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On multiple Walsh–Fourier coefficients of functions of $\phi - \Lambda$-bounded variation

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Abstract Here, we have estimated the order of magnitude of multiple Walsh–Fourier coefficients of functions of $\phi(\Lambda^1, \ldots, \Lambda^N)BV([0, 1]^N)$.

Mathematics Subject Classification 42C10 · 42B05 · 26B30 · 26D15

الملخص

قدرنا في هذه الورقة درجة المقدار لمعاملات والش – فورير متعددة للدوال $\emptyset(\Lambda^1, \cdots, \Lambda^N)BV([0,1]^N)$.

1 Introduction

In 2011, Móricz and Veres [6] obtained sufficiency condition for the absolute convergence of double Walsh–Fourier series. Recently, the order of magnitude of Walsh–Fourier coefficients of functions of the class $\Lambda BV(p(n) \uparrow \infty, \varphi, [0, 1])$ is estimated [2]. Here, we have estimated the order of magnitude of multiple Walsh–Fourier coefficients of functions of the class $\phi(\Lambda^1, \ldots, \Lambda^N)BV([0, 1]^N)$.

2 Notations and definitions

In the sequel $\mathbb{I} = [0, 1)$; $\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}$, $\mathbb{L}$ is a class of non-decreasing sequences $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ of positive numbers, such that $\sum_n \frac{1}{\lambda_n}$ diverges, and ϕ is an increasing convex function defined on the non-negative real numbers, such that $\phi(0) = 0$, $\frac{\phi(x)}{x} \to 0$ as $x \to 0$ and $\frac{\phi(x)}{x} \to \infty$ as $x \to \infty$.

The function ϕ is said to have property Δ_2 if there is a constant $d \geq 2$ such that $\phi(2x) \leq d\phi(x)$ for all $x \geq 0$.

Consider function f on $\mathbb{R}^k$. For $k = 1$ and $I = [a, b]$, define $\Delta f_a^b = f(I) = f(b) - f(a)$. For $k = 2$, $I = [a, b]$ and $J = [c, d]$, define

$$\Delta f_{(a,c)}^{(b,d)} = f(I \times J) = f(I, d) - f(I, c) = f(b, d) - f(a, d) - f(b, c) + f(a, c).$$

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Definition 2.1 For a given $\bigwedge = (\Lambda^1, \Lambda^2)$, where $\Lambda^k = \{\lambda_n^k\}_{n=1}^{\infty} \in \mathbb{L}$, for $k = 1, 2$, a complex valued measurable function f defined on a rectangle $R^2 := [a, b] \times [c, d]$ is said to be of $\phi - \bigwedge$-bounded variation (that is, $f \in \phi \bigwedge BV(R^2)$) if

$$V_{\bigwedge_\phi}(f, R^2) = \sup_{I_1, I_2} \left(\sum_j \sum_k \frac{\phi(|f(I_j \times I_k)|)}{\lambda_j^1 \lambda_k^2} \right) < \infty,$$

where I_1 and I_2 are finite collections of nonoverlapping subintervals $\{I_j\}$ and $\{I_k\}$ in $[a, b]$ and $[c, d]$, respectively.

Observe that a function $f \in \phi \bigwedge BV(R^2)$ need not be bounded.

Consider [4, Example 1.19(i), p. 23] $f : [0, 1]^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} \frac{1}{x} + \frac{1}{y}, & \text{if } x \neq 0 \text{ and } y \neq 0, \\ \frac{1}{x}, & \text{if } x \neq 0 \text{ and } y = 0, \\ \frac{1}{y}, & \text{if } x = 0 \text{ and } y \neq 0, \\ 0, & \text{if } x = 0 \text{ and } y = 0. \end{cases}$$

Then, $V_{\bigwedge_\phi}(f, [0, 1]^2) = 0$. Thus, unbounded function $f \in \phi \bigwedge BV([0, 1]^2)$.

If $f \in \phi \bigwedge BV(R^2)$ is such that the marginal functions $f(a, .) \in \phi\Lambda^2 BV([c, d])$ and $f(., c) \in \phi\Lambda^1 BV([a, b])$ (see [7, Definition 2, p. 770]), then f is said to be of $\phi - \bigwedge^*$-bounded variation (that is, $f \in \phi \bigwedge^* BV(R^2)$).

Observe that for $\phi(x) = x$ (and $\phi(x) = x^p$, $p = 1$), the conditions $\frac{\phi(x)}{x} \to 0$ as $x \to 0$ and $\frac{\phi(x)}{x} \to \infty$ as $x \to \infty$ are not valid.

Note that, for $\phi(x) = x$ and $\Lambda^1 = \Lambda^2 = \{1\}$ (that is, $\lambda_n^1 = \lambda_n^2 = 1$, for all n), the classes $\phi \bigwedge BV(R^2)$ and $\phi \bigwedge^* BV(R^2)$ reduce to the classes $BV_V(R^2)$ of functions of bounded variation in the sense of Vitali (refer [5, p. 279] for the definition of $BV_V(R^2)$) and $BV_H(R^2)$ of functions of bounded variation in the sense of Hardy (refer [5, p. 280] for the definition of $BV_H(R^2)$), respectively; for $\phi(x) = x$, the classes $\phi \bigwedge BV(R^2)$ and $\phi \bigwedge^* BV(R^2)$ reduce to the classes $\bigwedge BV(R^2)$ (see [1, Definition 2, p. 8]) and $\bigwedge^* BV(R^2)$ (see [3, Definition 2, p. 398]), respectively; and for $\phi(x) = x^p$ $(p \geq 1)$, the classes $\phi \bigwedge BV(R^2)$ and $\phi \bigwedge^* BV(R^2)$ reduce to the classes $\bigwedge BV^{(p)}(R^2)$ (see [8, Definition 1.2]) and $\bigwedge^* BV^{(p)}(R^2)$, respectively.

Let $\{\psi_m\}_{m \in \mathbb{N}_0}$ denote the complete orthonormal Walsh system defined on the interval [0,1] in the Paley enumeration, where the subscript denotes the number of zeros (that is, sign changes) in the interior of the interval [0,1].

Any $x \in \mathbb{I}$ can be written as

$$x = \sum_{k=0}^{\infty} x_k \, 2^{-(k+1)}, \quad \text{each } x_k = 0 \text{ or } 1.$$

For any $x \in \mathbb{I} \backslash Q$, there is only one expression of this form, where Q is a class of dyadic rationals in $\mathbb{I}$. When $x \in Q$, there are two expressions of this form, one which terminates in 0s and the other which terminates in 1s.

For any $x, y \in \mathbb{I}$, their dyadic sum is defined as

$$x \dotplus y = \sum_{k=0}^{\infty} |x_k - y_k| \, 2^{-(k+1)}.$$

Observed that, for each $m \in \mathbb{N}_0$, we have

$$\psi_m(x \dotplus y) = \psi_m(x) \, \psi_m(y), \quad x, y \in \mathbb{I}, \quad x \dotplus y \notin Q.$$

For a real-valued function $f \in L^1(\bar{\mathbb{I}}^2)$, where f is 1-periodic in each variable, its double Walsh–Fourier series is defined as

$$f(\mathbf{x}) = f(x, y) \sim \sum_{\mathbf{k} \in \mathbb{N}_0^2} \hat{f}(\mathbf{k}) \, \psi_m(x) \, \psi_n(y) = \sum_{m \in \mathbb{N}_0} \sum_{n \in \mathbb{N}_0} \hat{f}(m, n) \, \psi_m(x) \, \psi_n(y),$$

where

$$\hat{f}(\mathbf{k}) = \hat{f}(m,n) = \int\int_{\overline{\mathbb{I}}^2} f(x,y)\, \psi_m(x)\, \psi_n(y)\, \mathrm{d}x\, \mathrm{d}y$$

denotes the **k**th Walsh–Fourier coefficient of f.

3 New results for functions of two variables

We prove the following results.

Theorem 3.1 *If ϕ satisfies Δ_2 condition and $f \in \phi \bigwedge BV(\overline{\mathbb{I}}^2) \cap L^1(\overline{\mathbb{I}}^2)$, then*

$$\hat{f}(2^u, 2^v) = O\left(\phi^{-1}\left(\frac{1}{\sum_{j=1}^{2^u}\sum_{k=1}^{2^v}\frac{1}{\lambda_j^1\,\lambda_k^2}}\right)\right). \tag{3.1}$$

Corollary 3.2 *If ϕ satisfies Δ_2 condition and $f \in \phi \bigwedge^* BV(\overline{\mathbb{I}}^2)$, then (3.1) holds true.*

4 Proof of the results

Proof of Theorem 3.1. For fixed $u, v \in \mathbb{N}_0$, let $h_1 = \frac{1}{2^{u+1}}$ and $h_2 = \frac{1}{2^{v+1}}$. Take

$$\begin{aligned} g(x,y) &= f\left(x \dotplus \frac{1}{2^u} \dotplus \frac{1}{2^{u+1}}, y \dotplus \frac{1}{2^v} \dotplus \frac{1}{2^{v+1}}\right) - f\left(x, y \dotplus \frac{1}{2^v} \dotplus \frac{1}{2^{v+1}}\right) \\ &\quad - f\left(x \dotplus \frac{1}{2^u} \dotplus \frac{1}{2^{u+1}}, y\right) + f(x,y), \end{aligned}$$

for all $(x,y) \in \overline{\mathbb{I}}^2$.

For $m = 2^u$ and $n = 2^v$, $\psi_m(h_1) = \psi_n(h_2) = -1$ and $\psi_m\left(\frac{1}{2^u}\right) = \psi_n\left(\frac{1}{2^v}\right) = 1$ imply that

$$\begin{aligned} \hat{g}(m,n) &= \psi_m\left(\frac{1}{2^u}\right)\psi_m\left(\frac{1}{2^{u+1}}\right)\psi_n\left(\frac{1}{2^v}\right)\psi_n\left(\frac{1}{2^{v+1}}\right)\hat{f}(m,n) \\ &\quad - \psi_n\left(\frac{1}{2^v}\right)\psi_n\left(\frac{1}{2^{v+1}}\right)\hat{f}(m,n) - \psi_m\left(\frac{1}{2^u}\right)\psi_m\left(\frac{1}{2^{u+1}}\right)\hat{f}(m,n) + \hat{f}(m,n) \\ &= 4\hat{f}(m,n) \end{aligned}$$

and

$$\begin{aligned} 4|\hat{f}(m,n)| &\le \int\int_{\overline{\mathbb{I}}^2}\Bigg| f\left(x \dotplus \frac{1}{2^u} \dotplus \frac{1}{2^{u+1}}, y \dotplus \frac{1}{2^v} \dotplus \frac{1}{2^{v+1}}\right) - f\left(x, y \dotplus \frac{1}{2^v} \dotplus \frac{1}{2^{v+1}}\right) \\ &\quad - f\left(x \dotplus \frac{1}{2^u} \dotplus \frac{1}{2^{u+1}}, y\right) + f(x,y)\Bigg|\, \mathrm{d}x\, \mathrm{d}y \\ &= \int\int_{\overline{\mathbb{I}}^2}\Bigg| f\left(\left(x \dotplus \frac{1}{2^{u+1}}\right) \dotplus \left(\frac{1}{2^u} \dotplus \frac{1}{2^{u+1}}\right), \left(y \dotplus \frac{1}{2^{v+1}}\right) \dotplus \left(\frac{1}{2^v} \dotplus \frac{1}{2^{v+1}}\right)\right) \\ &\quad - f\left(x \dotplus \frac{1}{2^{u+1}}, \left(y \dotplus \frac{1}{2^{v+1}}\right) \dotplus \left(\frac{1}{2^v} \dotplus \frac{1}{2^{v+1}}\right)\right) \\ &\quad - f\left(\left(x \dotplus \frac{1}{2^{u+1}}\right) \dotplus \left(\frac{1}{2^u} \dotplus \frac{1}{2^{u+1}}\right), y \dotplus \frac{1}{2^{v+1}}\right) \\ &\quad + f\left(x \dotplus \frac{1}{2^{u+1}}, y \dotplus \frac{1}{2^{v+1}}\right)\Bigg|\, \mathrm{d}x\, \mathrm{d}y \end{aligned}$$

$$= \int\int_{\overline{\mathbb{I}}^2} \Bigg| f\left(x \dotplus \frac{1}{2^u}, y \dotplus \frac{1}{2^v}\right) - f\left(x \dotplus \frac{1}{2^{u+1}}, y \dotplus \frac{1}{2^v}\right) - f\left(x \dotplus \frac{1}{2^u}, y \dotplus \frac{1}{2^{v+1}}\right) + f\left(x \dotplus \frac{1}{2^{u+1}}, y \dotplus \frac{1}{2^{v+1}}\right) \Bigg| \,\mathrm{d}x\, \mathrm{d}y.$$

Similarly, we get

$$4|\hat{f}(m,n)| \le \int\int_{\overline{\mathbb{I}}^2} \Bigg| f\left(x \dotplus \frac{4}{2^{u+1}}, y \dotplus \frac{4}{2^{v+1}}\right) - f\left(x \dotplus \frac{3}{2^{u+1}}, y \dotplus \frac{4}{2^{v+1}}\right) - f\left(x \dotplus \frac{4}{2^{u+1}}, y \dotplus \frac{3}{2^{v+1}}\right) + f\left(x \dotplus \frac{3}{2^{u+1}}, y \dotplus \frac{3}{2^{v+1}}\right) \Bigg| \,\mathrm{d}x\, \mathrm{d}y$$

and in general we have

$$4|\hat{f}(m,n)| \le \int\int_{\overline{\mathbb{I}}^2} |\Delta f_{jk}(x,y)| \,\mathrm{d}x\, \mathrm{d}y, \tag{4.1}$$

where

$$\Delta f_{jk}(x,y) = f\left(x \dotplus \frac{2j}{2^{u+1}}, y \dotplus \frac{2k}{2^{v+1}}\right) - f\left(x \dotplus \frac{(2j-1)}{2^{u+1}}, y \dotplus \frac{2k}{2^{v+1}}\right) - f\left(x \dotplus \frac{2j}{2^{u+1}}, y \dotplus \frac{(2k-1)}{2^{v+1}}\right) + f\left(x \dotplus \frac{(2j-1)}{2^{u+1}}, y \dotplus \frac{(2k-1)}{2^{v+1}}\right),$$

for all $j = 1, \ldots, 2^u - 1$ and for all $k = 1, \ldots, 2^v - 1$.

For $c > 0$, by Jensen's inequality for integrals, we have

$$\phi(c|\hat{f}(2^u, 2^v)|) \le \int\int_{\overline{\mathbb{I}}^2} \phi(c|\Delta f_{jk}(x,y)|) \,\mathrm{d}x\, \mathrm{d}y.$$

Dividing both sides of the above inequality by $\lambda_j^1\, \lambda_k^2$ and then summing over $j = 1$ to $2^u - 1$ and $k = 1$ to $2^v - 1$, we get

$$\phi(c|\hat{f}(2^u, 2^v)|) \left(\sum_{j=1}^{2^u-1} \sum_{k=1}^{2^v-1} \frac{1}{\lambda_j^1\, \lambda_k^2} \right) \le \int\int_{\overline{\mathbb{I}}^2} \left(\sum_{j=1}^{2^u-1} \sum_{k=1}^{2^v-1} \frac{\phi(c|\Delta f_{jk}(x,y)|)}{\lambda_j^1\, \lambda_k^2} \right) \mathrm{d}x\, \mathrm{d}y.$$

For any $x, y \in \mathbb{R}$, all these points $x \dotplus 2jh_1,\ x \dotplus (2j-1)h_1,$ for $j = 1, \ldots, 2^u - 1$, and $y \dotplus 2kh_2,\ y \dotplus (2k-1)h_2,$ for $k = 1, \ldots, 2^v - 1$, lie in the interval of length 1. Thus,

$$\sum_{j=1}^{2^u-1} \sum_{k=1}^{2^v-1} \frac{\phi(c|\Delta f_{jk}(x,y)|)}{\lambda_j^1\, \lambda_k^2} \le V_{\bigwedge_\phi}(cf, \overline{\mathbb{I}}^2),$$

as ϕ satisfies Δ_2 condition implying $cf \in \phi \bigwedge BV(\overline{\mathbb{I}}^2)$.

Therefore,

$$\phi(c|\hat{f}(2^u, 2^v)|) \le \frac{V_{\bigwedge_\phi}(cf, \overline{\mathbb{I}}^2)}{\left(\sum_{j=1}^{2^u-1} \sum_{k=1}^{2^v-1} \frac{1}{\lambda_j^1\, \lambda_k^2}\right)}. \tag{4.2}$$

Since ϕ is convex and $\phi(0) = 0$, for $c \in (0, 1]$ we have $\phi(cx) \le c\phi(x)$ and hence we can choose sufficiently small $c \in (0, 1]$ such that $V_{\bigwedge_\phi}(cf, \overline{\mathbb{I}}^2) \le 1$. This together with $\sum_{j=1}^{2^u} \sum_{k=1}^{2^v} \frac{1}{\lambda_j^1\, \lambda_k^2} \approx \sum_{j=1}^{2^u-1} \sum_{k=1}^{2^v-1} \frac{1}{\lambda_j^1\, \lambda_k^2}$ and the above inequality (4.2) imply that

$$|\hat{f}(2^u, 2^v)| \le \frac{1}{c} \phi^{-1} \left(\frac{1}{\sum_{j=1}^{2^u} \sum_{k=1}^{2^v} \frac{1}{\lambda_j^1\, \lambda_k^2}} \right).$$

This completes the proof of the theorem.

Proof of Corollary 3.2 For any $f \in \phi \bigwedge^* BV(\bar{\mathbb{I}}^2)$,

$$\begin{aligned} |f(x,y)| &\le |f(0,0)| + |f(x,y) - f(0,y) - f(x,0) + f(0,0)| + |f(0,y) - f(0,0)| \\ &\quad + |f(x,0) - f(0,0)| \\ &\le |f(0,0)| + (\lambda_1^1 \lambda_1^2)\phi^{-1}(V_{\bigwedge_\phi}(f, \bar{\mathbb{I}}^2)) + (\lambda_1^2)\phi^{-1}(V_{\Lambda_\phi^2}(f(0,.), \bar{\mathbb{I}})) \\ &\quad + (\lambda_1^1)\phi^{-1}(V_{\Lambda_\phi^1}(f(.,0), \bar{\mathbb{I}})) \end{aligned}$$

implies that f is bounded on $\bar{\mathbb{I}}^2$.

Since $\phi \bigwedge^* BV(\bar{\mathbb{I}}^2) \subset \phi \bigwedge BV(\bar{\mathbb{I}}^2)$, the corollary follows from the Theorem 3.1.

5 Extension of the results for functions of several variables

Let $I^k = [a_k, b_k] \subset \mathbb{R}$, for $k = 1, 2, \ldots, N$. In Sect. 2 above, we defined $f(I^1)$ for a function f of one variable and $f(I^1 \times I^2)$ for a function f of two variables. Similarly, for a function f on $\mathbb{R}^N$, by induction, defining the expression $f(I^1 \times \cdots \times I^{N-1})$ for a function of $N-1$ variables, one gets

$$f(I^1 \times \cdots \times I^N) = f(I^1 \times \cdots \times I^{N-1}, b_N) - f(I^1 \times \cdots \times I^{N-1}, a_N).$$

Observe that $f(I^1 \times \cdots \times I^N)$ can also be expressed as

$$f(I^1 \times \cdots \times I^N) = \Delta f_{\mathbf{a}}^{\mathbf{b}} = \sum_{\mathbf{c}} k(\mathbf{c}) f(\mathbf{c}),$$

where $\mathbf{a} = (a_1, a_2, \ldots, a_N)$, $\mathbf{b} = (b_1, b_2, \ldots, b_N) \in \mathbb{R}^N$, the summation is over all $\mathbf{c} = (c_1, c_2, \ldots, c_N) \in \mathbb{R}^N$ such that $c_i \in \{a_i, b_i\}$, for $i = 1, \ldots, N$, and for any such $\mathbf{c}$, $k(\mathbf{c}) = k_1 \ldots k_N$, in which, for $1 \le i \le N$,

$$k_i = \begin{cases} 1, & \text{if } c_i = b_i, \\ -1, & \text{if } c_i = a_i. \end{cases}$$

Then, for $N = 1$, we get

$$f(I^1) = \Delta f_{\mathbf{a}}^{\mathbf{b}} = \Delta f_{a_1}^{b_1} = \sum_{c_1} k(\mathbf{c}) f(\mathbf{c}) = f(b_1) - f(a_1).$$

For $N = 2$, we get

$$\begin{aligned} f(I^1 \times I^2) = \Delta f_{\mathbf{a}}^{\mathbf{b}} = \Delta f_{(a_1,a_2)}^{(b_1,b_2)} &= \sum_{(c_1,c_2)} k(\mathbf{c}) f(\mathbf{c}) \\ &= f(b_1,b_2) + f(a_1,a_2) - f(b_1,a_2) - f(a_1,b_2). \end{aligned}$$

Similarly, for $N = 3$, we get

$$\begin{aligned} f(I^1 \times I^2 \times I^3) = \Delta f_{\mathbf{a}}^{\mathbf{b}} = \Delta f_{(a_1,a_2,a_3)}^{(b_1,b_2,b_3)} &= \sum_{(c_1,c_2,c_3)} k(\mathbf{c}) f(\mathbf{c}) \\ &= f(b_1,b_2,b_3) + f(b_1,a_2,a_3) + f(a_1,b_2,a_3) + f(a_1,a_2,b_3) \\ &\quad - f(b_1,b_2,a_3) - f(a_1,b_2,b_3) - f(b_1,a_2,b_3) - f(a_1,a_2,a_3). \end{aligned}$$

For a given $\bigwedge = (\Lambda^1, \ldots, \Lambda^N)$, where $\Lambda^k = \{\lambda_n^k\}_{n=1}^\infty \in \mathbb{L}$, for $k = 1, 2, \ldots, N$, a complex valued measurable function f defined on $R^N := \prod_{k=1}^N [a_k, b_k]$ is said to be of $\phi - \bigwedge$-bounded variation (that is, $f \in \phi \bigwedge BV(R^N)$) if

$$V_{\bigwedge_\phi}(f, R^N) = \sup_{I^1,\ldots,I^N} \left(\sum_{k_1} \cdots \sum_{k_N} \frac{\phi(|f(I_{k_1}^1 \times \cdots \times I_{k_N}^N)|)}{\lambda_{k_1}^1 \ldots \lambda_{k_N}^N} \right) < \infty,$$

where $I^1, \ldots, I^{N-1}$ and I^N are finite collections of nonoverlapping subintervals $\{I^1_{k_1}\}, \ldots, \{I^{N-1}_{k_{N-1}}\}$ and $\{I^N_{k_N}\}$ in $[a_1, b_1], \ldots, [a_{N-1}, b_{N-1}]$ and $[a_N, b_N]$, respectively.

Moreover, $f \in \phi \bigwedge BV(R^N)$ is said to be of $\phi - \bigwedge^*$-bounded variation (that is, $f \in \phi \bigwedge^* BV(R^N)$) if for each of its marginal functions

$$f(x_1, \ldots, x_{i-1}, a_i, x_{i+1}, \ldots, x_N) \in \phi(\Lambda^1, \ldots, \Lambda^{i-1}, \Lambda^{i+1}, \ldots, \Lambda^N)^* BV(R^N(a_i)),$$

for all $i = 1, 2, \ldots, N$, where

$$R^N(a_i) = \{(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N) \in \mathbb{R}^{N-1} : x_k \in [a_k, b_k] \quad \text{for } k = 1, \ldots, i-1, i+1, \ldots, N\}.$$

Note that, for $\phi(x) = x$ and $\Lambda^1 = \cdots = \Lambda^N = \{1\}$, the classes $\phi \bigwedge BV(R^N)$ and $\phi \bigwedge^* BV(R^N)$ reduce to the classes $BV_V(R^N)$ and $BV_H(R^N)$, respectively; for $\phi(x) = x$, the classes $\phi \bigwedge BV(R^N)$ and $\phi \bigwedge^* BV(R^N)$ reduce to the classes $\bigwedge BV(R^N)$ and $\bigwedge^* BV(R^N)$, respectively; and for $\phi(x) = x^p$ $(p \geq 1)$, the classes $\phi \bigwedge BV(R^N)$ and $\phi \bigwedge^* BV(R^N)$ reduce to the classes $\bigwedge BV^{(p)}(R^N)$ and $\bigwedge^* BV^{(p)}(R^N)$, respectively.

It is easy to prove that $f \in \phi \bigwedge^* BV(R^N)$ implies f is bounded on R^N.

For a real-valued function $f \in L^1(\overline{\mathbb{I}}^N)$, where f is 1-periodic in each variable, its multiple Walsh–Fourier series is defined as

$$\begin{aligned} f(\mathbf{x}) = f(x_1, \ldots, x_N) &\sim \sum_{\mathbf{k} \in \mathbb{N}_0^N} \hat{f}(\mathbf{k})\, \psi_{k_1}(x_1) \ldots \psi_{k_N}(x_N) \\ &= \sum_{k_1 \in \mathbb{N}_0} \cdots \sum_{k_N \in \mathbb{N}_0} \hat{f}(k_1, \ldots, k_N)\, \psi_{k_1}(x_1) \ldots \psi_{k_N}(x_N), \end{aligned}$$

where

$$\begin{aligned} \hat{f}(\mathbf{k}) &= \int \cdots \int_{\overline{\mathbb{I}}^N} f(\mathbf{x})\, \psi_{k_1}(x_1) \ldots \psi_{k_N}(x_N)\, d\mathbf{x} \\ &= \int \cdots \int_{\overline{\mathbb{I}}^N} f(x_1, \ldots, x_N)\, \psi_{k_1}(x_1) \ldots \psi_{k_N}(x_N)\, \mathrm{d}x_1 \ldots \mathrm{d}x_N \end{aligned}$$

denotes the **k**th Walsh–Fourier coefficient of f.

Now, we extend the above-mentioned results for higher-dimensional spaces in the following way.

Theorem 5.1 *If ϕ satisfies Δ_2 condition and $f \in \phi \bigwedge BV(\overline{\mathbb{I}}^N) \cap L^1(\overline{\mathbb{I}}^N)$, then*

$$\hat{f}(2^{u_1}, \ldots, 2^{u_N}) = O\left(\phi^{-1}\left(\frac{1}{\sum_{r_1=1}^{2^{u_1}} \cdots \sum_{r_N=1}^{2^{u_N}} \frac{1}{\lambda^1_{r_1} \ldots \lambda^N_{r_N}}}\right)\right). \tag{5.1}$$

Corollary 5.2 *If ϕ satisfies Δ_2 condition and $f \in \phi \bigwedge^* BV(\overline{\mathbb{I}}^N)$, then (5.1) holds true.*

The extended results of this section can be proved in the same way as we proved the results in Sect. 4.

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