ${}_k^s \sigma^2_{X,\alpha} = \sigma^2(X) = \int_a^b (\tau - E(X))^2 f(\tau)\, \mathrm{d}\tau$.
5. For $s = 0$ and $\alpha = k = 1$, the probability density function f satisfies ${}_k^s J_a^\alpha \left[f(t)\right]\big|_{t=b} = 1$.

In this paper, we give new integral inequalities for the (k, s)-fractional expectation and variance functions of a continuous random variable X having the probability density function $(p.d.f.)$ f.

The first main result is the following theorem:

Theorem 3.6 *Let X be a continuous random variable having a probability density function f defined on $[a, b]$. Then, we have*

(i) *for all* $a < t \le b$, $s \in \mathbb{R}\backslash\{-1\}$, $k > 0, \alpha \ge 0$,

$$ {}_k^sJ_a^\alpha\left[f(t)\right]\,{}_k^s\sigma^2_{X,\alpha}(t) - \left[{}_k^sE_{X-E(X),\alpha}(t)\right]^2 \le \|f\|^2_\infty\left(2\,{}_k^sJ_a^\alpha\left[1\right]\,{}_k^sJ_a^\alpha\left[t^2\right] - 2\left({}_k^sJ_a^\alpha\left[t\right]\right)^2\right), \tag{7}$$

where $f \in L_\infty[a, b]$.

(ii) *for all* $a < t \le b$, $k > 0$, $s \in \mathbb{R}\backslash\{-1\}$, $\alpha \ge 0$ *the following inequality holds*

$$\left({}_k^sJ_a^\alpha\left[f(t)\right]\right)\left({}_k^s\sigma^2_{X,\alpha}(t)\right) - \left({}_k^sE_{X-E(X),\alpha}(t)\right)^2 \le (t-a)^2\left({}_k^sJ_a^\alpha\left[f(t)\right]\right)^2. \tag{8}$$

Proof Let us define the following identity

$$H(\tau,\rho) := (g(\tau)-g(\rho))\,(h(\tau)-h(\rho))\,;\quad \tau,\rho\in(a,t)\,,\quad a<t\le b. \tag{9}$$

Taking a function $p : [a, b] \to \mathbb{R}^+$, multiplying (9) by

$$\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\tau^s p(\tau);\quad k>0,\quad s\in\mathbb{R}\backslash\{-1\}\,,\quad \tau\in(a,t)$$

and integrating the resulting identity with respect to τ from a to t, we can state that

$$\begin{aligned}&\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\tau^s p(\tau)H(\tau,\rho)\mathrm{d}\tau\\ &\quad= {}_k^sJ_a^\alpha\left[pgh(t)\right]-g(\rho)\,{}_k^sJ_a^\alpha\left[ph(t)\right]-h(\rho)\,{}_k^sJ_a^\alpha\left[pg(t)\right]+g(\rho)h(\rho)\,{}_k^sJ_a^\alpha\left[p(t)\right].\end{aligned} \tag{10}$$

Now, multiplying (10) by $\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\left(t^{s+1}-\rho^{s+1}\right)^{\frac{\alpha}{k}-1}\rho^s p(\rho); k>0, s\in\mathbb{R}\backslash\{-1\}\,, \rho\in(a,t)$ and integrating the resulting identity with respect to ρ from a to t , we can state that

$$\begin{aligned}&\frac{(s+1)^{2(1-\frac{\alpha}{k})}}{k^2\Gamma_k^2(\alpha)}\int_a^t\int_a^t\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1}-\rho^{s+1}\right)^{\frac{\alpha}{k}-1}\tau^s\rho^s p(\tau)p(\rho)H(\tau,\rho)\mathrm{d}\tau\mathrm{d}\rho\\ &\quad=2\left({}_k^sJ_a^\alpha\left[p(t)\right]\right)\left({}_k^sJ_a^\alpha\left[pgh(t)\right]\right)-2\,{}_k^sJ_a^\alpha\left[pg(t)\right]\,{}_k^sJ_a^\alpha\left[ph(t)\right].\end{aligned} \tag{11}$$

In (11), taking $p(t) = f(t)$, $g(t) = h(t) = t - E(X)$, $t \in (a, b)$, we have

$$\begin{aligned}&\frac{(s+1)^{2(1-\frac{\alpha}{k})}}{k^2\Gamma_k^2(\alpha)}\int_a^t\int_a^t\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1}-\rho^{s+1}\right)^{\frac{\alpha}{k}-1}\tau^s\rho^s f(\tau)f(\rho)\left(\tau-\rho\right)^2\mathrm{d}\tau\mathrm{d}\rho\\ &\quad=2\left({}_k^sJ_a^\alpha\left[f(t)\right]\right)\left({}_k^sJ_a^\alpha\left[f(t)\left(t-E(X)\right)^2\right]\right)-2\left({}_k^sJ_a^\alpha\left[f(t)\left(t-E(X)\right)\right]\right)^2.\end{aligned} \tag{12}$$

On the other hand, we have

$$\begin{aligned}&\frac{(s+1)^{2(1-\frac{\alpha}{k})}}{k^2\Gamma_k^2(\alpha)}\int_a^t\int_a^t\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1}-\rho^{s+1}\right)^{\frac{\alpha}{k}-1}\tau^s\rho^s f(\tau)f(\rho)\left(\tau-\rho\right)^2\mathrm{d}\tau\mathrm{d}\rho\\ &\quad\le\|f\|^2_\infty\frac{(s+1)^{2(1-\frac{\alpha}{k})}}{k^2\Gamma_k^2(\alpha)}\int_a^t\int_a^t\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1}-\rho^{s+1}\right)^{\frac{\alpha}{k}-1}\tau^s\rho^s\left(\tau-\rho\right)^2\mathrm{d}\tau\mathrm{d}\rho\\ &\quad=\|f\|^2_\infty\left(2\left({}_k^sJ_a^\alpha\left[1\right]\right)\left({}_k^sJ_a^\alpha\left[t^2\right]\right)-2\left({}_k^sJ_a^\alpha\left[t\right]\right)^2\right).\end{aligned} \tag{13}$$

Thanks to (12) and (13), we obtain Part (i) of Theorem 3.6.

For Part (ii) of this theorem, we have

$$\frac{(s+1)^{2(1-\frac{\alpha}{k})}}{k^2\Gamma_k^2(\alpha)}\int_a^t\int_a^t\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1}-\rho^{s+1}\right)^{\frac{\alpha}{k}-1}\tau^s\rho^s f(\tau)f(\rho)(\tau-\rho)^2\,\mathrm{d}\tau\mathrm{d}\rho$$
$$\leq \sup_{\tau,\rho\in[a,t]}|\tau-\rho|^2\left({}_k^sJ_a^\alpha[f(t)]\right)^2=(t-a)^2\left({}_k^sJ_a^\alpha[f(t)]\right)^2. \tag{14}$$

Then, by (12) and (14), we get the desired inequality (8). □

Remark 3.7 Applying Theorem 3.6, for $s=0$ and $k=1$, we obtain Theorem 3.1 in [7].

Also we give next theorem:

Theorem 3.8 *Let X be a continuous random variable having a probability density function f defined on $[a,b]$. Then we have:*

(i) *For all $a<t\leq b$, $k>0$, $s\in\mathbb{R}\setminus\{-1\}$, $\alpha\geq 0$, $\beta\geq 0$,*

$$\left({}_k^sJ_a^\alpha[f(t)]\right)\left({}_k^s\sigma^2_{X,\beta}(t)\right)+\left({}_k^sJ_b^\beta[f(t)]\right)\left({}_k^s\sigma^2_{X,\alpha}(t)\right)-2\left({}_k^sE_{X-E(X),\alpha}(t)\right)\left({}_k^sE_{X-E(X),\beta}(t)\right)$$
$$\leq\|f\|_\infty^2\left({}_k^sJ_a^\alpha[1]\left({}_k^sJ_a^\beta\left[t^2\right]\right)+{}_k^sJ_a^\beta[1]\left({}_k^sJ_a^\alpha\left[t^2\right]\right)-2\left({}_k^sJ_a^\alpha[f(t)]\right)\left({}_k^sJ_a^\beta[f(t)]\right)\right), \tag{15}$$

where $f\in L_\infty[a,b]$.

(ii) *For all $a<t\leq b$, $k>0$, $s\in\mathbb{R}\setminus\{-1\}$, $\alpha\geq 0$, $\beta\geq 0$,*

$$\left({}_k^sJ_a^\alpha[f(t)]\right)\left({}_k^s\sigma^2_{X,\beta}(t)\right)+\left({}_k^sJ_a^\beta[f(t)]\right)\left({}_k^s\sigma^2_{X,\alpha}(t)\right)-2\left({}_k^sE_{X-E(X),\alpha}(t)\right)\left({}_k^sE_{X-E(X),\beta}(t)\right)$$
$$\leq(t-a)^2\left({}_k^sJ_a^\alpha[f(t)]\right)\left({}_k^sJ_a^\beta[f(t)]\right). \tag{16}$$

Proof Using the identity(9) in the proof of Theorem 3.6, it follows that

$$\frac{(s+1)^{2(1-\frac{\alpha}{k})}}{k^2\Gamma_k^2(\alpha)}\int_a^t\int_a^t\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1}-\rho^{s+1}\right)^{\frac{\alpha}{k}-1}\tau^s\rho^s p(\tau)p(\rho)H(\tau,\rho)\mathrm{d}\tau\mathrm{d}\rho$$
$$=\left({}_k^sJ_a^\alpha[p(t)]\right)\left({}_k^sJ_a^\beta[pgh(t)]\right)+\left({}_k^sJ_a^\beta[p(t)]\right)\left({}_k^sJ_a^\alpha[pgh(t)]\right)$$
$$-\left({}_k^sJ_a^\alpha[ph(t)]\right)\left({}_k^sJ_a^\beta[pg(t)]\right)-\left({}_k^sJ_a^\alpha[ph(t)]\right)\left({}_k^sJ_a^\beta[pg(t)]\right). \tag{17}$$

Then, for all $t\in(a,b)$, choosing $p(t)=f(t)$, $g(t)=h(t)=t-E(X)$ in (17), we get

$$\frac{(s+1)^{\left(2-\frac{\alpha+\beta}{k}\right)}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^t\int_a^t\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1}-\rho^{s+1}\right)^{\frac{\beta}{k}-1}\tau^s\rho^s f(\tau)f(\rho)H(\tau,\rho)\,\mathrm{d}\tau\mathrm{d}\rho$$
$$=\left({}_k^sJ_a^\alpha[f(t)]\right)\left({}_k^sJ_a^\beta\left[f(t)(t-E(X))^2\right]\right)+\left({}_k^sJ_a^\beta[f(t)]\right)\left({}_k^sJ_a^\alpha\left[f(t)(t-E(X))^2\right]\right)$$
$$-2\left({}_k^sJ_a^\alpha[f(t)(t-E(X))]\right)\left({}_k^sJ_a^\beta[f(t)(t-E(X))]\right). \tag{18}$$

We have also

$$\frac{(s+1)^{\left(2-\frac{\alpha+\beta}{k}\right)}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^t\int_a^t\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1}-\rho^{s+1}\right)^{\frac{\beta}{k}-1}\tau^s\rho^s f(\tau)f(\rho)(\tau-\rho)^2\,\mathrm{d}\tau\mathrm{d}\rho$$
$$\leq\|f\|_\infty^2\frac{(s+1)^{\left(2-\frac{\alpha+\beta}{k}\right)}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^t\int_a^t(t-\tau)^{\frac{\alpha}{k}-1}(t-\rho)^{\frac{\beta}{k}-1}\tau^s\rho^s(\tau-\rho)^2\,\mathrm{d}\tau\mathrm{d}\rho$$
$$=\|f\|_\infty^2\left({}_k^sJ_a^\alpha[1]\left({}_k^sJ_a^\beta\left[t^2\right]\right)+{}_k^sJ_a^\beta[1]\left({}_k^sJ_a^\alpha\left[t^2\right]\right)-2\left({}_k^sJ_a^\alpha[f(t)]\right)\left({}_k^sJ_a^\beta[f(t)]\right)\right). \tag{19}$$

Thanks to (18) and (19), we obtain (i).
To prove (ii), we will use the fact that

$$\sup_{\tau,\rho\in[a,t]}|\tau-\rho|^2=(t-a)^2.$$

We obtain

$$\frac{1}{k^2\Gamma(\alpha)\Gamma(\beta)}\int_a^t\int_a^t (t-\tau)^{\frac{\alpha}{k}-1}(t-\rho)^{\frac{\beta}{k}-1}\,p(\tau)p(\rho)H(\tau,\rho)\mathrm{d}\tau\mathrm{d}\rho$$
$$\leq (t-a)^2\left({}_k^sJ_a^\alpha\left[f(t)\right]\right)\left({}_k^sJ_a^\beta\left[f(t)\right]\right). \tag{20}$$

By (18) and (20), we get the desired inequality (16) which completes the proof. □

Remark 3.9 Applying Theorem 3.8,

(1) for $\alpha=\beta$, we obtain Theorem 3.6.
(2) for $s=0, k=1$, we obtain Theorem 3.2 in [7].

We also give the following result for Riemann–Liouville (k,s)-fractional integrals:

Theorem 3.10 *Let X be a continuous random variable having a probability density function $f:[a,b]\to\mathbb{R}$. Assume that there exist constants φ, Φ such that $0\leq\varphi\leq f(t)\leq\Phi\leq 1$ a.e t on $[a,b]$. Then for all $\alpha\geq 0$, $\beta\geq 0$, $s\in\mathbb{R}_+/\{-1\}$, we have*

$${}_k^sJ_a^\alpha\left[f(t)\right]\,{}_k^s\sigma^2_{X,\alpha}(t)-\left({}_k^sE_{X-E(X),\alpha}(t)\right)^2\leq\frac{1}{4}(b-a)^2\left({}_k^sJ_a^\alpha\left[f(t)\right]\right)^2. \tag{21}$$

Proof Thanks to (5), we can state that:

$$\left|{}_k^sJ_a^\alpha\left[p(t)\right]\,{}_k^sJ_a^\alpha\left[pg^2(t)\right]-\left({}_k^sJ_a^\alpha\left[pg(t)\right]\right)^2\right|\leq\frac{1}{4}\left({}_k^sJ_a^\alpha\left[p(t)\right]\right)^2(\Phi-\varphi)^2. \tag{22}$$

In (22), if we take $p(t)=f(t)$, $g(t)=t-E(X)$, $t\in[a,b]$ and $\Phi=b-E(X)$, $\varphi=a-E(X)$, we can state the above inequality as follows:

$$0\leq\left({}_k^sJ_a^\alpha\left[f(t)\right]\right){}_k^sJ_a^\alpha\left[f(t)(t-E(X))^2\right]-\left({}_k^sJ_a^\alpha\left[f(t)(t-E(X))\right]\right)^2$$
$$\leq\frac{1}{4}\left({}_k^sJ_a^\alpha\left[f(t)\right]\right)^2(b-a)^2. \tag{23}$$

This implies that

$${}_k^sJ_a^\alpha\left[f(t)\right]\,{}_k^s\sigma^2_{X,\alpha}(t)-\left({}_k^sE_{X-E(X),\alpha}(t)\right)^2\leq\frac{1}{4}(b-a)^2\left({}_k^sJ_a^\alpha\left[f(t)\right]\right)^2. \tag{24}$$

So, the theorem is proved. □

Remark 3.11 Applying Theorem 3.10, for $s=0$ and $k=1$, we obtain Theorem 3.3 in [7].

Finally, the next inequality is hold.

Theorem 3.12 *Let X be a continuous random variable having a probability density function $f:[a,b]\to\mathbb{R}$. Assume that there exist constants φ, Φ such that $0\leq\varphi\leq f(t)\leq\Phi\leq 1$ a.e t on $[a,b]$, then for all $\alpha\geq 0$, $\beta\geq 0$, $s\in\mathbb{R}_+/\{-1\}$, we have*

$${}_k^sJ_a^\alpha\left[f(t)\right]\,{}_k^s\sigma^2_{X,\beta}(t)+{}_k^sJ_a^\beta\left[f(t)\right]\,{}_k^s\sigma^2_{X,\alpha}(t)+2\left(a-E(X)\right)\left(b-E(X)\right){}_k^sJ_a^\alpha\left[f(t)\right]\,{}_k^sJ_a^\beta\left[f(t)\right]$$
$$\leq\left(a+b-2E(X)\right)\left({}_k^sJ_a^\alpha\left[f(t)\right]\,{}_k^sE_{X-E(X),\beta}(t)+{}_k^sJ_a^\beta\left[f(t)\right]\,{}_k^sE_{X-E(X),\alpha}(t)\right). \tag{25}$$

Proof Thanks to (6), we can state that:

$$\left[{}_k^sJ_a^\alpha\left[p(t)\right]{}_k^sJ_a^\beta\left[pg^2(t)\right]+{}_k^sJ_a^\beta\left[p(t)\right]{}_k^sJ_a^\alpha\left[pg^2(t)\right]-2\,{}_k^sJ_a^\alpha\left[pg(t)\right]\,{}_k^sJ_a^\beta\left[pg(t)\right]\right]^2$$
$$\leq\left[\left(\Phi\,{}_k^sJ_a^\alpha\left[p(t)\right]-{}_k^sJ_a^\alpha\left[pg(t)\right]\right)\left({}_k^sJ_a^\beta\left[pg(t)\right]-\varphi\,{}_k^sJ_a^\beta\left[p(t)\right]\right)\right.$$
$$\left.+\left({}_k^sJ_a^\alpha\left[pg(t)\right]-\varphi\,{}_k^sJ_a^\alpha\left[p(t)\right]\right)\left(\Phi\,{}_k^sJ_a^\beta\left[p(t)\right]-{}_k^sJ_a^\beta\left[pg(t)\right]\right)\right]^2. \tag{26}$$

In (26), if we take $p(t) = f(t), g(t) = t - E(X), t \in [a, b]$, we obtain

$$\begin{aligned}&\left[{}_k^s J_a^\alpha [f(t)]\, {}_k^s J_a^\beta \left[f(t)(t - E(X))^2\right] + {}_k^s J_a^\beta [f(t)]\, {}_k^s J_a^\alpha \left[p(t)(t - E(X))^2\right]\right.\\&\quad \left. - 2\, {}_k^s J_a^\alpha [f(t)(t - E(X))]\, {}_k^s J_a^\beta [f(t)(t - E(X))]\right]^2\\&\quad \le \left[\left(\Phi\, {}_k^s J_a^\alpha [f(t)] - {}_k^s J_a^\alpha [f(t)(t - E(X))]\right)\left({}_k^s J_a^\beta [f(t)(t - E(X))] - \varphi\, {}_k^s J_a^\beta [f(t)]\right)\right.\\&\quad\quad \left. + \left({}_k^s J_a^\alpha [f(t)(t - E(X))] - \varphi\, {}_k^s J_a^\alpha [f(t)]\right)\left(\Phi\, {}_k^s J_a^\beta [f(t)] - {}_k^s J_a^\beta [f(t)(t - E(X))]\right)\right]^2. \qquad (27)\end{aligned}$$

Combining (22) and (27) and taking into account the fact that the left hand side of (22) is positive, we get:

$$\begin{aligned}&{}_k^s J_a^\alpha [f(t)]{}_k^s J_a^\beta \left[f(t)(t - E(X))^2\right] + {}_k^s J_a^\beta [f(t)]{}_k^s J_a^\alpha \left[f(t)(t - E(X))^2\right]\\&\quad - 2\, {}_k^s J_a^\alpha [f(t)(t - E(X))]\, {}_k^s J_a^\beta [f(t)(t - E(X))]\\&\quad \le \left(\Phi\, {}_k^s J_a^\alpha [f(t)] - {}_k^s J_a^\alpha [f(t)(t - E(X))]\right)\left({}_k^s J_a^\beta [f(t)(t - E(X))] - \varphi\, {}_k^s J_a^\beta [f(t)]\right)\\&\quad\quad + \left({}_k^s J_a^\alpha [f(t)(t - E(X))] - \varphi\, {}_k^s J_a^\alpha [f(t)]\right)\left(\Phi\, {}_k^s J_a^\beta [f(t)] - {}_k^s J_a^\beta [f(t)(t - E(X))]\right). \qquad (28)\end{aligned}$$

By a simple calculation, we get

$$\begin{aligned}&{}_k^s J_a^\alpha [f(t)]\, {}_k^s J_a^\beta \left[f(t)(t - E(X))^2\right] + {}_k^s J_a^\beta [f(t)]\, {}_k^s J_a^\alpha \left[f(t)(t - E(X))^2\right]\\&\quad \le \Phi\left({}_k^s J_a^\alpha [f(t)]\, {}_k^s E_{X-E(X),\beta}(t) + {}_k^s J_a^\beta [f(t)]\, {}_k^s E_{X-E(X),\alpha}(t)\right)\\&\quad\quad \times \varphi\left({}_k^s J_a^\alpha [f(t)]\, {}_k^s E_{X-E(X),\beta}(t) + {}_k^s J_a^\beta [f(t)]\, {}_k^s E_{X-E(X),\alpha}(t)\right) - 2\varphi\Phi\, {}_k^s J_a^\alpha [f(t)]\, {}_k^s J_a^\beta [f(t)]. \qquad (29)\end{aligned}$$

In (29), If $\Phi = b - E(X)$, $\varphi = a - E(X)$ are taken, we get the desired inequality (25). □

Remark 3.13 Applying Theorem 3.12, for $s = 0$ and $k = 1$, we obtain Theorem 3.4 in [7].

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