Arab. J. Math. (2016) 5:35–50
DOI 10.1007/s40065-015-0132-y

**Arabian Journal of Mathematics**

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# On the $|K^\lambda|$-summability of Fourier series and its conjugate series

Received: 15 July 2014 / Accepted: 6 August 2015 / Published online: 11 November 2015


**Abstract** The $K^\lambda$-means were first introduced by Karamata. Vučković first studied the $K^\lambda$-summability of a Fourier series and later on Lal studied the $K^\lambda$-summability of a conjugate series. In the present paper, we have studied the $|K^\lambda|$-summability of Fourier series and conjugate series.

**Mathematics Subject Classification** 42A28

**الملخص**

قدمت أوساط-$K^\lambda$ أول مرة من قبل كاراماتا. كان فوكوفيتش أول من درس قابلية-$K^\lambda$ لجمع متسلسلات فورير، ثم قام لال بدراسة قابلية-$K^\lambda$ لجمع متسلسلات مرافقة. درسنا في هذه الورقة قابلية-$|K^\lambda|$ لجمع متسلسلات فورير ومتسلسلات مرافقة.

## 1 Definitions and notations

For $n = 0, 1, 2, \ldots,$ define the numbers $[{n \atop k}]$ for $0 \le k \le n$ by

$$\Pi_{\nu=0}^{n-1}(x+\nu) = \sum_{k=0}^{n} [{n \atop k}] x^k, \quad x > 0 \tag{1.1}$$

where $\Pi_{\nu=0}^{n-1}(x+\nu) = x(x+1)\cdots(x+n-1) = \frac{\Gamma(x+n)}{\Gamma(x)}$. Clearly, $[{n \atop k}] = 0$ when $k < 1$ and $k > n$. We shall use the convention that $[{0 \atop 0}] = 1$. The numbers of $[{n \atop k}]$ are known as Stirling's number of first kind. We know [8, p. 43] the following recursion formula

$$[{n \atop k}] = [{n-1 \atop k-1}] + (n-1)[{n-1 \atop k}]. \tag{1.2}$$

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Let $\sum_{n=0}^{\infty} u_n$ be an infinite series with sequence of partial sums $\{s_n\}$ i.e., $s_n = \sum_{k=0}^{n} u_k$. Let $\lambda > 0$, the $K^\lambda$-mean $(t_n)$ of the sequence $\{s_n\}$ is defined by [2,5]

$$t_n = \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} \sum_{k=0}^{n} [{}^{n}_{k}]\lambda^k s_k. \tag{1.3}$$

If $\lim_{n\to\infty} t_n = s$, then we say that sequence $\{s_n\}$ (or the series $\sum u_n$) is summable $K^\lambda$ to $s$.

The series $\sum u_n$ (or the sequence $\{s_n\}$) is said to be absolutely $K^\lambda$-summable if $\{t_n\} \in BV$; i.e.,

$$\sum_{n=1}^{\infty} |t_n - t_{n-1}| < \infty. \tag{1.4}$$

Using (1.2) and (1.3), we obtain

$$\begin{aligned} t_n &= \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} \left[ \sum_{k=0}^{n} [{}^{n-1}_{k-1}]\lambda^k s_k + (n-1) \sum_{k=0}^{n} [{}^{n-1}_{k}]\lambda^k s_k \right] \\ &= \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} \left[ \sum_{k=1}^{n} [{}^{n-1}_{k-1}]\lambda^k s_k + (n-1) \sum_{k=0}^{n-1} [{}^{n-1}_{k}]\lambda^k s_k \right] \\ &= \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} \left[ \sum_{k=0}^{n-1} [{}^{n-1}_{k}]\lambda^{k+1} s_{k+1} + (n-1) \sum_{k=0}^{n-1} [{}^{n-1}_{k}]\lambda^k s_k \right], \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} t_{n-1} &= \frac{\Gamma(\lambda)}{\Gamma(n-1+\lambda)} \sum_{k=0}^{n-1} [{}^{n-1}_{k}]\lambda^k s_k \\ &= \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} \left[ \sum_{k=0}^{n-1} [{}^{n-1}_{k}]\lambda^{k+1} s_k + (n-1) \sum_{k=0}^{n-1} [{}^{n-1}_{k}]\lambda^k s_k \right]. \end{aligned} \tag{1.6}$$

From (1.5) and (1.6), it follows that

$$\begin{aligned} t_n - t_{n-1} &= \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} \sum_{k=0}^{n-1} [{}^{n-1}_{k}]\lambda^{k+1} (s_{k+1} - s_k) \\ &= \frac{\lambda}{n-1+\lambda} \left[ \frac{\Gamma(\lambda)}{\Gamma(n+\lambda-1)} \sum_{k=0}^{n-1} [{}^{n-1}_{k}]\lambda^k u_{k+1} \right] \\ &= \frac{\lambda \xi_{n-1}(u)}{n-1+\lambda}. \end{aligned} \tag{1.7}$$

We may derive the following useful identity (Proposition 1.1) which is similar to the Kogbetliantz identity [3] for the Cesàro mean, namely,

$$n(\sigma_n^\alpha - \sigma_{n-1}^\alpha) = \tau_n^\alpha,$$

where $\sigma_n^\alpha$ is the $(C, \alpha)$ mean of $\sum a_n$ and $\tau_n$ is the $(C, \alpha)$ mean of $\{na_n\}$.

**Proposition 1.1**

$$(n-1+\lambda)(t_n - t_{n-1}) = \xi_{n-1}(u)$$

*where $t_n$ is the $K^\lambda$ mean of $\sum u_n$ and $\xi_n$ is the $K^\lambda$ mean of $\{u_{n+1}\}$; i.e.,*

$$\xi_n(u) = \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} \sum_{k=0}^{n} [{}^{n}_{k}]\lambda^k u_{k+1}. \tag{1.8}$$

From Proposition 1.1, it follows that $\sum u_n$ is the $|K^\lambda|$-summable if and only if

$$\sum_{n=1}^{\infty} \frac{|\xi_{n-1}(u)|}{n} < \infty. \tag{1.9}$$

**Proposition 1.2** *The $K^\lambda$-method is absolutely conservative; that is* $|C, 0| \subset |K^\lambda|$.

*Proof* *We need the following result* [8, p. 43 problem 200]:

$$\sum_{n=k}^{\infty} \left[{n \atop k}\right] \frac{(1-u)^n}{n!} = \frac{1}{k!} \left(\log \frac{1}{u}\right)^k. \tag{1.10}$$

*From* (1.7), *it follows that*

$$t_n - t_{n-1} = \sum_{k=0}^{\infty} a_{n,k} u_k$$

*where*

$$a_{n,k} = \begin{cases} \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} \left[{n-1 \atop k-1}\right] \lambda^k, & 0 \le k \le n. \\ 0, & k > n. \end{cases}$$

*The $K^\lambda$-method is absolutely conservative if and only if*

$$\sum_{n=k}^{\infty} |a_{n,k}| < \infty;$$

*i.e.*,

$$\sum{}^{*} \equiv \sum_{n=k}^{\infty} \frac{\Gamma(\lambda)}{\Gamma(n+\lambda)} \left[{n-1 \atop k-1}\right] \lambda^k < \infty.$$

*We have*

$$\begin{aligned} \sum{}^{*} &= \lambda^k \sum_{n=k}^{\infty} \frac{1}{(n-1)!} \int_0^1 u^{\lambda-1} (1-u)^{n-1} \left[{n-1 \atop k-1}\right] \mathrm{d}u \\ &= \lambda^k \int_0^1 u^{\lambda-1} \left( \sum_{n=k-1}^{\infty} \left[{n \atop k-1}\right] \frac{(1-u)^n}{n!} \right) \mathrm{d}u \\ &= \lambda^k \int_0^1 u^{\lambda-1} \frac{1}{(k-1)!} (\log \frac{1}{u})^{k-1} \, \mathrm{d}u \quad \text{using (1.10)}, \\ &= \frac{\lambda^k}{(k-1)!} \int_0^\infty \theta^{k-1} \mathrm{e}^{-\lambda\theta} \, \mathrm{d}\theta = 1, \end{aligned}$$

*which shows that the $K^\lambda$-method is absolutely conservative.* □

## 2 Application to trigonometric Fourier series

Let $f$ be a $2\pi$-periodic function and integrable in the sense of Lebesgue over $(-\pi, \pi)$. Let the trigonometric Fourier series of $f$ at $x$ be given by

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x). \tag{2.1}$$

The conjugate series of (2.1) is given by

$$\begin{aligned}
&\sum_{n=1}^{\infty}(b_n \cos nx - a_n \sin nx) \equiv \sum_{n=1}^{\infty} B_n(x),\\
&\phi_x(t) = \frac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\},\\
&\psi_x(t) = \frac{1}{2}\{f(x+t) - f(x-t)\},\\
&\widetilde{f}(x;\epsilon) = \frac{2}{\pi}\int_{\epsilon}^{\pi} \psi_x(t)\frac{1}{2}\cot\frac{1}{2}t\,\mathrm{d}t,\\
&\text{and}\\
&\widetilde{f}(x) = \lim_{\epsilon\to 0^+}\widetilde{f}(x;\epsilon), \quad \text{whenever the limit exists.}
\end{aligned} \tag{2.2}$$

The $K^{\lambda}$-means were first introduced by Karamata [2]. Lototsky [5] reintroduced the special case $\lambda = 1$. Vučković [10] was the first to study the $K^{\lambda}$-summability of Fourier series and his result reads as follows.

**Theorem 2.1** *If*

$$(f(x+t) + f(x-t) - 2f(x))\log\frac{1}{t} = o(1) \quad \text{as } t \to 0^+,$$

*then the trigonometric Fourier series of $f$ at $x = t$ is $K^{\lambda}$-summable to $f(x)$.*

Later Lal [4] obtained the following result for the conjugate series.

**Theorem 2.2** *If $\int_0^t |\psi_x(u)|\,\mathrm{d}u = o(\frac{t}{\log t^{-1}})$ as $t \to 0^+$, then $\sum_{n=1}^{\infty} B_n(x)$ is $K^{\lambda}$-summable to $\widetilde{f}(x)$, whenever it exists.*

In the present paper, we study the absolute $K^{\lambda}$-summability of a Fourier series and its conjugate series. We prove

**Theorem 2.3** *Let $0 < \delta < \mathrm{e}^{-2}$. Then $\phi(t)\log t^{-1} \in BV(0,\delta) \Rightarrow \sum A_n(x) \in |K^{\lambda}|$.*

**Theorem 2.4** *Let $0 < \delta < \mathrm{e}^{-2}$. Then $\psi(t)\log t^{-1} \in BV(0,\delta)$, and $\frac{\psi(t)}{t} \in L(0,\delta) \Rightarrow \sum B_n(x) \in |K^{\lambda}|$.*

## 3 Notations and lemmas

For the proofs of the theorems, we need the following additional notations:

$$\begin{aligned}
K_n^{\lambda}(t) &= \frac{\Gamma(\lambda)}{\Gamma(n-1+\lambda)}\sum_{k=0}^{n-1}\left[{n-1 \atop k}\right]\lambda^k \sin(k+1)t,\\
\widetilde{K}_n^{\lambda}(t) &= \frac{\Gamma(\lambda)}{\Gamma(n-1+\lambda)}\sum_{k=0}^{n-1}\left[{n-1 \atop k}\right]\lambda^k \cos(k+1)t,\\
P_k(t) &= \left(\lambda^2 + 2\lambda k\cos t + k^2\right)^{\frac{1}{2}},
\end{aligned}$$

$$R(n,t)=\frac{\Gamma(\lambda)}{\Gamma(n-1+\lambda)}\Pi_{k=0}^{n-2}P_k(t),$$
$$\theta_k(t)=\tan^{-1}\left\{\frac{\lambda\sin t}{\lambda\cos t+k}\right\},$$
$$l(n)=2+\lambda\sum_{k=1}^{n-2}\frac{1}{\lambda+k}\sim\log n,$$
$$g(n,t)=\int_0^t\frac{\widetilde{K}_n^{\lambda}(u)}{\log u^{-1}}\,\mathrm{d}u,$$
$$h(n,t)=\int_t^{\delta}\frac{\widetilde{K}_n^{\lambda}(u)}{\log u^{-1}}\,\mathrm{d}u,$$
$$G(n,t)=\int_t^{\delta}\frac{u}{\log u^{-1}}\mathrm{e}^{-Au^2\log n}\,\mathrm{d}u,$$
$$H(n,t)=\int_t^{\delta}\frac{u^3}{\log u^{-1}}\mathrm{e}^{-Au^2\log n}\,\mathrm{d}u,$$
and
$$L(n,t)=\int_t^{\delta}\frac{\mathrm{e}^{-Au^2\log n}}{u(\log u^{-1})^2}\,\mathrm{d}u.$$

We need the following lemmas to prove our theorems:

**Lemma 3.1** *Let $\theta_k(t)$, $R(n,t)$, $K_n^{\lambda}(t)$ and $\widetilde{K}_n^{\lambda}(t)$ be defined as above. Then*

$$\text{(i) } K_n^{\lambda}(t)=R(n,t)\sin\left\{2t+\sum_{k=1}^{n-2}\theta_k(t)\right\},$$
$$\text{(ii) } \widetilde{K}_n^{\lambda}(t)=R(n,t)\cos\left\{2t+\sum_{k=1}^{n-2}\theta_k(t)\right\}.$$

*Proof* We have

$$\begin{aligned}\widetilde{K}_n^{\lambda}(t)+iK_n^{\lambda}(t)&=\frac{\Gamma(\lambda)}{\Gamma(n-1+\lambda)}\sum_{k=0}^{n-1}[{}^{n-1}_{\ \ k}]\lambda^k\mathrm{e}^{i(k+1)t}\\&=\frac{\Gamma(\lambda)}{\Gamma(n-1+\lambda)}\mathrm{e}^{it}\sum_{k=0}^{n-1}[{}^{n-1}_{\ \ k}](\lambda\mathrm{e}^{it})^k\\&=\frac{\Gamma(\lambda)\mathrm{e}^{it}}{\Gamma(n-1+\lambda)}\Pi_{k=0}^{n-2}(\lambda\mathrm{e}^{it}+k)\\&=\frac{\Gamma(\lambda)}{\Gamma(n-1+\lambda)}\mathrm{e}^{it}\Pi_{k=0}^{n-2}P_k(t)\exp(i\theta_k(t))\\&=\frac{\Gamma(\lambda)}{\Gamma(n-1+\lambda)}\left(\Pi_{k=0}^{n-2}P_k(t)\right)\exp\left\{i\left(2t+\sum_{k=1}^{n-2}\theta_k(t)\right)\right\}\\&=R(n,t)\exp\left\{i\left(2t+\sum_{k=1}^{n-2}\theta_k(t)\right)\right\},\end{aligned}$$

from which the lemma follows. □

**Lemma 3.2** [9, Chapter 5, Lemma 5.5] *Let* $R(n,t)$, $K_n^\lambda(t)$ *and* $\widetilde{K}_n^\lambda(t)$ *be defined as in Sect.* 3. *Then for some positive constant* $A$ *and all* $t \in (0, \pi)$

$$\text{(i) } R(n,t) = \begin{cases} 0(1) \\ 0(1) & \mathrm{e}^{-At^2 \log n}, \end{cases}$$
$$\text{(ii) } K_n^\lambda(t) = \begin{cases} 0(1) \\ 0(1) & \mathrm{e}^{-At^2 \log n}, \end{cases}$$

*and*

$$\text{(iii) } \widetilde{K}_n^\lambda(t) = \begin{cases} 0(1) \\ 0(1) & \mathrm{e}^{-At^2 \log n}. \end{cases}$$

*Proof* Note that $R(n,t)$ attains its maximum value for $t = 0$, and it is easy to see that $R(n,0) = 1$. This ensures the first estimates of (i)

Now

$$\begin{aligned} R(n,t) &= \frac{\Gamma(\lambda)}{\Gamma(n-1+\lambda)} \Pi_{k=0}^{n-2} \left(\lambda^2 + 2\lambda k \cos t + k^2\right)^{\frac{1}{2}} \\ &= \frac{\Gamma(\lambda)}{\Gamma(n-1+\lambda)} \Pi_{k=0}^{n-2} (\lambda + k) \left[1 - \frac{4\lambda k \sin^2 \frac{1}{2}t}{(\lambda+k)^2}\right]^{\frac{1}{2}} \\ &= \Pi_{k=0}^{n-2} \left[1 - \frac{4\lambda k \sin^2 \frac{1}{2}t}{(\lambda+k)^2}\right]^{\frac{1}{2}} \\ &= \exp\left[-\frac{1}{2} \sum_{k=0}^{n-2} \log \left\{1 - \frac{4\lambda k \sin^2 \frac{1}{2}t}{(\lambda+k)^2}\right\}^{-1}\right]. \end{aligned} \tag{3.1}$$

We observe that

$$0 < \frac{4\lambda k \sin^2 \frac{1}{2}t}{(\lambda+k)^2} < 1$$

for $k = 1, 2, 3, \ldots$ and $0 < t < \pi$. As $\log(1-\theta)^{-1} \geq \theta$ for $0 < \theta < 1$ and $\sin x \geq \frac{2x}{\pi}$, $o \leq x \leq \frac{\pi}{2}$, we have

$$\begin{aligned} \sum_{k=0}^{n-2} \log \left\{1 - \frac{4\lambda k \sin^2 \frac{1}{2}t}{(\lambda+k)^2}\right\}^{-1} &\geq \sum_{k=0}^{n-2} \frac{4\lambda k \sin^2 \frac{1}{2}t}{(\lambda+k)^2} \\ &\geq \frac{4\lambda t^2}{\pi^2} \sum_{k=0}^{n-2} \frac{k}{(\lambda+k)^2} \\ &\geq 4At^2 \log n \end{aligned} \tag{3.2}$$

where $A$ is a positive constant. Using (3.2) in (3.1), we obtain the second estimate of Lemma 3.2 (i). The proof of Lemma 3.2(ii) and (iii) follows from Lemma 3.2(i). ◻

**Lemma 3.3** [9, Chapter 5, Lemma 5.6] *Let* $0 < t \leq \frac{\pi}{4}$. *Then*

$$\text{(i) } \sin\left(2t + \sum_{k=1}^{n-2} \theta_k(t)\right) - \sin l(n)t = O(t^3 \log n),$$
$$\text{(ii) } \cos\left(2t + \sum_{k=1}^{n-2} \theta_k(t)\right) - \cos l(n)t = O(t^3 \log n).$$

*Proof* We have

$$\left|\sin\left(2t+\sum_{k=1}^{n-2}\theta_k(t)\right)-\sin l(n)t\right| \le \left|2t+\sum_{k=1}^{n-2}\theta_k - l(n)t\right|. \tag{3.3}$$

Next, we note that

$$0 < \frac{\lambda \sin t}{\lambda \cos t + k} < 1$$

whenever $0 < t \le \frac{\pi}{4}$ and $k \ge 1$. Thus for $0 < t \le \frac{\pi}{4}$

$$\begin{aligned}
\theta_k &= \left[\tan^{-1}\frac{\lambda \sin t}{\lambda \cos t + k} - \frac{\lambda \sin t}{\lambda \cos t + k}\right] + \left[\frac{\lambda \sin t}{\lambda \cos t + k} - \frac{\lambda t}{\lambda \cos t + k}\right] \\
&\quad + \left[\frac{\lambda t}{\lambda \cos t + k} - \frac{\lambda t}{\lambda + k}\right] + \frac{\lambda t}{\lambda + k} \\
&= O\left[\left(\frac{\lambda \sin t}{\lambda \cos t + k}\right)^3\right] + O\left(\frac{t^3}{\lambda \cos t + k}\right) + O\left[\frac{t^3}{(\lambda \cos t + k)\,(\lambda + k)}\right] + \frac{\lambda t}{\lambda + k} \\
&= O\left(\frac{t^3}{k^3}\right) + O\left(\frac{t^3}{k}\right) + O\left(\frac{t^3}{k^2}\right) + \frac{\lambda t}{\lambda + k} \\
&= \frac{\lambda t}{\lambda + k} + O\left(\frac{t^3}{k}\right), \quad 1 \le k \le n-2.
\end{aligned} \tag{3.4}$$

Using (3.4), we have

$$\begin{aligned}
2t + \sum_{k=1}^{n-2}\theta_k(t) &= t\left[2 + \lambda\sum_{k=1}^{n-2}\frac{1}{\lambda + k}\right] + O(t^3)\sum_{k=1}^{n-2}\frac{1}{k} \\
&= t l(n) + O(t^3 \log n).
\end{aligned} \tag{3.5}$$

Using (3.5) in (3.3), we obtain Lemma 3.3(i). The proof of Lemma 3.3(ii) is similar to that of Lemma 3.3(i). □

**Lemma 3.4** [9, Chapter 5, Lemma 5.7] *Let $0 < t \le \frac{\pi}{2}$. Then $R'(n,t) = O(1)t \log n\, R(n,t)$.*

*Proof* We have

$$R(n,t) = \frac{\Gamma(\lambda)}{\Gamma(n-1+\lambda)}\Pi_{k=0}^{n-2}P_k(t),$$

and so, by logarithmic differentiation, since $P_k(t) \ge k$

$$\begin{aligned}
R'(n,t) &= R(n,t)\sum_{k=0}^{n-2}\frac{P_k'(t)}{P_k(t)} \\
&= R(n,t)\sum_{k=1}^{n-2}\frac{(-\lambda k \sin t)}{(P_k(t))^2} \\
&= O(1)t R(n,t)\sum_{k=1}^{n-2}\frac{1}{k} \\
&= O(1)t \log n R(n,t),
\end{aligned}$$

from which the lemma follows. □

**Lemma 3.5** *Let $\alpha_n = \int_0^\delta \frac{\cos nu}{\log u^{-1}}\,\mathrm{d}u$. Then the series $\sum \alpha_n \in |K^\lambda|$; i.e., the series $\sum \frac{|g(n,\delta)|}{n}$ is convergent.*

*Proof* Integrating by parts, we have

$$\begin{aligned}\alpha_n &= \int_0^\delta \frac{\cos nu}{\log u^{-1}}\,\mathrm{d}u \\ &= \frac{\sin n\delta}{n\log\delta^{-1}} - \frac{1}{n}\int_0^\delta \frac{\sin nu}{u(\log u^{-1})^2}\,\mathrm{d}u \\ &= -\frac{1}{\log\delta^{-1}}\int_\delta^\pi \cos nu\,\mathrm{d}u - \frac{1}{n}\int_0^\delta \frac{\sin nu}{u(\log u^{-1})^2}\,\mathrm{d}u \\ &= -(\alpha_{n,1}+\alpha_{n,2}), \qquad \text{say.}\end{aligned}$$

It is known [1] that

$$\int_0^\delta \frac{\sin nu}{u(\log 1/u)^2}\,\mathrm{d}u = O\left(\frac{1}{(\log n)^2}\right)$$

and hence $\sum \alpha_{n,2}$ is absolutely convergent. And, since $|K^\lambda|$-method is absolutely conservative

$$\sum \alpha_{n,2} \in |K^\lambda|.$$

It remains to show that $\sum \alpha_{n,1} \in |K^\lambda|$.

By definition the series $\sum \alpha_{n,1} \in |K^\lambda|$ if

$$\sum \equiv \sum_{n=1}^\infty \frac{1}{n}\left|\frac{\Gamma(\lambda)}{\Gamma(n-1+\lambda)}\sum_{k=0}^{n-1}\left[{n-1 \atop k}\right]\lambda^k \int_\delta^\pi \cos(k+1)u\,\mathrm{d}u\right| < \infty.$$

Using the notation of Sect. 3 and Lemma 3.2(iii), we get

$$\begin{aligned}\sum &= \sum_{n=1}^\infty \frac{1}{n}\left|\int_\delta^\pi \widetilde{K}_n^\lambda(t)\,\mathrm{d}t\right| \\ &\le \sum_{n=1}^\infty \frac{1}{n}\int_\delta^\pi |\widetilde{K}_n^\lambda(t)|\,\mathrm{d}t \\ &\le O(1)\sum_{n=1}^\infty \frac{1}{n}\int_\delta^\pi \mathrm{e}^{-At^2\log n}\,\mathrm{d}t \\ &= O(1)\sum_{n=1}^\infty \frac{1}{n}\mathrm{e}^{-A\delta^2\log n} \\ &= O(1)\sum_{n=1}^\infty \frac{1}{n^{1+A\delta^2}} = O(1),\end{aligned}$$

which implies that $\sum \alpha_{n,1} \in |K^\lambda|$.

As $\sum \alpha_n \in |K^\lambda|$, and collecting the above results, it follows that

$$\sum_{n=1}^\infty \frac{1}{n}\left|\frac{\Gamma(\lambda)}{\Gamma(n+\lambda-1)}\sum_{k=0}^{n-1}\left[{n-1 \atop k}\right]\lambda^k \int_0^\delta \frac{\cos(k+1)u}{\log u^{-1}}\,\mathrm{d}u\right| < \infty;$$

that is

$$\sum_{n=1}^\infty \frac{1}{n}\left|\int_0^\delta \frac{\mathrm{d}u}{\log u^{-1}}\frac{\Gamma(\lambda)}{\Gamma(n-1+\lambda)}\sum_{k=0}^{n-1}\left[{n-1 \atop k}\right]\lambda^k \cos(k+1)u\,\mathrm{d}u\right| < \infty;$$

that is,

$$\sum_{n=1}^{\infty} \frac{1}{n} \left| \int_0^{\delta} \frac{\widetilde{K}_n^{\lambda}(u)}{\log u^{-1}} \, \mathrm{d}u \right| < \infty;$$

that is, $\sum \frac{|g(n,\delta)|}{n} < \infty$.

This completes the proof of the lemma. □

**Lemma 3.6** *For every positive* $\Delta$*, however large,*

$$\text{(i) } G(n,t) = O\left( \frac{\mathrm{e}^{-At^2 \log n}}{\log n \log t^{-1}} \right) + \frac{L(n,t)}{2A \log n},$$

$$\text{(ii) } H(n,t) = O\left( \frac{t^2 \mathrm{e}^{-At^2 \log n}}{\log t^{-1} \log n} \right) + O(1) \frac{G(n,t)}{\log n}.$$

*Proof of* (i) Integrating by parts we have,

$$\begin{aligned} G(n,t) &= -\frac{1}{2A \log n} \int_t^{\delta} \frac{\mathrm{d}}{\mathrm{d}u} (\mathrm{e}^{-Au^2 \log n}) \frac{\mathrm{d}u}{\log u^{-1}} \\ &= \frac{1}{2A \log n} \left[ \frac{\mathrm{e}^{-At^2 \log n}}{\log t^{-1}} - \frac{\mathrm{e}^{-A\delta^2 \log n}}{\log \delta^{-1}} \right] \\ &\quad + \frac{1}{2A \log n} \int_t^{\delta} \frac{\mathrm{e}^{-Au^2 \log n}}{u (\log u^{-1})^2} \\ &\le \frac{\mathrm{e}^{-At^2 \log n}}{2A (\log n) \log t^{-1}} + \frac{L(n,t)}{2A \log n}, \end{aligned}$$

from which (i) follows. □

*Proof of* (ii) Integrating by parts, we get

$$\begin{aligned} H(n,t) &= -\frac{1}{2A \log n} \int_t^{\delta} \frac{u^2}{(\log u^{-1})} \frac{\mathrm{d}}{\mathrm{d}u} (\mathrm{e}^{-Au^2 \log n}) \, \mathrm{d}u \\ &= \frac{1}{2A \log n} \left[ \frac{t^2 \mathrm{e}^{-At^2 \log n}}{\log t^{-1}} - \frac{\delta^2 \mathrm{e}^{-A\delta^2 \log n}}{\log \delta^{-1}} \right] \\ &\quad + \frac{1}{2A \log n} \int_t^{\delta} \mathrm{e}^{-Au^2 \log n} \frac{\mathrm{d}}{\mathrm{d}u} \left( \frac{u^2}{\log u^{-1}} \right) \mathrm{d}u \\ &< \frac{t^2 \mathrm{e}^{-At^2 \log n}}{2A \log n} + \frac{1}{A \log n} \left[ \int_t^{\delta} \frac{u \mathrm{e}^{-Au^2 \log n}}{\log u^{-1}} \, \mathrm{d}u + \frac{1}{2} \int_t^{\delta} \frac{u \mathrm{e}^{-Au^2 \log n}}{(\log u^{-1})^2} \, \mathrm{d}u \right] \\ &< \frac{t^2 \mathrm{e}^{-At^2 \log n}}{2A \log n} + \frac{1}{A \log n} \left[ \int_t^{\delta} \frac{u \mathrm{e}^{-Au^2 \log n}}{\log u^{-1}} \, \mathrm{d}u + \frac{1}{2 \log \delta^{-1}} \int_t^{\delta} \frac{u \mathrm{e}^{-Au^2 \log n}}{\log u^{-1}} \, \mathrm{d}u \right] \\ &= \frac{t^2 \mathrm{e}^{-At^2 \log n}}{2A \log n} + O(1) \frac{1}{\log n} \int_t^{\delta} \frac{u e^{-Au^2 \log n}}{\log u^{-1}} \, \mathrm{d}u, \end{aligned}$$

from which the result follows. □

**Lemma 3.7** *Let* $0 < t < \delta < \mathrm{e}^{-2}$. *Then*

$$\text{(i) } L(n,t) = O\left( \frac{1}{t^2 (\log t^{-1})^2 \log n} \right),$$

$$\text{(ii) } L(n,t) = O\left( \frac{1}{\log \log n} \right).$$

*Proof of* (i) Integrating by parts, we get

$$\begin{aligned}L(n,t) &= -\frac{1}{2A\log n}\int_t^{\delta}\frac{\mathrm{d}}{\mathrm{d}u}\left(\mathrm{e}^{-Au^2\log n}\right)\frac{\mathrm{d}u}{u^2(\log u^{-1})^2}\\ &= \frac{1}{2A\log n}\left[\frac{\mathrm{e}^{-At^2\log n}}{t^2(\log t^{-1})^2}-\frac{\mathrm{e}^{-A\delta^2\log n}}{\delta^2(\log\delta^{-1})^2}\right]\\ &\quad+\frac{1}{2A\log n}\int_t^{\delta}\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{1}{u^2(\log u^{-1})^2}\right)\mathrm{e}^{-Au^2\log n}\,\mathrm{d}u\\ &< \frac{\mathrm{e}^{-At^2\log n}}{2A\log nt^2(\log t^{-1})^2},\end{aligned}$$

since the last integral is negative, and this completes the proof of (i). □

*Proof of* (ii) Let $0<\beta<2$. By the simple computation, we get

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\mathrm{e}^{-Au^2\log n}}{(\log 1/u)^2}\right)=\frac{2\mathrm{e}^{-Au^2\log n}}{(\log 1/u)^3}\left\{1-Au^2\log\frac{1}{u}\log n\right\}.$$

The expression $\frac{\mathrm{e}^{-Au^2\log n}}{(\log\frac{1}{u})^2}$ is monotonic decreasing in $u$ whenever $1-Au^2\log\frac{1}{u}\log n<0$. It is easy to see that for $0<u<\delta<\mathrm{e}^{-2}$

$$\left(\frac{1}{u}\right)^{\frac{1-2\beta}{\beta}}\log\frac{1}{u}>1;$$

that is,

$$u^2\log\frac{1}{u}-u^{\frac{1}{\beta}}>0.$$

In view of this inequality, we get

$$\begin{aligned}1-Au^2\log\frac{1}{u}\log n &= 1-Au^{\frac{1}{\beta}}\log n-A\left\{u^2\log\frac{1}{u}-u^{\frac{1}{\beta}}\right\}\log n\\ &< 1-Au^{\frac{1}{\beta}}\log n<0,\end{aligned}$$

which holds for $u>(A\log n)^{-\beta}$. This ensures that $\frac{\mathrm{e}^{-Au^2\log n}}{(\log\frac{1}{u})^2}$ is monotonic decreasing for $u>(A\log n)^{-\beta}$. We shall consider the cases $(A\log n)^{-\beta}<\delta$ and $(A\log n)^{-\beta}\geq\delta$ separately. In case $(A\log n)^{-\beta}<\delta$ writing

$$L(n,t)=\left(\int_t^{1/(A\log n)^{\beta}}+\int_{1/(A\log n)^{\beta}}^{\delta}\right)\frac{\mathrm{e}^{-Au^2\log n}}{u(\log u^{-1})^2}\,\mathrm{d}u$$

and using the monotonicity of $\frac{\mathrm{e}^{-Au^2\log n}}{(\log u^{-1})^2}$ for the second integral, we get

$$\begin{aligned}L(n,t) &\leq \mathrm{e}^{-At^2\log n}\int_t^{1/(A\log n)^{\beta}}\frac{\mathrm{d}u}{u(\log u^{-1})^2}+\frac{\mathrm{e}^{-A^{1-2\beta}}(\log n)^{1-2\beta}}{(\beta\log(A\log n))^2}\int_{\frac{1}{(A\log n)^{\beta}}}^{\delta}\frac{\mathrm{d}u}{u}\\ &= O\left(\frac{\mathrm{e}^{-At^2\log n}}{\beta\log(A\log n)}\right)+O\left(\frac{1}{(\log n)^{\Delta}}\right),\quad \Delta>1\\ &= O\left(\frac{1}{\log\log n}\right).\end{aligned}$$

In case $(A\log n)^{-\beta} \geq \delta$, we have

$$L(n,t) = \int_t^{\delta} \frac{\mathrm{e}^{-Au^2\log n}}{u\left(\log\frac{1}{u}\right)^2}\,\mathrm{d}u, \\ \leq \int_t^{(A\log n)^{-\beta}} \frac{\mathrm{e}^{-Au^2\log n}}{u\left(\log\frac{1}{u}\right)^2}\,\mathrm{d}u,$$

which is same as the first integral in the first case discussed above. Lastly, in case $(A\log n)^{-\beta} < t$, $L(n,t)$ is majorized by the second integral $\int_{(A\log n)^{-\beta}}^{\delta}$ in the first case and this completes the proof of (ii). □

**Lemma 3.8** *Let* $0 < \delta < \mathrm{e}^{-2}$ *and* $\Delta > 1$, *however large. Then*

(i) $g(n,t) = O\left(\frac{t}{\log t^{-1}}\right)$,

(ii) $h(n,t) = O\left(\frac{1}{n^{A\delta^2}}\right) + O(1)\frac{\mathrm{e}^{-At^2\log n}}{(\log t^{-1})\log n} + O(1)\frac{t^2\mathrm{e}^{-At^2\log n}}{(\log t^{-1})} + \frac{L(n,t)}{\log n}$.

*Proof of* (i) As $\widetilde{K}_n^{\lambda}(u) = O(1)$ by Lemma 3.2(iii), the result follows. □

*Proof of* (ii) First using Lemma 3.1(ii) and thereafter applying Lemma 3.3(ii), we obtain

$$\begin{aligned} h(n,t) &= \int_t^{\delta} \frac{\widetilde{K}_n^{\lambda}(u)}{\log u^{-1}}\,\mathrm{d}u \\ &= \int_t^{\delta} \frac{R(n,u)}{\log u^{-1}}\cos\left\{2u + \sum_{k=1}^{n-2}\theta_k(u)\right\}\mathrm{d}u \\ &= \int_t^{\delta} \frac{1}{\log u^{-1}} R(n,u)\cos l(n)u\,\mathrm{d}u + \int_t^{\delta} \frac{R(n,u)}{\log u^{-1}}\left[\cos\left\{2u + \sum_{k=1}^{n-2}\theta_k(u)\right\} - \cos l(n)u\right]\mathrm{d}u \\ &= \int_t^{\delta} \frac{R(n,u)}{\log u^{-1}}\cos\{l(n)u\}\,\mathrm{d}u + O(1)\log n\int_t^{\delta}\frac{R(n,u)u^3\,\mathrm{d}u}{\log u^{-1}} \\ &= I_1 + O(1)I_2, \qquad \text{say.} \end{aligned} \tag{3.6}$$

Integrating by parts and using Lemma 3.4 and Lemma 3.2(i), we get

$$\begin{aligned} I_1 &= \left[\frac{R(n,u)\sin l(n)u}{\log u^{-1} l(n)}\right]_t^{\delta} - \int_t^{\delta}\left[\frac{R'(n,u)}{\log u^{-1}} + \frac{R(n,u)}{u(\log u^{-1})^2}\right]\frac{\sin l(n)u\,\mathrm{d}u}{l(n)} \\ &= O\left(\frac{\mathrm{e}^{-A\delta^2\log n}}{\log n}\right) + O\left(\frac{\mathrm{e}^{-At^2\log n}}{\log t^{-1}\log n}\right) + O(1)\int_t^{\delta}\mathrm{e}^{-Au^2\log n}\frac{u\,\mathrm{d}u}{\log u^{-1}} \\ &\quad + O(1)\frac{1}{\log n}\int_t^{\delta}\frac{\mathrm{e}^{-Au^2\log n}\,\mathrm{d}u}{u(\log u^{-1})^2} \\ &= O\left(\frac{1}{(n)^{A\delta^2}}\right) + O\left(\frac{\mathrm{e}^{-At^2\log n}}{\log n\log t^{-1}}\right) + O(1)G(n,t) + \frac{O(1)L(n,t)}{\log n}. \end{aligned} \tag{3.7}$$

Using Lemma 3.2 and Lemma 3.6(ii) , we have

$$\begin{aligned} I_2 &= \log n H(n,t) \\ &= O(1)\frac{t^2\mathrm{e}^{-At^2\log n}}{\log t^{-1}} + O(1)G(n,t). \end{aligned} \tag{3.8}$$

Collecting the results from (3.6) to (3.8) and using the estimate for $G(n,t)$ from Lemma 3.6(i), we obtain the desired estimate for $h(n,t)$. □

## 4 Proof of Theorem 2.3 Using [6]

For $n \geq 1$ and $0 < \delta < e^{-2}$, we write

$$A_n(x) = \frac{2}{\pi}\left(\int_0^{\delta} + \int_{\delta}^{\pi}\right)\phi(t)\cos nt \, dt$$
$$= \frac{2}{\pi}(P_n + Q_n), \qquad \text{say.} \tag{4.1}$$

Let $\xi_n(Q)$ be the $n$th $K^{\lambda}$-mean of the sequence $\{Q_{n+1}\}$.
The series $\sum Q_n \in |K^{\lambda}|$, if and only if

$$\sum \frac{|\xi_{n-1}(Q)|}{n} < \infty. \tag{4.2}$$

By simple computation and an appeal to Lemma 3.2(iii)

$$\xi_{n-1}(Q) = \int_{\delta}^{\pi}\phi(t)\left[\frac{\Gamma(\lambda)}{\Gamma(n-1+\lambda)}\sum_{k=0}^{n-1}\left[{n-1 \atop k}\right]\lambda^k\cos(k+1)t\right]dt$$
$$= \int_{\delta}^{\pi}\phi(t)\widetilde{K}_n^{\lambda}(t)\,dt$$
$$= O(1)\int_{\delta}^{\pi}|\phi(t)|e^{-At^2\log n}\,dt$$
$$= O(1)e^{-A\delta^2\log n}.$$

This ensures (4.2) and consequently vindicates that the $|K^{\lambda}|$-summability of trigonometric Fourier series is a local property. Writing $g(t) = \phi(t)\log t^{-1}$, $\alpha_n = \int_0^{\delta} g(u)\frac{\cos nu\,du}{\log u^{-1}}$ and integrating by parts, we obtain

$$P_n = g(\delta)\alpha_n - \int_0^{\delta} dg(t)\int_0^t \frac{\cos nu}{\log u^{-1}}\,du$$
$$= g(\delta)\alpha_n - \beta_n, \qquad \text{say.} \tag{4.3}$$

As $\sum \alpha_n \in |K^{\lambda}|$ by Lemma 3.5 it remains to prove that $\sum \beta_n \in |K^{\lambda}|$. Let $\xi_n(\beta)$ be the $n$th $K^{\lambda}$-mean of the sequence $\{\beta_{n+1}\}$. It is easily seen that

$$\xi_{n-1}(\beta) = \frac{\Gamma(\lambda)}{\Gamma(n-1+\lambda)}\sum_{k=0}^{n-1}\left[{n-1 \atop k}\right]\lambda^k\left[\int_0^{\delta} dg(t)\int_0^t \frac{\cos(k+1)u\,du}{\log u^{-1}}\right]$$
$$= \int_0^{\delta} g(n,t)\,dg(t).$$

By definition $\sum \beta_n \in |K^{\lambda}|$, if and only if

$$\sum \frac{|\xi_{n-1}(\beta)|}{n} < \infty;$$

that is,

$$\sum_{n=1}^{\infty}\left|\int_0^{\delta} g(n,t)\,dg(t)\right| < \infty. \tag{4.4}$$

As $\int_0^{\delta}|dg(t)|$ is finite, for the validity of (4.4), it is enough to show that uniformly in $0 < t \leq \delta$.

$$\sum = \sum_{n=1}^{\infty}\frac{|g(n,t)|}{n} = O(1). \tag{4.5}$$

Putting $T_1 = [\exp(t^{-1})]$ and $T_2 = [\exp(t^{-2})]$, we write

$$\sum = \left( \sum_{n=1}^{T_1} + \sum_{n=T_{1+1}}^{\infty} \right) \frac{|g(n,t)|}{n}. \tag{4.6}$$

By Lemma 3.8(i)

$$\sum_{n=1}^{T_1} = O\left(\frac{t}{\log t^{-1}}\right) \sum_{n=1}^{T_1} \frac{1}{n} = O(1). \tag{4.7}$$

By Lemma 3.5 and Lemma 3.8(ii)

$$\begin{aligned} \sum_{n=T_1+1}^{\infty} \frac{|g(n,t)|}{n} &\le \sum_{n=T_1+1}^{\infty} \frac{|g(n,\delta)|}{n} + \sum_{n=T_1+1}^{\infty} \frac{|h(n,t)|}{n} \\ &= O(1) + O(1) \sum_{n=T_1+1}^{\infty} \frac{1}{(n)^{1+A\delta^2}} + O(1) \frac{1}{(\log t^{-1})} \sum_{n=T_1+1}^{\infty} \frac{\mathrm{e}^{-At^2 \log n}}{n \log n} \\ &\quad + O(1) \frac{t^2}{\log t^{-1}} \sum_{n=T_1+1}^{\infty} \frac{\mathrm{e}^{-At^2 \log n}}{n} + O(1) \sum_{n=T_1+1}^{\infty} \frac{L(n,t)}{n \log n} \\ &= O(1) + O(1) \frac{1}{\log t^{-1}} \int_{T_1}^{\infty} \frac{\mathrm{e}^{-At^2 \log x}}{x \log x} \,\mathrm{d}x + O(1) \frac{t^2}{\log t^{-1}} \int_{T_1}^{\infty} \frac{\mathrm{e}^{-At^2 \log x} \,\mathrm{d}x}{x} \\ &\quad + O(1) \sum_{n=T_1+1}^{\infty} \frac{L(n,t)}{n \log n} \\ &= O(1) + O(1) \sum_{n=T_1+1}^{\infty} \frac{L(n,t)}{n \log n}, \end{aligned} \tag{4.8}$$

since $\int_{T_1}^{\infty} \frac{\mathrm{e}^{-At^2 \log x}}{x \log x} \,\mathrm{d}x = \int_t^{\infty} \frac{\mathrm{e}^{-A\theta}}{\theta} \,\mathrm{d}\theta = O(\log t^{-1})$ and $\int_{T_1}^{\infty} \frac{\mathrm{e}^{-At^2 \log x}}{x} \,\mathrm{d}x = t^{-2} \int_t^{\infty} \mathrm{e}^{-A\theta} \,\mathrm{d}\theta = O(t^{-2})$.

Now writing $\sum_{n=T_1+1}^{\infty} \frac{L(n,t)}{n \log n} = \left( \sum_{n=T_1+1}^{T_2} + \sum_{n=T_2+1}^{\infty} \right) \frac{L(n,t)}{n \log n}$ and employing Lemma 3.7(ii) and Lemma 3.7(i), respectively, for the first and second sums, we get

$$\begin{aligned} \sum_{n=T_1+1}^{\infty} \frac{L(n,t)}{n \log n} &= O(1) \sum_{n=T_1+1}^{T_2} \frac{1}{n \log n \log \log n} + O(1) \frac{1}{t^2 (\log t^{-1})^2} \sum_{n=T_2+1}^{\infty} \frac{1}{n (\log n)^2} \\ &= O(1) \log \frac{\log \log T_2}{\log \log T_1} + O(1) \frac{1}{t^2 (\log t^{-1})^2 \log T_2} \\ &= O(1). \end{aligned} \tag{4.9}$$

Collecting the results from (4.6)–(4.9), we get (4.5) and this completes the proof of Theorem 2.3. □

## 5 Proof of Theorem 2.4

We need the following additional lemmas for the proof of Theorem 2.4.

**Lemma 5.1** [7, p. 314, Lemma 10] $\psi(t) \log t^{-1} \in BV(0,\delta)$, *and* $\frac{\psi(t)}{t} \in L(0,\delta)$ *if and only if* $\psi(+0) = 0$ *and* $\int_0^{\delta} \log t^{-1} |\mathrm{d}\psi(t)| < \infty$.

**Lemma 5.2** *Let* $T_1 = [\exp(t^{-1})]$. *Then*

$$\text{(i)} \int_t^{\delta} K_n^{\lambda}(u)\,\mathrm{d}u = \mathrm{e}^{-At^2\log n}\left[O(t^2) + O\left(\frac{1}{\log n}\right)\right],$$

$$\text{(ii)} \int_t^{\delta} K_n^{\lambda}(u)\,\mathrm{d}u = O\left(\frac{1}{\log n}\right), \quad \textit{when } n \le T_1.$$

*Proof* By Lemma 3.3(i) and Lemma 3.2(i)

$$\begin{aligned}\int_t^{\delta} K_n^{\lambda}(u)\,\mathrm{d}u &= \int_t^{\delta} R(n,u)\sin l(n)u\,\mathrm{d}u + O(1)\log n \int_t^{\delta} u^3 R(n,u)\,\mathrm{d}u \\ &= \int_t^{\delta} R(n,u)\sin l(n)u\,\mathrm{d}u + O(1)\log n \int_t^{\delta} u^3 \mathrm{e}^{-Au^2\log n}\,\mathrm{d}u \\ &= J_1 + O(1)J_2, \qquad \text{say}\end{aligned} \tag{5.1}$$

Integrating by parts, we get

$$\begin{aligned}J_2 &= \frac{\log n}{2}\int_{t^2}^{\delta^2} \nu \mathrm{e}^{-A\upsilon\log n}\,\mathrm{d}\upsilon \\ &= \frac{1}{2A}\left(t^2\mathrm{e}^{-At^2\log n} - \delta^2\mathrm{e}^{-A\delta^2\log n}\right) \\ &\quad + \frac{1}{2A^2\log n}\left(\mathrm{e}^{-At^2\log n} - \mathrm{e}^{-A\delta^2\log n}\right) \\ &\le \frac{1}{2A}\left(t^2 + \frac{1}{2A\log n}\right)\mathrm{e}^{-At^2\log n}.\end{aligned} \tag{5.2}$$

As $R(n,t)$ is monotonic non-increasing in $t$, by second mean value theorem, we have for $t < t' < \delta$.

$$\begin{aligned}J_1 &= R(n,t)\left[\frac{\cos l(n)t - \cos l(n)t'}{l(n)}\right], \\ &= O\left(\frac{\mathrm{e}^{-At^2\log n}}{\log n}\right),\end{aligned} \tag{5.3}$$

using Lemma 3.2(i). □

Part (i) follows from (5.1), (5.2) and (5.3). When $n \le T_1$, it is easily seen that $t^2$ is dominated by $(\log n)^{-1}$ and hence (ii) follows from (i).

By Lemma 5.1, Theorem 2.4 takes the following equivalent form:

**Theorem 5.3** *If* $\psi(+0) = 0$ *and* $\int_0^{\delta} |\mathrm{d}\psi(t)| \log t^{-1} < \infty$, *then* $\sum_{n=1}^{\infty} B_n(x) \in |K^{\lambda}|$.

*Proof of Theorem 5.3* For $n \ge 1$, we write

$$\begin{aligned}B_n(x) &= \frac{2}{\pi}\left[\int_0^{\delta} + \int_{\delta}^{\pi}\right]\psi(t)\sin nt\,\mathrm{d}t \\ &= \frac{2}{\pi}(p_n + q_n), \qquad \text{say.}\end{aligned}$$

By adopting the argument used in proving $\sum Q_n \in |K^{\lambda}|$ (see the proof of Theorem 2.3) it can be shown that $\sum q_n \in |K^{\lambda}|$. Integrating by parts and using the fact that $\psi(+0) = 0$, we get

$$p_n = \int_0^{\delta} \mathrm{d}\psi(t) \int_t^{\delta} \sin nu\,\mathrm{d}u. \tag{5.4}$$

Let $\xi_n(p)$ denote the $n$th $K^\lambda$-mean of the sequence $\{p_{n+1}\}$. By routine simplification, we have

$$\xi_{n-1}(p) = \int_0^\delta d\psi(t) \int_t^\delta \frac{\Gamma(\lambda)}{\Gamma(n+\lambda-1)} \left( \sum_{k=0}^{n-1} \left[{n-1 \atop k}\right] \lambda^k \sin(k+1)u \, du \right)$$
$$= \int_0^\delta d\psi(t) \int_t^\delta K_n^\lambda(u)\, du. \tag{5.5}$$

By definition $\sum p_n \in |K^\lambda|$, if and only if

$$\sum_{n=1}^{\infty} \frac{|\xi_{n-1}(p)|}{n} < \infty;$$

$$\text{that is,} \quad \sum_{n=1}^{\infty} \frac{\left| \int_0^\delta d\psi(t) \int_t^\delta K_n^\lambda(u)\, du \right|}{n} < \infty. \tag{5.6}$$

As $\int_0^\delta |d\psi(t)| \log t^{-1}$ is finite, for the validity of (5.6), it suffices to show that uniformly in $0 < t \le \delta$

$$\sum{}^{*} \equiv \sum_{n=1}^{\infty} \frac{1}{n} \left| \int_t^\delta K_n^\lambda(u)\, du \right| = O(\log t^{-1}). \tag{5.7}$$

Writing $\sum^* = \sum_{n=1}^{T_1} + \sum_{n=T_1+1}^{\infty}$ and using Lemma 5.2(ii) and Lemma 5.2(i), respectively, for the first sum and second sum, we get

$$\begin{aligned}
\sum{}^{*} &= O(1) \sum_{n=1}^{T_1} \frac{1}{n \log n} + O(t^2) \sum_{n=T_1+1}^{\infty} \frac{e^{-At^2 \log n}}{n} \\
&\quad + O(1) \sum_{n=T_1+1}^{\infty} \frac{e^{-At^2 \log n}}{n \log n} \\
&= O(1) \log\log T_1 + O(t^2) \int_{T_1}^{\infty} \frac{e^{-At^2 \log x}}{x}\, dx \\
&\quad + O(1) \int_{T_1}^{\infty} \frac{e^{-At^2 \log x}}{x \log x}\, dx \\
&= O(\log t^{-1}),
\end{aligned}$$

which ensures (5.7) and this completes the proof of Theorem 5.3. □

**Acknowledgments** The authors are thankful to the referee for his valuable suggestions and criticisms which led to the improvement of the paper.



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