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Multiple solutions for a BVP on $(0, +\infty)$ via Morse theory and $H^1_{0,p}(\mathbb{R}^+)$ versus $C^1_p(\mathbb{R}^+)$ local minimizers

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Abstract This work is concerned with the existence of at least three nonzero solutions for a boundary value problem posed on the half-line. The method we employ is based upon Morse theory and uses $H^1_{0,p}$ versus C^1_p local minimizers.

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الملخص

يتعلق هذا العمل بوجود ثلاثة حلول غير صفرية على الأقل لمسألة قيمة حدية مطروحة على نصف الخط. الطريقة التي نطبقها مبنية على نظرية مورس وتستعمل المصغرات المحلية في $H^1_{0,p}$ مقابل C^1_p

1 Introduction and main result

Many problems arising in industry, epidemiology, population dynamics, and chemistry are governed by boundary value problems (BVPs for short) posed on the half-line: the propagation of a flame reacting in a long combustion tube, the concentration of some products in a long-period chemical reaction (see, e.g., [1] and some references therein). These problems have been the object of a large amount of research papers over the last three decades. In particular, second-order boundary value problems can be considered as interesting models both from the mathematical and the physical points of view. Regarding the recent mathematical results for BVPs set on unbounded domains, we refer the reader to, e.g., [12–14], and the references therein. The behavior of the nonlinearity involved in the ordinary differential equation, which refers to the physical source term, represents the main difficulty when dealing with such BVPs: several methods have been employed so far such that iterative and topological methods as well as techniques based on monotonicity and comparison principles.

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In this work, we are concerned with the following second-order model BVP:

$$\begin{cases} -(p(t)u'(t))' = f(t, u(t)), \quad t > 0, \\ u(0) = u(+\infty) = 0. \end{cases} \tag{1.1}$$

We will assume that the function $f : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a function $q : (0, +\infty) \longrightarrow \mathbb{R}$ such that $\frac{1}{q} \in L^1(0, +\infty)$ and $q(\cdot)f(\cdot, \cdot)$ sends $\mathbb{R}^+ \times B(\mathbb{R})$ into $B(\mathbb{R})$. $B(\mathbb{R})$ stands for the set of all bounded subsets of $\mathbb{R}$. Moreover, f satisfies the sign condition $f(t, u)u \geq 0$, for all $(t, u) \in (0, +\infty) \times \mathbb{R}$. The weight function $p : (0, +\infty) \longrightarrow (0, +\infty)$ is such that $\frac{1}{p} \in L^1(0, +\infty)$ and

$$M_1 = \int_0^{+\infty} \left(\int_t^{+\infty} \frac{\mathrm{d}s}{p(s)} \right) \mathrm{d}t < \infty. \tag{1.2}$$

We show the existence of at least three nonzero solutions to Problem (1.1), two of which having constant signs. A variational approach combining Morse theory and $H^1_{0,p}(\mathbb{R}^+)$ versus $C^1_p(\mathbb{R}^+)$ local minimizers is developed. The main existence result of this paper is

Theorem 1.1 *Suppose that:*
(H_1) *there exist* $\tau \in (1, 2)$, $\theta \in (1, +\infty)$, *and positive functions* $c_0, c_1 \in L^1(0, +\infty)$ *such that*

$$F(t, u) \geq c_0(t)|u|^\tau - c_1(t)|u|^{\theta+1}, \quad \textit{for all } t \in [0, +\infty) \textit{ and all } u \in \mathbb{R},$$

where $F(t, u) = \int_0^u f(t, v)\mathrm{d}v$.
(H_2) *There exist* $\delta > 0$ *and a positive function* $c_2 \in L^1(0, +\infty)$ *such that*

$$\left| F(t, u) - \frac{1}{\tau} f(t, u)u \right| \leq c_2(t)|u|^{\theta+1}, \quad \textit{for all } t \in [0, +\infty) \textit{ and all } |u| \leq \delta,$$

where θ *is that introduced in* (H_1).
(H_3) *There exist two positive functions* $a, b \in L^1(0, +\infty)$ *such that*

$$|f(t, u)| \leq a(t)|u| + b(t), \quad \textit{for all } (t, u) \in [0, +\infty) \times \mathbb{R}$$

with

$$M_2 = \int_0^{+\infty} a(t) \left(\int_t^{+\infty} \frac{\mathrm{d}s}{p(s)} \right) \mathrm{d}t < 1.$$

Then Problem (1.1) *has at least three nontrivial solutions.*

The following example of application illustrates the existence result.

Example 1.2 Consider the BVP

$$\begin{cases} -(\mathrm{e}^t u'(t))' = sgn(u(t)) \exp(-t)\sqrt{|u(t)|}, \quad t > 0, \\ u(0) = u(+\infty) = 0. \end{cases} \tag{1.3}$$

Set $M_1 = 1$, $M_2 = 1/2$, and $p(t) = q(t) = \mathrm{e}^t$. For $f(t, u) = sgn(u) \exp(-t)\sqrt{|u|}$, we have

$$F(t, u) = \frac{2}{3} \exp(-t)|u|^{\frac{3}{2}}$$

and

$$|f(t, u)| \leq \exp(-t)|u| + \exp(-t),$$

so (H_3) is satisfied. In addition, for $\tau = \frac{3}{2}$, $c_0(t) = \frac{2}{3} \exp(-t)$, $\theta = 2$, $c_1(t) = \frac{1}{2} \exp(-t)$, we have

$$F(t, u) = \frac{2}{3} \exp(-t)|u|^{\frac{3}{2}} \geq c_0(t)|u|^\tau - c_1(t)|u|^{\theta+1}, \quad \text{for all } (t, u) \in [0, +\infty) \times \mathbb{R},$$

whence (H_1). Finally, we check (H_2). For all $\delta \in \mathbb{R}$ and for every integrable nonnegative function $c_2(.)$, we have

$$\left|F(t,u) - \frac{2}{3} f(t,u)u\right| = \left|\frac{2}{3}\exp(-t)|u|^{\frac{3}{2}} - \frac{2}{3}\exp(-t)|u|^{\frac{3}{2}}\right| = 0 \leq c_2(t)|u|^3,$$

for all $|u| \leq \delta$.

Therefore, all the assumptions of Theorem 1.1 are met and then Problem (1.3) has at least three nontrivial solutions.

The proof of Theorem 1.1 is postponed in Sect. 3. It requires some basic notions in critical point theory which are collected in next section together with several technical lemmas.

2 Preliminaries

2.1 Critical point theory

Since we need to use variational techniques, we first recall some fundamental notions on minimization principal, critical groups, and also Morse theory. More detail can be found, e.g., in [3,4,10,18,20,21,23].

For a topological pair (A, B), i.e., a topological space A and a subset B of A, we denote by $H_k(A, B)$ the kth-relative singular homology group of (A, B) with coefficients in a ring $\mathbb{F}$ with characteristic zero (see [20]), and by $\beta_k = \dim H_k(A, B)$, the kth-Betti number. In algebraic topology, the kth-Betti number denotes the rank of the kth-homology group. Each Betti number is a natural number or equal to $+\infty$. They are topological invariants.

Let $\mathcal{H}$ be a Hilbert space endowed with a norm $||.||$ and $I \in C^1(\mathcal{H}, \mathbb{R})$ a functional. Let u_0 be an isolated critical point of I, i.e., $I'(u_0) = 0$, $I(u_0) = c \in \mathbb{R}$ and U a neighborhood of u_0 such that I has only u_0 as a critical point in U. The critical groups of I at u_0 are defined by

$$C_k(I, u_0) = H_k(I^c \cap U, I^c \cap U \backslash \{u_0\}), \quad \text{for all } k \in \mathbb{N},$$

where

$$I^c = \{u \in \mathcal{H} : I(u) \leq c\}$$

is the sub-level set at $c \in \mathbb{R}$. Let

$$K^c = \{u \in \mathcal{H} : I'(u) = 0, I(u) = c\}$$

be the set of critical points at level c. By the excision property of the singular homology theory, the definition of $C_k(I, u_0)$ does not depend on the choice of the neighborhood U.

When $I \in C^2(\mathcal{H}, \mathbb{R})$ and u_0 is a critical point of I, the Morse index of u_0 is defined as the supremum of dimensions of the vector subspaces of $\mathcal{H}$ on which $I''(u_0)$ is negative definite (it can be equal to ∞).

We say that u_0 is nondegenerate if the Hessian matrix $I''(u_0)$ is invertible.

Recall that $\beta_k(a, b) = \dim H_k(I^b, I^a)$ is the kth-Betti number of I with respect to the interval (a, b). Critical groups are needed to distinguish critical points of energy functionals.

Definition 2.1 Let $I \in C^1(\mathcal{H}, \mathbb{R})$ and $c \in \mathbb{R}$. The functional I satisfies the Palais–Smale condition at the level c (shortly $(\text{PS})_c$) if any sequence $(u_n)_{n \in \mathbb{N}} \subset \mathcal{H}$ such that

$$I(u_n) \to c, \quad I'(u_n) \to 0, \quad \text{as } n \to +\infty \tag{2.1}$$

has a convergent subsequence. I satisfies the Palais–Smale condition ((PS) in brief) if it satisfies the Palais–Smale condition at every level $c \in \mathbb{R}$.

$a \in \mathbb{R}$ is called a regular value for the functional I whenever $I'(a) \neq 0$.

Definition 2.2 Given two regular values $-\infty < a < b < +\infty$, assume that I has only isolated critical values $c_1 < c_2 < \cdots$ in (a, b) such that each of them corresponds to a finite number of critical points at each level. If further I satisfies the compactness condition $(\mathrm{PS})_c$, for all $c \in [a, b]$, then the Morse-type numbers of I with respect to the interval (a, b) are defined by

$$M_k(a,b) = \sum_i \dim H_k(I^{a_{i+1}}, I^{a_i}), \quad k \in \mathbb{N},$$

where $a = a_1 < c_1 < a_2 < c_2 < \cdots < c_l < a_l = b$.

They are independent of the a_i by the second deformation lemma (see, e.g., [22, Lemma 1.1.2]), and are related to the critical groups by the formula:

$$M_k(a,b) = \sum_{i=1}^{l} \sum_{u \in K^{c_i}} \dim C_k(I, u), \quad k \in \mathbb{N}.$$

By Ekeland's Variational Principle, we know that if a C^1-functional I is bounded from below, then there exists a minimizing sequence for I. If further I satisfies the (PS) condition, then it is coercive, hence achieves its lower bounds (see, e.g., [10, Corollary 4.8.4], [20, Corollary 5.21]). The relationship between coercivity and (PS) condition is well explained in [8].

Using a cutoff technique, this principle will be applied to the Euler functional associated with problem (1.1) and yield existence of two solutions with constant signs.

The following lemma from Morse Theory provides a relationship between Morse-type numbers and Betti numbers. It will be crucial in the proof of existence of a third nontrivial solution to Problem (1.1).

Lemma 2.3 [9,18] *Assume that $I \in C^1(\mathcal{H}, \mathbb{R})$ satisfies the (PS) condition and let $a < b$ be two regular values of I. Suppose that I has at most finitely many critical points on $I^{-1}[a, b]$ and that the dimension of the critical group for every critical point is finite. Then the following inequality holds:*

$$\sum_{j=0}^{k} (-1)^{k-j} M_j \geq \sum_{j=0}^{k} (-1)^{k-j} \beta_j, \quad k = 0, 1, 2 \ldots,$$

where for each $k \in \mathbb{N}$, M_k and β_k denote the Morse-type number and the kth-Betti number, respectively. Moreover,

$$\sum_{k=0}^{\infty} (-1)^k M_k = \sum_{k=0}^{\infty} (-1)^k \beta_k,$$

whenever the left-hand series of the equality converges.

2.2 Embedding lemmas

Let $C_0 = \{u \in C([0, +\infty), \mathbb{R}) : \lim_{t \to +\infty} u(t) = 0\}$ be endowed with the sup-norm $||u||_\infty = \sup_{t \geq 0} |u(t)|$. Given $a \in L^1((0, +\infty), (0, +\infty))$, consider the Hilbert space:

$$L_a^2 = \{u : (0, +\infty) \longrightarrow \mathbb{R} \text{ measurable}, \sqrt{a}u \in L^2((0, +\infty), \mathbb{R})\}$$

with the norm

$$||u||_{L_a^2} = \left(\int_0^{+\infty} a(t) u^2(t) \mathrm{d}t \right)^{1/2}.$$

Let $\mathrm{AC}([0, +\infty), \mathbb{R})$ denote the set of all real absolutely continuous functions on $[0, +\infty)$. Given the weight function p defined in Sect. 1, let

$$H_{0,p}^1 = \{u \in \mathrm{AC}([0, +\infty), \mathbb{R}) : u(0) = u(+\infty) = 0 \text{ and } \sqrt{p}u' \in L^2(0, +\infty)\}$$

with the norm:

$$||u||_{H^1_{0,p}} = \left(\int_0^{+\infty} p(t)u'(t)^2 \mathrm{d}t + \int_0^{+\infty} u(t)^2\,\mathrm{d}t\right)^{1/2}.$$

Concerning these spaces, we have

Lemma 2.4 [7] (a) *On $H^1_{0,p}$, the quantity $||u|| = (\int_0^{+\infty} p(t)u'(t)^2\,\mathrm{d}t)^{1/2}$ defines a norm which is equivalent to the $H^1_{0,p}$ norm.*

(b) *$H^1_{0,p}$ is continuously embedded in C_0; more precisely there exists a constant $d > 0$ such that for every $u \in H^1_{0,p}$, one has $||u||_\infty \leq d||u||$ with $d = (\int_0^{+\infty} \frac{1}{p(s)})^{\frac{1}{2}}$.*

(c) *The embedding*

$$H^1_{0,p} \hookrightarrow C_0$$

is compact.

Since $||u||^2_{L^2_a} \leq ||u||^2_\infty ||a||_{L^1}$, we infer that

Lemma 2.5 *C_0 is continuously embedded in L^2_a.*

Corollary 2.6 *$H^1_{0,p}$ is compactly embedded in L^2_a.*

Define the spaces

$$C^1_p = \{u \in C_0([0,+\infty),\mathbb{R}) : u(0) = 0,\ u \text{ derivable},\ pu' \in C([0,+\infty),\mathbb{R}), \text{and} \lim_{t\to+\infty} p(t)u'(t) \text{exists}\}$$

and

$$C^2_{p,q} = \{u \in C^1_p :\ pu' \text{ derivable and } \sup_{t\geq 0} |q(t)(p(t)u'(t))'| < +\infty\}$$

equipped with the norms

$$||u||_{C^1_p} = ||u||_\infty + ||pu'||_\infty \tag{2.2}$$

and

$$||u||_{C^2_{p,q}} = ||u||_{C^1_p} + ||q(pu')'||_\infty, \tag{2.3}$$

respectively. From the estimates

$$||u||^2 = \int_0^{+\infty} p(t)u'^2(t)\mathrm{d}t = \int_0^{+\infty} \frac{1}{p(t)} p^2(t)u'^2(t)\mathrm{d}s \leq ||pu||^2_\infty \int_0^{+\infty} \frac{1}{p(t)}\mathrm{d}t \leq ||u||^2_{C^1_p} \left\| \frac{1}{p} \right\|_{L^1},$$

we deduce

Lemma 2.7 *C^1_p is continuously embedded in $H^1_{0,p}$.*

We need the following variant of the classical Corduneanu compactness criterion:

Lemma 2.8 [11] *Let $D \subset C^1_p$ be a bounded set. Then D is relatively compact if the following conditions hold:*

(a) *D is equicontinuous on every compact interval of $\mathbb{R}^+$, i.e.,*

$$\begin{gathered}\forall J \subset [0,+\infty) \text{ compact},\quad \forall \varepsilon > 0,\quad \exists \delta > 0,\quad \forall t_1, t_2 \in J :\\ |t_1 - t_2| < \delta \Rightarrow |u(t_1) - u(t_2)| \leq \varepsilon \text{ and } |p(t_1)u'(t_1) - p(t_2)u'(t_2)| \leq \varepsilon,\quad \forall u \in D,\end{gathered}$$

(b) *D is equiconvergent at $+\infty$, i.e.,*

$$\begin{gathered}\forall \varepsilon > 0,\quad \exists T = T(\varepsilon) > 0 \text{ such that } \forall t_1, t_2 :\\ t_1, t_2 \geq T(\varepsilon) \Rightarrow |u(t_1) - u(t_2)| \leq \varepsilon \text{ and } |p(t_1)u'(t_1) - p(t_2)u'(t_2)| \leq \varepsilon,\quad \forall u \in D.\end{gathered}$$

Then we deduce

Lemma 2.9 *The embedding $C^2_{p,q} \hookrightarrow C^1_p$ is compact.*

Proof The norms $||\cdot||_{C^1_p}$ and $||\cdot||_{C^2_{p,q}}$ are as defined by (2.2) and (2.3), respectively. Note that $||u||_{C^1_p} \leq ||u||_{C^2_{p,q}}$, which means that $C^2_{p,q}$ is continuously embedded in C^1_p. To apply Lemma 2.8, let $D \subset C^2_{p,q}$ be a bounded set, hence bounded in C^1_p.

Notice that, by the Cauchy criterion, we have

$$\lim_{t\to+\infty}\int_t^{+\infty}\frac{1}{p(s)}\mathrm{d}s = \lim_{t\to+\infty}\int_t^{+\infty}\frac{1}{q(s)}\mathrm{d}s = 0.$$

(a) *D is equicontinuous.* There exists $R > 0$ such that $||u||_{C^2_{p,q}} \leq R$, for all $u \in D$. Let $t_1, t_2 > 0$. Noting that $1/p, 1/q \in L^1(0, \infty)$, we get

$$\begin{aligned}|u(t_1) - u(t_2)| &= \left|\int_{t_1}^{t_2} u'(s)\mathrm{d}s\right| \\ &\leq \int_{t_1}^{t_2}\frac{1}{p(s)}p(s)|u'(s)|\,\mathrm{d}s \\ &\leq ||u||_{C^2_{p,q}}\int_{t_1}^{t_2}\frac{1}{p(s)}\,\mathrm{d}s \\ &\leq R\int_{t_1}^{t_2}\frac{1}{p(s)}\,\mathrm{d}s \to 0, \quad \text{as } |t_1 - t_2| \to 0\end{aligned}$$

as well as

$$\begin{aligned}|p(t_1)u'(t_1) - p(t_2)u'(t_2)| &\leq \int_{t_1}^{t_2}(p(s)u'(s))'\mathrm{d}s \\ &\leq \int_{t_1}^{t_2}\frac{1}{q(s)}q(s)|(p(s)u'(s))'|\,\mathrm{d}s \\ &\leq R\int_{t_1}^{t_2}\frac{1}{q(s)}\,\mathrm{d}s \to 0, \quad \text{as } |t_1 - t_2| \to 0.\end{aligned}$$

(b) *D is equiconvergent.* For every positive ε, there exists T such that for all $t_1, t_2 > T$ and $u \in D$, we have

$$\begin{aligned}|u(t_1) - u(t_2)| &\leq \int_{t_1}^{t_2}|u'(s)|\,\mathrm{d}s \\ &\leq \int_{t_1}^{t_2}\frac{1}{p(s)}p(s)|u'(s)|\,\mathrm{d}s \\ &\leq R\int_{t_1}^{t_2}\frac{1}{p(s)}\,\mathrm{d}s \to 0\end{aligned}$$

and

$$\begin{aligned}|p(t_1)u'(t_1) - p(t_2)u'(t_2)| &\leq \int_{t_1}^{t_2}(p(s)u'(s))'\mathrm{d}s \\ &\leq \int_{t_1}^{t_2}\frac{1}{q(s)}q(s)(p(s)u'(s))'\mathrm{d}s \\ &\leq R\int_{t_1}^{t_2}\frac{1}{q(s)}\,\mathrm{d}s \to 0.\end{aligned}$$

□

2.3 Auxiliary lemmas

Let $\lambda_1(a)$ be the first eigenvalue of the linear problem:

$$\begin{cases} -(p(t)u'(t))' = \lambda a(t)u(t), & t > 0 \\ u(0) = u(+\infty) = 0, \end{cases} \tag{2.4}$$

where $a \in L^1((0,\infty),(0,\infty))$. Then

$$\lambda_1 = \inf_{u \in H^1_{0,p}\setminus\{0\}} \frac{||u||^2}{||u||_{L^2_a}} \geq \frac{1}{M_2}. \tag{2.5}$$

By Poincaré's inequality, we have

$$\begin{aligned} |u(t)|^2 &= \left| \int_t^{+\infty} u'(s)\mathrm{d}s \right|^2 \\ &= \left| \int_t^{+\infty} \sqrt{p(s)}u'(s)\frac{1}{\sqrt{p(s)}}\,\mathrm{d}s \right|^2 \\ &\leq \left(\int_t^{+\infty} p(s)u'^2(s)\mathrm{d}s \right) \left(\int_t^{+\infty} \frac{\mathrm{d}s}{p(s)} \right) \\ &\leq \left(\int_0^{+\infty} p(s)u'^2(s)\mathrm{d}s \right) \left(\int_t^{+\infty} \frac{\mathrm{d}s}{p(s)} \right). \end{aligned}$$

Hence,

$$\int_0^{+\infty} a(t)u(t)^2\mathrm{d}t \leq \left(\int_0^{+\infty} p(s)u'^2(s)\mathrm{d}s \right) \left(\int_0^{+\infty} a(t) \left(\int_t^{+\infty} \frac{\mathrm{d}s}{p(s)} \right) \mathrm{d}t \right),$$

i.e.,

$$||u||^2_{L^2_a} \leq M_2||u||^2.$$

Then (2.5) follows by taking the infimum of the Rayleigh quotient. We even know, from [2, Lemma 2.5], that λ_1 is achieved by some positive eigenfunction $\phi_1 \in H^1_{0,p}\setminus\{0\}$.

The Euler functional $I : H^1_{0,p} \to \mathbb{R}$ associated to Problem (1.1) is defined by

$$I(u) = \frac{1}{2}||u||^2 - \int_0^{+\infty} F(t,u(t))\mathrm{d}t, \quad \forall\, u \in H^1_{0,p},$$

where $F(t,u) = \int_0^u f(t,v)\mathrm{d}v$. By virtue of (H_3) and Lebesgue Dominated Convergence Theorem, one can easily prove that $I \in C^1(H^1_{0,p},\mathbb{R})$ with first derivative given by

$$(I'(u),v) = \int_0^{+\infty} p(t)u'(t)v'(t)\mathrm{d}t - \int_0^{+\infty} f(t,u(t))v(t)\mathrm{d}t, \quad \forall\, v \in H^1_{0,p}.$$

However, the fixed point operator associated to Problem (1.1) is given by

$$Au(t) = \int_0^{+\infty} G(t,s)f(s,u(s))\mathrm{d}s,$$

where

$$G(t,s) = \frac{1}{||\frac{1}{p}||_{L^1}} \begin{cases} \varphi_1(t)\varphi_2(s), & t \leq s \\ \varphi_1(s)\varphi_2(t), & s \leq t \end{cases}$$

and $\varphi_1(t) = \int_0^t \frac{\mathrm{d}s}{p(s)},\ \varphi_2(t) = \int_t^{+\infty} \frac{\mathrm{d}s}{p(s)}$.

Lemma 2.10 [2] *We have*

$$(a) \text{ the operator } A : H^1_{0,p} \to H^1_{0,p} \text{ is compact.}$$
$$(b) \; I' = Id - A,$$

Remark 2.11 Notice that from the sign condition $f(t,u)u \geq 0$, for all $(t,u) \in \mathbb{R}^+ \times \mathbb{R}$ and by the continuity of f, we have $f(t,0) = 0$, for all $t \in \mathbb{R}^+$; so 0 is a critical point of I, and thus $C_k(I,0)$ is well defined.

Next, we compute the critical groups of I at zero. Arguing as in the proof of [15, Proposition 2.1] and [19, Theorem 1], we prove that all the critical groups at the origin are trivial.

Lemma 2.12 *Under Assumptions* (H_1) *and* (H_2), *all critical groups of the functional* I *at* 0 *are trivial:*

$$C_k(I,0) = 0, \quad \text{for all } k \in \mathbb{N}.$$

Proof From (H_1), we obtain that for $u \in H^1_{0,p} \backslash \{0\}$ and $s > 0$:

$$\begin{aligned} I(su) &= \frac{s^2}{2}||u||^2 - \int_0^{+\infty} F(t, su(t))\mathrm{d}t \\ &\leq \frac{s^2}{2}||u||^2 - s^\tau \int_0^{+\infty} c_0(t)|u(t)|^\tau \,\mathrm{d}t + s^{\theta+1} \int_0^{+\infty} c_1(t)|u(t)|^{\theta+1}\,\mathrm{d}t \\ &= \frac{s^2}{2}||u||^2 - s^\tau || \sqrt[\tau]{c_0} u||^\tau_{L^\tau} + s^{\theta+1} || \sqrt[\theta+1]{c_1} u||^{\theta+1}_{L^{\theta+1}}. \end{aligned}$$

Notice that all norms are equivalent in the one-dimensional space spanned by u (see [16, page 86]). In addition, since $1 < \tau < 2 < \theta + 1$, then for sufficiently small s, the leading term of the right-hand polynomial is s^τ. Hence, there exists $s_0 \in (0,1)$ such that

$$I(su) < 0, \quad \text{for all } s \in (0, s_0). \tag{2.6}$$

In addition, (H_2) implies

$$\int_0^{+\infty} \left(F(t,u(t)) - \frac{1}{\tau} f(t,u(t))u(t) \right) \mathrm{d}t = o(||u||^2), \quad \text{as } ||u|| \to 0.$$

Hence,

$$\begin{aligned} \frac{1}{\tau} \frac{\mathrm{d}I(su)}{\mathrm{d}s}\bigg|_{s=1} &= \frac{1}{\tau}(I'(su), u)|_{s=1} \\ &= \frac{1}{\tau}||u||^2 - \int_0^{+\infty} F(t,u(t))\mathrm{d}t + o(||u||^2) \\ &= I(u) + \left(\frac{1}{\tau} - \frac{1}{2}\right)||u||^2 + o(||u||^2), \quad \text{as } ||u|| \to 0. \end{aligned}$$

Therefore, there exists $\rho > 0$ such that

$$\frac{\mathrm{d}I(su)}{\mathrm{d}s}\bigg|_{s=1} > 0, \quad \forall u \in I^{-1}([0,+\infty)) \cap B_\rho \backslash \{0\}, \tag{2.7}$$

where B_ρ refers to the ball centered at the origin with a radius ρ in $H^1_{0,p}$. By virtue of (2.7), we deduce that

$$I(su) < 0, \quad \text{for } s \in (0,1) \text{ and } u \in I^{-1}(-\infty, 0) \cap B_\rho. \tag{2.8}$$

Indeed, if $||u|| \leq \rho$ and $I(u) < 0$ then, by the continuity of I, there exists $s' \in (0,1)$ such that $I(su) < 0$, for all $s \in (1-s', 1]$. Suppose that there is some $s_0 \in (0, 1-s']$ such that $I(s_0 u) = 0$ and $I(su) < 0$, for $s_0 < s < 1$ and let $u_0 = s_0 u$. Then from (2.7), we find

$$\frac{\mathrm{d}I(su_0)}{\mathrm{d}s}\bigg|_{s=1} > 0.$$

However, $I(su) - I(s_0u) < 0$ implies that

$$\left.\frac{\mathrm{d}I(su)}{\mathrm{d}s}\right|_{s=s_0} = \left.\frac{\mathrm{d}I(su_0)}{\mathrm{d}s}\right|_{s=1} \leq 0,$$

leading to a contradiction. Hence, (2.8) holds. From (2.6), (2.7), and (2.8), if $I(u) > 0$ then there is a unique $T(u) \in (0, 1)$ such that

$$I(T(u)u) = 0, \quad I(su) < 0, \quad \text{for all } \ s \in (0, T(u)) \tag{2.9}$$

and

$$I(su) > 0, \quad \text{for all } \ s \in (T(u), 1). \tag{2.10}$$

If $I(u) \leq 0$, then we set $T(u) = 1$. From (2.7), (2.9), (2.10), and appealing to the implicit function theorem, we deduce that T is continuous in u. Define the map $\eta : [0, 1] \times B_\rho \to B_\rho$ by

$$\eta(s, u) = (1 - s)u + sT(u)u, \quad s \in [0, 1], \quad \text{for } \ u \in B_\rho. \tag{2.11}$$

η is a continuous map which defines two retraction maps, the first one from B_ρ to $B_\rho \cap I^0$ and the second one from $B_\rho \backslash \{0\}$ to $B_\rho \cap I^0 \backslash \{0\}$; indeed when either $u \in B_\rho \cap I^0$ or $u \in B_\rho \cap I^0 \backslash \{0\}$, we have that $\eta(s, u) = u$.

As B_ρ and $B_\rho \backslash \{0\}$ are contractible sets and since a retract of a contractible set is also a contractible set (see [5, Theorem 13.2]), then the sets $B_\rho \cap I^0$ and $B_\rho \cap I^0 \backslash \{0\}$ are contractible. We conclude that

$$C_k(I, 0) = H_k(B_\rho \cap I^0, \quad B_\rho \cap I^0 \backslash \{0\}) \simeq 0, \quad \text{for all } k \in \mathbb{N}.$$

□

To use a truncation technique, we define the function

$$f_+(t, u) = \begin{cases} f(t, u), & \text{if } u \geq 0, \\ 0, & \text{if } u < 0 \end{cases}$$

and its primitive $F_+(t, u) = \int_0^u f_+(t, v)\mathrm{d}v$. Then the critical points of the functional $I_+ : H^1_{0,p} \to \mathbb{R}$ given by

$$I_+(u) = \frac{1}{2}\int_0^{+\infty} p(t)u'(t)^2 \mathrm{d}t - \int_0^{+\infty} F_+(t, u(t))\mathrm{d}t$$

are weak solutions of the BVP:

$$\begin{cases} -(p(t)u'(t))' = f_+(t, u(t)), & t > 0 \\ u(0) = u(+\infty) = 0. \end{cases} \tag{2.12}$$

Also if we let

$$f_-(t, u) = \begin{cases} 0, & \text{if } u \geq 0, \\ f(t, u), & \text{if } u < 0 \end{cases}$$

and $F_-(t, u) = \int_0^u f_-(t, v)\mathrm{d}v$, then the critical points of the functional $I_- : H^1_{0,p} \to \mathbb{R}$ given by

$$I_-(u) = \frac{1}{2}\int_0^{+\infty} p(t)u'(t)^2 \mathrm{d}t - \int_0^{+\infty} F_-(t, u(t))\mathrm{d}t$$

are weak solutions of the BVP:

$$\begin{cases} -(p(t)u'(t))' = f_-(t, u(t)), & t > 0 \\ u(0) = u(+\infty) = 0. \end{cases} \tag{2.13}$$

Lemma 2.13 *Suppose that* (H_1) *holds. Then 0 is not a local minimum of* I_+.

Proof Let ϕ_1 be the first eigenfunction of Problem (2.4). From (H_1), for all $s > 0$, we have

$$\begin{aligned} I_+(s\phi_1) &= \frac{s^2}{2}||\phi_1||^2 - \int_0^{+\infty} F_+(t, s\phi_1(t))\mathrm{d}t \\ &\leq \frac{s^2}{2}||\phi_1||^2 - s^\tau \int_0^{+\infty} c_0(t)\phi_1^\tau(t)\mathrm{d}t + s^{\theta+1}\int_0^{+\infty} c_1(t)\phi_1^{\theta+1}(t)\mathrm{d}t \\ &= \frac{s^2}{2}||\phi_1||^2 - s^\tau||\sqrt[\tau]{c_0}\phi_1||_{L^\tau}^\tau + s^{\theta+1}||\sqrt[\theta+1]{c_1}\phi_1||_{L^{\theta+1}}^{\theta+1}. \end{aligned}$$

Since all norms are equivalent in the one-dimensional space spanned by ϕ_1 and arguing as in the beginning of the proof of Lemma 2.12, we can let $s \to 0$ to find some $s_0 > 0$ such that $I_+(s\phi_1) < 0 = I_+(0),$ for all $s \in (0, s_0)$. Hence, 0 is not a local minimizer of I_+.

□

In an identical manner, we can prove that 0 is not a local minimizer of I_-.

Lemma 2.14 *Suppose that* (H_3) *holds. Then* I_+ *is bounded from below and satisfies the (PS) condition.*

Proof (a) *I_+ is bounded from below.* From (H_3) and (2.5), we have the estimates

$$\begin{aligned} I_+(u) &= \frac{1}{2}||u||^2 - \int_0^{+\infty} F_+(t, u(t))\mathrm{d}t \\ &\geq \frac{1}{2}||u||^2 - \int_0^{+\infty}\left(\frac{1}{2}a(t)|u(t)|^2 + b(t)|u(t)|\right)\mathrm{d}t \\ &= \frac{1}{2}||u||^2 - \frac{1}{2}\int_0^{+\infty} a(t)u(t)^2\mathrm{d}t - \int_0^{+\infty} b(t)|u(t)|\mathrm{d}t \\ &\geq \frac{1}{2}||u||^2 - \frac{1}{2}||u||_{L_a^2}^2 - d||u||||b||_{L^1} \\ &= \frac{1}{2}||u||^2 - \frac{1}{2}\frac{\lambda_1(a)}{\lambda_1(a)}||u||_{L_a^2}^2 - d||u||||b||_{L^1} \\ &\geq \frac{1}{2}||u||^2 - \frac{1}{2\lambda_1(a)}||u||^2 - d||u||||b||_{L^1} \\ &= \frac{1}{2}\left(1 - \frac{1}{\lambda_1(a)}\right)||u||^2 - d||u|||b||_{L^1}, \end{aligned}$$

proving the claim.

(b) *I_+ satisfies the (PS) condition.* Suppose that $(u_n) \subset H_{0,p}^1$ and there exists $M > 0$ such that $|I_+(u_n)| \leq M$ and $I'_+(u_n) = u_n - A_+u_n \to 0$ in $H_{0,p}^1$, as $n \to \infty$, where

$$A_\pm u(t) = \int_0^{+\infty} G(t, s)f_\pm(s, u(s))\mathrm{d}s.$$

Since I_+ is bounded from below, the sequence (u_n) is bounded in $H_{0,p}^1$. By the compactness of $A_+ : H_{0,p}^1 \to H_{0,p}^1$ there exists a subsequence $(A_+(u_{n_k}))$ which converges to some limit w. Hence,

$$||u_{n_k} - w|| \leq ||u_{n_k} - A_+u_{n_k}|| + ||A_+u_{n_k} - w||$$

and since $u_{n_k} - A_+u_{n_k} \to 0$ in $H_{0,p}^1$, as $k \to \infty$, we deduce that (u_n) has a convergent subsequence (u_{n_k}) which converges to w. Therefore, I_+ satisfies the (PS) condition on $H_{0,p}^1$.

□

Similarly, we can prove that I_- is bounded from below and satisfies the (PS) condition. Our last technical result is next stated and proved in the spirit of Brézis–Nirenberg Theorem (see [6, Theorem 1]).

Lemma 2.15 *Under assumption* (H_3), *assume that* $u_0 \in H^1_{0,p}$ *is a local minimizer of* I *in the* C^1_p*-topology, which means that there is some* $r > 0$ *such that*

$$I(u_0) \leq I(v), \quad \forall\, v \in C^1_p \text{ with } ||v - u_0||_{C^1_p} \leq r. \tag{2.14}$$

Then u_0 *is a local minimizer of* I *in the* $H^1_{0,p}$*-topology, i.e., there exists* $\varepsilon_0 > 0$ *such that*

$$I(u_0) \leq I(v), \quad \forall\, v \in H^1_{0,p} \text{ with } ||v - u_0|| \leq \varepsilon_0. \tag{2.15}$$

Remark 2.16 By Proposition 2.7, notice that a C^1_p-neighborhood of u is smaller than an $H^1_{0,p}$-neighborhood of u.

Proof Without loss of generality, suppose that $u_0 = 0$ and argue by contradiction assuming that (2.15) does not hold. Then there exists a sequence $(v_k) \subset H^1_{0,p}$ such that

$$||v_k|| \leq 1/k \text{ and } I(v_k) < I(0), \ \forall\, k \in \{1, 2, \ldots\}. \tag{2.16}$$

Since the functional I is coercive and weakly lower semi-continuous, the minimum $\min_{B_k} I$ is achieved, where B_k is the ball centered at the origin and with radius $1/k$ in $H^1_{0,p}$. For this, let $(u_n) \subset H^1_{0,p}$ be a sequence such that (u_n) is weakly convergent to some limit u. By (H_3) and Lebesgue Dominated Convergence Theorem, we deduce that

$$\lim_{n\to+\infty} \int_0^{+\infty} F(t, u_n(t))\mathrm{d}t = \int_0^{+\infty} F(t, u(t))\mathrm{d}t.$$

By the weak lower semi-continuity of the norm, we infer that

$$I(u) \leq \liminf_{n\to+\infty} I(u_n)$$

which means that I is weakly lower semi-continuous. Thus, $\min_{B_k} I$ is achieved at some point still denoted v_k. Since (v_k) lies in $H^1_{0,p}$, then $(v_k) \subset C^2_{p,q}$. In fact, $v_k(0) = 0$ and by Lemma 2.4, part (b) we know that $(v_k) \subset C_0$, v_k satisfies Problem (1.1), hence derivable, and that pv_k' is also derivable, for all $k \in \{1, 2, \ldots\}$. Also, we have that

$$\lim_{t\to+\infty} p(t)v_k'(t) = p(0)v_k'(0) - \int_0^{+\infty} f(s, v_k(s))\mathrm{d}s;$$

so $\lim_{t\to+\infty} p(t)v_k'(t)$ exists as a finite limit. Furthermore, since (v_k) is bounded in $H^1_{0,p}$, then (v_k) is bounded on $C^2_{p,q}$. To see this, assume that there exists $r_0 > 0$ such that $||v_k|| \leq r_0, \forall\, k \in \{1, 2, \ldots\}$. By Lemma 2.4, part (b),

$$||v_k||_\infty \leq d||v_k|| \leq dr_0,$$

independently of $k \in \{1, 2, \ldots\}$. From Problem (1.1) and since $q(\cdot)f(\cdot, \cdot)$ sends $\mathbb{R}^+ \times B(\mathbb{R})$ into $B(\mathbb{R})$, we have on one hand

$$||q(pv_k')'||_\infty = \sup_{t\geq 0} |q(t)(p(t)v_k'(t))'| < \infty$$

and on the other one

$$\begin{aligned} |p(t)v_k'(t)| &\leq |\lim_{t\to+\infty} p(t)v_k'(t)| + \int_t^{+\infty} |f(s, v_k(s))|\mathrm{d}s \\ &= |\lim_{t\to+\infty} p(t)v_k'(t)| + \int_t^{+\infty} \left|\frac{1}{q(s)}\right| |q(s)f(s, v_k(s))|\mathrm{d}s \\ &\leq |\lim_{t\to+\infty} p(t)v_k'(t)| + d_1 \left|\left|\frac{1}{q}\right|\right|_{L^1}, \end{aligned}$$

where

$$|q(t)f(t,v_k)| \le ||q(pv_k')'||_\infty \le d_1, \quad \forall t \ge 0.$$

Hence,

$$||pv_k'||_\infty = \sup_{t\ge 0}|p(t)v_k'(t)| \le d_2,$$

where

$$d_2 = |\lim_{t\to+\infty} p(t)v_k'(t)| + d_1||1/q||_{L^1}$$

is independent of k. By Lemma 2.9, there is a subsequence, still denoted (v_k), such that $v_k \to w$ in C_p^1, as $k \to \infty$. Moreover, Lemma 2.7 yields some positive constant d_3 such that $\|v_k - w\| \le d_3\|v_k - w\|_{C_p^1} \to 0$. This shows that (v_k) also converges strongly to w in $H_{0,p}^1$, as $k \to \infty$ and from the definition of (v_k), we then deduce that $w = 0$. Combining (2.16) and (2.14), we get

$$I(0) \le I(v_k) < I(0),$$

a contradiction.

□

3 Proof of Theorem 1.1

Step 1. By Lemma 2.14 and Ekeland's Variational Principal, there exists $u_+ \in H_{0,p}^1$ such that

$$I_+(u_+) = \min_{u\in H_{0,p}^1} I_+(u). \tag{3.1}$$

Lemma 2.13 tells us that $u_+ \not\equiv 0$, hence u_+ is a nonzero solution of Problem (2.12). Then u_+ can be written as

$$u_+(t) = \int_0^{+\infty} G(t,s)f_+(s,u_+(s))\mathrm{d}s.$$

By the sign condition $f(t,u)u \ge 0$, for all $(t,u) \in [0,+\infty) \times \mathbb{R}$, we have that $f_+(t,u_+) \ge 0$. Since the Green function is positive, then u_+ is a positive nonzero solution for Problem (2.12).

Moreover, u_+ is also a positive nonzero solution of Problem (1.1) for $f_+(t,u) = f(t,u)$, as $u \ge 0$.

Notice that

$$\lim_{t\to+\infty} p(t)u_+'(t) \ne 0.$$

Indeed, since u_+ is a solution of Problem (2.12) and $f_+(.,.)$ is positive, we deduce that $p(.)u_+'(.)$ is decreasing. If further

$$\lim_{t\to+\infty} p(t)u_+'(t) = 0,$$

then $p(.)u_+'(.)$ is positive. Since p is positive too, then u_+ is increasing and satisfies the conditions $u_+(0) = u_+(+\infty) = 0$, a contradiction.

By (3.1), u_+ is a local minimum of I_+ in C_p^1 and there exists $r > 0$ such that

$$I_+(u_+) \le I_+(v), \quad \text{for all } v \in C_p^1 \text{ with } ||v - u_+||_{C_p^1} \le r. \tag{3.2}$$

Next, we argue as in the proof of [17, Theorem 1.1]. Since $u_+ > 0$, then u_+ is an interior point of $(C_p^1)^+$, the set of positive functions of C_p^1, with respect to the C_p^1-topology. Therefore, we can choose $r_1 < r$ such that for $v \in C_p^1$ with $||v - u_+||_{C_p^1} \le r_1$, we have $v(t) > 0$, for all positive t.

However, if $u > 0$, then $F(t, u(t)) = F_+(t, u(t))$, hence $I(u) = I_+(u)$. Then for $v \in C^1_p$ with $||v - u_+||_{C^1_p} \leq r_1$, we have

$$I(u_+) = I_+(u_+) \leq I_+(v) = I(v).$$

Then u_+ is a local minimizer of I on C^1_p in the C^1_p-topology.

Appealing to Lemma 2.15, we can find some $\rho > 0$ such that for $v \in H^1_{0,p}$ with $||v - u_+|| \leq \rho$, we have $I(u_+) \leq I(v)$. Hence, u_+ is a local minimizer of I in $H^1_{0,p}$.

Step 2. Arguing as in Step 1, we can show that u_- is a local minimizer of I in $H^1_{0,p}$ which is a negative nonzero solution for Problem (1.1). As a consequence

$$C_k(I, u_+) = \begin{cases} \mathbb{F}, & \text{for } k = 0 \\ 0, & \text{for } k \neq 0 \end{cases} \tag{3.3}$$

and

$$C_k(I, u_-) = \begin{cases} \mathbb{F}, & \text{for } k = 0 \\ 0, & \text{for } k \neq 0. \end{cases} \tag{3.4}$$

Thus, we have obtained two nonzero solutions for Problem (1.1), namely u_+ and u_-.

Step 3. To show the existence of a third nontrivial solution, we assume, by contradiction, that $0, u_+, u_-$ are the unique critical points of I and we let

$$a < \inf_{u \in H^1_{0,p}} I(u) \quad \text{and} \quad b > \max\{0, I(u_+), I(u_-)\}.$$

Then $I^a = \emptyset$ and I^b is a strong deformation retraction of the space $H^1_{0,p}$. Hence,

$$H_k(I^b, I^a) = \begin{cases} \mathbb{F}, & \text{for } k = 0 \\ 0, & \text{for } k \neq 0 \end{cases}$$

and

$$\sum_{k=0}^{+\infty} (-1)^k \beta_k(a, b) = (-1)^0 = 1.$$

By (3.4), (3.4), and Lemma 2.12, we finally arrive at the conclusion that

$$\sum_{k=0}^{+\infty} (-1)^k M_k(a, b) = 0 + (-1)^0 + (-1)^0 = 2,$$

leading to a contradiction with Lemma 2.3 and ending the proof of the main existence theorem.

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