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Non-linear differential polynomials sharing small function with finite weight

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Abstract The purpose of the paper is to study the uniqueness of meromorphic functions sharing a small function with weight. The results of the paper improve and extend some recent results due to Banerjee and Sahoo (Sarajevo J Math 20:69–89, 2012), which in turn radically improve, extend and supplement some results of Dyavanal (J Math Anal Appl 372(1):252–261, 2010; 374(1):334, 2011; 374(1):345–355, 2011).

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الملخص

هدف هذه الورقة هو دراسة وحدانية الدوال جزئية التَّشَكّل التي تتشارك دالة صغيرة مع وزن. تحسن وتمدد نتائج هذه الورقة بعض النتائج التي تعزى إلى أ. بانيرجي و ب. ساهو [5]، والتي تعمم وتمدد وتكمل جذريا بعض نتائج ديفانال [6] – [8].

1 Introduction, definitions and results

In this paper by meromorphic functions we will always mean meromorphic functions in the complex plane.

Let f and g be two non-constant meromorphic functions and let a be a finite complex number. We say that f and g share a CM, provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. In addition we say that f and g share ∞ CM, if $1/f$ and $1/g$ share 0 CM, and we say that f and g share ∞ IM, if $1/f$ and $1/g$ share 0 IM.

We adopt the standard notations of value distribution theory (see [11]). We denote by $T(r)$ the maximum of $T(r, f)$ and $T(r, g)$. The notation $S(r)$ denotes any quantity satisfying $S(r) = o(T(r))$ as $r \longrightarrow \infty$, outside of a possible exceptional set of finite linear measure.

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Throughout this paper, we need the following definition.

$$\Theta(a; f) = 1 - \limsup_{r \longrightarrow \infty} \frac{\overline{N}(r, a; f)}{T(r, f)},$$

where a is a value in the extended complex plane.

So far to the knowledge of the authors the inquisition for the possible relationship between two meromorphic functions related to value sharing of non-linear differential polynomials first highlighted by Lahiri [12] which ushers a new era in the uniqueness theory. In [12], Lahiri asked the following question.

What can be said if two non linear differential polynomials generated by two meromorphic functions share 1 CM?

It is to be noted that earlier Yang and Hua [25] made some progress in the direction of the above question for some specific type of non-linear differential polynomials namely differential monomials. Below we are stating their result.

Theorem A. [25] *Let f and g be two non-constant meromorphic functions, $n \geq 11$ be a positive integer and $a \in \mathbb{C} - \{0\}$. If $f^n f'$ and $g^n g'$ share a CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$ or $f \equiv tg$ for a constant t such that $t^{n+1} = 1$.*

The introduction of the new notion of scaling between CM and IM, known as weighted sharing of values by Lahiri [13,14] in 2001 further influenced the investigations remarkably in the above direction. To verify the above statement readers are requested to go through the references (see [2–5,16–18,21,23]).

Below we are giving the definition of weighted sharing.

Definition 1.1 [13,14] Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k + 1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k then z_0 is an a-point of f with multiplicity m $(\leq k)$ if and only if it is an a-point of g with multiplicity m $(\leq k)$ and z_0 is an a-point of f with multiplicity m $(> k)$ if and only if it is an a-point of g with multiplicity n $(> k)$, where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k), then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively. If a is a small function with respect to f and g we define that f and g share (a, l) which means f and g share a with weight l if $f - a$ and $g - a$ share $(0, l)$.

In 2004, Lin and Yi [22] further improved the result of Fang and Hong [9] in the following manner.

Theorem B. [22] *Let f and g be two non-constant meromorphic functions satisfying $\Theta(\infty, f) > \frac{2}{(n+1)}$, $n(\geq 12)$ an integer. If $f^n(f-1)f^{'}$ and $g^n(g-1)g^{'}$ share $(1, \infty)$, then $f \equiv g$.*

Theorem C. [22] *Let f and g be two non-constant meromorphic functions and $n(\geq 13)$ be an integer. If $f^n(f-1)^2 f^{'}$ and $g^n(g-1)^2 g^{'}$ share $(1, \infty)$, then $f \equiv g$.*

In 2010 Dyavanal [6] proved the following result in which for the value sharing of differential polynomials multiplicities of zeros and poles of f and g are taken into consideration.

Theorem D. [6] *Let f and g be two non-constant meromorphic functions, whose zeros and poles are of multiplicities at least s, where s is a positive integer. Let $n \geq 2$ be an integer satisfying $(n + 1)s \geq 12$. If $f^n f'$ and $g^n g'$ share $(1, \infty)$, then either $f = dg$, for some $(n + 1)$-th root d of unity 1 or $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$.*

In 2011 Dyavanal further obtained the following results:

Theorem E. [7,8] *Let f and g be two non-constant meromorphic functions, whose zeros and poles are of multiplicities at least s, where s is a integer. Let n be an integer satisfying $(n - 2)s \geq 10$. If $f^n(f-1)f^{'}$ and $g^n(g-1)g^{'}$ share $(1, \infty)$, then $g = \frac{(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})}$, $f = \frac{(n+2)h(1-h^{n+1})}{(n+1)(1-h^{n+2})}$, where h is a non-constant meromorphic function.*

Theorem F. [7,8] *Under the condition of* Theorem E *if* $(n-3)s \geq 10$ *and* $f^n(f-1)^2 f'$ *and* $g^n(g-1)^2 g'$ *share* $(1, \infty)$*, then* $f \equiv g$.

For the last couple of years the main trend in the value sharing of nonlinear differential polynomials has been replaced mainly towards that of the k-th derivative of some linear expression of f and g.

Recently A. Banerjee and P. Sahoo [5] obtained the following results which improve, extend and rectify the results of Dyavanal [6,8] to a large extent.

Theorem G. [5] *Let f and g be two transcendental meromorphic functions, whose zeros and poles are of multiplicities at least s, where s is a positive integer. Let $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share (b, l), where $n(\geq 3)$, $k(\geq 1)$ and $l(\geq 0)$ are integers, $b(\neq 0)$ is a constant and one of the following conditions holds:*

(i) $l \geq 2$ *and* $n > \frac{3k+8}{s}$;
(ii) $l = 1$ *and* $n > \frac{4k+9}{s}$;
(iii) $l = 0$ *and* $n > \frac{9k+14}{s}$.

then either $(f^n)^{(k)}(g^n)^{(k)} \equiv b^2$ *or* $f(z) \equiv dg(z)$ *for some* $(n+1)$*-th root* d *of unity* 1.

If $k = 1$*, then* $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$*, where* c, c_1, c_2 *are constants satisfying* $(c_1 c_2)^n c^2 = -\frac{b^2}{n^2}$.

Theorem H. [5] *Let f and g be two transcendental meromorphic functions, whose zeros and poles are of multiplicities at least s, where s is a positive integer and* $\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n}$*. Let* $[f^n(a_1 f + a_2)]^{(k)}$ *and* $[g^n(a_1 g + a_2)]^{(k)}$ *share* (b, l)*, where* $k(\geq 1)$ *and* $l(\geq 0)$ *are integers,* a_1, a_2, b *are non-zero constants and one of the following conditions holds:*

(i) $l \geq 2$ *and* $n > \max\{\frac{3k+8}{s} + 1, 3 + \frac{2}{s}\}$;
(ii) $l = 1$ *and* $n > \max\{\frac{4k+9}{s} + \frac{3}{2}, 3 + \frac{2}{s}\}$;
(iii) $l = 0$ *and* $n > \max\{\frac{9k+14}{s} + 4, 3 + \frac{2}{s}\}$.

then either $[f^n(a_1 f + a_2)]^{(k)}[g^n(a_1 g + a_2)]^{(k)} \equiv b^2$ *or* $f(z) \equiv g(z)$.

The possibility $[f^n(a_1 f + a_2)]^{(k)}[g^n(a_1 g + a_2)]^{(k)} \equiv b^2$ *does not occur for* $k = 1$.

Theorem I. [5] *Let f and g be two transcendental meromorphic functions, whose zeros and poles are of multiplicities at least s, where s is a positive integer. Let* $[f^n(a_1 f^2 + a_2 f + a_3)]^{(k)}$ *and* $[g^n(a_1 g^2 + a_2 g + a_3)]^{(k)}$ *share* (b, l)*, where* $k(\geq 1)$ *and* $l(\geq 0)$ *are integers,* a_1, a_2, b *are non-zero constants and one of the following conditions holds:*

(i) $l \geq 2$ *and* $n > \max\{\frac{3k+8}{s} + 2, 4 + \frac{4}{s}\}$;
(ii) $l = 1$ *and* $n > \max\{\frac{4k+9}{s} + 3, 4 + \frac{4}{s}\}$;
(iii) $l = 0$ *and* $n > \max\{\frac{9k+14}{s} + 8, 4 + \frac{4}{s}\}$.

Then either $[f^n(a_1 f^2 + a_2 f + a_3)]^{(k)}[g^n(a_1 g^2 + a_2 g + a_3)]^{(k)} \equiv b^2$ *or* $f(z) \equiv g(z)$ *or* f, g *satisfy the algebraic equation* $R(f, g) = 0$*, where*

$$R(x, y) = x^n(a_1 x^2 + a_2 x + a_3) - y^n(a_1 y^2 + a_2 y + a_3).$$

The possibility $[f^n(a_1 f^2 + a_2 f + a_3)]^{(k)}[g^n(a_1 g^2 + a_2 g + a_3)]^{(k)} \equiv b^2$ *does not occur for* $k = 1$.

Now from the above discussion the following questions are inevitable.

Question 1.2 What can be said if the sharing value b is replaced by a small function in the above Theorems G, H, I?

Question 1.3 Are the Theorems G, H, I also true for non-constant meromorphic functions ?

In this paper, taking the possible answer of the above questions into background we obtain the following results.

First let t_1 be the number of distinct roots of the equation $P_*(w) = 0$, where $P_*(w)$ be defined by

$$P_*(w) = a_m(n+m)w^m + a_{m-1}(n+m-1)w^{m-1} + \cdots + a_1(n+1)w + a_0 n, \tag{1.1}$$

where $a_0(\neq 0)$, $a_1,\ldots, a_m(\neq 0)$ are complex constants. Also we define k_1 by

$$k_1 = \frac{2m(s+1)}{st_1} - (m-1) + 1, \tag{1.2}$$

where m, s and t_1 are three positive integers such that $t_1 \leq m$.

For the sake of simplicity, for any positive integer k we also use the notation

$$\chi_k = \begin{cases} 0, & if \;\; k \geq 2 \\ 1, & if \;\; k = 1. \end{cases}$$

Theorem 1.4 *Let f and g be two non-constant meromorphic functions such that either the zeros and poles of f and g are of multiplicities at least s, where s is a positive integer or they have no zeros and poles and $a(z)(\not\equiv 0, \infty)$ be a small function with respect to f and g. Let $P(w) = a_m w^m + a_{m-1}w^{m-1} + \cdots + a_1 w + a_0$, for a positive integer m or $P(w) \equiv c_0$ where $a_0(\neq 0), a_1, \ldots, a_{m-1}, a_m(\neq 0), c_0(\neq 0)$ are complex constants. Also we suppose that $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share (a, l), where $n(\geq 1)$, $k(\geq 1)$ and $l(\geq 0)$ are integers. Now (I) when $P(w) = a_m w^m + a_{m-1}w^{m-1} + \cdots + a_1 w + a_0$, and one of the following conditions holds:*

(a) $l \geq 2$ *and* $n > \max\{\frac{3k+8}{s} + m, k_1^*\}$;
(b) $l = 1$ *and* $n > \max\{\frac{4k+9}{s} + \frac{3m}{2}, k_1^*\}$;
(c) $l = 0$ *and* $n > \max\{\frac{9k+14}{s} + 4m, k_1^*\}$,

where $k_1^* = \chi_k . k_1$, k_1 is given by (1.2) with t_1 as the number of distinct roots of $P_*(w) = 0$ where $P_*(w)$ is given by (1.1),

then one of the following three cases holds:

(I1) *$f(z) \equiv tg(z)$ for a constant t such that $t^{d_1} = 1$, where $d_1 = gcd(n+m, \ldots, n+m-i, \ldots, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, 2, \ldots, m$;*
(I2) *f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^n(a_m\omega_1^m + a_{m-1}\omega_1^{m-1} + \cdots + a_0) - \omega_2^n(a_m\omega_2^m + a_{m-1}\omega_2^{m-1} + \cdots + a_0)$, except for $P(w) = a_1 w + a_2$ and $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n}$;*
(I3) *$[f^n P(f)]^{(k)}[g^n P(g)]^{(k)} \equiv a^2$, except for $k = 1$;*

(II) when $P(w) \equiv c_0$, and one of the following conditions holds:

(a) $l \geq 2$ *and* $n > \frac{3k+8}{s}$;
(b) $l = 1$ *and* $n > \frac{4k+9}{s}$;
(c) $l = 0$ *and* $n > \frac{9k+14}{s}$,

then one of the following two cases holds:

(II1) *$f \equiv tg$ for some constant t such that $t^n = 1$,*
(II2) *$c_0^2[f^n]^{(k)}[g^n]^{(k)} \equiv a^2$. In particular when $n > 2k$ and $a(z) = d_2 = constant$, we get $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are constants satisfying $(-1)^k c_0^2 (c_1 c_2)^n (nc)^{2k} = d_2^2$.*

Let t_2 be the number of distinct roots of the equation $P(w) = 0$, where $P(w)$ be defined by

$$P(w) = a_m w^m + a_{m-1}w^{m-1} + \cdots + a_1 w + a_0, \tag{1.3}$$

where $a_0(\neq 0)$, $a_1,\ldots, a_m(\neq 0)$ are complex constants. Also we define k_2 by

$$k_2 = \frac{2m(s+1)}{st_2} - (m-1) \tag{1.4}$$

where m, s and t_2 are three positive integers such that $t_2 \leq m$.

Theorem 1.5 *Let f and g be two non-constant meromorphic functions such that either the zeros and poles of f and g are of multiplicities at least s, where s is a positive integer or they have no zeros and poles and $a(z)(\not\equiv 0, \infty)$ be a small function with respect to f and g. Let m be a positive integer and t_2 denotes the number of distinct roots of the equation $P(w) = 0$, where $P(w)$ be defined as in (1.3). If $f^n P(f) f'$, $g^n P(g) g'$ share (a, l) where $n(\geq 1)$, $k(\geq 1)$ and $l(\geq 0)$ are integers and one of the following conditions holds:*

(a) $l \geq 2$ *and* $n > \max\{\frac{11}{s} + m - 1, k_2\}$;
(b) $l = 1$ *and* $n > \max\{\frac{13}{s} + \frac{3m}{2} - 1, k_2\}$;
(c) $l = 0$ *and* $n > \max\{\frac{23}{s} + 4m - 1, k_2\}$,

where k_2 *is defined by (1.4), then one of the following two cases holds:*

(I) $f(z) \equiv tg(z)$ *for a constant* t *such that* $t^{d_3} = 1$, *where* $d_3 = gcd(n+m+1, \ldots, n+m+1-i, \ldots, n+1)$, $a_{m-i} \neq 0$ *for some* $i = 0, 1, 2, \ldots, m$,
(II) f *and* g *satisfy the algebraic equation* $R(f, g) \equiv 0$, *where* $R(\omega_1, \omega_2) = \omega_1^{n+1}(\frac{a_m\omega_1^m}{n+m+1} + \frac{a_{m-1}\omega_1^{m-1}}{n+m} + \cdots + \frac{a_0}{n+1}) - \omega_2^{n+1}(\frac{a_m\omega_2^m}{n+m+1} + \frac{a_{m-1}\omega_2^{m-1}}{n+m} + \cdots + \frac{a_0}{n+1})$.

We now explain following definitions and notations which are used in the paper.

Definition 1.6 [18] Let p be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$.

(i) $N(r, a; f \mid\geq p)$ ($\overline{N}(r, a; f \mid\geq p)$)denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not less than p.
(ii) $N(r, a; f \mid\leq p)$ ($\overline{N}(r, a; f \mid\leq p)$)denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not greater than p.

Definition 1.7 ([1], cf. [26]) For $a \in \mathbb{C} \cup \{\infty\}$ and a positive integer p we denote by $N_p(r, a; f)$ the sum $\overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2) + \cdots + \overline{N}(r, a; f \mid\geq p)$. Clearly $N_1(r, a; f) = \overline{N}(r, a; f)$.

Definition 1.8 Let $a, b \in \mathbb{C} \cup \{\infty\}$. Let p be a positive integer. We denote by $\overline{N}(r, a; f \mid \geq p \mid g = b)$ ($\overline{N}(r, a; f \mid \geq p \mid g \neq b)$) the reduced counting function of those a-points of f with multiplicities $\geq p$, which are the b-points (not the b-points) of g.

Definition 1.9 (cf. [1,2]) Let f and g be two non-constant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multiplicity p, a 1-point of g with multiplicity q. We denote by $\overline{N}_L(r, 1; f)$ the counting function of those 1-points of f and g where $p > q$, by $N_E^{1)}(r, 1; f)$ the counting function of those 1-points of f and g where $p = q = 1$ and by $\overline{N}_E^{(2}(r, 1; f)$ the counting function of those 1-points of f and g where $p = q \geq 2$, each point in these counting functions is counted only once. In the same way we can define $\overline{N}_L(r, 1; g)$, $N_E^{1)}(r, 1; g)$, $\overline{N}_E^{(2}(r, 1; g)$.

Definition 1.10 (cf. [1,2]) Let k be a positive integer. Let f and g be two non-constant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multiplicity p, a 1-point of g with multiplicity q. We denote by $\overline{N}_{f>k}(r, 1; g)$ the reduced counting function of those 1-points of f and g such that $p > q = k$. $\overline{N}_{g>k}(r, 1; f)$ is defined analogously.

Definition 1.11 [13,14] Let f, g share a value a IM. We denote by $\overline{N}_*(r, a; f, g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.

Clearly $\overline{N}_*(r, a; f, g) \equiv \overline{N}_*(r, a; g, f)$ and $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.

Definition 1.12 Let $a, b_1, b_2, \ldots, b_q \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f| g \neq b_1, b_2, \ldots, b_q)$ the counting function of those a-points of f, counted according to multiplicity, which are not the b_i-points of g for $i = 1, 2, \ldots, q$.

2 Lemmas

Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$. We denote by H the function as follows:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right). \tag{2.1}$$

Lemma 2.1 [18] *Let f be a non-constant meromorphic function and let $a_n(z)(\not\equiv 0)$, $a_{n-1}(z), \ldots, a_0(z)$ be meromorphic functions such that $T(r, a_i(z)) = S(r, f)$ for $i = 0, 1, 2, \ldots, n$. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \cdots + a_1 f + a_0) = nT(r, f) + S(r, f).$$

Lemma 2.2 [30] *Let f be a non-constant meromorphic function, and p, k be positive integers. Then*

$$N_p\left(r, 0; f^{(k)}\right) \leq T\left(r, f^{(k)}\right) - T(r, f) + N_{p+k}(r, 0; f) + S(r, f), \tag{2.2}$$

$$N_p\left(r, 0; f^{(k)}\right) \leq k\overline{N}(r, \infty; f) + N_{p+k}(r, 0; f) + S(r, f). \tag{2.3}$$

Lemma 2.3 [15] *If $N(r, 0; f^{(k)} \mid f \neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity then*

$$N(r, 0; f^{(k)} \mid f \neq 0) \leq k\overline{N}(r, \infty; f) + N(r, 0; f| < k) + k\overline{N}(r, 0; f| \geq k) + S(r, f).$$

Lemma 2.4 [20] *Let f_1 and f_2 be two non-constant meromorphic functions satisfying $\overline{N}(r, 0; f_i) + \overline{N}(r, \infty; f_i) = S(r; f_1, f_2)$ for $i = 1, 2$. If $f_1^s f_2^t - 1$ is not identically zero for arbitrary integers s and $t(|s| + |t| > 0)$, then for any positive ε, we have*

$$N_0(r, 1; f_1, f_2) \leq \varepsilon T(r) + S(r; f_1, f_2),$$

where $N_0(r, 1; f_1, f_2)$ denotes the deduced counting function related to the common 1-points of f_1 and f_2 and $T(r) = T(r, f_1) + T(r, f_2)$, $S(r; f_1, f_2) = o(T(r))$ as $r \longrightarrow \infty$ possibly outside a set of finite linear measure.

Lemma 2.5 [10] *Let f be a non-constant entire function, $k \geq 2$ be a positive integer. If $ff^{(k)} \neq 0$ then $f = e^{az+b}$, where $a \neq 0$, b are constants.*

Lemma 2.6 [28] *Let f be a non-constant meromorphic function, and let k be a positive integer. Suppose that $f^{(k)} \not\equiv 0$, then*

$$N(r, 0; f^{(k)}) \leq N(r, 0; f) + k\overline{N}(r, \infty; f) + S(r, f).$$

Lemma 2.7 *Let f, g be two non-constant meromorphic functions and n and k be two positive integers such that*

$$[f^n]^{(k)}[g^n]^{(k)} \equiv 1.$$

Then $T(r, f) = O(T(r, g))$ and $T(r, g) = O(T(r, f))$.

Proof From the given condition we have

$$[f^n]^{(k)} \equiv \frac{1}{[g^n]^{(k)}}.$$

Also $T(r, g^{(j)}) = O(T(r, g))$ holds for every positive integer j. Noting the fact that $[g^n]^{(k)}$ is a differential polynomial in $g, g^{'}, \ldots, g^{(k)}$, using the first fundamental theorem we have $T(r, f) = O(T(r, g))$. Similarly we can get $T(r, g) = O(T(r, f))$. This completes the proof of the Lemma. □

Lemma 2.8 *Let f, g be two non-constant meromorphic functions such that either the zeros and poles of f and g are of multiplicities at least s, where s is a positive integer or they have no zeros and poles. Let n, k be two positive integers such that $n > 2k$. Suppose $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share d_2 CM. If $[f^n]^{(k)}[g^n]^{(k)} \equiv d_2^2$, then $f = c_1 e^{cz}$, $g = c_2 e^{-cz}$, where c_1, c_2 and c are constants such that $(-1)^k (c_1 c_2)^n (nc)^{2k} = d_2^2$.*

Proof Without loss of generality we may assume that $d = 1$, since otherwise we may start with $f_1 = \frac{f}{d_2}$, $g_1 = \frac{g}{d_2}$.

Suppose,

$$[f^n]^{(k)}[g^n]^{(k)} \equiv 1. \tag{2.4}$$

Let us assume that the zeros and poles of f and g are of multiplicities at least s, where s is a positive integer.

Let z_0 be a zero of f with multiplicity q. Then z_0 be a zero of $[f^n]^{(k)}$ with multiplicity $nq - k$. Now one of the following possibilities holds:

(i) z_0 will be neither a zero of $[g^n]^{(k)}$ nor a pole of g,
(ii) z_0 will be a zero of g,
(iii) z_0 will be a zero of $[g^n]^{(k)}$ but not a zero of g and
(iv) z_0 will be a pole of g.

We now explain only the above two possibilities (i) and (iv) because other two possibilities follow from (i).

For the possibility (i): Note that since $n \geq 2k+1$, we must have

$$nq - k \geq n - k \geq k + 1.$$

Thus z_0 must be a zero of $[f^n]^{(k)}$ with multiplicity at least $k+1$, which is impossible and so f has no zero in this case.

For the possibility (iv): Let z_0 be a pole of g with multiplicity q_1. Clearly z_0 will be pole of $[g^n]^{(k)}$ with multiplicity $nq_1 + k$. Obviously $q > q_1$ and $nq - k = nq_1 + k$. Now

$$nq - k = nq_1 + k$$

implies that

$$n(q - q_1) = 2k. \tag{2.5}$$

Since $n \geq 2k+1$, we get a contradiction from (2.5).

Hence f has no zero. Similarly we can prove that g has no zero. Thus we arrive at a contradiction. Therefore the case " zeros of f and g are of multiplicities at least s, where s is a positive integer" is discarded automatically. Hence one can easily conclude that f and g have no zeros.

Also we know that

$$N(r,\infty;[f^n]^{(k)}) = nN(r,\infty;f) + k\overline{N}(r,\infty;f).$$

Also by Lemma 2.6 we have

$$N(r,0;[g^n]^{(k)}) \leq nN(r,0;g) + k\overline{N}(r,\infty;g) + S(r,g) \leq k\overline{N}(r,\infty;g) + S(r,g).$$

From (2.4) we get

$$N(r,\infty;[f^n]^{(k)}) = N(r,0;[g^n]^{(k)}),$$

i.e

$$nN(r,\infty;f) + k\overline{N}(r,\infty;f) \leq k\overline{N}(r,\infty;g) + S(r,g). \tag{2.6}$$

Similarly we get

$$nN(r,\infty;g) + k\overline{N}(r,\infty;g) \leq k\overline{N}(r,\infty;f) + S(r,f). \tag{2.7}$$

Combining (2.6) and (2.7) yields

$$N(r,\infty;f) + N(r,\infty;g) = S(r,f) + S(r,g).$$

By Lemma 2.7 we have $S(r,f) = S(r,g)$. So we obtain

$$N(r,\infty;f) = S(r,f), \quad N(r,\infty;g) = S(r,g). \tag{2.8}$$

Let

$$F_1 = [f^n]^{(k)}, \quad G_1 = [g^n]^{(k)}. \tag{2.9}$$

Clearly in view of Lemma 2.2, $S(r,f)$ and $S(r,g)$ can be replaced by $S(r,F_1)$ and $S(r,G_1)$ respectively. From (2.4) we get

$$F_1 G_1 \equiv 1. \tag{2.10}$$

Also from (2.10) we see that F_1 and G_1 share -1 IM.

If $F_1 \equiv cG_1$, where c is a nonzero constant, then F_1 is a constant and so f is a polynomial, which is impossible as f has no zero. Hence $F_1 \not\equiv cG_1$.

Note that $T(r,F_1) \leq n(k+1)T(r,f) + S(r,f)$ and so $T(r,F_1) = O(T(r,f))$. Also by Lemma 2.2, one can obtain $T(r,f) = O(T(r,F_1))$. Hence $S(r,F_1) = S(r,f)$. Similarly we get $S(r,G_1) = S(r,g)$. Hence we get $S(r,F_1) = S(r,G_1)$.

Now by Lemma 2.6 we have

$$N(r,0;F_1)\le nN(r,0;f)+k\overline{N}(r,\infty;f)+S(r,f)\le S(r,F_1).$$

Similarly we have

$$N(r,0;G_1)\le nN(r,0;g)+k\overline{N}(r,\infty;g)+S(r,g)\le S(r,G_1).$$

We see that

$$N(r,\infty;F_1)=S(r,F_1),\quad N(r,\infty;G_1)=S(r,G_1).$$

Also it is clear that $T(r,F_1)=T(r,G_1)+S(r,F_1)$. Let

$$f_1=\frac{F_1}{G_1}.$$

and

$$f_2=\frac{F_1-1}{G_1-1}.$$

Clearly f_1 is non-constant. If f_2 is a nonzero constant then F_1 and G_1 share ∞ CM and so from (2.10) we conclude that F_1 and G_1 have no poles.

Next we suppose that f_2 is non-constant. Also we note that

$$F_1=\frac{f_1(1-f_2)}{f_1-f_2},\quad G_1=\frac{1-f_2}{f_1-f_2}.$$

Clearly

$$T(r,F_1)\le 2[T(r,f_1)+T(r,f_2)]+O(1)$$

and

$$T(r,f_1)+T(r,f_2)\le 4T(r,F_1)+O(1).$$

These give $S(r,F_1)=S(r;f_1,f_2)$. It is clear that

$$\overline{N}(r,0;f_i)+\overline{N}(r,\infty;f_i)=S(r;f_1,f_2)$$

for $i=1,2$.

Next we suppose $\overline{N}(r,-1;F_1)\ne S(r,F_1)$, since otherwise noting that $N(r,0;F_1)=N(r,\infty;F_1)=S(r,F_1)$, from the second fundamental theorem we can deduce that F_1 is a constant.

Also we see that

$$\overline{N}(r,-1;F_1)\le N_0(r,1;f_1,f_2).$$

Thus we have

$$T(r,f_1)+T(r,f_2)\le 4\,N_0(r,1;f_1,f_2)+S(r,F_1).$$

Hence by Lemma 2.4 there exist two mutually prime integers s and $t\,(|s|+|t|>0)$ such that

$$f_1^s f_2^t\equiv 1,$$

i.e.,

$$\left[\frac{F_1}{G_1}\right]^s\left[\frac{F_1-1}{G_1-1}\right]^t\equiv 1. \tag{2.11}$$

If either s or t is zero then we arrive at a contradiction and so $st\ne 0$.

We now consider following cases:

Case (i): Suppose $s > 0$ and $t = -t_1$, where $t_1 > 0$. Then we have

$$\left[\frac{F_1}{G_1}\right]^s \equiv \left[\frac{F_1 - 1}{G_1 - 1}\right]^{t_1}. \tag{2.12}$$

Let z_1 be a pole of F_1 of multiplicity p. Then from (2.10) we see that z_1 must be a zero of G_1 of multiplicity p. Now from (2.12) we get $2s = t_1$, which is impossible. Hence F_1 has no pole. Similarly we can prove that G_1 also has no poles.

Case (ii): Suppose either $s > 0$ and $t > 0$ or $s < 0$ and $t < 0$. Then from (2.12) one can easily prove that F_1 and G_1 have no poles.

Consequently from (2.10) we see that F_1 and G_1 have no zeros.

We deduce from (2.9) that both f and g have no pole, which is a contradiction. Therefore the case " poles of f and g are of multiplicities at least s, where s is a positive integer" is discarded automatically. Hence one can easily conclude that f and g no poles.

Finally both f and g have no zeros and poles and so we can take f and g as follows:

$$f = e^{\alpha}, \quad g = e^{\beta}. \tag{2.13}$$

Moreover we see that

$$N(r, 0; [f^n]^{(k)}) = 0, \quad N(r, 0; [g^n]^{(k)}) = 0. \tag{2.14}$$

We consider the following cases:

Subcase 1: Let $k \geq 2$. Then from (2.14) and Lemma 2.5 we must have

$$f(z) = c_1 e^{cz}, g(z) = c_2 e^{-cz}, \tag{2.15}$$

where c, c_1 and c_2 are constants such that $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$.

Subcase 2: Let $k = 1$. Suppose that α and β are both transcendental. Then from (2.4) we get

$$AB\alpha^{'}\beta^{'}e^{n(\alpha+\beta)} \equiv 1, \tag{2.16}$$

where $AB = n^2$

Let $\alpha + \beta = \gamma$. From (2.16) we know that γ is not a constant since in that case we get a contradiction. Then from (2.16) we get

$$AB\alpha^{'}(\gamma^{'} - \alpha^{'})e^{n\gamma} \equiv 1. \tag{2.17}$$

We have $T(r, \gamma^{'}) = m(r, \gamma^{'}) = m(r, \frac{(e^{n\gamma})^{'}}{e^{n\gamma}}) = S(r, e^{n\gamma})$. Thus from (2.17) we get

$$\begin{aligned} T(r, e^{n\gamma}) &\leq T(r, \frac{1}{\alpha^{'}(\gamma^{'} - \alpha^{'})}) + O(1) \\ &\leq T(r, \alpha^{'}) + T(r, \gamma^{'} - \alpha^{'}) + O(1) \\ &\leq 2\, T(r, \alpha^{'}) + S(r, \alpha^{'}) + S(r, e^{n\gamma}), \end{aligned}$$

which implies that $T(r, e^{n\gamma}) = O(T(r, \alpha^{'}))$ and so $S(r, e^{n\gamma})$ can be replaced by $S(r, \alpha^{'})$. Thus we get $T(r, \gamma^{'}) = S(r, \alpha^{'})$ and so $\gamma^{'}$ is a small with respect to $\alpha^{'}$. In view of (2.17) and by the second fundamental theorem for small functions we get

$$\begin{aligned} T(r, \alpha^{'}) &\leq \overline{N}(r, \infty; \alpha^{'}) + \overline{N}(r, 0; \alpha^{'}) + \overline{N}(r, 0; \alpha^{'} - \gamma^{'}) + S(r, \alpha^{'}) \\ &\leq S(r, \alpha^{'}), \end{aligned}$$

which shows that $\alpha^{'}$ is a non-zero constant and so α is a polynomial. Similarly we can prove that β is also a polynomial. This contradicts the fact that α and β are transcendental.

Next suppose without loss of generality that α is a polynomial and β is a transcendental entire function. Then γ is transcendental. So in view of (2.17) we can obtain

$$\begin{aligned} nT(r, e^{\gamma}) &\leq T(r, \frac{1}{\alpha^{'}(\gamma^{'}-\alpha^{'})}) + O(1) \\ &\leq T(r, \alpha^{'}) + T(r, \gamma^{'} - \alpha^{'}) + S(r, e^{\gamma}) \\ &\leq T(r, \gamma^{'}) + S(r, e^{\gamma}) = S(r, e^{\gamma}), \end{aligned}$$

which leads to a contradiction. Thus α and β are both polynomials. Also from (2.16) we can conclude that $\alpha(z) + \beta(z) \equiv C$ for a constant C and so $\alpha^{'}(z) + \beta^{'}(z) \equiv 0$. Again from (2.16) we get $n^2 e^{nC} \alpha^{'} \beta^{'} \equiv 1$. By computation we get

$$\alpha^{'} = c, \beta^{'} = -c. \tag{2.18}$$

Hence

$$\alpha = cz + b_1, \beta = -cz + b_2, \tag{2.19}$$

where b_1, b_2 are constants. Finally we take f and g as

$$f(z) = c_1 e^{cz}, g(z) = c_2 e^{-cz},$$

where c_1, c_2 and c are constants such that $(-1)(nc)^2(c_1c_2)^n = 1$. This completes the proof of the Lemma. □

Lemma 2.9 *Let f and g be two non-constant meromorphic functions such that either the zeros and poles of f and g are of multiplicities at least s, where s is a positive integer or they have no zeros and poles. Let $P(w)$ be defined as in Theorem 1.4 and k, m, $n(> \frac{3k}{s} + m)$ be three positive integers. If $[f^n P(f)]^{(k)} \equiv [g^n P(g)]^{(k)}$, then $f^n P(f) \equiv g^n P(g)$.*

Proof By the assumption $[f^n P(f)]^{(k)} \equiv [g^n P(g)]^{(k)}$.

When $k \geq 2$, integrating we get

$$[f^n P(f)]^{(k-1)} \equiv [g^n P(g)]^{(k-1)} + c_{k-1}.$$

If possible we suppose $c_{k-1} \neq 0$.

Now in the view of the Lemma 2.2 for $p = 1$ and using the second fundamental theorem we get

$$\begin{aligned} &(n+m)T(r, f) \\ &\leq T(r, [f^n P(f)]^{(k-1)}) - \overline{N}(r, 0; [f^n P(f)]^{(k-1)}) + N_k(r, 0; f^n P(f)) + S(r, f) \\ &\leq \overline{N}(r, 0; [f^n P(f)]^{(k-1)}) + \overline{N}(r, \infty; f) + \overline{N}(r, c_{k-1}; [f^n P(f)]^{(k-1)}) \\ &\quad -\overline{N}(r, 0; [f^n P(f)]^{(k-1)}) + N_k(r, 0; f^n P(f)) + S(r, f) \\ &\leq \overline{N}(r, \infty; f) + \overline{N}(r, 0; [g^n P(g)]^{(k-1)}) + k\overline{N}(r, 0; f) + N(r, 0; P(f)) + S(r, f) \\ &\leq \left\{\frac{k+1}{s} + m\right\} T(r, f) + (k-1)\overline{N}(r, \infty; g) + N_k(r, 0; g^n P(g)) + S(r, f) \\ &\leq \left\{\frac{k+1}{s} + m\right\} T(r, f) + (k-1)\, \overline{N}(r, \infty; g) + k\, \overline{N}(r, 0; g) + N(r, 0; P(g)) + S(r, f) \\ &\leq \left\{\frac{k+1}{s} + m\right\} T(r, f) + \left\{\frac{2k-1}{s} + m\right\} T(r, g) + S(r, f) + S(r, g) \\ &\leq \left\{\frac{3k}{s} + 2m\right\} T(r) + S(r). \end{aligned}$$

Similarly we get

$$(n+m)\, T(r, g) \leq \left\{\frac{3k}{s} + 2m\right\} T(r) + S(r),$$

where $T(r) = \max\{T(r, f), T(r, g)\}$ and $S(r) = \max\{S(r, f), S(r, g)\}$.

Combining these we get

$$\left(n-m-\frac{3k}{s}\right)\ T(r)\leq S(r),$$

which is a contradiction since $n>\frac{3k}{s}+m$.

Therefore $c_{k-1}=0$ and so $[f^nP(f)]^{(k-1)}\equiv[g^nP(g)]^{(k-1)}$. Repeating $k-1$ times, we obtain

$$f^nP(f)\equiv g^nP(g)+c_0.$$

If $k=1$, clearly integrating once we obtain the above. If possible suppose $c_0\neq 0$.

Now using the second fundamental theorem we get

$$\begin{aligned}(n+m)T(r,f)&\\ \leq\ &\overline{N}(r,0;f^nP(f))+\overline{N}(r,\infty;f^nP(f))+\overline{N}(r,c_0;f^nP(f))+S(r,f)\\ \leq\ &\overline{N}(r,0;f)+mT(r,f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g^nP(g))+S(r,f)\\ \leq\ &\left(m+\frac{2}{s}\right)\ T(r,f)+\overline{N}(r,0;g)+m\ T(r,g)+S(r,f)+S(r,g)\\ \leq\ &\left(m+\frac{2}{s}\right)\ T(r,f)+\left(m+\frac{1}{s}\right)\ T(r,g)+S(r,f)+S(r,g)\\ \leq\ &\left\{\frac{3}{s}+2m\right\}\ T(r)+S(r).\end{aligned}$$

Similarly we get

$$(n+m)\ T(r,g)\leq\left\{\frac{3}{s}+2m\right\}\ T(r)+S(r).$$

Combining these we get

$$\left(n-m-\frac{3}{s}\right)\ T(r)\leq S(r),$$

which is a contradiction since $n>\frac{3}{s}+m$.

Therefore $c_0=0$ and so

$$f^nP(f)\equiv g^nP(g).$$

This completes the Lemma. □

Lemma 2.10 [27, Lemma 6] *If $H\equiv 0$, then F, G share 1 CM. If further F, G share ∞ IM then F, G share ∞ CM.*

Lemma 2.11 *Let f, g be two non-constant meromorphic functions such that either the zeros and poles of f and g are of multiplicities at least s, where s is a positive integer or they have no zeros and poles and $F=\frac{[f^nP(f)]^{(k)}}{a}$, $G=\frac{[g^nP(g)]^{(k)}}{a}$, where $a(z)(\not\equiv 0,\infty)$ be a small function with respect to f and g, $n(\geq 1)$, $k(\geq 1)$, $m(\geq 0)$ are positive integers such that $n>\frac{3k+3}{s}+m$ and $P(w)$ be defined as in Theorem 1.4. If $H\equiv 0$ then*

(I) *when $P(w)=a_mw^m+a_{m-1}w^{m-1}+\cdots+a_1w+a_0$, one of the following three cases holds:*
- (I1) *$f(z)\equiv tg(z)$ for a constant t such that $t^{d_1}=1$, where $d_1=gcd(n+m,\ldots,n+m-i,\ldots,n)$, $a_{m-i}\neq 0$ for some $i=1,2,\ldots,m$;*
- (I2) *f and g satisfy the algebraic equation $R(f,g)\equiv 0$, where $R(\omega_1,\omega_2)=\omega_1^n(a_m\omega_1^m+a_{m-1}\omega_1^{m-1}+\cdots+a_0)-\omega_2^n(a_m\omega_2^m+a_{m-1}\omega_2^{m-1}+\cdots+a_0)$, except for $P(w)=a_1w+a_2$ and $\Theta(\infty;f)+\Theta(\infty;g)>\frac{4}{n}$;*
- (I3) *$[f^nP(f)]^{(k)}[g^nP(g)]^{(k)}\equiv a^2$;*

(II) *when $P(w)\equiv c_0$, one of the following two cases holds:*
- (II1) *$f\equiv tg$ for some constant t such that $t^n=1$,*

(II2) $c_0^2[f^n]^{(k)}[g^n]^{(k)} \equiv a^2$. *In particular when* $n > 2k$ *and* $a(z) = d_2$ *we get* $f(z) = c_1 e^{cz}$ *and* $g(z) = c_2 e^{-cz}$, *where* c_1, c_2 *and* c *are constants satisfying* $(-1)^k c_0^2 (c_1 c_2)^n (nc)^{2k} = d_2^2$.

Proof Since $H \equiv 0$, by Lemma 2.10 we get F and G share 1 CM.

On integration we get

$$\frac{1}{F-1} \equiv \frac{bG + a - b}{G-1}, \tag{2.20}$$

where a, b are constants and $a \neq 0$. We now consider the following cases.

Case 1. Let $b \neq 0$ and $a \neq b$.

If $b = -1$, then from (2.20) we have

$$F \equiv \frac{-a}{G - a - 1}.$$

Therefore

$$\overline{N}(r, a+1; G) = \overline{N}(r, \infty; F) = \overline{N}(r, \infty; f).$$

So in view of Lemma 2.2 and the second fundamental theorem we get

$$\begin{aligned}
&(n+m)\,T(r,g)\\
&\le T(r,G) + N_{k+1}(r,0;g^nP(g)) - \overline{N}(r,0;G)\\
&\le \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,a+1;G) + N_{k+1}(r,0;g^nP(g)) - \overline{N}(r,0;G) + S(r,g)\\
&\le \overline{N}(r,\infty;g) + N_{k+1}(r,0;g^nP(g)) + \overline{N}(r,\infty;f) + S(r,g)\\
&\le \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + N_{k+1}(r,0;g^n) + N_{k+1}(r,0;P(g)) + S(r,g)\\
&\le \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + (k+1)\overline{N}(r,0;g) + T(r,P(g)) + S(r,g)\\
&\le \frac{1}{s}\,T(r,f) + \left\{\frac{k+2}{s} + m\right\}\,T(r,g) + S(r,f) + S(r,g).
\end{aligned}$$

Without loss of generality, we suppose that there exists a set I with infinite measure such that $T(r, f) \le T(r, g)$ for $r \in I$.

So for $r \in I$ we have

$$\left\{n - \frac{k+3}{s}\right\}\,T(r,g) \le S(r,g),$$

which is a contradiction since $n > \frac{k+3}{s}$.

If $b \neq -1$, from (2.20) we obtain that

$$F - \left(1 + \frac{1}{b}\right) \equiv \frac{-a}{b^2[G + \frac{a-b}{b}]}.$$

So

$$\overline{N}\left(r, \frac{(b-a)}{b}; G\right) = \overline{N}(r,\infty;F) = \overline{N}(r,\infty;f)$$

Using Lemma 2.2 and the same argument as used in the case when $b = -1$ we can get a contradiction.

Case 2. Let $b \neq 0$ and $a = b$.

If $b = -1$, then from (2.20) we have

$$FG \equiv 1,$$

i.e.,

$$[f^nP(f)]^{(k)}[g^nP(g)]^{(k)} \equiv a^2(z),$$

where $[f^nP(f)]^{(k)}$ and $[g^nP(g)]^{(k)}$ share $a(z)$ CM.

Note that if $P(w) \equiv c_0$ then we have

$$c_0^2[f^n]^{(k)}[g^n]^{(k)} \equiv a^2(z).$$

In particular when $n > 2k$ and $a(z) = d_2$ then we get by Lemma 2.8 that $f(z) = c_1 e^{cz}$ and $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are constants satisfying $(-1)^k c_0^2 (c_1 c_2)^n (nc)^{2k} = d_2^2$.

If $b \neq -1$, from (2.20) we have

$$\frac{1}{F} \equiv \frac{bG}{(1+b)G - 1}.$$

Therefore

$$\overline{N}\left(r, \frac{1}{1+b}; G\right) = \overline{N}(r, 0; F).$$

So in view of Lemma 2.2 and the second fundamental theorem we get

$$\begin{aligned}
&(n+m)\, T(r, g) \\
&\quad \le T(r, G) + N_{k+1}(r, 0; g^n P(g)) - \overline{N}(r, 0; G) + S(r, g) \\
&\quad \le \overline{N}(r, \infty; G) + \overline{N}(r, 0; G) + \overline{N}\left(r, \frac{1}{1+b}; G\right) + N_{k+1}(r.0; g^n P(g)) - \overline{N}(r, 0; G) + S(r, g) \\
&\quad \le \overline{N}(r, \infty; g) + (k+1)\overline{N}(r, 0; g) + T(r, P(g)) + \overline{N}(r, 0; F) + S(r, g) \\
&\quad \le \overline{N}(r, \infty; g) + (k+1)\overline{N}(r, 0; g) + T(r, P(g)) + (k+1)\overline{N}(r, 0; f) + T(r, P(f)) \\
&\qquad + k\overline{N}(r, \infty; f) + S(r, f) + S(r, g) \\
&\quad \le \left\{\frac{k+2}{s} + m\right\}\, T(r, g) + \left\{\frac{2k+1}{s} + m\right\}\, T(r, f) + S(r, f) + S(r, g).
\end{aligned}$$

So for $r \in I$ we have

$$\left\{n - \frac{3k+3}{s} - m\right\}\, T(r, g) \le S(r, g),$$

which is a contradiction since $n > \frac{3k+3}{s} + m$.

Case 3. Let $b = 0$. From (2.20) we obtain

$$F \equiv \frac{G + a - 1}{a}. \tag{2.21}$$

If $a \neq 1$ then from (2.21) we obtain

$$\overline{N}(r, 1-a; G) = \overline{N}(r, 0; F).$$

We can similarly deduce a contradiction as in Case 2. Therefore $a = 1$ and from (2.21) we obtain

$$F \equiv G,$$

i.e.,

$$[f^n P(f)]^{(k)} \equiv [g^n P(g)]^{(k)}.$$

Note that

$$n > \frac{3k+3}{s} + m > \frac{3k}{s} + m.$$

So by Lemma 2.9 we have

$$f^n P(f) \equiv g^n P(g). \tag{2.22}$$

Let $h = \frac{f}{g}$. If h is a constant, putting $f = gh$ in (2.22) we get

$$a_m g^{n+m}(h^{n+m} - 1) + a_{m-1} g^{n+m-1}(h^{n+m-1} - 1) + \cdots + a_0 g^n(h^n - 1) = 0,$$

which implies $h^{d_1} = 1$, where $d_1 = gcd(n+m, \ldots, n+m-i, \ldots, n+1, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$. Thus $f = tg$ for a constant t such that $t^{d_1} = 1$, $d_1 = gcd(n+m, \ldots, n+m-i, \ldots, n+1, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$.

If h is not a constant, then from (2.22) we can say that f and g satisfy the algebraic equation $R(f, g) = 0$, where $R(\omega_1, \omega_2) = \omega_1^n(a_m\omega_1^m + a_{m-1}\omega_1^{m-1} + \cdots + a_0) - \omega_2^n(a_m\omega_2^m + a_{m-1}\omega_2^{m-1} + \cdots + a_0)$. In particular when $P(w) = a_1 w + a_2$ and $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n}$ then following the same procedure as adopted in the proof of Theorem H in [5] one can prove that $f \equiv g$.

Note that when $P(w) \equiv c_0$ then we must have $f \equiv tg$ for some constant t such that $t^n = 1$. □

Lemma 2.12 *Let f and g be two non-constant meromorphic functions such that either the zeros and poles of f and g are of multiplicities at least s, where s is a positive integer or they have no zeros and poles and $a(z)(\not\equiv 0, \infty)$ be small function of f and g. Let n and m be two positive integers such that $n > k_2$, where k_2 be defined by (1.4), t_2 denotes the number of distinct roots of the equation $P(w) = 0$, where $P(w)$ is defined as in (1.3). Then*

$$f^n P(f) f^{'} g^n P(g) g^{'} \not\equiv a^2,$$

Proof First suppose that

$$f^n P(f) f^{'} g^n P(g) g^{'} \equiv a^2(z). \tag{2.23}$$

Let d_i be the distinct zeros of $P(w) = 0$ with multiplicity p_i, where $i = 1, 2, \ldots, t_2$, $1 \leq t_2 \leq m$ and $\sum_{i=1}^{t_2} p_i = m$.

Now by the second fundamental theorem for f and g we get respectively

$$t_2 T(r, f) \leq \overline{N}(r, 0; f) + \overline{N}(r, \infty; f) + \sum_{i=1}^{t_2} \overline{N}(r, d_i; f) - \overline{N}_0(r, 0; f^{'}) + S(r, f), \tag{2.24}$$

and

$$t_2 T(r, g) \leq \overline{N}(r, 0; g) + \overline{N}(r, \infty; g) + \sum_{i=1}^{t_2} \overline{N}(r, d_i; g) - \overline{N}_0(r, 0; g^{'}) + S(r, g), \tag{2.25}$$

where $\overline{N}_0(r, 0; f^{'})$ denotes the reduced counting function of those zeros of $f^{'}$ which are not the zeros f and $f - d_i$, $i = 1, 2, \ldots, t_2$ and $\overline{N}_0(r, 0; g^{'})$ can be similarly defined.

Let z_0 be a zero of f with multiplicity p but $a(z_0) \neq 0, \infty$. Clearly z_0 must be a pole of g with multiplicity q. Then from (2.23) we get $np + p - 1 = nq + mq + q + 1$. This gives

$$mq + 2 = (n+1)(p-q). \tag{2.26}$$

From (2.26) we get $p - q \geq 1$ and so $q \geq \frac{n-1}{m}$. Now $np + p - 1 = nq + mq + q + 1$ gives $p \geq \frac{n+m-1}{m}$. Thus we have

$$\overline{N}(r, 0; f) \leq \frac{m}{n+m-1} N(r, 0; f) \leq \frac{m}{n+m-1} T(r, f). \tag{2.27}$$

Let $z_1(a(z_1) \neq 0, \infty)$ be a zero of $f - d_i$ with multiplicity q_i, $i = 1, 2, \ldots, t_2$. obviously z_1 must be a pole of g with multiplicity $r(\geq s)$. Then from (2.23) we get $q_i p_i + q_i - 1 = (n+m+1)r + 1 \geq (n+m+1)s + 1$. This gives $q_i \geq \frac{(n+m+1)s+2}{p_i+1}$ for $i = 1, 2, \ldots, t_2$ and so we get

$$\overline{N}(r, d_i; f) \leq \frac{p_i + 1}{(n+m+1)s+2} N(r, d_i; f) \leq \frac{p_i + 1}{(n+m+1)s+2} T(r, f).$$

Clearly

$$\sum_{i=1}^{t_2} \overline{N}(r, d_i; f) \le \frac{m + t_2}{(n + m + 1)s + 2} T(r, f). \tag{2.28}$$

Similarly we have

$$\overline{N}(r, 0; g) \le \frac{m}{n + m - 1} T(r, g), \tag{2.29}$$

and

$$\sum_{i=1}^{t_2} \overline{N}(r, d_i; g) \le \frac{m + t_2}{(n + m + 1)s + 2} T(r, g). \tag{2.30}$$

Also it is clear that

$$\begin{aligned} &\overline{N}(r, \infty; f) \\ &\le \overline{N}(r, 0; g) + \sum_{i=1}^{t_2} \overline{N}(r, d_i; g) + \overline{N}_0(r, 0; g^{'}) + S(r, f) + S(r, g) \\ &\le \left(\frac{m}{n + m - 1} + \frac{m + t_2}{(n + m + 1)s + 2}\right) T(r, g) + \overline{N}_0(r, 0; g^{'}) + S(r, f) + S(r, g), \end{aligned} \tag{2.31}$$

by (2.29) and (2.30).

Then by (2.24), (2.27), (2.28) and (2.31) we get

$$\begin{aligned} &t_2\, T(r, f) \\ &\le \left(\frac{m}{n + m - 1} + \frac{m + t_2}{(n + m + 1)s + 2}\right) \{T(r, f) + T(r, g)\} + \overline{N}_0(r, 0; g^{'}) \\ &-\overline{N}_0(r, 0; f^{'}) + S(r, f) + S(r, g). \end{aligned} \tag{2.32}$$

Similarly we have

$$\begin{aligned} &t_2\, T(r, g) \\ &\le \left(\frac{m}{n + m - 1} + \frac{m + t_2}{(n + m + 1)s + 2}\right) \{T(r, f) + T(r, g)\} + \overline{N}_0(r, 0; f^{'}) \\ &-\overline{N}_0(r, 0; g^{'}) + S(r, f) + S(r, g). \end{aligned} \tag{2.33}$$

Then from (2.32) and (2.33) we get

$$t_2\{T(r, f) + T(r, g)\} \le 2 \left(\frac{m}{n + m - 1} + \frac{m + t_2}{(n + m + 1)s + 2}\right) \{T(r, f) + T(r, g)\} + S(r, f) + S(r, g),$$

i.e

$$\left(t_2 - \frac{2m}{n + m - 1} - \frac{2(m + t_2)}{(n + m + 1)s + 2}\right) \{T(r, f) + T(r, g)\} \le S(r, f) + S(r, g). \tag{2.34}$$

Since

$$\begin{aligned} &\left(t_2 - \frac{2m}{n + m - 1} - \frac{2(m + t_2)}{(n + m + 1)s + 2}\right) \\ &= \frac{(n + m - 1)^2 s t_2 + 2(n + m - 1)(s t_2 - s m - m) - 4m(s + 1)}{(n + m - 1)((n + m + 1)s + 2)}, \end{aligned}$$

we note that when $n + m - 1 > \frac{2m}{st_2} + \frac{2m}{t_2}$, i.e., when $n > \frac{2m(s+1)}{st_2} - (m - 1) = k_2$, then clearly $t_2 - \frac{2m}{n+m-1} - \frac{2(m+t_2)}{(n+m+1)s+2} > 0$ and so (2.34) leads to a contradiction. This completes the proof. □

Lemma 2.13 *Let f and g be two non-constant meromorphic functions such that either the zeros and poles of f and g are of multiplicities at least s, where s is a positive integer or they have no zeros and poles and $a(z)(\not\equiv 0, \infty)$ be small function of f and g. Let n and m be two positive integers such that $n > k_1$, where k_1 be defined by (1.2), t_1 denotes the number of distinct roots of the equation $P_*(w) = 0$, where $P_*(w)$ is defined as in (1.1). Then*

$$[f^n P(f)]^{'}[g^n P(g)]^{'} \not\equiv a^2,$$

Proof Clearly $[f^n P(f)]^{'} = f^{n-1}P_*(f)f^{'}$ and $[g^n P(g)]^{'} = g^{n-1}P_*(g)g^{'}$. The remaining part follows from Lemma 2.12. □

Lemma 2.14 *Let f, g be two non-constant meromorphic functions such that either the zeros and poles of f and g are of multiplicities at least s, where s is a positive integer or they have no zeros and poles and $F = \frac{f^n P(f) f^{'}}{a}$, $G = \frac{g^n P(g) g^{'}}{a}$, where $P(w)$ is defined as in the (1.1), $a = a(z)(\not\equiv 0, \infty)$ is a small function with respect to f and g, and n is a positive integer such that $n > \frac{6}{s} + m - 1$. If $H \equiv 0$ then one of the following three cases holds:*

1. *$f^n P(f) f^{'} g^n P(g) g^{'} \equiv a^2(z)$,*
2. *$f(z) \equiv tg(z)$ for a constant t such that $t^{d_3} = 1$, where $d_3 = gcd(n+m+1, \ldots, n+m+1-i, \ldots, n+1)$, $a_{m-i} \neq 0$ for some $i = 1, 2, \ldots, m$,*
3. *f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^{n+1}(\frac{a_m\omega_1^m}{n+m+1} + \frac{a_{m-1}\omega_1^{m-1}}{n+m} + \cdots + \frac{a_0}{n+1}) - \omega_2^{n+1}(\frac{a_m\omega_2^m}{n+m+1} + \frac{a_{m-1}\omega_2^{m-1}}{n+m} + \cdots + \frac{a_0}{n+1})$.*

Proof Clearly

$$F = \left[f^{n+1}\left\{\frac{a_m}{n+m+1}f^m + \frac{a_{m-1}}{n+m}f^{m-1} + \cdots + \frac{a_0}{n+1}\right\}\right]^{'} /a = [f^{n+1}P_1(f)]^{'}/a,$$

and

$$G = \left[g^{n+1}\left\{\frac{a_m}{n+m+1}g^m + \frac{a_{m-1}}{n+m}g^{m-1} + \cdots + \frac{a_0}{n+1}\right\}\right]^{'} /a = [g^{n+1}P_1(g)]^{'}/a,$$

where

$$P_1(w) = \frac{a_m}{n+m+1}w^m + \frac{a_{m-1}}{n+m}w^{m-1} + \cdots + \frac{a_0}{n+1},$$

Proceeding in the same way as the proof of Lemma 2.11, taking $k = 1$ and considering $n + 1$ instead of n we get either

$$f^n P(f) f^{'} g^n P(g) g^{'} \equiv a^2(z)$$

or

$$f^n P(f) f^{'} \equiv g^n P(g) g^{'}. \tag{2.35}$$

Let $h = \frac{f}{g}$. If h is a constant, by putting $f = hg$ in (2.35) we get

$$a_m g^m (h^{n+m+1} - 1) + a_{m-1} g^{m-1}(h^{n+m} - 1) + \cdots + a_1 g(h^{n+2} - 1) + a_0(h^{n+1} - 1) \equiv 0,$$

which implies that $h^{d_3} = 1$, where $d_3 = gcd(n+m+1, \ldots, n+m+1-i, \ldots, n+1)$, $a_{m-i} \neq 0$ for some $i \in \{0, 1, \ldots, m\}$. Thus $f \equiv tg$ for a constant t such that $t^{d_3} = 1$, where $d_3 = gcd(n+m+1, \ldots, n+m+1-i, \ldots, n+1)$, $a_{m-i} \neq 0$ for some $i \in \{0, 1, \ldots, m\}$.

If h is not constant then f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^{n+1}(\frac{a_m\omega_1^m}{n+m+1} + \frac{a_{m-1}\omega_1^{m-1}}{n+m} + \cdots + \frac{a_0}{n+1}) - \omega_2^{n+1}(\frac{a_m\omega_2^m}{n+m+1} + \frac{a_{m-1}\omega_2^{m-1}}{n+m} + \cdots + \frac{a_0}{n+1})$. □

Lemma 2.15 [1] *If f, g be two non-constant meromorphic functions such that they share* $(1, 1)$. *Then*

$$2\overline{N}_L(r,1;f)+2\overline{N}_L(r,1;g)+\overline{N}_E^{(2}(r,1;f)-\overline{N}_{f>2}(r,1;g) \\ \leq N(r,1;g)-\overline{N}(r,1;g).$$

Lemma 2.16 [2] *Let f and g be the same as in Lemma 2.15. Then*

$$\overline{N}_{f>2}(r,1;g)\leq \frac{1}{2}\overline{N}(r,0;f)+\frac{1}{2}\overline{N}(r,\infty;f)-\frac{1}{2}N_0(r,0;f^{'})+S(r,f),$$

where $N_0(r,0;f^{'})$ is the counting function of those zeros of $f^{'}$ which are not the zeros of $f(f-1)$.

Lemma 2.17 [2] *Let f and g be two non-constant meromorphic functions sharing* $(1, 0)$. *Then*

$$\overline{N}_L(r,1;f)+2\overline{N}_L(r,1;g)+\overline{N}_E^{(2}(r,1;f)-\overline{N}_{f>1}(r,1;g)-\overline{N}_{g>1}(r,1;f) \\ \leq N(r,1;g)-\overline{N}(r,1;g).$$

Lemma 2.18 [2] *Let f and g be the same as in Lemma 2.17. Then*

$$\overline{N}_L(r,1;f)\leq \overline{N}(r,0;f)+\overline{N}(r,\infty;f)+S(r,f)$$

Lemma 2.19 [2] *Let f and g be the same as in Lemma 2.17. Then*

$$(i)\quad \overline{N}_{f>1}(r,1;g)\leq \overline{N}(r,0;f)+\overline{N}(r,\infty;f)-N_0(r,0;f^{'})+S(r,f)$$
$$(ii)\quad \overline{N}_{g>1}(r,1;f)\leq \overline{N}(r,0;g)+\overline{N}(r,\infty;g)-N_0(r,0;g^{'})+S(r,g).$$

3 Proof of the Theorem

Proof of Theorem 1.4 Let $F=[f^nP(f)]^{(k)}/a$ and $G=[g^nP(g)]^{(k)}/a$. It follows that F and G share $(1,l)$ except for the zeros and poles of $a(z)$.

Case 1. Let $H\not\equiv 0$.

Subcase 1.1. $l\geq 1$.

From (2.1) it can be easily calculated that the possible poles of H occur at (i) multiple zeros of F and G, (ii) those 1 points of F and G whose multiplicities are different, (iii) poles of F and G, (iv) zeros of $F^{'}(G^{'})$ which are not the zeros of $F(F-1)(G(G-1))$, (v) the zeros and poles of $a(z)$.

Since H has only simple poles we get

$$N(r,\infty;H)\leq \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}_*(r,1;F,G)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2) \\ +\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})+S(r,f)+S(r,g), \tag{3.1}$$

where $\overline{N}_0(r,0;F^{'})$ is the reduced counting function of those zeros of $F^{'}$ which are not the zeros of $F(F-1)$ and $\overline{N}_0(r,0;G^{'})$ is similarly defined.

Let z_0 be a simple zero of $F-1$ but $a(z_0)\neq 0,\infty$. Then z_0 is a simple zero of $G-1$ and a zero of H. So

$$N(r,1;F|=1)\leq N(r,0;H)\leq N(r,\infty;H)+S(r,f)+S(r,g). \tag{3.2}$$

While $l\geq 2$, using (3.1) and (3.2) we get

$$\overline{N}(r,1;F)\leq N(r,1;F|=1)+\overline{N}(r,1;F|\geq 2) \\ \leq \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+\overline{N}_*(r,1;F,G) \\ +\overline{N}(r,1;F|\geq 2)+\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})+S(r,f)+S(r,g). \tag{3.3}$$

Now in the view of Lemma 2.3 we get

$$
\begin{aligned}
&\overline{N}_0(r,0;G^{'})+\overline{N}(r,1;F\mid\geq 2)+\overline{N}_*(r,1;F,G)\\
&\quad\leq \overline{N}_0(r,0;G^{'})+\overline{N}(r,1;F\mid\geq 2)+\overline{N}(r,1;F\mid\geq 3)\\
&\quad= \overline{N}_0(r,0;G^{'})+\overline{N}(r,1;G\mid\geq 2)+\overline{N}(r,1;G\mid\geq 3)\\
&\quad\leq \overline{N}_0(r,0;G^{'})+N(r,1;G)-\overline{N}(r,1;G)\\
&\quad\leq N(r,0;G^{'}\mid G\neq 0)\leq \overline{N}(r,0;G)+\overline{N}(r,\infty;g)+S(r,g).
\end{aligned} \tag{3.4}
$$

Hence using (3.3), (3.4), Lemmas 2.1 and 2.2 we get from second fundamental theorem that

$$
\begin{aligned}
&(n+m)T(r,f)\\
&\quad\leq T(r,F)+N_{k+2}(r,0;f^nP(f))-N_2(r,0;F)+S(r,f)\\
&\quad\leq \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,1;F)+N_{k+2}(r,0;f^nP(f))-N_2(r,0;F)-N_0(r,0;F^{'})\\
&\qquad+S(r,f)\\
&\quad\leq 2\,\overline{N}(r,\infty,f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F)+N_{k+2}(r,0;f^nP(f))+\overline{N}(r,0;F\mid\geq 2)\\
&\qquad+\overline{N}(r,0;G\mid\geq 2)+\overline{N}(r,1;F\mid\geq 2)+\overline{N}_*(r,1;F,G)+\overline{N}_0(r,0;G^{'})-N_2(r,0;F)\\
&\qquad+S(r,f)+S(r,g)\\
&\quad\leq 2\,\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+N_{k+2}(r,0;f^nP(f))+N_2(r,0;G)+S(r,f)+S(r,g)\\
&\quad\leq 2\,\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+N_{k+2}(r,0;f^nP(f))+k\,\overline{N}(r,\infty;g)+N_{k+2}(r,0;g^nP(g))\\
&\qquad+S(r,f)+S(r,g)\\
&\quad\leq 2\,\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g\}+(k+2)\,\overline{N}(r,0;f)+T(r,P(f))+(k+2)\,\overline{N}(r,0;g)\\
&\qquad+T(r,P(g))+k\,\overline{N}(r,\infty;g)+S(r,f)+S(r,g)\\
&\quad\leq \left(\frac{k+4}{s}+m\right)T(r,f)+\left(\frac{2k+4}{s}+m\right)T(r,g)+S(r,f)+S(r,g)\\
&\quad\leq \left(\frac{3k+8}{s}+2m\right)T(r)+S(r).
\end{aligned} \tag{3.5}
$$

In a similar way we can obtain

$$
(n+m)\,T(r,g)\leq\left(\frac{3k+8}{s}+2m\right)T(r)+S(r). \tag{3.6}
$$

Combining (3.5) and (3.6) we see that

$$
(n+m)\,T(r)\leq\left(\frac{3k+8}{s}+2m\right)T(r)+S(r),
$$

i.e.,

$$
\left(n-\frac{3k+8}{s}-m\right)T(r)\leq S(r). \tag{3.7}
$$

Since $n>\frac{3k+8}{s}+m$, (3.7) leads to a contradiction. While $l=1$, using Lemmas 2.3, 2.15, 2.16, (3.1) and (3.2) we get

$$
\begin{aligned}
&\overline{N}(r,1;F)\\
&\quad\leq N(r,1;F\mid=1)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)\\
&\quad\leq \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F\mid\geq 2)+\overline{N}(r,0;G\mid\geq 2)+\overline{N}_*(r,1;F,G)\\
&\qquad+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)+\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})\\
&\qquad+S(r,f)+S(r,g)\\
&\quad\leq \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F\mid\geq 2)+\overline{N}(r,0;G\mid\geq 2)+2\overline{N}_L(r,1;F)
\end{aligned}
$$

$$
\begin{aligned}
&+2\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)+\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})+S(r,f)+S(r,g)\\
&\le \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F|\ge 2)+\overline{N}(r,0;G|\ge 2)+\overline{N}_{F>2}(r,1;G)\\
&+N(r,1;G)-\overline{N}(r,1;G)+\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})+S(r,f)+S(r,g)\\
&\le \frac{3}{2}\,\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F|\ge 2)+\frac{1}{2}\,\overline{N}(r,0;F)+\overline{N}(r,0;G|\ge 2)\\
&+N(r,1;G)-\overline{N}(r,1;G)+\overline{N}_0(r,0;G^{'})+\overline{N}_0(r,0;F^{'})+S(r,f)+S(r,g)\\
&\le \frac{3}{2}\,\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F|\ge 2)+\frac{1}{2}\,\overline{N}(r,0;F)+\overline{N}(r,0;G|\ge 2)\\
&+N(r,0;G^{'}|G\ne 0)+\overline{N}_0(r,0;F^{'})+S(r,f)+S(r,g)\\
&\le \frac{3}{2}\,\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+\overline{N}(r,0;F|\ge 2)+\frac{1}{2}\,\overline{N}(r,0;F)+N_2(r,0;G)\\
&+\overline{N}_0(r,0;F^{'})+S(r,f)+S(r,g).
\end{aligned}\tag{3.8}
$$

Hence using (3.8), Lemmas 2.1 and 2.2 we get from second fundamental theorem that

$$
\begin{aligned}
&(n+m)T(r,f)\\
&\le T(r,F)+N_{k+2}(r,0;f^nP(f))-N_2(r,0;F)+S(r,f)\\
&\le \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,1;F)+N_{k+2}(r,0;f^nP(f))-N_2(r,0;F)-N_0(r,0;F^{'})\\
&+S(r,f)\\
&\le \frac{5}{2}\,\overline{N}(r,\infty,f)+2\overline{N}(r,\infty;g)+N_2(r,0;F)+\frac{1}{2}\overline{N}(r,0;F)+N_{k+2}(r,0;f^nP(f))\\
&+N_2(r,0;G)-N_2(r,0;F)+S(r,f)+S(r,g)\\
&\le \frac{5}{2}\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+N_{k+2}(r,0;f^nP(f))+\frac{1}{2}\overline{N}(r,0;F)+N_2(r,0;G)\\
&+S(r,f)+S(r,g)\\
&\le \frac{5}{2}\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+N_{k+2}(r,0;f^nP(f))+k\,\overline{N}(r,\infty;g)+N_{k+2}(r,0;g^nP(g))\\
&+\frac{1}{2}\{k\overline{N}(r,\infty;f)+\overline{N}_{k+1}(r,0;f^nP(f))\}+S(r,f)+S(r,g)\\
&\le \frac{5+k}{2}\,\overline{N}(r,\infty;f)+(k+2)\overline{N}(r,\infty;g)+\frac{3k+5}{2}\,\overline{N}(r,0;f)+\frac{3}{2}\,T(r,P(f))\\
&+(k+2)\,\overline{N}(r,0;g)+T(r,P(g))+S(r,f)+S(r,g)\\
&\le \left(\frac{2k+5}{s}+\frac{3m}{2}\right)\,T(r,f)+\left(\frac{2k+4}{s}+m\right)\,T(r,g)+S(r,f)+S(r,g)\\
&\le \left(\frac{4k+9}{s}+\frac{5m}{2}\right)\,T(r)+S(r).
\end{aligned}\tag{3.9}
$$

In a similar way we can obtain

$$
(n+m)\,T(r,g)\le\left(\frac{4k+9}{s}+\frac{5m}{2}\right)\,T(r)+S(r). \tag{3.10}
$$

Combining (3.9) and (3.10) we see that

$$
(n+m)\,T(r)\le\left(\frac{4k+9}{s}+\frac{5m}{2}\right)\,T(r)+S(r),
$$

i.e.,

$$\left(n-\frac{4k+9}{s}-\frac{3m}{2}\right)\ T(r)\leq S(r). \tag{3.11}$$

Since $n>\frac{4k+9}{s}+\frac{3m}{2}$, (3.11) leads to a contradiction.

Subcase 1.2. $l=0$. Here (3.2) changes to

$$N_E^{1)}(r,1;F\mid=1)\leq N(r,0;H)\leq N(r,\infty;H)+S(r,F)+S(r,G). \tag{3.12}$$

Using Lemmas 2.3, 2.17, 2.18, 2.19, (3.1) and (3.12) we get

$$\begin{aligned}
&\overline{N}(r,1;F)\\
&\leq N_E^{1)}(r,1;F)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)\\
&\leq \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+\overline{N}_*(r,1;F,G)\\
&\quad+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)+\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})\\
&\quad+S(r,f)+S(r,g)\\
&\leq \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+2\overline{N}_L(r,1;F)\\
&\quad+2\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)+\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})+S(r,f)+S(r,g)\\
&\leq \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+\overline{N}_{F>1}(r,1;G)\\
&\quad+\overline{N}_{G>1}(r,1;F)+\overline{N}_L(r,1;F)+N(r,1;G)-\overline{N}(r,1;G)+\overline{N}_0(r,0;F^{'})\\
&\quad+\overline{N}_0(r,0;G^{'})+S(r,f)+S(r,g)\\
&\leq 3\ \overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+N_2(r,0;F)+\overline{N}(r,0;F)+N_2(r,0;G)\\
&\quad+N(r,1;G)-\overline{N}(r,1;G)+\overline{N}_0(r,0;G^{'})+\overline{N}_0(r,0;F^{'})+S(r,f)+S(r,g)\\
&\leq 3\ \overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+N_2(r,0;F)+\overline{N}(r,0;F)+N_2(r,0;G)\\
&\quad+N(r,0;G^{'}|G\neq 0)+\overline{N}_0(r,0;F^{'})+S(r,f)+S(r,g)\\
&\leq 3\overline{N}(r,\infty;f)+3\overline{N}(r,\infty;g)+N_2(r,0;F)+\overline{N}(r,0;F)+N_2(r,0;G)\\
&\quad+\overline{N}(r,0;G)+\overline{N}_0(r,0;F^{'})+S(r,f)+S(r,g).
\end{aligned} \tag{3.13}$$

Hence using (3.13), Lemmas 2.1 and 2.2 we get from second fundamental theorem that

$$\begin{aligned}
&(n+m)T(r,f)\\
&\leq T(r,F)+N_{k+2}(r,0;f^nP(f))-N_2(r,0;F)+S(r,f)\\
&\leq \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,1;F)+N_{k+2}(r,0;f^nP(f))-N_2(r,0;F)-N_0(r,0;F^{'})\\
&\quad+S(r,f)\\
&\leq 4\overline{N}(r,\infty,f)+3\overline{N}(r,\infty;g)+N_2(r,0;F)+2\overline{N}(r,0;F)+N_{k+2}(r,0;f^nP(f))\\
&\quad+N_2(r,0;G)+\overline{N}(r,0;G)-N_2(r,0;F)+S(r,f)+S(r,g)\\
&\leq 4\overline{N}(r,\infty;f)+3\overline{N}(r,\infty;g)+N_{k+2}(r,0;f^nP(f))+2\overline{N}(r,0;F)+N_2(r,0;G)\\
&\quad+\overline{N}(r,0;G)+S(r,f)+S(r,g)\\
&\leq 4\overline{N}(r,\infty;f)+3\overline{N}(r,\infty;g)+N_{k+2}(r,0;f^nP(f))+2k\overline{N}(r,\infty;f)+2N_{k+1}(r,0;f^nP(f))\\
&\quad+k\ \overline{N}(r,\infty;g)+N_{k+2}(r,0;g^nP(g))+k\overline{N}(r,\infty;g)+\overline{N}_{k+1}(r,0;g^nP(g))+S(r,f)+S(r,g)\\
&\leq (2k+4)\ \overline{N}(r,\infty;f)+(2k+3)\overline{N}(r,\infty;g)+(3k+4)\overline{N}(r,0;f)+3T(r,P(f))\\
&\quad+(2k+3)\ \overline{N}(r,0;g)+2T(r,P(g))+S(r,f)+S(r,g)\\
&\leq \left(\frac{5k+8}{s}+3m\right)\ T(r,f)+\left(\frac{4k+6}{s}+2m\right)\ T(r,g)+S(r,f)+S(r,g)\\
&\leq \left(\frac{9k+14}{s}+5m\right)\ T(r)+S(r).
\end{aligned} \tag{3.14}$$

In a similar way we can obtain

$$(n+m)\,T(r,g) \leq \left(\frac{9k+14}{s} + 5m\right)\,T(r) + S(r). \tag{3.15}$$

Combining (3.14) and (3.15) we see that

$$(n+m)\,T(r) \leq \left(\frac{9k+14}{s} + 5m\right)\,T(r) + S(r),$$

i.e.,

$$\left(n - \frac{9k+14}{s} - 4m\right)\,T(r) \leq S(r). \tag{3.16}$$

Since $n > \frac{9k+14}{s} + 4m$, (3.16) leads to a contradiction.

Case 2. Let $H \equiv 0$. Then the theorem follows from Lemma 2.11. □

Proof of Theorem 1.5 Let $F = \frac{f^n P(f) f'}{a(z)}$ and $G = \frac{g^n P(g) g'}{a(z)}$. Then F, G share $(1, l)$, except the zeros and poles of $a(z)$.

Clearly

$$F = \left[f^{n+1}\left\{\frac{a_m}{n+m+1} f^m + \frac{a_{m-1}}{n+m} f^{m-1} + \cdots + \frac{a_0}{n+1}\right\}\right]' / a = [f^{n+1} P_1(f)]' / a,$$

and

$$G = \left[g^{n+1}\left\{\frac{a_m}{n+m+1} g^m + \frac{a_{m-1}}{n+m} g^{m-1} + \cdots + \frac{a_0}{n+1}\right\}\right]' / a = [g^{n+1} P_1(g)]' / a,$$

where

$$P_1(w) = \frac{a_m}{n+m+1} w^m + \frac{a_{m-1}}{n+m} w^{m-1} + \cdots + \frac{a_0}{n+1},$$

Case 1. Let $H \not\equiv 0$.

Now following the same procedure as adopted in the proof of **Case 1** of Theorem 1.4 we can easily deduce a contradiction.

Case 2. Let $H \equiv 0$. Since $n > k_1$ and $n > \frac{6}{s} + m - 1$ the theorem follows from Lemmas 2.12 and 2.14. □

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References

1. Alzahary, T.C.; Yi, H.X.: Weighted value sharing and a question of I. Lahiri. Complex Var. Theory Appl. **49**(15), 1063–1078 (2004)
2. Banerjee, A.: Meromorphic functions sharing one value. Int. J. Math. Math. Sci. **22**, 3587–3598 (2005)
3. Banerjee, A.: On uniqueness for non-linear differential polynomials sharing the same 1-points. Ann. Polon. Math. **89**(3), 259–272 (2006)
4. Banerjee, A.: A uniqueness result on some differential polynomials sharing 1 points. Hiroshima Math. J. **37**(3), 397–408 (2007)
5. Banerjee, A.; Sahoo, P.: Uniqueness of meromorphic functions when two differential polynomials share one value. Sarajevo.J. Math. **20**, 69–89 (2012)
6. Dyavanal, R.S.: Uniqueness and value sharing of differential polynomials of meromorphic functions. J. Math. Anal. Appl. **372**(1), 252–261 (2010)

7. Dyavanal, R.S.: Erratum to uniqueness and value sharing of differential polynomials of meromorphic functions. J. Math. Anal. Appl. **374**(1), 334 (2011)
8. Dyavanal, R.S.: Uniqueness and value sharing of differential polynomials of meromorphic functions. J. Math. Anal. Appl. **374**(1), 345–355 (2011)
9. Fang, M.L.; Hong, W.: A unicity theorem for entire functions concerning differential polynomials. Indian J. Pure Appl. Math. **32**(9), 1343–1348 (2001)
10. Frank, G.: Eine Vermutung von Hayman über Nullstellen meromorpher Funktionen. Math. Z. **149**, 29–36 (1976)
11. Hayman, W.K.: Meromorphic Functions. The Clarendon Press, Oxford (1964)
12. Lahiri, I.: Uniqueness of meromorphic functions when two linear differential polynomials share the same 1-points. Ann. Polon. Math. **71**(2), 113–128 (1999)
13. Lahiri, I.: Weighted sharing and uniqueness of meromorphic functions. Nagoya Math. J. **161**, 193–206 (2001)
14. Lahiri, I.: Weighted value sharing and uniqueness of meromorphic functions. Complex Var. Theory Appl. **46**, 241–253 (2001)
15. Lahiri, I.; Dewan, S.: Value distribution of the product of a meromorphic function and its derivative. Kodai Math. J. **26**(1), 95–100 (2003)
16. Lahiri, I.; Mandal, N.: Uniqueness of nonlinear differential polynomials sharing simple and double 1-points. Int. J. Math. Math. Sci. **12**, 1933–1942 (2005)
17. Lahiri, I.; Pal, R.: Nonlinear differential polynomials sharing 1-points. Bull. Korean Math. Soc. **43**(1), 161–168 (2006)
18. Lahiri, I.; Sarkar, A.: Nonlinear differential polynomials sharing 1-points with weight two. Chinese J. Contemp. Math. **25**(3), 325–334 (2004)
19. Lahiri, I.; Sahoo, P.: Uniqueness of non-linear differential polynomials sharing 1-points. Georgian Math. J. **12**(1), 131–138 (2005)
20. Li, P.; Yang, C.C.: On the characteristics of meromorphic functions that share three values CM. J. Math. Anal. Appl. **220**, 132–145 (1998)
21. Lin, W.C.: Uniqueness of differential polynomials and a question of Lahiri. Pure Appl. Math. **17**(2), 104–110 (2001) (in Chinese)
22. Lin, W.C.; Yi, H.X.: Uniqueness theorems for meromorphic function. Indian J. Pure Appl. Math. **35**(2), 121–132 (2004)
23. Meng, C.: On unicity of meromorphic functions when two differential polynomials share one value. Hiroshima Math. J. **39**, 163–169 (2009)
24. Nevanlinna, R.: Einige Eindeutigkeitssätze in der Theorie der Meromorphen Funktionen. Acta Math. **48**, 367–391 (1926)
25. Yang, C.C.; Hua, X.H.: Uniqueness and value sharing of meromorphic functions. Ann. Acad. Sci. Fenn. Math. **22**, 395–406 (1997)
26. Yi, H.X.: On characteristic function of a meromorphic function and its derivative. Indian J. Math. **33**(2), 119–133 (1991)
27. Yi, H.X.: Meromorphic functions that share one or two values II. Kodai Math. J. **22**, 264–272 (1991)
28. Yi, H.X.; Yang, C.C.: Uniqueness Theory of Meromorphic Functions. Science Press, Beijing (1995)
29. Yamanoi, K.: The second main theorem for small functions and related problems. Acta Math. **192**, 225–294 (2004)
30. Zhang, Q.C.: Meromorphic function that shares one small function with its derivative. J. Inequal. Pure Appl. Math. **6**(4)(2005), Art.116. http://jipam.vu.edu.au/