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# Ulam stabilities for partial Hadamard fractional integral equations

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**Abstract** This paper deals with some existence and Ulam stability results for a class of partial integral equations via Hadamard's fractional integral, by applying Schauder's fixed-point theorem.

**Mathematics Subject Classification** 34A08 · 34K05

## المخلص

تتعامل هذه الورقة مع نتائج وجود واستقرار أولام لصف من المعادلات التكاملية الجزئية باستخدام تكامل هادامارد الكسري بتطبيق مبرهنة شاور للنفطة الثابتة.

## 1 Introduction

The fractional calculus represents a powerful tool in applied mathematics to study many problems from different fields of science and engineering, with many breakthroughs in mathematical physics, finance, hydrology, biophysics, thermodynamics, control theory, statistical mechanics, astrophysics, cosmology and bioengineering [13, 27]. There has been a significant development in ordinary and partial fractional differential and integral equations in recent years; see the monographs of Abbas et al. [3, 4], Kilbas et al. [19], Miller and Ross [20], the papers of Abbas and Benchohra [1, 2], Abbas et al. [5], Benchohra et al. [6], Vityuk [29], Vityuk and Golushkov [30] and the references therein.

In [8], Butzer et al. investigated properties of the Hadamard fractional integral and the derivative. In [9], they obtained the Mellin transforms of the Hadamard fractional integral and differential operators and in [22], Pooseh et al. obtained expansion formulas of the Hadamard operators in terms of integer-order derivatives.

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Many other interesting properties of those operators and others are summarized in [26] and the references therein.

The stability of functional equations was originally raised by Ulam in 1940 in a talk given at Wisconsin University. The problem posed by Ulam was the following: Under what conditions does there exist an additive mapping near an approximately additive mapping? (for more details see [28]). The first answer to Ulam's question was given by Hyers in 1941 in the case of Banach spaces in [14]. Thereafter, this type of stability is called the Ulam–Hyers stability. In 1978, Rassias [23] provided a remarkable generalization of the Ulam–Hyers stability of mappings by considering variables. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus, the stability question of functional equations is how do the solutions of the inequality differ from those of the given functional equation? Considerable attention has been given to the study of the Ulam–Hyers and Ulam–Hyers–Rassias stability of all kinds of functional equations; one can see the monographs [15, 16]. Bota-Boriceanu and Petrusel [7], Petru et al. [21], and Rus [24, 25] discussed the Ulam–Hyers stability for operatorial equations and inclusions. Castro and Ramos [10] and Jung [18] considered the Hyers–Ulam–Rassias stability for a class of Volterra integral equations. More details from historical point of view, and recent developments of such stabilities are reported in [17, 24].

This paper deals with the existence for the Ulam stability of solutions to the following Hadamard partial fractional integral equation of the form

$$u(x, y) = \mu(x, y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{f(s, t, u(s, t))}{st} dt ds; \quad \text{if } (x, y) \in J, \quad (1)$$

where  $J := [1, a] \times [1, b]$ ,  $a, b > 1$ ,  $r_1, r_2 > 0$ ,  $\mu : J \rightarrow \mathbb{R}$ ,  $f : J \times \mathbb{R} \rightarrow \mathbb{R}$  are given continuous functions.

We present two results for the integral equation (1). The first one is based on Banach's contraction principle and the second one on the nonlinear alternative of Leray–Schauder type.

The present paper initiates the Ulam stability for integral equations involving the Hadamard fractional integral.

## 2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Denote by  $C := C(J, \mathbb{R})$  the Banach space of continuous functions  $u : J \rightarrow \mathbb{R}$  with the norm

$$\|u\|_C = \sup_{(x,y) \in J} |u(x, y)|.$$

$L^1(J, \mathbb{R})$  the Banach space of functions  $u : J \rightarrow \mathbb{R}$  that are Lebesgue integrable with norm

$$\|u\|_{L^1} = \int_1^a \int_1^b |u(x, y)| dy dx.$$

**Definition 2.1** [12, 19] The Hadamard fractional integral of order  $q > 0$  for a function  $g \in L^1([1, a], \mathbb{R})$ , is defined as

$$({}^H I_1^q g)(x) = \frac{1}{\Gamma(q)} \int_1^x \left(\log \frac{x}{s}\right)^{q-1} \frac{g(s)}{s} ds,$$

where  $\Gamma(\cdot)$  is the Euler gamma function.

**Definition 2.2** Let  $r_1, r_2 \geq 0$ ,  $\sigma = (1, 1)$  and  $r = (r_1, r_2)$ . For  $w \in L^1(J, \mathbb{R})$ , define the Hadamard partial fractional integral of order  $r$  by the expression

$$({}^H I_\sigma^r w)(x, y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{w(s, t)}{st} dt ds.$$

Now, we consider the Ulam stability for the integral equation (1). Consider the operator  $N : C \rightarrow C$  defined by:



$$(Nu)(x, y) = \mu(x, y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{f(s, t, u(s, t))}{st} dt ds. \tag{2}$$

Clearly, the fixed points of the operator  $N$  are solution of the integral equation (1). Let  $\epsilon > 0$  and  $\Phi : J \rightarrow [0, \infty)$  be a continuous function. We consider the following inequalities

$$|u(x, y) - (Nu)(x, y)| \leq \epsilon; \quad (x, y) \in J. \tag{3}$$

$$|u(x, y) - (Nu)(x, y)| \leq \Phi(x, y); \quad (x, y) \in J \tag{4}$$

$$|u(x, y) - (Nu)(x, y)| \leq \epsilon\Phi(x, y); \quad (x, y) \in J. \tag{5}$$

**Definition 2.3** [3,24] Equation (1) is Ulam–Hyers stable if there exists a real number  $c_N > 0$  such that for each  $\epsilon > 0$  and for each solution  $u \in C$  of the inequality (3) there exists a solution  $v \in C$  of Eq. (1) with

$$|u(x, y) - v(x, y)| \leq \epsilon c_N; \quad (x, y) \in J.$$

**Definition 2.4** [3,24] Equation (1) is generalized Ulam–Hyers stable if there exists  $c_N : C([0, \infty), [0, \infty))$  with  $c_N(0) = 0$  such that for each  $\epsilon > 0$  and for each solution  $u \in C$  of the inequality (3) there exists a solution  $v \in C$  of Eq. (1) with

$$|u(x, y) - v(x, y)| \leq c_N(\epsilon); \quad (x, y) \in J.$$

**Definition 2.5** [3,24] Equation (1) is Ulam–Hyers–Rassias stable with respect to  $\Phi$  if there exists a real number  $c_{N,\Phi} > 0$  such that for each  $\epsilon > 0$  and for each solution  $u \in C$  of the inequality (5) there exists a solution  $v \in C$  of Eq. (1) with

$$|u(x, y) - v(x, y)| \leq \epsilon c_{N,\Phi} \Phi(x, y); \quad (x, y) \in J.$$

**Definition 2.6** [3,24] Equation (1) is generalized Ulam–Hyers–Rassias stable with respect to  $\Phi$  if there exists a real number  $c_{N,\Phi} > 0$  such that for each solution  $u \in C$  of the inequality (4) there exists a solution  $v \in C$  of Eq. (1) with  $|u(x, y) - v(x, y)| \leq c_{N,\Phi} \Phi(x, y); \quad (x, y) \in J.$

*Remark 2.7* It is clear that (i) Definition 2.3  $\Rightarrow$  Definition 2.4, (ii) Definition 2.5  $\Rightarrow$  Definition 2.6, (iii) Definition 2.5 for  $\Phi(., .) = 1 \Rightarrow$  Definition 2.3.

One can have similar remarks for the inequalities (3) and (5).

### 3 Existence and Ulam stabilities results

In this section, we discuss the existence of solutions and present conditions for the Ulam stability for the Hadamard integral equation (1).

The following hypotheses will be used in the sequel.

(H<sub>1</sub>) There exist functions  $p_1, p_2 \in C(J, \mathbb{R}_+)$  such that for any  $u \in \mathbb{R}$  and  $(x, y) \in J$ ,

$$|f(x, y, u)| \leq p_1(x, y) + \frac{p_2(x, y)}{1 + |u(x, y)|} |u(x, y)|,$$

with

$$p_i^* = \sup_{(x,y) \in J} p_i(x, y); \quad i = 1, 2,$$

(H<sub>2</sub>) There exists  $\lambda_\Phi > 0$  such that for each  $(x, y) \in J$ , we have

$$({}^H I_\sigma^r \Phi)(x, y) \leq \lambda_\Phi \Phi(x, y).$$

**Theorem 3.1** Assume that the hypothesis (H<sub>1</sub>) holds. If

$$\frac{(\log a)^{r_1} (\log b)^{r_2}}{\Gamma(1 + r_1)\Gamma(1 + r_2)} p_2^* < 1, \tag{6}$$

then the integral equation (1) has a solution defined on  $J$ .

*Proof* Let  $\rho > 0$  be a constant such that

$$\rho > \frac{\|\mu\|_\infty + \frac{(\log a)^{r_1} (\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} P_1^*}{1 - \frac{(\log a)^{r_1} (\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} P_2^*}.$$

We shall use Schauder's theorem [11], to prove that the operator  $N$  defined in (2) has a fixed point. The proof will be given in four steps.

**Step 1:**  $N$  transforms the ball  $B_\rho := \{u \in \mathcal{C} : \|u\|_{\mathcal{C}} \leq \rho\}$  into itself.

For any  $u \in B_\rho$  and each  $(x, y) \in J$ , we have

$$\begin{aligned} |(Nu)(x, y)| &\leq |\mu(x, y)| + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left| \log \frac{x}{s} \right|^{r_1-1} \left| \log \frac{y}{t} \right|^{r_2-1} \\ &\quad \times \frac{p_1(s, t) + p_2(s, t) \|u\|_{\mathcal{C}}}{st} dt ds \\ &\leq \|\mu\|_\infty + \frac{(\log a)^{r_1} (\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} (P_1^* + P_2^* \rho). \end{aligned}$$

Thus, by (6) and the definition of  $\rho$  we get  $\|(Nu)\|_{\mathcal{C}} \leq \rho$ . This implies that  $N$  transforms the ball  $B_\rho$  into itself.

**Step 2:**  $N : B_\rho \rightarrow B_\rho$  is continuous.

Let  $\{u_n\}_{n \in \mathbb{N}}$  be a sequence such that  $u_n \rightarrow u$  in  $B_\rho$ . Then

$$\begin{aligned} |(Nu_n)(x, y) - (Nu)(x, y)| &\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left| \log \frac{x}{s} \right|^{r_1-1} \left| \log \frac{y}{t} \right|^{r_2-1} \\ &\quad \times \frac{|f(s, t, u_n(s, t)) - f(s, t, u(s, t))|}{st} dt ds \\ &\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left| \log \frac{x}{s} \right|^{r_1-1} \left| \log \frac{y}{t} \right|^{r_2-1} \\ &\quad \times \frac{\sup_{(s,t) \in J} |f(s, t, u_n(s, t)) - f(s, t, u(s, t))|}{st} dt ds \\ &\leq \frac{(\log a)^{r_1} (\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \|f(\cdot, \cdot, u_n(\cdot, \cdot)) - f(\cdot, \cdot, u(\cdot, \cdot))\|_{\mathcal{C}}. \end{aligned}$$

From Lebesgue's dominated convergence theorem and the continuity of the function  $f$ , we get

$$|(Nu_n)(x, y) - (Nu)(x, y)| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

**Step 3:**  $N(B_\rho)$  is bounded.

This is clear since  $N(B_\rho) \subset B_\rho$  and  $B_\rho$  is bounded.

**Step 4:**  $N(B_\rho)$  is equicontinuous.

Let  $(x_1, y_1), (x_2, y_2) \in J$ ,  $x_1 < x_2$ ,  $y_1 < y_2$ . Then

$$\begin{aligned} |(Nu)(x_2, y_2) - (Nu)(x_1, y_1)| &\leq |\mu(x_1, y_1) - \mu(x_2, y_2)| \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{x_1} \int_1^{y_1} \left[ \left| \log \frac{x_2}{s} \right|^{r_1-1} \left| \log \frac{y_2}{t} \right|^{r_2-1} - \left| \log \frac{x_1}{s} \right|^{r_1-1} \left| \log \frac{y_1}{t} \right|^{r_2-1} \right] \\ &\quad \times \frac{|f(s, t, u(s, t))|}{st} dt ds \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2} \int_{y_1}^{y_2} \left| \log \frac{x_2}{s} \right|^{r_1-1} \left| \log \frac{y_2}{t} \right|^{r_2-1} \frac{|f(s, t, u(s, t))|}{st} dt ds \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{x_1} \int_{y_1}^{y_2} \left| \log \frac{x_2}{s} \right|^{r_1-1} \left| \log \frac{y_2}{t} \right|^{r_2-1} \frac{|f(s, t, u(s, t))|}{st} dt ds \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2} \int_1^{y_1} \left| \log \frac{x_2}{s} \right|^{r_1-1} \left| \log \frac{y_2}{t} \right|^{r_2-1} \frac{|f(s, t, u(s, t))|}{st} dt ds. \end{aligned}$$



Thus

$$\begin{aligned}
 |(Nu)(x_2, y_2) - (Nu)(x_1, y_1)| &\leq |\mu(x_1, y_1) - \mu(x_2, y_2)| \\
 &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{x_1} \int_1^{y_1} \left[ \left| \log \frac{x_2}{s} \right|^{r_1-1} \left| \log \frac{y_2}{t} \right|^{r_2-1} - \left| \log \frac{x_1}{s} \right|^{r_1-1} \left| \log \frac{y_1}{t} \right|^{r_2-1} \right] \\
 &\times \frac{p_1^* + p_2^* \rho}{st} dt ds \\
 &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2} \int_{y_1}^{y_2} \left| \log \frac{x_2}{s} \right|^{r_1-1} \left| \log \frac{y_2}{t} \right|^{r_2-1} \frac{p_1^* + p_2^* \rho}{st} dt ds \\
 &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{x_1} \int_{y_1}^{y_2} \left| \log \frac{x_2}{s} \right|^{r_1-1} \left| \log \frac{y_2}{t} \right|^{r_2-1} \frac{p_1^* + p_2^* \rho}{st} dt ds \\
 &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2} \int_1^{y_1} \left| \log \frac{x_2}{s} \right|^{r_1-1} \left| \log \frac{y_2}{t} \right|^{r_2-1} \frac{p_1^* + p_2^* \rho}{st} dt ds \\
 &\leq \frac{p_1^* + p_2^* \rho}{\Gamma(1+r_1)\Gamma(1+r_2)} \\
 &\times [2(\log y_2)^{r_2}(\log x_2 - \log x_1)^{r_1} + 2(\log x_2)^{r_1}(\log y_2 - \log y_1)^{r_2} \\
 &+ (\log x_1)^{r_1}(\log y_1)^{r_2} - (\log x_2)^{r_1}(\log y_2)^{r_2} - 2(\log x_2 - \log x_1)^{r_1}(\log y_2 - \log y_1)^{r_2}].
 \end{aligned}$$

As  $x_1 \rightarrow x_2$  and  $y_1 \rightarrow y_2$ , the right-hand side of the above inequality tends to zero.

As a consequence of steps 1–4 together with the Arzelà–Ascoli theorem, we can conclude that  $N$  is continuous and compact. From an application of Schauder’s theorem [11], we deduce that  $N$  has a fixed point  $u$  which is a solution of the integral equation (1). □

Now, we are concerned with the stability of solutions for the integral equation (1).

**Theorem 3.2** Assume that  $(H_1)$ ,  $(H_2)$  and the condition (6) hold. Furthermore, suppose that there exist  $q_i \in C(J, \mathbb{R}_+)$ ;  $i = 1, 2$  such that for each  $(x, y) \in J$  we have

$$p_i(x, y) \leq q_i(x, y)\Phi(x, y).$$

Then, the integral equation (1) is generalized Ulam–Hyers–Rassias stable.

*Proof* Let  $u$  be a solution of the inequality (4). By Theorem 3.1, there exists  $v$  which is a solution of the integral equation (1). Hence

$$v(x, y) = \mu(x, y) + \int_1^x \int_1^y \left( \log \frac{x}{s} \right)^{r_1-1} \left( \log \frac{y}{t} \right)^{r_2-1} \frac{f(s, t, v(s, t))}{st\Gamma(r_1)\Gamma(r_2)} dt ds.$$

By the inequality (4) for each  $(x, y) \in J$ , we have

$$\left| u(x, y) - \mu(x, y) - \int_1^x \int_1^y \left( \log \frac{x}{s} \right)^{r_1-1} \left( \log \frac{y}{t} \right)^{r_2-1} \frac{f(s, t, u(s, t))}{st\Gamma(r_1)\Gamma(r_2)} dt ds \right| \leq \Phi(x, y).$$

Set

$$q_i^* = \sup_{(x,y) \in J} q_i(x, y); \quad i = 1, 2.$$

For each  $(x, y) \in J$ , we have

$$\begin{aligned}
 & |u(x, y) - v(x, y)| \\
 & \leq \left| u(x, y) - \mu(x, y) - \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{f(s, t, u(s, t))}{st\Gamma(r_1)\Gamma(r_2)} dt ds \right| \\
 & \quad + \int_1^x \int_1^y \left| \log \frac{x}{s} \right|^{r_1-1} \left| \log \frac{y}{t} \right|^{r_2-1} \frac{|f(s, t, u(s, t)) - f(s, t, v(s, t))|}{st\Gamma(r_1)\Gamma(r_2)} dt ds \\
 & \leq \Phi(x, y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left| \log \frac{x}{s} \right|^{r_1-1} \left| \log \frac{y}{t} \right|^{r_2-1} \\
 & \quad \times \left( 2q_1^* + \frac{q_2^*|u(s, t)|}{1+|u|} + \frac{q_2^*|v(s, t)|}{1+|v|} \right) \frac{\Phi(s, t)}{st} dt ds \\
 & \leq \Phi(x, y) + 2(q_1^* + q_2^*)({}^H I_{\sigma}^r \Phi)(x, y) \\
 & \leq [1 + 2(q_1^* + q_2^*)\lambda_{\Phi}] \Phi(x, y) \\
 & := c_{N, \Phi} \Phi(x, y).
 \end{aligned}$$

Hence, the integral equation (1) is generalized Ulam–Hyers–Rassias stable.  $\square$

#### 4 An example

As an application of our results, we consider the following partial Hadamard integral equation

$$\begin{aligned}
 u(x, y) &= \mu(x, y) \\
 &+ \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{f(s, t, u(s, t))}{st\Gamma(r_1)\Gamma(r_2)} dt ds; \quad (x, y) \in [1, e] \times [1, e], \quad (7)
 \end{aligned}$$

where

$$r_1, r_2 > 0, \quad \mu(x, y) = x + y^2; \quad (x, y) \in [1, e] \times [1, e],$$

and

$$f(x, y, u(x, y)) = cxy^2 \left( e^{-4} + \frac{u(x, y)}{e^{x+y+5}} \right); \quad (x, y) \in [1, e] \times [1, e],$$

with

$$c := \frac{e^4}{2} \Gamma(1+r_1)\Gamma(1+r_2).$$

For each  $u \in \mathbb{R}$  and  $(x, y) \in [1, e] \times [1, e]$  we have

$$|f(x, y, u(x, y))| \leq ce^{-4}(1+|u|).$$

Hence, condition (6) is satisfied with  $p_1^* = p_2^* = ce^{-4}$ . We shall show that condition (6) holds with  $a = b = e$ . Indeed,

$$\frac{(\log a)^{r_1} (\log b)^{r_2} p_2^*}{\Gamma(1+r_1)\Gamma(1+r_2)} = \frac{c}{e^4\Gamma(1+r_1)\Gamma(1+r_2)} = \frac{1}{2} < 1.$$

Consequently, Theorem 3.1 implies that the Hadamard integral equation (7) has a solution defined on  $[1, e] \times [1, e]$ .

Also, the hypothesis  $(H_2)$  is satisfied with

$$\Phi(x, y) = e^3, \quad \text{and} \quad \lambda_{\Phi} = \frac{1}{\Gamma(1+r_1)\Gamma(1+r_2)}.$$



Indeed, for each  $(x, y) \in [1, e] \times [1, e]$  we get

$$\begin{aligned}({}^H I_{\sigma}^r \Phi)(x, y) &\leq \frac{e^3}{\Gamma(1+r_1)\Gamma(1+r_2)} \\ &= \lambda_{\Phi} \Phi(x, y).\end{aligned}$$

Consequently, Theorem 3.2 implies that the equation (7) is generalized Ulam–Hyers–Rassias stable.

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