Arab. J. Math. (2015) 4:153–157
DOI 10.1007/s40065-014-0123-4

**Arabian Journal of Mathematics**

CrossMark

**Zujin Zhang · Pingzhou Hong · Dingxing Zhong · Shulin Qiu**

# A regularity criterion for the 3D MHD equations in terms of the gradient of the pressure in the multiplier spaces

Received: 4 July 2014 / Accepted: 7 December 2014 / Published online: 19 December 2014


**Abstract** In this paper, we consider the regularity criterion for the 3D MHD equations and prove that if the gradient of the pressure belongs to $L^{\frac{2}{2-r}}(0, T; \dot{X}_r(\mathbb{R}^3))$ with $0 \le r \le 1$, then the solution is smooth. Notice that we extend the result given by Gala (Appl Anal 92:96–103, 2013).

**الملخص**

نعتبر في هذه الورقة معيار الانتظام لمعادلات 3D MHD، ونثبت أنه إذا كان تدرُّج الضغط ينتمي إلى $L^{\frac{2}{2-r}}(0, T; \dot{X}_r(\mathbb{R}^3))$ حيث $0 \le r \le 1$، فإن الحل مصقول. لاحظ أننا نمدد النتيجة التي حصل عليها جالا [س. جالا، ملاحظات على معيار الانتظام للحلول الضعيفة لمعادلات نافيير – ستوكس بدلالة تدرُّج الضغط، مجلة التحليل التطبيقي، **92** (2013)، 103 – 96].

**Mathematics Subject Classification** 35B65 · 35Q35 · 76D03

## 1 Introduction

In this paper, we consider the following three-dimensional (3D) magnetohydrodynamic (MHD) equations:

$$\begin{cases} \boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - (\boldsymbol{b} \cdot \nabla)\boldsymbol{b} - \triangle \boldsymbol{u} + \nabla \pi = \boldsymbol{0}, \\ \boldsymbol{b}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{b} - (\boldsymbol{b} \cdot \nabla)\boldsymbol{u} - \triangle \boldsymbol{b} = \boldsymbol{0}, \\ \nabla \cdot \boldsymbol{u} = 0, \\ \nabla \cdot \boldsymbol{b} = 0, \\ \boldsymbol{u}(0) = \boldsymbol{u}_0, \ \boldsymbol{b}(0) = \boldsymbol{b}_0, \end{cases} \tag{1}$$

where $\boldsymbol{u} = (u_1, u_2, u_3)$ is the fluid velocity field, $\boldsymbol{b} = (b_1, b_2, b_3)$ is the magnetic field, $\pi$ is a scalar pressure, and $(\boldsymbol{u}_0, \boldsymbol{b}_0)$ are the prescribed initial data satisfying $\nabla \cdot \boldsymbol{u}_0 = \nabla \cdot \boldsymbol{b}_0 = 0$ in the distributional sense. Physically, (1) governs the dynamics of the velocity and magnetic fields in electrically conducting fluids, such as plasmas,

Z. Zhang (✉) · P. Hong · D. Zhong · S. Qiu
School of Mathematics and Computer Science, Gannan Normal University, Ganzhou 341000, Jiangxi,
People's Republic of China
E-mail: zhangzujin361@163.com

P. Hong
E-mail: pzhong-66@163.com

D. Zhong
E-mail: zhongdingxing678@sina.com

S. Qiu
E-mail: qiushulin2003@163.com

liquid metals, and salt water. Moreover, $(1)_1$ reflects the conservation of momentum, $(1)_2$ is the induction equation, and $(1)_3$ specifies the conservation of mass.

Besides its physical applications, the MHD system (1) is also mathematically significant. Duvaut and Lions [3] constructed a global weak solution to (1) for initial data with finite energy. However, the issue of regularity and uniqueness of such a given weak solution remains a challenging open problem. Many sufficient conditions (see e.g., [1,2,4–7,9,11–16,18,20–28] and the references therein) were derived to guarantee the regularity of the weak solution.

We are interested in the regularity condition in terms of the pressure, the pressure gradient or its partial components. Let us now list some finest results up to date.

- In [6], the author improved [25], and established the fundamental Serrin-type regularity criterion in terms of the pressure,

$$\pi \in L^p(0, T; L^q(\mathbb{R}^3)), \quad \frac{2}{p} + \frac{3}{q} = 2, \quad \frac{3}{2} < q \le \infty;$$

  or the pressure gradient,

$$\nabla\pi \in L^p(0, T; L^q(\mathbb{R}^3)), \quad \frac{2}{p} + \frac{3}{q} = 3, \quad 1 < q \le \infty;$$

  that is, if one of the above two conditions holds on $(0, T)$ with $0 < T < \infty$, then the solution is smooth on $(0, T)$.
- Jia and Zhou [13] used intricate decomposition technique and delicate inequalities to obtain the following regularity criterion:

$$\partial_3\pi \in L^p(0, T; L^q(\mathbb{R}^3)), \quad \frac{2}{p} + \frac{3}{q} = 2, \quad 3 \le q < \infty. \tag{2}$$

  Applying a more subtle decomposition technique (see [23, Remark 3]), Zhang et al. [23] consider the range $3/2 \le q \le 3$.

In this paper, we would like to make a further contribution in this direction. We shall extend the smoothness condition

$$\nabla\pi \in L^{\frac{2}{3-r}}(0, T; \dot{X}_r), \quad 0 \le r \le 1$$

for the Navier–Stokes equations (see [8]) to the MHD equations (1), where $\dot{X}_r$ is the multiplier spaces (see Sect. 2 below). However, due to strong coupling of the velocity field with the magnetic field, we could only be able to prove the following regularity condition for (1),

$$\nabla\pi \in L^{\frac{2}{2-r}}(0, T; \dot{X}_r), \quad 0 \le r \le 1.$$

Before stating the main result, let us recall the weak formulation of the MHD equations (1).

**Definition 1.1** Let $(\boldsymbol{u}_0, \boldsymbol{b}_0) \in L^2(\mathbb{R}^3)$ with $\nabla \cdot \boldsymbol{u}_0 = \nabla \cdot \boldsymbol{b}_0 = 0$, and $T > 0$. A measurable $\mathbb{R}^3$-valued pair $(\boldsymbol{u}, \boldsymbol{b})$ is called a weak solution to (1) with initial data $(\boldsymbol{u}_0, \boldsymbol{b}_0)$, provided that the following three conditions hold:

(1) $\boldsymbol{u} \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3))$, $\boldsymbol{b} \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3))$;
(2) $(1)_{1,2,3,4}$ are satisfied in the distributional sense;
(3) the energy inequality

$$\|(\boldsymbol{u}, \boldsymbol{b})\|_{L^2}^2 (t) + 2\int_0^t \|\nabla(\boldsymbol{u}, \boldsymbol{b})\|_{L^2}^2 \, \mathrm{d}s \le \|(\boldsymbol{u}_0, \boldsymbol{b}_0)\|_{L^2}^2 ,$$

holds for almost every $t \ge 0$.

Now, we are ready to announce the main result of the paper.

**Theorem 1.2** *Let* $(\boldsymbol{u}_0, \boldsymbol{b}_0) \in L^2(\mathbb{R}^3) \cap L^4(\mathbb{R}^3)$ *with* $\nabla \cdot \boldsymbol{u}_0 = \nabla \cdot \boldsymbol{b}_0 = 0$*, and* $T > 0$*. Assume that* $(\boldsymbol{u}, \boldsymbol{b})$ *is a given weak solution pair of the MHD system* (1) *with initial data* $(\boldsymbol{u}_0, \boldsymbol{b}_0)$ *on* $(0, T)$*. If*

$$\nabla \pi \in L^{\frac{2}{3-r}}(0, T; \dot{X}_r), \quad 0 \leq r \leq 1, \tag{3}$$

*then, the solution pair* $(\boldsymbol{u}, \boldsymbol{b})$ *is smooth on* $(0, T)$.

Noticing that $\dot{X}_0 \cong BMO$ (see Sect. 2 for the definition, and [10,19] for the equivalence relation), we have the following corollary.

**Corollary 1.3** *Let* $(\boldsymbol{u}_0, \boldsymbol{b}_0) \in L^2(\mathbb{R}^3) \cap L^4(\mathbb{R}^3)$ *with* $\nabla \cdot \boldsymbol{u}_0 = \nabla \cdot \boldsymbol{b}_0 = 0$*, and* $T > 0$*. Assume that* $(\boldsymbol{u}, \boldsymbol{b})$ *is a given weak solution pair of the MHD system* (1) *with initial data* $(\boldsymbol{u}_0, \boldsymbol{b}_0)$ *on* $(0, T)$*. If*

$$\nabla \pi \in L^1(0, T; BMO), \tag{4}$$

*then the solution pair* $(\boldsymbol{u}, \boldsymbol{b})$ *is smooth on* $(0, T)$.

In the rest of the paper, we make some preliminaries in Sect. 2 and prove Theorem 1.2 in Sect. 3.

## 2 Preliminaries

In this section, we recall the definition and fine properties of the multiplier spaces $\dot{X}_r$ (see e.g., [10,19]).

**Definition 2.1** For $0 \leq r < 3/2$, the homogeneous space $\dot{X}_r$ is defined as the space of $f \in L^2_{loc}(\mathbb{R}^3)$ such that

$$\|f\|_{\dot{X}_r} \equiv \sup_{\|g\|_{\dot{H}^r} \leq 1} \|fg\|_{L^2} < \infty,$$

where $\dot{H}^r(\mathbb{R}^3)$ is the space of distributions $u$ such that

$$\|u\|_{\dot{H}^r} \equiv \left\|(-\triangle)^{\frac{r}{2}} u\right\|_{L^2} < \infty.$$

We have the following scaling properties:

$$\|f(\cdot + x_0)\|_{\dot{X}_r} = \|f\|_{\dot{X}_r}, \ \forall \ x_0 \in \mathbb{R}^3,$$
$$\|f(\lambda \cdot)\|_{\dot{X}_r} = \frac{1}{\lambda^r} \|f\|_{\dot{X}_r}, \ \forall \ \lambda > 0.$$

When $r = 0$, we have

$$\dot{X}_0 \cong BMO,$$

where $BMO$ is the homogeneous space of bounded mean oscillations associated with semi-norm (see [17])

$$\|f\|_{BMO} = \sup_{x \in \mathbb{R}^3, r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} \left| f(x) - \frac{1}{B_r(y)} \int_{B_r(y)} f(z)\, \mathrm{d}z \right| \mathrm{d}y.$$

Furthermore, for $0 < r < \dfrac{3}{2}$, we have the following strict imbeddings:

$$L^{\frac{3}{r}}(\mathbb{R}^3) \subsetneq \dot{X}_r(\mathbb{R}^3),$$

which could be justified simply as

$$\begin{aligned} \|fg\|_{L^2} &\leq \|f\|_{L^{\frac{3}{r}}} \|g\|_{L^{\frac{6}{3-2r}}} \quad \text{(Hölder inequality)} \\ &\leq C \|f\|_{L^{\frac{3}{r}}} \|g\|_{\dot{H}^r} \quad \text{(Sobolev imbeddings)} \\ &\leq C \|f\|_{L^{\frac{3}{r}}} \quad \left(\forall \ g \in \dot{H}^r(\mathbb{R}^3) \text{ with } \|g\|_{\dot{H}^r} \leq 1\right). \end{aligned}$$

## 3 Proof of Theorem 1.2

In this section, we are ready to prove Theorem 1.2.

First, let us convert (1) into a symmetric form. Writing

$$\boldsymbol{\omega}^{\pm} = \boldsymbol{u} \pm \boldsymbol{b},$$

we find by adding and subtracting $(1)_1$ with $(1)_2$,

$$\begin{cases} \boldsymbol{\omega}_t^{+} + (\boldsymbol{\omega}^{-} \cdot \nabla)\boldsymbol{\omega}^{+} - \triangle \boldsymbol{\omega}^{+} + \nabla \pi = \mathbf{0}, \\ \boldsymbol{\omega}_t^{-} + (\boldsymbol{\omega}^{+} \cdot \nabla)\boldsymbol{\omega}^{-} - \triangle \boldsymbol{\omega}^{-} + \nabla \pi = \mathbf{0}, \\ \nabla \cdot \boldsymbol{\omega}^{+} = \nabla \cdot \boldsymbol{\omega}^{-} = 0, \\ \boldsymbol{\omega}^{+}(0) = \boldsymbol{\omega}_0^{+} \equiv \boldsymbol{u}_0 + \boldsymbol{b}_0, \quad \boldsymbol{\omega}^{-}(0) = \boldsymbol{\omega}_0^{-} \equiv \boldsymbol{u}_0 - \boldsymbol{b}_0. \end{cases} \tag{5}$$

Multiplying $(5)_1$ with $|\boldsymbol{\omega}^{+}|^2\boldsymbol{\omega}^{+}$ and $(5)_2$ with $|\boldsymbol{\omega}^{-}|^2\boldsymbol{\omega}^{-}$, and integrating over $\mathbb{R}^3$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\boldsymbol{\omega}^{\pm}\right\|_{L^2}^2 + \frac{1}{2}\left\|\nabla|\boldsymbol{\omega}^{\pm}|^2\right\|_{L^2}^2 + \left\||\boldsymbol{\omega}^{\pm}| \cdot |\nabla \boldsymbol{\omega}^{\pm}|\right\|_{L^2}^2 \\ &\quad = \int_{\mathbb{R}^3} \nabla \pi \cdot |\boldsymbol{\omega}^{\pm}|^2 \boldsymbol{\omega}^{\pm}\,\mathrm{d}x \\ &\quad \equiv I. \end{aligned} \tag{6}$$

We may now estimate $I$, applying Hölder inequality,

$$\begin{aligned} I &\le \left\|\nabla \pi \cdot |\boldsymbol{\omega}^{\pm}|^2\right\|_{L^2}\left\|\boldsymbol{\omega}^{\pm}\right\|_{L^2} \\ &\le \|\nabla\pi\|_{\dot{X}_r}\left\||\boldsymbol{\omega}^{\pm}|^2\right\|_{\dot{H}^r}\left\|\boldsymbol{\omega}^{\pm}\right\|_{L^2} \\ &\le \|\nabla\pi\|_{\dot{X}_r}\left\||\boldsymbol{\omega}^{\pm}|\right\|_{L^2}^{1-r}\left\|\nabla|\boldsymbol{\omega}^{\pm}|^2\right\|_{L^2}^{r}\left\|\boldsymbol{\omega}^{\pm}\right\|_{L^2} \\ &\le C\,\|\nabla\pi\|_{\dot{X}_r}^{\frac{2}{2-r}}\left\|\boldsymbol{\omega}^{\pm}\right\|_{L^2}^{\frac{2}{2-r}}\left\||\boldsymbol{\omega}^{\pm}|^2\right\|_{L^2}^{\frac{2(1-r)}{2-r}} + \frac{1}{4}\left\|\nabla|\boldsymbol{\omega}^{\pm}|\right\|_{L^2}^2. \end{aligned}$$

Notice that the weak solution $(\boldsymbol{u}, \boldsymbol{b}) \in L^{\infty}(0, T; L^2(\mathbb{R}^3))$, then we have $\boldsymbol{\omega}^{\pm} \in L^{\infty}(0, T; L^2(\mathbb{R}^3))$. Also, since $\dfrac{2(1-r)}{2-r} \le 2$, we may invoke Young inequality to dominate $I$ further as

$$I \le C\,\|\nabla\pi\|_{\dot{X}_r}^{\frac{2}{2-r}}\left[\left\||\boldsymbol{\omega}^{\pm}|^2\right\|_{L^2}^2 + 1\right] + \frac{1}{4}\left\|\nabla|\boldsymbol{\omega}^{\pm}|^2\right\|_{L^2}^2. \tag{7}$$

Substituting (7) into (6), and applying Gronwall inequality, we deduce

$$\left\|\boldsymbol{\omega}^{\pm}(t)\right\|_{L^4}^4 \le \left\|\boldsymbol{\omega}_0^{\pm}\right\|_{L^4}^4 \exp\left\{C\int_0^t \|\nabla\pi(s)\|_{\dot{X}_r}^{\frac{2}{2-r}}\,\mathrm{d}s\right\} < \infty.$$

Thus,

$$\boldsymbol{\omega}^{\pm} \in L^{\infty}(0, T; L^4(\mathbb{R}^3)) \Rightarrow \boldsymbol{u} = \frac{1}{2}(\boldsymbol{\omega}^{+} + \boldsymbol{\omega}^{-}) \in L^{\infty}(0, T; L^4(\mathbb{R}^3)).$$

The classical Serrin-type regularity criterion, as in [11], then concludes the Proof of Theorem 1.2.

**Acknowledgments** Zujin Zhang was partially supported by the National Natural Science Foundation of China (11326138) and the Youth Natural Science Foundation of Jiangxi Province (20132BAB211007). Dingxing Zhong was partially supported by the Natural Science Foundation of Jiangxi Province (20122BAB201014), the Science Foundation of Jiangxi Provincial Department of Education (GJJ13659), and the National Natural Science Foundation of China (11361004).

## References


1. Benbernou, S.; Gala, S.; Ragusa, M.A.: On the regularity criteria for the 3D magnetohydrodynamic equations via two components in terms of BMO spaces. Math. Meth. Appl. Sci. (2013). doi:10.1002/mma.2981
2. Benbernou, S.; Samia, Terbeche, M.; Ragusa, M.A.; Zhang, Z.J.: A note on the regularity criterion for 3D MHD equations in $\dot{B}^{-1}_{\infty,\infty}$ space. Appl. Math. Comput. **238**, 245–249 (2014)
3. Duvaut, G.; Lions, J.L.: Inéquations en thermoélasticité et magnétohydrodynamique. Arch. Rational Mech. Anal. **46**, 241–279 (1972)
4. Cao, C.S.; Wu, J.H.: Two regularity criteria for the 3D MHD equations. J. Differ. Equ. **248**, 2263–2274 (2010)
5. Chen, Q.L.; Miao, C.X.; Zhang, Z.F.: On the regularity criterion of weak solutions for the 3D viscous magneto-hydrodynamics equations. Commun. Math. Phys. **284**, 919–930 (2008)
6. Duan, H.L.: On regularity criteria in terms of pressure for the 3D viscous MHD equations. Appl. Anal. **91**, 947–952 (2012)
7. Fan, J.S.; Jiang, S.; Nakamura, G.; Zhou, Y.: Logarithmically improved regularity criteria for the Navier–Stokes and MHD Equations. J. Math. Fluid Mech. **13**, 557–571 (2011)
8. Gala, S.: Remarks on regularity criterion for weak solutions to the Navier–Stokes equations in terms of the gradient of the pressure. Appl. Anal. **92**, 96–103 (2013)
9. Gala, S.; Guo, Z.G.; Ragusa, M.A.: A remark on the regularity criterion of Boussinesq equations with zero heat conductivity. Appl. Math. Lett. **27**, 70–73 (2014)
10. Gala, S.; Lemarié-Rieusset, P.G.: Multipliers between Sobolev spaces and fractional differentiation. J. Math. Anal. Appl. **322**, 1030–1054 (2006)
11. He, C.; Xin, Z.P.: On the regularity of weak solutions to the magnetohydrodynamic equations. J. Differ. Equ. **213**, 235–254 (2005)
12. Ji, E.; Lee, J.: Some regularity criteria for the 3D incompressible magnetohydrodynamics. J. Math. Anal. Appl. **369**, 317–322 (2010)
13. Jia, X.J.; Zhou, Y.: A new regularity criterion for the 3D incompressible MHD equations in terms of one component of the gradient of pressure. J. Math. Anal. Appl. **396**, 345–350 (2012)
14. Jia, X.J.; Zhou, Y.: Regularity criteria for the 3D MHD equations involving partial components. Nonlinear Anal. Real World Appl. **13**, 410–418 (2012)
15. Jia, X.J., Zhou, Y.: Regularity criteria for the 3D MHD equations via partial derivatives. Kinet. Relat. Models **5**, 505–516 (2012)
16. Jia, X.J.; Zhou, Y.: Regularity criteria for the 3D MHD equations via partial derivatives II. Kinet. Relat. Models. **7**, 291–304 (2014)
17. John, F.; Nirenberg, L.: On functions of bounded mean oscillation. Commun. Pure Appl. Math. **14**, 415–426 (1961)
18. Ni, L.D.; Guo, Z.G.; Zhou, Y.: Some new regularity criteria for the 3D MHD equations. J. Math. Anal. Appl. **396**, 108–118 (2012)
19. Lemarié-Rieusset, P.G.: Recent Developments in the Navier–Stokes Problem. Chapman and Hall, London (2002)
20. Wang, Y.Z.; Zhao, J.; Wang, Y.X.: Regularity criteria for weak solutions to the 3-d magnetohydrodynamic equations. ScienceAsia **38**, 108–112 (2012)
21. Yuan, B.Q.: Regularity criterion of weak solutions to the MHD system based on vorticity and electric current in negative index Besov spaces. Adv. Math. (China) **37**, 451–458 (2008)
22. Zhang, Z.J.: Remarks on the regularity criteria for generalized MHD equations. J. Math. Anal. Appl. **375**, 799–802 (2011)
23. Zhang, Z.J.; Li, P.; Yu, G.H.: Regularity criteria for the 3D MHD equations via one directional derivative of the pressure. J. Math. Anal. Appl. **401**, 66–71 (2013)
24. Zhou, Y.: Remaks on regularities for the 3D MHD equations. Discrete Contin. Dyn. Syst. **12**, 881–886 (2005)
25. Zhou, Y.: Regularity criteria for the 3D MHD equations in terms of the pressure. Int. J. Non-Linear Mech. **41**, 1174–1180 (2006)
26. Zhou, Y.: Regularity criteria for the generalized viscous MHD equations. Ann. Inst. H. Poincaré Anal. Non Linéaire. **24**, 491–505 (2007)
27. Zhou, Y.; Fan, J.S.: Logarithmically improved regularity criteria for the 3D viscous MHD equations. Forum Math. **24**, 691–708 (2012)
28. Zhou, Y.; Gala, S.: Regularity criterion for the solutions to the 3D MHD equations in the multiplier space. Z. Angew. Math. Phys. **61**, 193–199 (2011)