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Cyclic subgradient extragradient methods for equilibrium problems

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Abstract In this paper, we introduce a cyclic subgradient extragradient algorithm and its modified form for finding a solution of a system of equilibrium problems for a class of pseudomonotone and Lipschitz-type continuous bifunctions. The main idea of these algorithms originates from several previously known results for variational inequalities. The proposed algorithms are extensions of the subgradient extragradient method for variational inequalities to equilibrium problems and the hybrid (outer approximation) method. The paper can help in the design and analysis of practical algorithms and gives us a generalization of the most convex feasibility problems.

Mathematics Subject Classification 65J15 · 47H05 · 47J25 · 91B50

الملخص

في هذه الورقة نقدم خوارزمية مستوفية التدرج جزئية التدرج دورية وصيغتها المعدَّلة لإيجاد حل لنظام من مسائل التوازن لصف من الدوال الثنائية المستمرة من نوع ليبشيتز. تنبع الفكرة الرئيسية لتلك الخوارزميات من عدة نتائج معروفة مسبقا للمتباينات التغيُّريَّة. الخوارزميات المقترحة امتدادات للطريقة مستوفية التدرج جزئية التدرج للمتباينات التغيرية لمسائل التوازن وطريقة الهجين (التقريب الخارجي). يمكن أن تساعد هذه الورقة في تصميم وتحليل خوارزميات عملية، وتعطينا تعميما، لمسائل قابلية التنفيذ ذات أكبر تحدب ممكن.

1 Introduction

Let H be a real Hilbert space and $C_i,\ i = 1, \ldots, N$ be closed convex subsets of H such that $C = \cap_{i=1}^N C_i \neq \emptyset$. Let $f_i : H \times H \to \Re,\ i = 1, \ldots, N$ be bifunctions with $f_i(x, x) = 0$ for all $x \in C_i$. The common solutions to equilibriums problem (CSEP) [14] for the bifunctions $f_i,\ i = 1, \ldots, N$ is to find $x^* \in C$ such that

$$f_i(x^*, y) \geq 0, \quad \forall y \in C_i, \quad i = 1, \ldots, N. \tag{1}$$

We denote $F = \cap_{i=1}^N EP(f_i, C_i)$ by the solution set of CSEP (1), where $EP(f_i, C_i)$ is the solution set of each equilibrium subproblem for f_i on C_i. CSEP (1) is very general in the sense that it includes, as special cases, many mathematical models: common solutions to variational inequalities, convex feasibility problems, common fixed point problems, see for instance [2,8,10,11,14,21,34,37]. These problems have been widely studied both theoretically and algorithmically over the past decades due to their applications to other fields [5,10,15,29]. The following are three very special cases of CSEP. Firstly, if $f_i(x, y) = 0$ then CSEP is reduced to the following *convex feasibility problem* (CFP):

$$\text{find } x^* \in C = \cap_{i=1}^N C_i \neq \emptyset,$$

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that is to find an element in the intersection of a family of given closed convex sets. CFP has received a lot of attention because of its broad applicable ability to mathematical fields, most notably, as image reconstruction, signal processing, approximation theory and control theory, see in [5,10,15,29] and the references therein.

Next, if $f_i(x, y) = \langle x - S_i x, y - x\rangle$ for all $x, y \in C$ where $S_i : C \to C$ is a mapping for each $i = 1, \ldots, N$ then CSEP becomes the following *common fixed point problem* (CFPP) [8] for a family of the mappings S_i, i.e.,

$$\text{find } x^* \in F := \cap_{i=1}^N F(S_i),$$

where $F(S_i)$ is the fixed point set of S_i. Finally, if $f_i(x, y) = \langle A_i(x), y - x\rangle$, where $A_i : H \to H$ is a nonlinear operator for each $i = 1, \ldots, N$, then CSEP becomes the following *common solutions to variational inequalities problem* (CSVIP): find $x^* \in C = \cap_{i=1}^N C_i$ such that

$$\left\langle A_i(x^*), y - x^*\right\rangle \geq 0, \quad \forall y \in C_i,\ i = 1, \ldots, N \tag{2}$$

which was introduced and studied in [11,21,36].

In 2005, Combettes and Hirstoaga [14] introduced a general procedure for solving CSEPs. After that, many methods were also proposed for solving CSVIPs and CSEPs, see for instance [4,21,30,32–35] and the references therein. However, the general procedure in [14] and the most existing methods are frequently based on the proximal point method (PPM) [22,28], i.e., at the current step, given x_n, the next approximation x_{n+1} is the solution of the following regularized equilibrium problem (REP).

$$\text{Find } x \in C \text{ such that: } f(x, y) + \frac{1}{r_n}\langle y - x, x - x_n\rangle \geq 0, \quad \forall y \in C, \tag{3}$$

or $x_{n+1} = J_{r_n f}(x_n)$ where r_n is a suitable parameter, J_f is the resolvent [14] of the bifunction f and C is a nonempty closed convex subset of H. Note that, when f is monotone, REP (3) is strongly monotone, hence its solution exists and is unique. However, if the bifunction f is generally monotone [7], for instance, pseudomonotone then REP (3), in general, is not strongly monotone. So, the existence and uniqueness of the solution of (3) is not guaranteed. In addition, its solution set is not necessarily convex. Therefore, PPM can not be applied to the class of equilibrium problems for pseudomonotone bifunctions.

In 1976, Korpelevich [23] introduced the following extragradient method (or double projection method) for solving saddle point problem for L-Lipschitz continuous and monotone operators in Euclidean spaces,

$$\begin{cases} y_n = P_C(x_n - \lambda A(x_n)), \\ x_{n+1} = P_C(x_n - \lambda A(y_n)), \end{cases} \tag{4}$$

where $\lambda \in (0, \frac{1}{L})$. In 2008, Quoc et al. [30] extended Korpelevich's extragradient method to equilibrium problems for pseudomonotone and Lipschitz-type continuous bifunctions in which two strongly convex optimization programs are solved at each iteration. The advantage of extragradient method is that two optimization problems are numerically easier than non-linear inequality (3) in PPM.

In 2011, in order to improve the second projection in Korpelevich's extragradient method on the feasible set C, Censor et al. [13] proposed the following subgradient extragradient method,

$$\begin{cases} y_n = P_C(x_n - \lambda A(x_n)), \\ x_{n+1} = P_{T_n}(x_n - \lambda A(y_n)), \end{cases} \tag{5}$$

where the second projection is performed on the specially constructed half-space T_n as $T_n = \Big\{v \in H : \langle (x_n - \lambda A(x_n)) - y_n, v - y_n\rangle \leq 0\Big\}$. It is clear that the second projection on the half-space T_n in the subgradient extragradient method is inherently explicit. Figures 1 and 2 (see [13]) illustrate the iterative steps of Korpelevich's extragradient method and the subgradient extragradient method, respectively.

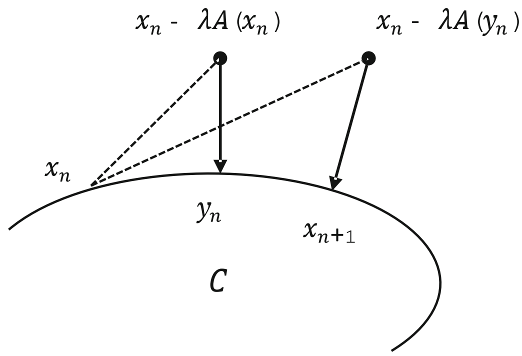

Fig. 1 Iterative step of the Korpelevich's extragradient method

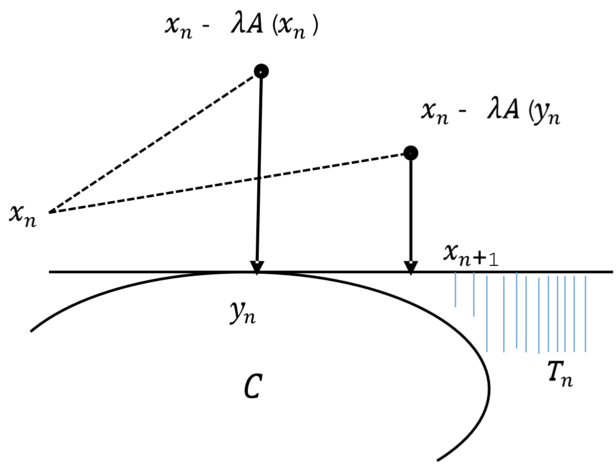

Fig. 2 Iterative step of the subgradient extragradient method

For the special case, when CSEP (1) is CSVIP (2), Censor et al. [11] used Korpelevich's extragradient method and the hybrid (outer approximation) method to propose the following hybrid method for CSVIPs,

$$\begin{cases} y_n^i = P_{C_i}(x_n - \lambda_n^i A_i(x_n)), \ i = 1, \ldots, N, \\ z_n^i = P_{C_i}(x_n - \lambda_n^i A_i(y_n^i)), , \ i = 1, \ldots, N, \\ H_n^i = \left\{z \in H : \left\langle x_n - z_n^i, z - x_n - \gamma_n^i(z_n^i - x_n)\right\rangle \le 0\right\}, \\ H_n = \cap_{i=1}^N H_n^i, \\ W_n = \{z \in H : \langle x_1 - x_n, z - x_n\rangle \le 0\}, \\ x_{n+1} = P_{H_n \cap W_n} x_1. \end{cases} \tag{6}$$

Then, they proved that the sequence $\{x_n\}$ generated by (6) converges strongly to the projection of x_1 on the solution set of CSVIP.

The purpose of this paper is triple. Firstly, we extend the subgradient extragradient method [13] to equilibrium problems, i.e., REP (3) is replaced by two optimization programs

$$y_n = \operatorname{argmin}\left\{\lambda_n f(x_n, y) + \frac{1}{2}||x_n - y||^2 : y \in C\right\}, \tag{7}$$

$$x_{n+1} = \operatorname{argmin}\left\{\lambda_n f(y_n, y) + \frac{1}{2}||x_n - y||^2 : y \in T_n\right\}, \tag{8}$$

where $\{\lambda_n\}$ is a suitable parameter sequence and T_n is the specially constructed half-space as

$$T_n = \{v \in H : \langle (x_n - \lambda_n w_n) - y_n, v - y_n\rangle \le 0\},$$

and $w_n \in \partial_2 f(x_n, y_n) := \partial f(x_n, .)(y_n)$. The advantages of the subgradient extragradient method (7)–(8) are that two optimization problems are not only numerically solved more easily than non-linear inequality (3), but also optimization program (8) is performed onto the half-space T_n. There are many class of bifunctions in which the program (8) can be effectively solved in many cases, for example, if $f(x,.)$ is a convex quadratic function then problem (8) can be computed by using the available methods of convex quadratic programming [9, Chapter 8] or if $f(x,y) = \langle A(x), y - x\rangle$ then problem (8) is an explicit projection on the halfspace T_n

Secondly, based on the subgradient extragradient method (7)–(8) and hybrid method (6) we introduce a cyclic algorithm for CSEPs, so-called the cyclic subgradient extragradient method (see, Algorithm 3.1 in Sect. 3). Note that, hybrid method (6) is parallel in the sense that the intermediate approximations y_n^i are simultaneously computed at each iteration, and z_n^i are too. A disadvantage of hybrid method (6) is that in order to compute the next iteration x_{n+1} we must solve a distance optimization program onto the intersection of $N+1$ sets $H_n^1, H_n^2, \ldots, H_n^N, W_n$. This might be costly if the number of subproblems N is large. This is the reason which explains why we design the cyclic algorithm in which x_{n+1} is expressed by an explicit formula (see, Remarks 3.2 and 3.7 in Sect. 3). Finally, we present a modification of the cyclic subgradient extragradient method for finding a common element of the solution set of CSEP and the fixed point set of a nonexpansive mapping. Strongly convergent theorems are established under standard assumptions imposed on bifunctions. Some numerical experiments are implemented to illustrate the convergence of the proposed algorithm and compare it with a parallel hybrid extragradient method.

The paper is organized as follows: in Sect. 2, we collect some definitions and preliminary results for proving the convergence theorems. Section 3 deals with the proposed cyclic algorithms and analyzing their convergence. In Sect. 4, we illustrate the efficiency of the proposed cyclic algorithm in comparison with a parallel hybrid extragradient method by considering some preliminary numerical experiments.

2 Preliminaries

In this section, we recall some definitions and results for further use. Let C be a nonempty closed convex subset of a real Hilbert space H. A mapping $S : C \to H$ is called nonexpansive on C if $||S(x) - S(y)|| \leq ||x - y||$ for all $x, y \in C$. The fixed point set of S is denoted by $F(S)$. We begin with the following properties of a nonexpansive mapping.

Lemma 2.1 [17] *Assume that $S : C \to H$ is a nonexpansive mapping. If S has a fixed point, then*

(i) *$F(S)$ is closed convex subset of C.*
(ii) *$I - S$ is demiclosed, i.e., whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I - S)x_n\}$ strongly converges to some y, it follows that $(I - S)x = y$.*

Next, we present some concepts of the monotonicity of a bifunction and an operator (see [8,26]).

Definition 2.2 A bifunction $f : C \times C \to \Re$ is said to be

(i) strongly monotone on C if there exists a constant $\gamma > 0$ such that

$$f(x,y) + f(y,x) \leq -\gamma ||x - y||^2, \quad \forall x, y \in C;$$

(ii) monotone on C if

$$f(x,y) + f(y,x) \leq 0, \quad \forall x, y \in C;$$

(iii) pseudomonotone on C if

$$f(x,y) \geq 0 \Longrightarrow f(y,x) \leq 0, \quad \forall x, y \in C.$$

From definitions above, it is clear that a strongly monotone bifunction is monotone and a monotone bifunction is pseudomonotone.

Definition 2.3 [23] An operator $A : C \to H$ is called

(i) monotone on C if

$$\langle A(x) - A(y), x - y\rangle \geq 0, \quad \forall x, y \in C;$$

(ii) pseudomonotone on C if

$$\langle A(x), y - x\rangle \geq 0 \Longrightarrow \langle A(y), x - y\rangle \leq 0, \ \forall x, y \in C;$$

(iii) L-Lipschitz continuous on C if there exists a positive number L such that

$$||A(x) - A(y)|| \leq L||x - y||, \quad \forall x, y \in C.$$

For solving CSEP (1), we assume that the bifunction $f : H \times H \to \Re$ satisfies the following conditions, see [30].

(A1) f is pseudomonotone on C and $f(x, x) = 0$ for all $x, y \in C$;
(A2) f is Lipschitz-type continuous on H, i.e., there exist two positive constants c_1, c_2 such that

$$f(x, y) + f(y, z) \geq f(x, z) - c_1||x - y||^2 - c_2||y - z||^2, \quad \forall x, y, z \in H;$$

(A3) f is weakly continuous on $H \times H$;
(A4) $f(x, .)$ is convex and subdifferentiable on H for every fixed $x \in H$.

Hypothesis (A2) was introduced by Mastroeni [25]. It is necessary to imply the convergence of the auxiliary principle method for equilibrium problems. Now, we give some cases for bifunctions satisfying hypotheses (A1) and (A2). Firstly, we consider the following optimization problem,

$$\min \{\varphi(x) : x \in C\},$$

where $\varphi : H \to \Re$ is a convex function. Then, the bifunction $f(x, y) = \varphi(y) - \varphi(x)$ satisfies conditions (A1) and (A2) automatically. Secondly, let $A : H \to H$ be a L-Lipschitz continuous and pseudomonotone operator. Then, the bifunction $f(x, y) = \langle A(x), y - x\rangle$ also satisfies conditions (A1) $-$ (A2). Indeed, hypothesis (A1) is automatically fulfilled. From the L-Lipschitz continuity of A, we have

$$\begin{aligned} f(x, y) + f(y, z) - f(x, z) &= \langle A(x) - A(y), y - z\rangle \geq -||A(x) - A(y)||||y - z|| \\ &\geq -L||x - y||||y - z|| \geq -\frac{L}{2}||x - y||^2 - \frac{L}{2}||y - z||^2. \end{aligned}$$

This implies that f satisfies condition (A2) with $c_1 = c_2 = L/2$. Finally, a class of other bifunctions, which is generalized from the Cournot–Nash equilibrium model [30] as

$$f(x, y) = \langle F(x) + Qy + q, y - x\rangle, \; x, y \in \Re^n,$$

where $F : \Re^n \to \Re^n$, $Q \in \Re^{n\times n}$ is a symmetric positive semidefinite matrix and $q \in \Re^n$ also satisfies condition $(A2)$ under some suitable assumptions on the mapping F [30].

Note that, from assumption (A2) with $x = z$ we obtain

$$f(x, y) + f(y, x) \geq -(c_1 + c_2)||x - y||^2, \quad \forall x, y \in H.$$

This does not imply the monotonicity, even pseudomonotonicity, of the bifunction f.

The metric projection $P_C : H \to C$ is defined by $P_C(x) = \arg\min \{\|y - x\| : y \in C\}$. Since C is non-empty, closed and convex, $P_C(x)$ exists and is unique. It is also known that P_C has the following characteristic properties, see [18].

Lemma 2.4 *Let $P_C : H \to C$ be the metric projection from H onto C. Then*

(i) *P_C is firmly nonexpansive, i.e.,*

$$\langle P_C(x) - P_C(y), x - y\rangle \geq \|P_C(x) - P_C(y)\|^2, \quad \forall x, y \in H.$$

(ii) *For all $x \in C, y \in H$,*

$$\|x - P_C(y)\|^2 + \|P_C(y) - y\|^2 \leq \|x - y\|^2. \tag{9}$$

(iii) *$z = P_C(x)$ if and only if*

$$\langle x - z, z - y\rangle \geq 0, \quad \forall y \in C. \tag{10}$$

Note that any closed convex subset C of H can be represented as the sublevel set of an appropriate convex function $c : H \to \Re$,

$$C = \{v \in H : c(v) \leq 0\}.$$

The subdifferential of c at x is defined by

$$\partial c(x) = \{w \in H : c(y) - c(x) \geq \langle w, y - x \rangle, \quad \forall y \in H\}.$$

For each $z \in H$ and $w \in \partial c(z)$, we denote $T(z) = \{v \in H : c(z) + \langle w, v - z \rangle \leq 0\}$. If $z \notin \mathrm{int}C$ then $T(z)$ is a half-space whose bounding hyperplane separates the set C from the point z. Otherwise, $T(z)$ is the entire space H. We recall that the normal cone of C at $x \in C$ is defined as follows:

$$N_C(x) = \{w \in H : \langle w, y - x \rangle \leq 0, \quad \forall y \in C\}.$$

Lemma 2.5 *[16] Let C be a nonempty convex subset of a real Hilbert space H and $g : C \to \Re$ be a convex, subdifferentiable, lower semicontinuous function on C. Then, x^* is a solution to the following convex problem* $\min\{g(x) : x \in C\}$ *if and only if* $0 \in \partial g(x^*) + N_C(x^*)$, *where $\partial g(.)$ denotes the subdifferential of g and $N_C(x^*)$ is the normal cone of C at x^*.*

3 Main results

In this section, we present a cyclic subgradient extragradient algorithm for solving CSEP for the pseudomonotone bifunctions f_i, $i = 1, \ldots, N$ and its modified algorithm and analyze the strong convergence of the obtained iteration sequences. In the sequel, we assume that the bifunctions f_i are Lipschitz-type continuous with the same constants c_1 and c_2, i.e.,

$$f_i(x, y) + f_i(y, z) \geq f_i(x, z) - c_1||x - y||^2 - c_2||y - z||^2$$

for all $x, y, z \in H$ and the solution set $F = \cap_{i=1}^N EP(f_i, C_i)$ is nonempty. It is easy to show that if f_i satisfies conditions (A1) − (A4) then $EP(f_i, C_i)$ is closed and convex (see, for instance [30]). Thus, F is also closed and convex. We denote $[n] = n(mod\ N) + 1$ to stand for the mod function taking the values in $\{1, 2, \ldots, N\}$. We have the following cyclic algorithm:

Algorithm 3.1 (Cyclic Subgradient Extragradient Method)
Initialization. *Choose $x_0 \in H$ and two parameter sequences $\{\lambda_n\}, \{\gamma_n\}$ satisfying the following conditions* $0 < \alpha \leq \lambda_n \leq \beta < \min\left(\frac{1}{2c_1}, \frac{1}{2c_2}\right)$, $\gamma_n \in [\epsilon, \frac{1}{2}]$, *for some* $\epsilon \in (0, \frac{1}{2}]$.
Step 1 *Solve two strongly convex programs*

$$y_n = \mathrm{argmin}\left\{\lambda_n f_{[n]}(x_n, y) + \frac{1}{2}||x_n - y||^2 : y \in C_{[n]}\right\},$$
$$z_n = \mathrm{argmin}\left\{\lambda_n f_{[n]}(y_n, y) + \frac{1}{2}||x_n - y||^2 : y \in T_n\right\},$$

where T_n is the half-space whose bounding hyperplane supported on $C_{[n]}$ at y_n, i.e.,

$$T_n = \{v \in H : \langle (x_n - \lambda_n w_n) - y_n, v - y_n \rangle \leq 0\},$$

and $w_n \in \partial_2 f_{[n]}(x_n, y_n) := \partial f_{[n]}(x_n, .)(y_n)$.
Step 2 *Compute* $x_{n+1} = P_{H_n \cap W_n}(x_0)$, *where*

$$H_n = \{z \in H : \langle x_n - z_n, z - x_n - \gamma_n(z_n - x_n) \rangle \leq 0\};$$
$$W_n = \{z \in H : \langle x_0 - x_n, z - x_n \rangle \leq 0\}.$$

Set $n := n + 1$ and go back ***Step 1***.

Remark 3.2 Two sets H_n and W_n in Algorithm 3.1 are either the half-spaces or the space H. Therefore, using the same techniques as in [30], we can define the explicit formula of the projection x_{n+1} of x_0 onto the intersection $H_n \cap W_n$. Indeed, let $v_n = x_n + \gamma_n(z_n - x_n)$, we rewrite the set H_n as follows:

$$H_n = \{z \in H : \langle x_n - z_n, z - v_n \rangle \leq 0\}.$$

Therefore, by the same arguments as in [30], we obtain

$$x_{n+1} := P_{H_n} x_0 = x_0 - \frac{\langle x_n - z_n, x_0 - v_n \rangle}{||x_n - z_n||^2}(x_n - z_n)$$

if $P_{H_n} x_0 \in W_n$. Otherwise,

$$x_{n+1} = x_0 + t_1(x_n - z_n) + t_2(x_0 - x_n),$$

where t_1, t_2 is the solution of the system of linear equations with two unknowns

$$\begin{cases} t_1||x_n - z_n||^2 + t_2 \langle x_n - z_n, x_0 - x_n \rangle = -\langle x_0 - v_n, x_n - z_n \rangle, \\ t_1 \langle x_n - z_n, x_0 - x_n \rangle + t_2||x_0 - x_n||^2 = -||x_0 - x_n||^2. \end{cases}$$

We need the following results for proving the convergence of Algorithm 3.1.

Lemma 3.3 *Assume that $x^* \in F$. Let $\{x_n\}, \{y_n\}, \{z_n\}$ be the sequences defined as in Algorithm 3.1. Then, there holds the relation*

$$||z_n - x^*||^2 \leq ||x_n - x^*||^2 - (1 - 2\lambda_n c_1)\,||y_n - x_n||^2 - (1 - 2\lambda_n c_2)\,||z_n - y_n||^2.$$

Proof Since $z_n \in T_n$, we have

$$\langle (x_n - \lambda_n w_n) - y_n, z_n - y_n \rangle \leq 0.$$

Thus

$$\langle x_n - y_n, z_n - y_n \rangle \leq \lambda_n \langle w_n, z_n - y_n \rangle. \tag{11}$$

From $w_n \in \partial_2 f_{[n]}(x_n, y_n)$ and the definition of subdifferential, we obtain

$$f_{[n]}(x_n, y) - f_{[n]}(x_n, y_n) \geq \langle w_n, y - y_n \rangle, \quad \forall y \in H.$$

The last inequality with $y = z_n$ and (11) imply that

$$\lambda_n \left\{ f_{[n]}(x_n, z_n) - f_{[n]}(x_n, y_n) \right\} \geq \langle x_n - y_n, z_n - y_n \rangle. \tag{12}$$

By Lemma 2.5 and

$$z_n = \text{argmin} \left\{ \lambda_n f_{[n]}(y_n, y) + \frac{1}{2}||x_n - y||^2 : y \in T_n \right\},$$

one has

$$0 \in \partial_2 \left\{ \lambda_n f_{[n]}(y_n, y) + \frac{1}{2}||x_n - y||^2 \right\}(z_n) + N_{T_n}(z_n).$$

Thus, there exist $w \in \partial_2 f_{[n]}(y_n, z_n)$ and $\bar{w} \in N_{T_n}(z_n)$ such that

$$\lambda_n w + z_n - x_n + \bar{w} = 0. \tag{13}$$

From the definition of the normal cone and $\bar{w} \in N_{T_n}(z_n)$, we get $\langle \bar{w}, y - z_n \rangle \leq 0$ for all $y \in T_n$. This together with (13) implies that

$$\lambda_n \langle w, y - z_n \rangle \geq \langle x_n - z_n, y - z_n \rangle$$

for all $y \in T_n$. Since $x^* \in T_n$,

$$\lambda_n \left\langle w, x^* - z_n \right\rangle \geq \left\langle x_n - z_n, x^* - z_n \right\rangle \tag{14}$$

By $w \in \partial_2 f_{[n]}(y_n, z_n)$,

$$f_{[n]}(y_n, y) - f_{[n]}(y_n, z_n) \geq \langle w, y - z_n \rangle, \quad \forall y \in H.$$

This together with (14) implies that

$$\lambda_n \left\{ f_{[n]}(y_n, x^*) - f_{[n]}(y_n, z_n) \right\} \geq \left\langle x_n - z_n, x^* - z_n \right\rangle. \tag{15}$$

Note that $x^* \in EP(f_{[n]}, C_{[n]})$ and $y_n \in C_{[n]}$, so $f_{[n]}(x^*, y_n) \geq 0$. The pseudomonotonicity of $f_{[n]}$ implies that $f_{[n]}(y_n, x^*) \leq 0$. From (15), we get

$$\left\langle x_n - z_n, z_n - x^* \right\rangle \geq \lambda_n f_{[n]}(y_n, z_n). \tag{16}$$

The Lipschitz-type continuity of $f_{[n]}$ leads to

$$f_{[n]}(y_n, z_n) \geq f_{[n]}(x_n, z_n) - f_{[n]}(x_n, y_n) - c_1 ||x_n - y_n||^2 - c_2 ||z_n - y_n||^2. \tag{17}$$

Combining relations (16) and (17), we obtain

$$\begin{aligned} \left\langle x_n - z_n, z_n - x^* \right\rangle \geq \lambda_n &\left\{ f_{[n]}(x_n, z_n) - f_{[n]}(x_n, y_n) \right\} \\ &- \lambda_n c_1 ||x_n - y_n||^2 - \lambda_n c_2 ||z_n - y_n||^2. \end{aligned} \tag{18}$$

By (12), (18), we obtain

$$\begin{aligned} \left\langle x_n - z_n, z_n - x^* \right\rangle \geq &\langle x_n - y_n, z_n - y_n \rangle - \lambda_n c_1 ||x_n - y_n||^2 \\ &- \lambda_n c_2 ||z_n - y_n||^2. \end{aligned} \tag{19}$$

We have the following facts

$$2 \langle x_n - z_n, z_n - x^* \rangle = ||x_n - x^*||^2 - ||z_n - x_n||^2 - ||z_n - x^*||^2. \tag{20}$$

$$2 \langle x_n - y_n, z_n - y_n \rangle = ||x_n - y_n||^2 + ||z_n - y_n||^2 - ||x_n - z_n||^2. \tag{21}$$

Relations (19)–(21) lead to the desired conclusion of Lemma 3.3. □

Lemma 3.4 *Let* $\{x_n\}, \{y_n\}, \{z_n\}$ *be the sequences generated by Algorithm* 3.1. *Then*

(i) $F \subset W_n \cap H_n$ *and* x_{n+1} *is well-defined for all* $n \geq 0$.
(ii) $\lim_{n\to\infty} ||x_{n+1} - x_n|| = \lim_{n\to\infty} ||y_n - x_n|| = \lim_{n\to\infty} ||z_n - x_n|| = 0$.

Proof (i). From the definitions of H_n, W_n, we see that these sets are closed and convex. We now show that $F \subset H_n \cap W_n$ for all $n \geq 0$. For each $i = 1, \ldots, N$, let

$$B_n = \left\{ z \in H : \left\langle x_n - z_n, z - x_n - \frac{1}{2}(z_n - x_n) \right\rangle \leq 0 \right\}.$$

By $\gamma_n \in [\epsilon, \frac{1}{2}]$, $B_n \subset H_n$. From Lemma 3.3 and the assumption of λ_n, we obtain $||z_n - x^*|| \leq ||x_n - x^*||$ for all $x^* \in F$. This inequality is equivalent to the following inequality

$$\left\langle x_n - z_n, x^* - x_n - \frac{1}{2}(z_n - x_n) \right\rangle \leq 0, \quad \forall x^* \in F.$$

Therefore, $F \subset B_n$ for all $n \geq 0$. Next, we show that $F \subset B_n \cap W_n$ for all $n \geq 0$ by the induction. Indeed, we have $F \subset B_0 \cap W_0$. Assume that $F \subset B_n \cap W_n$ for some $n \geq 0$. From $x_{n+1} = P_{H_n \cap W_n}(x_0)$ and (10), we obtain

$$\langle x_0 - x_{n+1}, x_{n+1} - z \rangle \geq 0, \quad \forall z \in H_n \cap W_n.$$

Since $F \subset (B_n \cap W_n) \subset (H_n \cap W_n)$,

$$\langle x_0 - x_{n+1}, x_{n+1} - z \rangle \geq 0, \quad \forall z \in F.$$

This together with the definition of W_{n+1} implies that $F \subset W_{n+1}$, and so $F \subset (B_n \cap W_n) \subset (H_n \cap W_n)$ for all $n \geq 0$. Since F is nonempty, x_{n+1} is well-defined.

(ii). From the definition W_n, we have $x_n = P_{W_n}(x_0)$. For each $u \in F \subset W_n$, from (9), one obtains

$$||x_n - x_0|| \leq ||u - x_0||. \tag{22}$$

Thus, the sequence $\{||x_n - x_0||\}$ is bounded, and so $\{x_n\}$ is. Moreover, the projection $x_{n+1} = P_{H_n \cap W_n}(x_0)$ implies $x_{n+1} \in W_n$. From (9) and $x_n = P_{W_n}(x_0)$, we see that

$$||x_n - x_0|| \leq ||x_{n+1} - x_0||.$$

So, the sequence $\{||x_n - x_0||\}$ is non-decreasing. Hence, there exists the limit of the sequence $\{||x_n - x_0||\}$. By $x_{n+1} \in W_n$, $x_n = P_{W_n}(x_0)$ and relation (9), we also have

$$||x_{n+1} - x_n||^2 \leq ||x_{n+1} - x_0||^2 - ||x_n - x_0||^2. \tag{23}$$

Passing to the limit in inequality (23) as $n \to \infty$, one gets

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{24}$$

From the definition of H_n and $x_{n+1} \in H_n$, we have

$$\gamma_n ||z_n - x_n||^2 \leq \langle x_n - z_n, x_n - x_{n+1}\rangle \leq ||x_n - z_n|| ||x_n - x_{n+1}||.$$

Thus, $\gamma_n ||z_n - x_n|| \leq ||x_n - x_{n+1}||$. From $\gamma_n \geq \epsilon > 0$ and (24), one has

$$\lim_{n\to\infty} ||z_n - x_n|| = 0. \tag{25}$$

From Lemma 3.3 and the triangle inequality, we have

$$\begin{aligned}(1 - 2\lambda_n c_1)\, ||y_n - x_n||^2 &\leq ||x_n - x^*||^2 - ||z_n - x^*||^2\\ &\leq (||x_n - x^*|| + ||z_n - x^*||)(||x_n - x^*|| - ||z_n - x^*||)\\ &\leq (||x_n - x^*|| + ||z_n - x^*||)||x_n - z_n||.\end{aligned}$$

The last inequality together with (25), the hypothesis of λ_n and the boundedness of $\{x_n\}$, $\{z_n\}$ implies that

$$\lim_{n\to\infty} ||y_n - x_n|| = 0.$$

The proof of Lemma 3.4 is complete. □

Theorem 3.5 *Let C_i, $i = 1, 2, \ldots, N$ be nonempty closed convex subsets of a real Hilbert space H such that $C = \cap_{i=1}^N C_i \neq \emptyset$. Assume that the bifunctions $f_i, i = 1, \ldots, N$ satisfy all conditions* (A1) − (A4). *In addition, the solution set F is nonempty. Then, the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ generated by Algorithm 3.1 converge strongly to $P_F(x_0)$.*

Proof By Lemma 3.4, we see that the sets H_n, W_n are closed and convex for all $n \geq 0$. Besides, the sequence $\{x_n\}$ is bounded. Assume that p is some weak cluster point of the sequence $\{x_n\}$. From Lemma 3.4(ii) and [6, Theorem 5.3], for each fixed $i \in \{1, 2, \ldots, N\}$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ weakly converging to p, i.e., $x_{n_j} \rightharpoonup p$ as $j \to \infty$ such that $[n_j] = i$ for all j. We now show that $p \in F$. Indeed, from the definition of y_{n_j} and Lemma 2.5, one gets

$$0 \in \partial_2 \left\{\lambda_{n_j} f_{[n_j]}(x_{n_j}, y) + \frac{1}{2}||x_{n_j} - y||^2\right\}(y_{n_j}) + N_{C_{[n_j]}}(y_{n_j}).$$

Thus, there exist $\bar{w} \in N_{C_{[n_j]}}(y_{n_j})$ and $w \in \partial_2 f_{[n_j]}(x_{n_j}, y_{n_j})$ such that

$$\lambda_{n_j} w + x_{n_j} - y_{n_j} + \bar{w} = 0. \tag{26}$$

From the definition of the normal cone $N_{C_{[n_j]}}(y_{n_j})$, we have $\langle \bar{w}, y - y_{n_j}\rangle \leq 0$ for all $y \in C_{[n_j]}$. Taking into account (26), we obtain

$$\lambda_{n_j} \langle w, y - y_{n_j}\rangle \geq \langle y_{n_j} - x_{n_j}, y - y_{n_j}\rangle \tag{27}$$

for all $y \in C_{[n_j]}$. Since $w \in \partial_2 f_{[n_j]}(x_{n_j}, y_{n_j})$,

$$f_{[n_j]}(x_{n_j}, y) - f_{[n_j]}(x_{n_j}, y_{n_j}) \geq \left\langle w, y - y_{n_j} \right\rangle, \quad \forall y \in H. \tag{28}$$

Combining (27) and (28), one has

$$\lambda_{n_j} \left(f_{[n_j]}(x_{n_j}, y) - f_{[n_j]}(x_{n_j}, y_{n_j}) \right) \geq \left\langle y_{n_j} - x_{n_j}, y - y_{n_j} \right\rangle \tag{29}$$

for all $\forall y \in C_{[n_j]}$. From Lemma 3.4(ii) and $x_{n_j} \rightharpoonup p$, we also have $y_{n_j} \rightharpoonup p$. Passing to the limit in inequality (29) and employing assumption (A3), we conclude that $f_{[n_j]}(p, y) \geq 0$ for all $y \in C_{[n_j]}$. Since $[n_j] = i$ for all j, $p \in EP(f_i, C_i)$. This is true for all $i = 1, \ldots, N$. Thus, $p \in F$. Finally, we show that $x_{n_j} \to p$. Let $x^\dagger = P_F(x_0)$. Using inequality (22) with $u = x^\dagger$, we get

$$||x_{n_j} - x_0|| \leq ||x^\dagger - x_0||.$$

By the weak lower semicontinuity of the norm $||.||$ and $x_{n_j} \rightharpoonup p$, we have

$$||p - x_0|| \leq \lim \inf_{j \to \infty} ||x_{n_j} - x_0|| \leq \lim \sup_{j \to \infty} ||x_{n_j} - x_0|| \leq ||x^\dagger - x_0||.$$

By the definition of $x^\dagger$, $p = x^\dagger$ and $\lim_{j \to \infty} ||x_{n_j} - x_0|| = ||x^\dagger - x_0||$. Since $x_{n_j} - x_0 \rightharpoonup x^\dagger - x_0$ and the Kadec–Klee property of the Hilbert space H, we have $x_{n_j} - x_0 \to x^\dagger - x_0$. Thus $x_{n_j} \to x^\dagger = P_F(x_0)$ as $j \to \infty$. Now, assume that $\bar{p}$ is any weak cluster point of the sequence $\{x_n\}$. By above same arguments, we also get $\bar{p} = x^\dagger$. Therefore, $x_n \to P_F(x_0)$ as $n \to \infty$. From Lemma 3.4(ii), we also see that $\{y_n\}, \{z_n\}$ converge strongly to $P_F(x_0)$. This completes the proof of Theorem 3.5. □

Remark 3.6 The proof of Theorem 3.5 is different from one of Theorem 3.3(ii) in [14]. We emphasize that the proof of Theorem 3.3(ii) in [14] is based on the resolvent $J_{rf} : H \to 2^C$ of the bifunction rf as

$$J_{rf}(x) = \{z \in C : rf(z, y) + \langle z - x, y - z \rangle \geq 0, \quad \forall y \in C\}, \quad x \in H,$$

where $r > 0$. If f is monotone then J_f is single valued, strongly monotone and firmly nonexpansive, i.e.,

$$||J_{rf}(x) - J_{rf}(y)||^2 \leq \left\langle J_{rf}(x) - J_{rf}(y), x - y \right\rangle,$$

which implies that J_{rf} is nonexpansive. However, if f is pseudomonotone then J_{rf}, in general, is set-valued. Moreover, J_{rf} is not necessarily convex and nonexpansive. Thus, the arguments in the proof of Theorem 3.3(ii) in [14] which use the characteristic properties of J_{rf} can not be applied to the proof of Theorem 3.5.

Remark 3.7 In the special case, CSEP (1) is CSVIP (2) then Algorithm 3.1 becomes the following cyclic algorithm,

$$\begin{cases} y_n = P_{C_{[n]}}(x_n - \lambda_n A_{[n]}(x_n)), \\ z_n = P_{T_n}(x_n - \lambda_n A_{[n]}(y_n)), \\ H_n = \{z \in H : \langle x_n - z_n, z - x_n - \gamma_n(z_n - x_n) \rangle \leq 0\}, \\ W_n = \{z \in H : \langle x_1 - x_n, z - x_n \rangle \leq 0\}, \\ x_{n+1} = P_{H_n \cap W_n}(x_0), \end{cases} \tag{30}$$

where $T_n = \left\{ v \in H : \left\langle (x_n - \lambda_n A_{[n]}(x_n)) - y_n, v - y_n \right\rangle \leq 0 \right\}$. The character of the projection z_n is explicit and it is defined by

$$z_n = \begin{cases} u_n & \text{if } u_n \in T_n, \\ u_n + \frac{v_n - y_n}{||v_n - y_n||^2} \langle v_n - y_n, y_n - u_n \rangle & \text{if } u_n \notin T_n, \end{cases}$$

where $u_n = x_n - \lambda_n A_{[n]}(y_n)$ and $v_n = x_n - \lambda_n A_{[n]}(x_n)$). Using the same techniques as in [19] then x_{n+1} in (30) is also expressed by an explicit formula and we rewrite the algorithm (30) as follows:

$$\begin{cases} y_n = P_{C_{[n]}}(x_n - \lambda_n A_{[n]}(x_n)), \\ \text{set } u_n = x_n - \lambda_n A_{[n]}(y_n), \ v_n = x_n - \lambda_n A_{[n]}(x_n)), \\ z_n = \begin{cases} u_n & \text{if } \langle v_n - y_n, u_n - y_n \rangle \le 0, \\ u_n + \frac{v_n - y_n}{||v_n - y_n||^2} \langle v_n - y_n, y_n - u_n \rangle & \text{if } \langle v_n - y_n, u_n - y_n \rangle > 0, \end{cases} \\ \text{set } \pi_n = \langle x_0 - x_n, \gamma_n (x_n - z_n) \rangle, \ \mu_n = ||x_0 - x_n||^2, \\ \nu_n = ||\gamma_n (x_n - z_n)||^2, \text{ and } \rho_n = \mu_n \nu_n - \pi_n^2. \\ x_{n+1} = \begin{cases} \gamma_n (x_n + z_n), & \text{if } \rho_n = 0 \text{ and } \pi_n \ge 0, \\ x_0 + \gamma_n \left(1 + \frac{\pi_n}{\nu_n}\right)(z_n - x_n), & \text{if } \rho_n > 0 \text{ and } \pi_n \nu_n \ge \rho_n, \\ y_n + \frac{\nu_n}{\rho_n} \left(\pi_n (x_0 - x_n) + \gamma_n \mu_n (z_n - x_n)\right), & \text{if } \rho_n > 0 \text{ and } \pi_n \nu_n < \rho_n. \end{cases} \end{cases} \tag{31}$$

Thus, algorithm (30) (or (31)) can be considered as an improvement of Algorithm 3.1 in [11] for CSVIPs.

Next, we propose a modification of Algorithm 3.1 which combines the subgradient extragradient method and Mann's iteration for finding a common solution of CSEP which is also a fixed point of a nonexpansive mapping S. Some algorithms for finding a common element of the solution set of EPs (or VIPs) and the fixed point set of nonexpansive mappings can be found, for example, in [1, Algorithm 1], [4, Methods A and B], [13, Algorithm 6.1], [35, Algorithms 1, 2 and 3], [31, Theorem 3.2], [32, Theorems 3.1, 3.6 and 3.7], [38, Theorems 3.1 and 3.6].

Algorithm 3.8 (Modified Cyclic Subgradient Extragradient Method)
Initialization *Choose $x_0 \in H$ and three control parameter sequences $\{\lambda_n\}$, $\{\gamma_n\}$, $\{\alpha_n\}$ satisfying the following conditions.*

(i) $0 < \alpha \le \lambda_n \le \beta < \min\left(\frac{1}{2c_1}, \frac{1}{2c_2}\right)$, $\gamma_n \in [\epsilon, \frac{1}{2}]$, *for some* $\epsilon \in (0, \frac{1}{2}]$.
(ii) $\{\alpha_n\} \subset (0, 1)$ *such that* $\lim_{n\to\infty} \sup \alpha_n < 1$.

Step 1 *Solve two strongly convex programs*

$$y_n = \operatorname{argmin}\left\{\lambda_n f_{[n]}(x_n, y) + \frac{1}{2}||x_n - y||^2 : y \in C_{[n]}\right\}.$$

$$z_n = \operatorname{argmin}\left\{\lambda_n f_{[n]}(y_n, y) + \frac{1}{2}||x_n - y||^2 : y \in T_n\right\},$$

where T_n is defined as in Algorithm 3.1.
Step 2 *Calculate $u_n = \alpha_n x_n + (1 - \alpha_n) S z_n$.*
Step 3 *Compute $x_{n+1} = P_{H_n \cap W_n}(x_0)$, where*

$$H_n = \{z \in H : \langle x_n - u_n, z - x_n - \gamma_n(u_n - x_n) \rangle \le 0\};$$
$$W_n = \{z \in H : \langle x_0 - x_n, z - x_n \rangle \le 0\}.$$

*Set $n := n + 1$ and go back **Step 1**.*

Three algorithms in [35] used the extragradient method [30] for equilibrium problems while the idea of Algorithm 3.8 comes from the subgradient extragradient method. The hybrid step for finding projection $x_{n+1} = P_{H_n \cap W_n}(x_0)$ in Algorithm 3.8 is explicit, but that one for the algorithms in [35] still deals with the feasible set C. The approximation z_n in Step 1 belongs to the halfspace T_n and it, in general, is not in C. Thus, we assume here that S is defined on the whole space H. For $N = 1$, the author in [1] proposed a strongly convergent hybrid extragradient algorithm for an equilibrium problem and a fixed point problem which does not use cutting-halfspaces. However, its convergence requires a strong assumption that $||x_{n+1} - x_n|| \to 0$ as $n \to \infty$. We have the following result for the convergence of Algorithm 3.8.

Theorem 3.9 *Let C_i, $i = 1, \ldots, N$ be nonempty closed convex subsets of a real Hilbert space H such that $C = \cap_{i=1}^N C_i \ne \emptyset$. Assume that the bifunctions f_i, $i = 1, \ldots, N$ satisfy all conditions* (A1) − (A4) *and $S : H \to H$ is a nonexpansive mapping. In addition, the solution set $F \cap F(S)$ is nonempty. Then, the sequences $\{x_n\}, \{y_n\}, \{z_n\}, \{u_n\}$ generated by Algorithm 3.8 converge strongly to $P_{F \cap F(S)}(x_0)$.*

Proof From Lemma 2.1, $F(S)$ is closed and convex. Therefore, the sets $F \cap F(S)$, H_n, W_n are closed and convex for all $n \geq 0$. By arguing similarly to the proof of Lemma 3.4, we also have $F \cap F(S) \subset H_n \cap W_n$ for all $n \geq 0$. We next show that

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = \lim_{n\to\infty} ||y_n - x_n|| = \lim_{n\to\infty} ||z_n - x_n|| = 0,$$
$$\lim_{n\to\infty} ||u_n - x_n|| = \lim_{n\to\infty} ||S(x_n) - x_n|| = 0.$$

Indeed, by arguing similarly to (24), (25) we obtain

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = \lim_{n\to\infty} ||u_n - x_n|| = 0. \tag{32}$$

By the triangle inequality, we have $\left| ||x_n - x^*||^2 - ||u_n - x^*||^2 \right| \leq ||x_n - u_n||(||x_n - x^*|| + ||u_n - x^*||)$. The last inequality together with (32), the boundedness of $\{x_n\}$, $\{u_n\}$ one has

$$\lim_{n\to\infty} \left(||x_n - x^*||^2 - ||u_n - x^*||^2 \right) = 0. \tag{33}$$

For each $x^* \in F \cap F(S)$, from the convexity of $||.||^2$ and Lemma 3.3 we get

$$\begin{aligned}
||u_n - x^*||^2 &= ||\alpha_n(x_n - x^*) + (1-\alpha_n)(Sz_n - x^*)||^2 \\
&\leq \alpha_n ||x_n - x^*||^2 + (1-\alpha_n)||Sz_n - x^*||^2 \\
&\leq \alpha_n ||x_n - x^*||^2 + (1-\alpha_n)||z_n - x^*||^2 \\
&= ||x_n - x^*||^2 + (1-\alpha_n)\left\{ z_n - x^*||^2 - ||x_n - x^*||^2 \right\} \\
&\leq ||x_n - x^*||^2 - (1-\alpha_n)\left\{ (1 - 2\lambda_n c_1)||x_n - y_n||^2 + (1 - 2\lambda_n c_2)||z_n - y_n||^2 \right\}.
\end{aligned}$$

Therefore,

$$(1 - 2\lambda_n c_1)||x_n - y_n||^2 + (1 - 2\lambda_n c_2)||z_n - y_n||^2 \leq \frac{||x_n - x^*||^2 - ||u_n - x^*||^2}{1 - \alpha_n}.$$

Combining this inequality with relation (33) and the hypotheses (i), (ii), we obtain

$$\lim_{n\to\infty} ||x_n - y_n|| = \lim_{n\to\infty} ||z_n - y_n|| = 0. \tag{34}$$

Thus, from $||x_n - z_n|| \leq ||x_n - y_n|| + ||y_n - z_n||$ and (34), we obtain

$$\lim_{n\to\infty} ||x_n - z_n|| = 0.$$

Moreover, from $u_n = \alpha_n x_n + (1-\alpha_n)Sz_n$, we obtain

$$||u_n - x_n|| = (1-\alpha_n)||x_n - Sz_n||. \tag{35}$$

From (32), (35) and the hypothesis $\lim_{n\to\infty} \sup \alpha_n < 1$, we conclude that

$$\lim_{n\to\infty} ||x_n - Sz_n|| = 0.$$

This together with the inequality $||x_n - Sx_n|| \leq ||x_n - Sz_n|| + ||Sz_n - Sx_n|| \leq ||x_n - Sz_n|| + ||z_n - x_n||$ implies that

$$\lim_{n\to\infty} ||x_n - Sx_n|| = 0. \tag{36}$$

Note that $\{x_n\}$ is bounded. Assume that p is any weak cluster point of the sequence $\{x_n\}$. From Lemma 3.4(ii) and [6, Theorem 5.3] (or [3, Lemma 6]), for each fixed $i \in \{1, 2, \ldots, N\}$, there exists a subsequence $\left\{x_{n_j}\right\}$ of $\{x_n\}$ converging weakly to p, i.e., $x_{n_j} \rightharpoonup p$ as $j \to \infty$ such that $[n_j] = i$ for all j. Lemma 2.1 and relation (36) ensure that $p \in F(S)$. Repeating the proof of Theorem 3.5, we conclude that $p \in F$, hence $p \in F \cap F(S)$ and $x_n \to p$ as $n \to \infty$. The proof of Theorem 3.9 is complete. □

Theorem 3.9 with $N = 1$ gives us the following result.

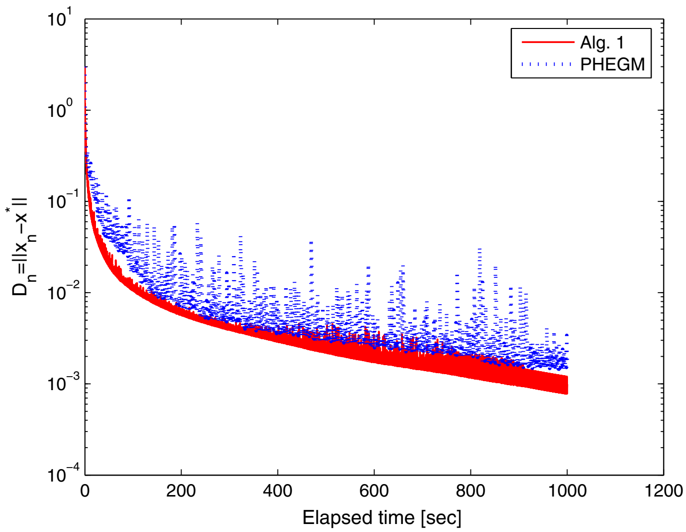

Fig. 3 Behavior of D_n in Experiment 1 for Algorithm 3.1 and PHEGM with $\lambda_n = 1/4c_1$

Corollary 3.10 *Let C be a nonempty closed convex subset of a real Hilbert space H. Assume that the bifunction f satisfies all conditions* (A1) − (A4) *and $S : H \to H$ is a nonexpansive mapping. In addition, the solution set $EP(f, C) \cap F(S)$ is nonempty. Let $\{x_n\}, \{y_n\}, \{z_n\}, \{u_n\}$ be the sequences generated by the following manner*

$$\begin{cases} x_0 \in H, \\ y_n = \text{argmin}\{\lambda_n f(x_n, y) + \frac{1}{2}||x_n - y||^2 : y \in C\}, \\ z_n = \text{argmin}\{\lambda_n f(y_n, y) + \frac{1}{2}||x_n - y||^2 : y \in T_n\}, \\ u_n = \alpha_n x_n + (1-\alpha_n)Sz_n, \\ H_n = \{z \in H : \langle x_n - u_n, z - x_n - \gamma_n(u_n - x_n)\rangle \le 0\}, \\ W_n = \{z \in H : \langle x_0 - x_n, z - x_n\rangle \le 0\}, \\ x_{n+1} = P_{H_n \cap W_n}(x_0), \end{cases}$$

where T_n is defined as in Algorithm 3.1 with $w_n \in \partial_2 f(x_n, y_n)$ and $0 < \alpha \le \lambda_n \le \beta < \min\left(\frac{1}{2c_1}, \frac{1}{2c_2}\right)$, $0 < \epsilon \le \gamma_n \le 1/2$, $0 < \alpha_n < 1$, $\lim_{n\to\infty} \sup \alpha_n < 1$. Then, the sequences $\{x_n\}, \{y_n\}, \{z_n\}, \{u_n\}$ converge strongly to $P_{EP(f,C)\cap F(S)}x_0$.

4 Numerical experiments

We consider the feasible sets $C_i = C$ for all $i = 1, \dots, N$ and a family of bifunctions $f_i : C \times C \to \Re$ in $\Re^m$ ($m = 10$) by

$$f_i(x, y) = \langle P_i x + Q_i y + q_i, y - x\rangle, \ i = 1, 2, \dots, N, \ (N = 10),$$

where P_i, Q_i are matrices of order m such that Q_i is symmetric positive semidefinite and $Q_i - P_i$ is negative semidefinite, q_i is a vector in $\Re^m$ for each i. The starting point x_0 is $x_0 = (1, 1, \dots, 1)^T \in \Re^m$. We compare Algorithm 3.1 with the parallel hybrid extragradient method (PHEGM) [35, Algorithm 1]. The advantage of the proposed algorithms is a computational modification of an optimization program over each iteration. Thus, we use the function $D_n = ||x_n - x^*||$, $n = 0, 1, \dots$ to check the convergence of $\{x_n\}$ generated by the algorithms when execution time elapses, where $x^* = P_F(x_0)$ is a solution of the considered problem. The convergence of $\{D_n\}$ to 0 implies that the sequence $\{x_n\}$ converges to the solution of the problem. We do not compare the numbers of iterations of the algorithms because this seems to be not fair. In fact, per each step Algorithm 3.1 only computes a bifunction while PHEGM computes simultaneously N bifunctions.

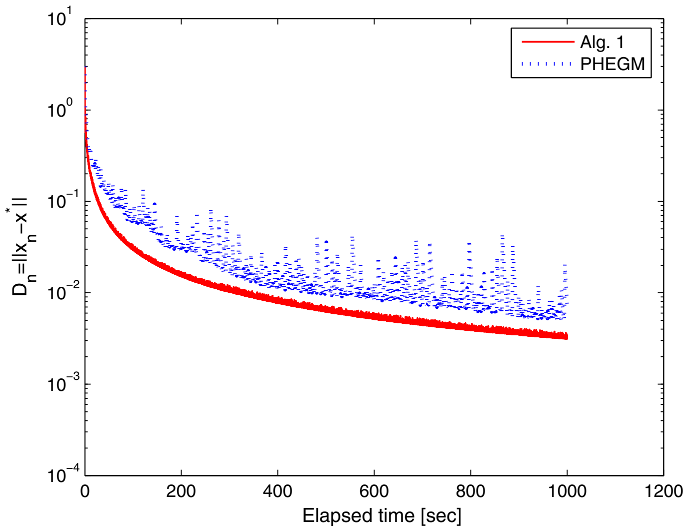

Fig. 4 Behavior of D_n in Experiment 1 for Algorithm 3.1 and PHEGM with $\lambda_n = 1/10c_1$

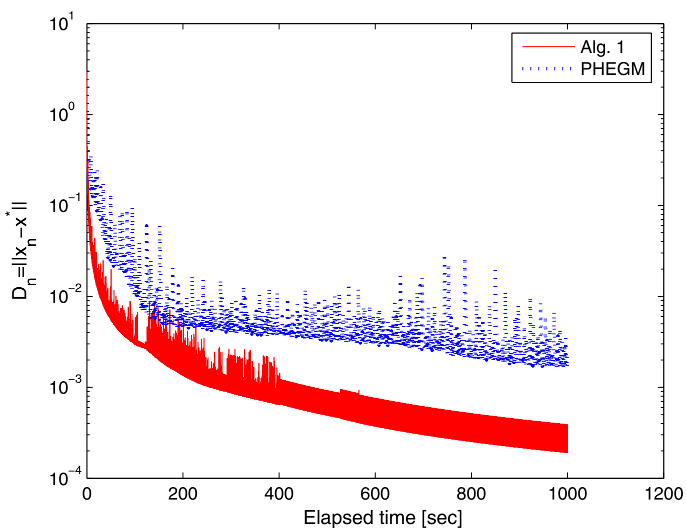

Fig. 5 Behavior of D_n in Experiment 1 for Algorithm 3.1 and PHEGM with $\lambda_n = 1/2.01c_1$

All the convex optimization problems over C and quadratic convex ones over polyhedral convex sets are solved, respectively, by the functions *fmincon* and *quadprog* in Matlab 7.0 Optimization Toolbox. All the projections onto the intersection of C and halfspaces in [35, Algorithm 1] are rewritten equivalently to distance optimization problems while ones onto the intersection of two halfspaces in Algorithm 3.1 are explicit. The program is written in Matlab 7.0 and performed on a PC Desktop Intel(R) Core(TM) i5-3210M CPU @ 2.50 GHz 2.50 GHz, RAM 2.00 GB.

Experiment 1 Suppose that $C = B_1 \cap B_2$, where $B_1 = \left\{x \in \Re^m : ||x||^2 \leq 4\right\}$ and $B_2 = \left\{x \in \Re^m : ||x - (2, 0, \ldots, 0)||^2 \leq 1\right\}$ and $q_i = 0$ for all i. With each i, we chose $P_i = Q_i$ is a diagonal matrix with the first diagonal entry being 1 and other diagonal ones being generated randomly and uniformly in $[2, m]$. The bifunctions f_i satisfy all conditions (A1)–(A4) for all Lipschitz-type constants $c_1, c_2 > 0$ and we chose here $c_1 = c_2 = 5$. By a straightforward computation, the exact solution of the problem is $x^* = (1, 0, \ldots, 0)$. Three fixed stepsizes of λ_n are chosen as $\lambda_n = \lambda$, where $\lambda \in \left\{\frac{1}{4c_1}, \frac{1}{10c_1}, \frac{1}{2.01c_1}\right\}$ and the parameter γ_n in Algorithm 3.1 is $\gamma_n = \frac{1}{2}$ for all $n \geq 0$.

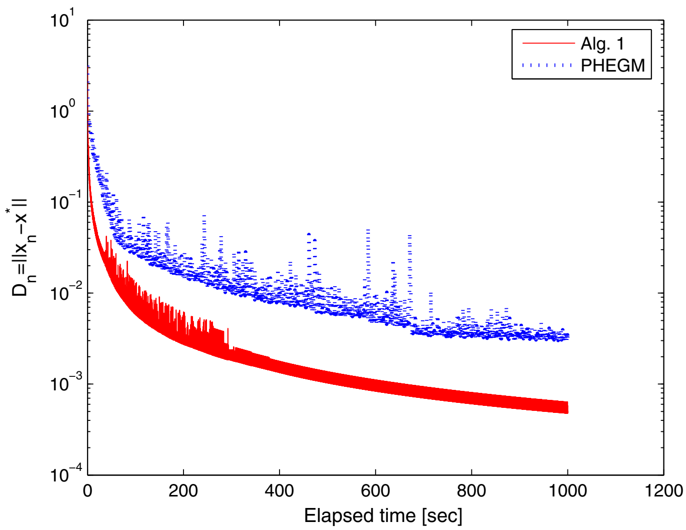

Fig. 6 Behavior of D_n in Experiment 2 for Algorithm 3.1 and PHEGM with $\lambda_n = 1/4c_1$

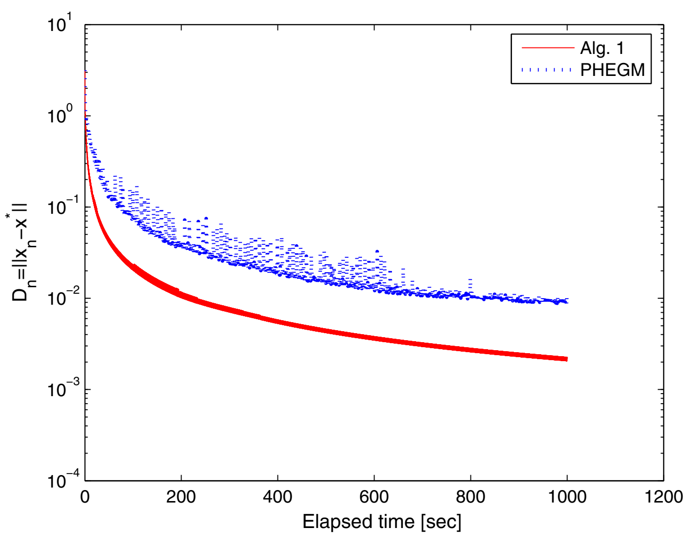

Fig. 7 Behavior of D_n in Experiment 2 for Algorithm 3.1 and PHEGM with $\lambda_n = 1/10c_1$

Figures 3, 4 and 5 show the results for $\{D_n\}$ generated by Algorithm 3.1 and PHEGM [35] for the chosen stepsizes of λ_n. In these figures, the y-axes represent the value of D_n while the x-axes represent elapsed time (in second). From these figures, we see that D_n with Algorithm 3.1 decreases faster than that one with PHEGM after the first 1000s elapses. Besides, $\{D_n\}$ generated by the algorithms, in general, is not monotone and the behavior of it also depends on the stepsize of λ_n.

Experiment 2 The feasible set C is the intersection of six balls with the same radius $r = 2$ and the centers as $a_1 = (1, 0, 0, \ldots, 0)$, $a_2 = (-1, 0, 0, \ldots, 0)$, $a_3 = (0, 1, 0, \ldots, 0)$, $a_4 = (0, -1, 0, \ldots, 0)$, $a_5 = (0, 0, 1, 0, \ldots, 0)$, $a_6 = (0, 0, -1, 0, \ldots, 0)$. Note that $C \neq \emptyset$ because $0 \in C$. In this experiment, we chose q_i is the zero vector for all i. For each $i = 2, \ldots, N$, two matrices P_i, Q_i are randomly generated[1] satisfying the conditions of the problem. Two matrices P_1, Q_1 are made similarly such that $Q_1 - P_1$ is negative definite. Thus f_1 is strongly monotone. From the properties of P_i and Q_i, $EP(f_1, C) = \{0\}$ and $0 \in EP(f_i, C)$ for all $i = 2, 3, \ldots, N$. Hence, $F = \cap_{i=1}^N EP(f_i, C) = \{0\}$. Each bifunction f_i satisfies conditions (A1)-(A4) with

[1] We randomly chose $\lambda_{1k}^i \in [-m, 0]$, $\lambda_{2k}^i \in [1, m]$, $k = 1, \ldots, m$, $i = 2 \ldots, N$. Set $\widehat{Q}_1^i$, $\widehat{Q}_2^i$ as two diagonal matrixes with eigenvalues $\{\lambda_{1k}^i\}_{k=1}^m$ and $\{\lambda_{2k}^i\}_{k=1}^m$, respectively. Then, we make a positive definite matrix Q_i and a negative semidefinite matrix T_i by using random orthogonal matrixes with $\widehat{Q}_2^i$ and $\widehat{Q}_1^i$, respectively. Finally, set $P_i = Q_i - T_i$.

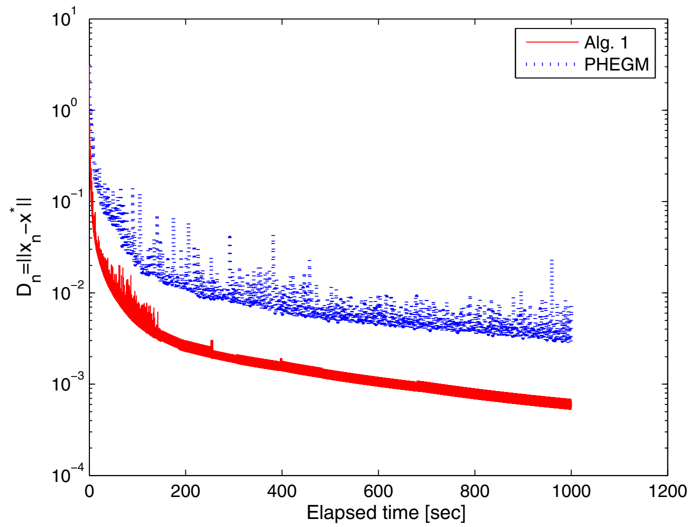

Fig. 8 Behavior of D_n in Experiment 2 for Algorithm 3.1 and PHEGM with $\lambda_n = 1/2.01c_1$

$c_1^i = c_2^i = ||P_i - Q_i||/2$ [30, Lemma 6.1]. We chose $c_1 = c_2 = \max\left\{c_1^i : i = 1, 2, \ldots, N\right\}$. The parameters γ_n and λ_n are chosen as in Experiment 1. Figures 6, 7 and 8 describe the behaviors of $\{D_n\}$ generated by the algorithms with $\lambda_n = \frac{1}{4c_1}$, $\lambda_n = \frac{1}{10c_1}$ and $\lambda_n = \frac{1}{2.01c_1}$, respectively. The obtained results are similar to those in Experiment 1.

5 Conclusions

The paper extends the subgradient extragradient method for variational inequalities to equilibrium problems. Based on this extension, some cyclic iterative algorithms are proposed for finding a particular solution of a system of equilibrium problems. The algorithms can be considered as modifications of the extragradient method. Some preliminary numerical experiments are implemented to illustrate the convergence of the proposed algorithm and compare it with the parallel hybrid extragradient method.

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