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Morse relations and Fredholm deformations of v-convex contact forms

Per Diana Nunziante, un caro ricordo "Te voglio bene assaie". Grazie, grazie tanto!

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Abstract We address in this paper the Fredholm and compactness issues for the variational problem (J, C_β), Bahri (Pitman Research Notes in Mathematics Series No. 173. Scientific and Technical, London, 1988), Bahri (C. R. Acad. Sci. Paris 299, Serie I, 15, 757–760, 1984). We prove that the intersection operator restricted to periodic orbits of the Reeb vector-field ∂_{per} does not mix with the intersection operator ∂_∞ of the critical points at infinity. The Fredholm issues are extensively discussed in the Introduction and solved in Bahri (Arab J Maths, 2014). We also address in this paper the issue of existence of periodic orbits for three-dimensional Reeb vector-fields, the Weinstein conjecture (Weinstein, J Differ Equ 33:353–358, 1979) on S^3, solved in dimension 3 throughout the works of Rabinowitz (Commun Pure Appl Math 31:157–184, 1978) and Hofer (Invent Math 114:515–563, 1993); see also Hutchings (Proc. 2010 ICM 46:1022–1041, 2010) and Taubes (Geom Topol 11:2117–2202, 2007) for the full Weinstein conjecture in dimension 3. Following our previous work (Bahri, Adv Nonlinear Stud 8:1–17, 2008), we devise a new method to find these periodic orbits when they are of odd index. We conjecture that this method, when combined with the other results described above about the intersection operator, gives rise to a homology that is specific of the contact structure and that is invariant by deformation. The existence result, as derived here, is weaker than the one announced by Taubes (Geom Topol 11:2117–2202, 2007). After appropriate generalization, it provides a new proof, via variational theory, of the Weinstein conjecture on three-dimensional closed contact manifolds with finite fundamental group.

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الملخص

نتناول في هذه الورقة مسائل فريدهولم والتراص لمسألة التغايرية (J, C_β) [1] وَ [2]. نثتب أن مؤثر التقاطع المحصور على المدارات الدورية لحقل ريب المتجهي ∂_{per} لا يختلط مع مؤثر التقاطع ∂_∞ للنقاط الحرجة عند مالانهاية. تتم مناقشة مسائل فريدهولم، والتي تم حلها في [8]، على نطاق واسع في المقدمة. نتناول في هذه الورقة أيضاً مسألة وجود مدارات دورية لحقول ريب متجهية ثلاثية الأبعاد، وحدس ڤاينشتاين [39] الذي تم حله في البعد 3 من خلال أبحاث پ. هـ. رابينوفنتش [29] وَ هـ. هوفر [17] وَ م. هتشنجز [20] وَ تاوبز [32]. متابعة لبحثنا السابق [7]، نصمم طريقة جديدة لإيجاد مدارات دورية عندما يكون مؤشرها فردياً. نحدس أن هذه الطريقة، جنباً إلى جنب مع النتائج الأخرى حول مؤثر التقاطع والتي تم وصفها أعلاه، تؤدي إلى هومولوجية محددة لبناء الاتصال تكون ثابتة تحت التشوُّه. نتيجة الوجود التي تم اشتقاقها هنا أضعف من نتيجة تاوبز [32]. بعد تعميم مناسب، تقدم إثباتاً جديداً، باستخدام نظرية التغايرية، لحدس ڤاينشتاين على متنوعات الاتصال المغلقة ثلاثية الأبعاد التي تكون زمرتها الأساسية منتهية.

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1 Introduction

This paper is concerned with two issues; it is, therefore, sub-divided into two distinct parts. The first part is concerned with Fredholm and compactness issues for the variational formulation (J, C_β) [1,2], of the periodic orbit problem for three-dimensional Reeb vector-fields. The second part is concerned with the issue of existence of periodic orbits for these Reeb vector fields on three-dimensional closed contact manifolds. These two parts are addressed in this Introduction. They are to a large extent independent and a reader that is not interested in the first issue can go directly, after reading this Introduction, to Part II of this paper.

1.1 Fredholm and compactness issues in contact form geometry; the intersections operators ∂_{per} and ∂_∞

Let us start with a symplectic Hamiltonian framework and recognize some of its features. These features will gradually lead us to the content of the present paper.

The L^2-"pseudo-gradient" in the standard Hamiltonian framework has the familiar form:

$$\mathbf{J}\dot{z} + H'(z)$$

z is usually a closed curve of $\mathbb{R}^{2n}$, with some mild regularity (e.g. $H^{\frac{1}{2}}$, H is the Hamiltonian and $\mathbf{J}$ is the symplectic matrix of $\mathbb{R}^{2n}$).

This operator is special; it is usually seen—in Nonlinear Analysis to the least—as a Fredholm operator $T + K$, with T invertible bi-continuous and K compact.

Without entering into required details, T may be viewed as a small modification of the operator $H^{\frac{1}{2}} \longrightarrow H^{\frac{-1}{2}}$, which to z assigns $\mathbf{J}\dot{z}$ once $H^{\frac{1}{2}}$ is identified with $H^{\frac{-1}{2}}$ using the natural duality operator. K is essentially the Caratheodory operator $H'(z)$.

From this point of view, all Hamiltonian problems (we could include the time-dependent ones), whether they pertain to sub-linear problems (associated with the Arnold conjecture) or to super-linear problems (associated with extended forms of the Weinstein conjecture), are all compact perturbations of the fundamental operator $\mathbf{J}\dot{z}$.

What is $\mathbf{J}\dot{z}$?

Considering the action functional

$$J(z) = \int_0^1 \alpha_z(\dot{z})\mathrm{d}t,\ z \in H^{\frac{1}{2}}(S^1, \mathbb{R}^{2n}),$$

where

$$\alpha_{(p,q)} = \sum_{i=1}^{n}(p_i dq_i - q_i dp_i),$$

we compute for a variation h of z:

$$J'(z).h = -\int_0^1 \mathrm{d}\alpha_z(\dot{z}, h) = \int_0^1 < \mathbf{J}\dot{z}, h >_{\mathbb{R}^{2n}} \mathrm{d}t,$$

where $<, >_{\mathbb{R}^{2n}}$ denotes the standard dot product of $\mathbb{R}^{2n}$.

Clearly $h = -\mathbf{J}\dot{z}$ decreases J. The operator $-\mathbf{J}\dot{z}$ is a generalized pseudo-gradient (with very little existence of (semi)-flow-lines) for the action functional with $H = 0$. The operator $-\mathbf{J}\dot{z} - H'(z)$ is then a compact perturbation of this operator.

This understanding—for which we claim no originality—extends to the framework of a general symplectic manifold (M^{2n}, ω). Using a taming almost complex structure $\mathbf{J} = \mathbf{J}(z)$ (Gromov [15]) and a Hamiltonian that grows at most linearly, we find a similar framework, with a non-linear operator $-\mathbf{J}(z)\dot{z}$. $H'(z)$ or $H'(t, z)$ introduces a compact perturbation (we are viewing this through the linearized operator). This is the framework for the Arnold conjecture [30] and also the framework used for the so-called Floer–Rabinowitz homology ([10,28] and more).

There have been attempts to generalize these techniques to the exotic, non-symplectic framework through the technique of pseudo-holomorphic curves. These attempts were successful for establishing the existence of one periodic orbit for over-twisted contact forms on three-dimensional closed manifolds [17]. This proof required also a preliminary work of classification, etc. [11]. The existence of the periodic orbit followed from the rigid framework provided by a special two-dimensional disk bounded by a Legendrian curve. This disk is used to support pseudo-holomorphic disks that eventually have to blow-up, yielding then the periodic orbit [17]. Such a disk does exist for every over-twisted contact structure: it is part of the definition.

Then, there has been the additional attempt to fit all this work in a very general framework [12], where moduli spaces of pseudo-holomorphic curves were defined, invariants followed from various constructions, etc.

Written in a very long paper published at the turn of the century, in an issue of GAFA called "Visions" [12], these ideas were studied over a full year at the Institute of Advanced Study (2001–2002).

Some of the claims of this paper were later dismissed in a short paper [39] by Yau for over-twisted contact forms. Since then, the emphasis for this area has come back to the more classical ideas of the Floer–Rabinowitz homology, also to variants and more general forms of this using cylindrical homology, etc., see, e.g. Bourgeois [9] for an early form of this homology.

Let us observe that, if we analyze the roots of the idea of pseudo-holomorphic curves

$$\frac{\partial u}{\partial s} = -\mathbf{J}\frac{\partial u}{\partial t}$$

we find a " generalized pseudo-gradient" $-\mathbf{J}\frac{\partial u}{\partial t}$, only that all ideas of cylinders associated with flow-lines are abandoned and replaced by more general Riemann surfaces (these Riemann surfaces are "capped" in the cylindrical homology, which, therefore, following a long route, comes closer to flow-lines of pseudo-gradients).

Assuming that the equation above and the associated moduli spaces are understood for a given $\mathbf{J}(u) = \mathbf{J}_0(u)$, the hope is to prove that some invariants do not change as $\mathbf{J}(u)$ is deformed into $\mathbf{J}_\tau(u)$, $\tau \in [0, 1]$, along a deformation of contact forms (obviously of the same contact structure).

Yau [39] has proven that the contact homology of [12] (Eliashberg provides another proof of the same result in an Appendix to [39]) was zero in the case of the over-twisted contact forms. The hope is, therefore, that some appropriate modification of the tool of pseudo-holomorphic curves can be attached to them and that this object or "structure" will not change as α_0 is deformed into another contact form α_t and $\mathbf{J}_0$ into $\mathbf{J}_\tau$. The thesis of Schwartz [31] is an important and rigorous work of [31] that should be useful for general contact structures.

A modification of this tool has been introduced by Hutchings [18–24] and used by Taubes [32–37], in combination with other results about the Seiberg–Witten equations, to prove the Weinstein conjecture. This method is called "Embedded Contact Homology". It starts from the same concept, the concept of moduli spaces of pseudo-holomorphic curves, but it introduces restrictions on the ∂ operator and requires on the "Morse relations" (these are not Morse relations, they are more complicated forms of them) to be embedded through the use of an appropriate index [18,19]. The grading of the moduli spaces is different in Embedded Contact Homology from the grading in Symplectic Field Theory. The idea uses the positivity of intersection for pseudo-holomorphic curves due to Gromov [15] and proven by Mc Duff [26] and Micallef and White [27].

The proof of the Weinstein conjecture [38] should then follow from these results, see Taubes [32].

However, the Embedded Contact Homology is not specific of the contact structure; it is an invariant of the three-dimensional manifold (Taubes). Therefore, the program of defining an invariant attached to a contact structure, as well as the program of understanding all the Morse relations of the variational problem defined by the action functional, remains largely open.

This short summary describes several frameworks that have been developed to define invariants of contact structures. Let us observe here that the pseudo-holomorphic technique, for example, starts with a Fredholm operator that changes in a continuous manner , usually through compact perturbations or deformations that will respect the Fredholm framework (in particular, the index of this Fredholm operator is unchanged along these deformations, as long as the definition of the moduli space does not change and no blow-up occurs).

Let us move away from the ideas originating in the symplectic framework and come to our very own framework for these problems [1–5]. α is given, a contact form on M^3, a three-dimensional closed and orientable manifold.

We assume for simplicity that $\ker\alpha \longrightarrow M^3$ is trivial and we find, therefore, two independent non-zero vector-fields v and $\bar{w}$ in $\ker\alpha$ such that $d\alpha(v, \bar{w}) = 1$. ξ is the Reeb vector field of α.

We consider the loop space $H^1(S^1, M)$ and the functional $J(x) = \int_0^1 \alpha_x(\dot{x})dt$ for $x \in H^1(S^1, M)$.

$\dot{x}$ reads as $a\xi + bv + c\bar{w}$. If $z \in T_x H^1(S^1, M)$ reads

$$z = \lambda\xi + \mu v + \eta w, \lambda, \mu, \eta \in H^1(S^1, \mathbb{R},)$$

then

$$J'(x).z = \int_0^1 (c\mu - b\eta)\mathrm{d}t$$

We, therefore, recognize the "decreasing pseudo-gradient" for J

$$z = -cw + bv = -\text{“}\mathbf{J}\dot{x}\text{”}$$

Here, $\mathbf{J}\dot{x}$ is limited to its projection onto kerα. One could get back the full operator "$\mathbf{J}\dot{x}$" by considering $\mathbb{R} \times M$, with $\mathbb{R} \oplus \mathbb{R}\xi$ invariant by $\mathbf{J}$ (see [17]). Considering then $H^1(\mathbb{R} \times M)$ and a slight modification of J, we can create a framework where the full operator "$\mathbf{J}\dot{x}$" (for a $\mathbf{J}$ adapted to $(v, \bar{w})$) is a "decreasing pseudo-gradient" for the modified functional.

Our more special $z = -cv + bw$ is, however, quite convenient on $H^1(S^1, M)$; we are going in fact to make it more particular.

We introduce as in [1,3,4] a generalization of the Legendre duality derived under the assumption:

$$(A)\beta = \mathrm{d}\alpha(v, .)$$

is a contact form with the same orientation than α.

We will say that a contact form α of a given contact structure is v**-convex** if it verifies (A) for a suitable v in kerα.

(A) is verified by convex Hamiltonians of $\mathbb{R}^{2n}$ (and their associated contact forms, with v a vector-field in their kernel defining a Hopf fibration). It is also verified [16] for some contact forms of the first exotic (thereby) over-twisted contact structure of Gonzalo and Varela [14], with v the vector-field of Vittorio Martino [25].

Under (A), we can restrict the variations of J so that they take place in

$$C_\beta = \{x \in H^1(S^1, M); \beta_x(\dot{x}) = 0, \alpha_x(\dot{x}) = C \gneq 0\}$$

C is not prescribed.

$\dot{x}$ then reads

$$\dot{x} = a\xi + bv$$

If $z = \lambda\xi + \mu v + \eta\bar{w}$ belongs to $T_x C_\beta$, then

$$J'(x).z = -\int_0^1 b\eta\mathrm{d}t$$

In view of this formula, there is a "natural pseudo-gradient" that can be derived by taking $\eta = b$ in the formula above. It is tempting to write then that $z = b\bar{w}$, which would be our $\mathbf{J}\dot{x}$ from above with $c = 0$. However, $z \in T_x C_\beta$ and, therefore, z has a more complicated form. If w is the contact vector field of β and $\alpha(w) = \bar{\mu}$, then [3,4]

$$z = \left(\int_0^t b^2 - t\int_0^1 b^2 - \bar{\mu}b\right)\xi + \frac{\dot{b} + b\left(\int_0^t b^2 - t\int_0^1 b^2 - \bar{\mu}\right)}{a}v + bw$$

The evolution equation $\frac{\partial x}{\partial s} = z(x), x(0) = x_0 \in C_\beta$ is very close to the mean or normal curvature equation on one-dimensional curves.

This flow has several remarkable geometric properties [3,4,6]. However, it does have several "undesired" blow-ups (in [3], a flow that "corrects" the defects of the flow described above is defined and studied) and it is, therefore, difficult to define a homology related to the periodic orbits of the Reeb vector field ξ with this "pseudo-gradient" that can be thought as a $\mathbf{J}\dot{x}$ ($\mathbf{J}$ should not be thought of as in the symplectic framework where it tames a global ω on N^{2n}. $\mathbf{J}$ here tames only $d\alpha_{|\ker\alpha}$).

In this new framework, which includes cases of exotic contact forms and structures, there is no obvious need to limit ourselves to the natural $\mathbf{J}\dot{x}$ that we found above. Freeing ourselves of any special form, we think of $\mathbf{J}\dot{x}$ as a general decreasing "pseudo-gradient" for the functional J on C_β.

Two natural questions then arise, in view of our discussion above: the first one can be formulated as follows: is there a reference decreasing pseudo-gradient Z for J, for which a homology or, a weaker statement, Morse relations specifically attached to the periodic orbits of the Reeb vector field can be defined?

The second one follows then: assuming that we now define the contact form through an isotopy of contact forms α_t, is it possible to deform Z into Z_t, a decreasing pseudo-gradient for (J_t, C_{β_t}) (under (A_t)) and prove that this homology or these Morse relations defined for Z are also defined and unchanged for Z_t?

The present paper, which is a continuation of earlier papers [3–7] answers positively to both questions: given a deformation of contact forms, there is a continuous family of special pseudo-gradients Z_t, such that the Morse relations between periodic orbits are not changed by the tangencies with the critical points at infinity. The critical points at infinity can affect the Morse relations between periodic orbits only as these periodic orbits are created or eliminated. Furthermore, the cycles of the intersection operator ∂_{per} at the odd order which are "minimal", see below, survive the deformation.

There are, in fact, more restrictions on the Morse relations at infinity that can change the Morse relations between periodic orbits. The former must include some "point to circle" Morse–Bott relation between a combination of critical points at infinity and a periodic orbit of odd index, see Sect. 2.5 of this paper for more details.

In the first part, Part I, of this paper, we require the deformation of Z into Z_t to be "Fredholm" (or "symplectic" using Definition 2.9 in Sect. 2.4 of the present paper, below). This assumption is removed in [8]; however, this is a major issue in this variational problem and in other variational problems as well.

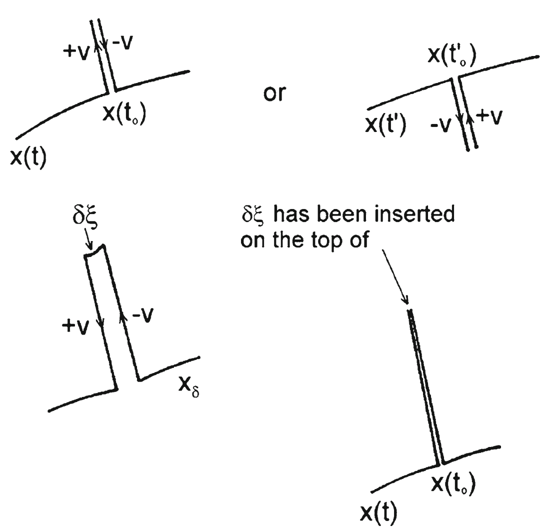
Let us clarify, therefore, in this Introduction, what is meant by "Fredholm" or "symplectic" deformation of Z into Z_t.

The variational problem (J, C_β) is not a Fredholm variational problem: the linearized operator of J' does not read as $T + K$, with T invertible, bi-continuous and K compact (in a suitable separable space).

This drastic feature has important consequences on J, its critical points, its critical points at infinity etc. These consequences have been drawn and analyzed in great detail in our earlier works [3], pp 236–239, [4], pp 151–178.

We describe again this phenomenon in the present paper, in Sect. 2.4 and in an appendix (Appendix 1, Sect. 2.8) to this paper, written for the convenience of the reader.

Summarizing in this introduction the phenomenon, we can add to every curve x of C_β a back or forth and back run along v at a time t_0. The value of J is not changed and even if the new derived curves are not in C_β anymore, they are "almost" in this space. Cutting details, once this "Dirac mass" along v is inserted along the curve, it can be "opened" up at its top or at its bottom depending on the cases and a small piece of ξ-orbit can be inserted:



If the "Dirac mass" is chosen with the appropriate length and location, $J(x_\epsilon)$ can be made smaller than $J(x)$.

Because of this phenomenon (which may be traced back to the fact that this variational problem in not Fredholm), a periodic orbit of ξ, w_m of index m, can have a companion "shadow critical point at infinity" $(\delta + w_m)^\infty$, made of the combination of a "Dirac mass" as above and of w_m, of index $(m + 1)$. $(\delta + w_m)^\infty$ "cancels" w_m topologically, see [4], pp 151–178 for more details, Lemma 16 (i) and (iv) p 161 in particular.

Furthermore, $J((\delta + w_m)^\infty) = J(w_m)$; $(\delta + w_m)^\infty$ is, relative to w_m and its index, a "mountain pass" critical point, which is exactly at the same critical level.

The phenomenon is subtle because if the "Dirac mass" is small in size, $J(x_\epsilon)$ or $J(w_{m\epsilon})$ is always more than $J(x)$ or $J(w_m)$; but this might change (see Sects. 2.4 and 2.8, Appendix 1) with a larger "Dirac mass".

In the case of the first exotic contact structure of Gonzalo and Varela [14], every w_m has a companion $(\delta + w_m)^\infty$ that cancels it. In terms of Morse Theory, this has a fundamental consequence: given a periodic orbit w_{m+1} of index $(m + 1)$ such that $J(w_{m+1})$ is larger than $J(w_m)$ and such that no other critical point (at infinity) of J has a critical value in $(J(w_m), J(w_{m+1})$, the intersection number $i(w_{m+1}, w_m)$ is not defined intrinsically: it depends on the choice of a pseudo-gradient.

Given two pseudo-gradients Z_0 and Z_1 for (J, C_β) such that, as we deform Z_0 into Z_1, a generic tangency occurs between the unstable manifold $W_u^t(w_{m+1})$ with the stable manifold $W_s^t((\delta + w_m)^\infty)$ along the deformation of pseudo-gradients Z_t, $i(w_{m+1}, w_m)$ will increase or decrease by 1.

There is no way that a stable homology related to the periodic orbits can then be defined. The project seems hopeless.

It is not unreasonable to think then that restrictions on the pseudo-gradient Z would not allow this to happen. In our constructions [3–5], Z has important properties: along its (semi)-flow-lines, the number of zeros of b, the v-component of $\dot{x}$ that is the tangent vector to the curve x under deformation, never increases; the L^1-norm of b is bounded; all (semi)-flow-lines end either at critical points (periodic orbits) or critical points at infinity ([3–5]). In addition, see [5], Proposition 2.2, p 469 and the first section of the present paper, the behavior of Z in the vicinity of periodic orbits can be prescribed.

All these properties allow us in fact to define a homology related to the periodic orbits for Z.

It is, however, true that, e.g. not only in the case of the first exotic contact structure of Gonzalo and Varela [14], but also in more general cases, we can find two different pseudo-gradients Z_0 and Z_1 for J, both verifying **all** of the above-stated properties, such that any deformation Z_t from Z_0 to Z_1 among "pseudo-gradients" for J will involve a tangency between $W_u^{t_0}(w_m)$ and $W_s^{t_0}((\delta + w_{m-1})^\infty)$ at a certain time t_0 of the deformation and for a suitable w_m.

Such restrictions are, therefore, not sufficient to "stabilize" the homology.

In what follows, we will limit ourselves to the case of the standard contact structure of S^3 and to the case of the first exotic contact structure of Gonzalo and Varela [14]. A large part of our arguments extend to more general contact structures. There is one restriction: sequences of variational dominations with non-zero intersection numbers $w_{2k+2} - w_{2k+1}^\infty - w_{2k}$ should not be present at the time zero of the deformation. This holds for the standard contact **form** α_0 of S^3 because all periodic orbits are, transversally to the S^1-action, of odd index. It also holds for the first exotic contact **form** α_1 of Gonzalo and Varela [14] because of a basic circle symmetry that allows to "unravel" these Morse relations, reducing all indexes by 1 and rendering them thereby impossible, see Sect. 2.10 of the present paper. This basic circle symmetry for a special contact form in a given contact structure exists in several cases, see in particular all the exotic contact structures of Gonzalo and Varela [14].

The result that we prove in this paper reads as follows:

Theorem 1.1 *Considering a deformation of contact forms α_t under $(A)_t$, there is a corresponding deformation of decreasing "pseudo-gradients" Z_t for the variational problems (J_t, Z_t)—we assume both to be in general position—so that the following hold: (i) Let ∂_{per} be the intersection operator restricted to periodic orbits. This operator is not modified by tangencies and creations or cancellations involving the critical points at infinity of (J, C_β). It can only be modified through tangencies between stable and unstable manifolds of periodic orbits, or by creations and cancellations of periodic orbits. (ii)Considering a deformation that starts from the standard contact forms for the standard contact structure of S^3 and for the first exotic contact structure of Gonzalo and Varela on S^3 [14], a homology related only to the periodic orbits of the Reeb vector fields ξ_t can be defined for each time t of the deformation as long as no creation and cancellation occurs between periodic corbits. (iii) If no tangency between $W_u^t(w_m^t)$ and $W_s^t((\delta + w_{m-1}^t)^\infty)$ occurs at any time t and for any w_m^t and*

w^t_{m-1} periodic orbits of respective indexes m and $(m-1)$ (m arbitrary ≥ 1) and if no creation of cancellation of periodic orbits occurs along this deformation, then this homology does not change as t changes.

In other words, if the deformation of "pseudo-gradients" Z_t is "Fredholm" or "symplectic" in that it does not change, via Fredholm violation, (use of $(\delta + w^t_{m-1})^\infty$), the intersection numbers between periodic orbits and if there is no creation or cancellation of periodic orbits, then a homology solely related to periodic orbits is well defined and it does not change as t changes.

It then follows for example that

Theorem 1.2 (i) *This homology has one generator in each odd dimension larger than or equal to 3, under the assumptions of Theorem 1.1 (which include (A) and $(A)_t$), for the standard contact structure* $\ker\alpha_0$ *on* S^3.

(ii) *under the same assumptions, there exists a sequence k_m tending to ∞ such that the homology has at least two generators at each odd index $2k_m - 1$, for the first exotic contact structure* $\ker\alpha_1$ *on* S^3.

The assumption that the deformation is "Fredholm" or "symplectic" is removed in [8] as we prove that, for a suitable flow that can be continuously tracked along the deformation , no tangency between $W^t_u(w^t_m)$ and $W^t_s((\delta + w^t_{m-1})^\infty)$ occurs at any time t and for any w^t_m and w^t_{m-1} periodic orbits of respective indexes m and $(m-1)$ (m arbitrary ≥ 1).

The homology that we define here uses (part) the Morse relations of the variational problem (J, C_β); it is also transverse to the S^1-action by time translation on curves: any S^1-equivariant cycle, equivariant for the time translation on C_β, collapses for this homology.

The proof of Theorem 1.2 (i) follows from the understanding of the homology for the **standard** contact **form** (with generators only in **odd** dimensions, so that, in this case, the homology for Z_0 and J_0 does not depend on Z_0). The proof of (ii) of Theorem 1.2 is based on sections 2 through 7, section 8, subsections 3 and 4 and section 9 of [6]. The fact that there are two generators for each large index $2k - 1$ follows from the basic symmetry $(x_1, x_2) \longrightarrow (x_3, x_4)$ that (α_1, v) exhibits on $S^3 = \{(x_1, x_2, x_3, x_4), x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$, see [6] for more details, also Sect. 2.10 of the present paper.

It makes sense, to conclude the first part of this long Introduction, to ask ourselves about the precise content of Theorem 1.1 (and of its application, in Theorem 1.2). These theorems state that the Morse relations between periodic orbits depend to a large extent from themselves and do not depend on the critical points at infinity in that tangencies of stable and unstable manifolds, as pseudo-gradient flows are deformed, involving the critical points at infinity do not affect these Morse relations.

One could claim that there is no point in discussing the stability of Morse relations in between creation and cancellation of periodic orbits. However, we are also interested in the full variations and we believe that these Fredholm and compactness issues have other, deeper, roots. Therefore, we think that this result is meaningful.

Let us now address, in a second part, the issue of existence of periodic orbits:

1.2 Elliptic periodic orbits and the Fadell–Rabinowitz index: existence theorems

As stated above, the issue of existence of one and more periodic orbits for three-dimensional Reeb vector fields on closed manifolds (the three-dimensional Weinstein conjecture) has been solved for S^3 through the work of Rabinowitz [29] and Hofer [17]; see Hutchings [19] and Taubes [32] for the general three dimensional Weinstein conjecture, also for multiplicity issues.

The full understanding of the Fredholm and compactness issues and the full understanding of all the variations of the corresponding variational problem(s) or variants/extensions of these are not exhausted by these results and we have explained above and stated two theorems, Theorems 1.1 and 1.2, that show how the evolution of the intersection operator related to the periodic orbit ∂_{per} did not depend, in between creations and cancellations of periodic orbits, on the critical points at infinity.

We are not able to prove, however, that, through a creation or a cancellation of periodic orbits, ∂_{per} does not change and "mix" with ∂_∞, thereby destroying the relation $\partial_{\text{per}} \circ \partial_{\text{per}} = 0$. This relation can be seen to hold, e.g. for all the exotic contact forms of Gonzalo and Varela [14] on S^3. It of course holds also for the standard contact form on S^3.

It is especially interesting to study this "mixing" at the odd indexes $(2k - 1)$. Indeed, Lemma 2.5 of [4] implies that no tangency between the stable manifold of a periodic orbit ofindex $(2k - 1)$ and the unstable manifold of another periodic orbit of the same index may occur. This result holds true through creations and cancellations of periodic orbits, see Lemma 2.13, in Sect. 2.5 of the present paper.

It is natural then to conjecture that, through a creation/cancellation of periodic orbits, the operator ∂_{per} stays unchanged at the odd indexes and the value of $\partial_{\text{per}} \circ \partial_{\text{per}}$ would remain unaltered and a constant homology would be attached to this intersection operator for the odd indexes.

A weaker result turns out to be true. It is stated for cycles c_{2k-1} of ∂_{per} that are "minimal" in the sense that they cannot be decomposed into two non-trivial smaller cycles. We then claim that the following holds; the "Fredholm" assumption of Part I is not required here:

Theorem 1.3 (i) *The Morse relation, with σ a collection of unstable manifolds of dimension $2k$ (therefore of constant classifying map into $\mathbb{PC}^\infty$), $\partial\sigma = c_{2k-1} + h^t_{2k-1,\infty}$, with c_{2k-1} not empty and minimal and with $\partial_{\text{per}} c_{2k-1} = 0$ is impossible.*

(ii) *Let L^+ be the set of curves of C_β or $\cup\Gamma_{2m}$ such that the v-component of their tangent vector is non-negative and non-zero; L^- is defined in a similar way, $\partial L^\pm$ are the "boundaries" of these sets and $J_{\infty,\epsilon}$ is the sub-level set $\{x \in \cup\Gamma_{2m};\ J_\infty(x) \le \epsilon\}$. Assume that $\partial(c_{2k-1} + h^t_{2k-1,\infty}) = 0$, with $\partial_{\text{per}} c_{2k-1} = 0$ and c_{2k-1} not empty. Let $L^\pm_d = W_u(c_{2k-1}) \cap L^\pm$. Assume that the Fadell–Rabinowitz index [13] γ_{FR} of each of L^+ and of L^- is at most $(k-2)$. Then the cycle $(c_{2k-1} + h^t_{2k-1,\infty} - (L^+ \cup L^-), (c_{2k-1} + h^t_{2k-1,\infty}) \cap (\partial L^+ \cup \partial L^- \cup J_{\infty,\epsilon}))$ maps under the (modified) classifying map $(b - \int_0^1 b, \psi(\int_0^1 b))$ onto a generator of the homology group with rational coefficients $H_{2k-1}(\mathbb{PC}^{k-1} \times [-1,1], \mathbb{PC}^{k-1} \times \{-1,1\} \cup \mathbb{PC}^{k-2} \times [-1,1])$. The function $\psi(x)$ is equal to $Min(1,|x|)sgn(x)$.*

(iii) *Assume now that the Fadell–Rabinowitz indexes of γ_{FR} of L^+_d and of L^-_d are both equal to $(k-1)$. Then, the cycle $c_{2k-1} + h^t_{2k-1,\infty}$ defined as in* (ii) *cannot be a boundary in the homology of the pair $(C_\beta, L^+ \cup L^- \cup J^{-1}_\infty(\epsilon))$.*

(iv) *Assume that, at the time zero of the deformation, $\beta = \mathrm{d}\alpha(v,.)$ "turns well" [1] along ξ. Then, $\gamma_{FR}(L^+_d)$ and $\gamma_{FR}(L^-_d)$ are equal and each non-empty cycle c_{2k-1} of ∂_{per} that cannot be written as a sum of two smaller cycles survives the deformation of the contact form.*

We cannot prove the same result for the even indexes. We do conjecture, however, that this conclusion should be true in broad generality.

Let us here go with some detail into the proof of invariance at the odd indexes: let us assume that a creation of two periodic orbits of indexes $(2k+1/2k)$, y_{2k+1}/y_{2k} occurs and some "mixing" of the two operators ∂_{per} and ∂_∞ takes place.

If this occurs, we find that a "rhombus" must arise that mixes periodic orbits and critical points at infinity. The "rhombus" gives rise to the Morse relation:

$$\partial y^{(\infty)}_{2k} = c_{2k-1} + x^{\infty,t}_{2k-1} (*)$$

Here, we pause and we describe with some precision the meaning of each of the terms involved in this relation:

$y^{(\infty)}_{2k}$ stands for a critical point, typically at infinity, of index $2k$. Its unstable manifold is achieved with $(2k+1)$)(not $2k$!) or more $\pm v$-jumps. c_{2k-1} is a cycle for ∂_{per} of index $(2k-1)$. $x^{\infty,t}_{2k-1}$ is a collection of critical points at infinity of index $(2k-1)$ whose unstable manifolds can be followed (collectively) with the use of $2k$ trackable $\pm v$-jumps. They are in particular achieved in $\Gamma_{4k} = \{\text{curves made of } 2k\xi\text{-pieces alternating with } 2k \pm v\text{-pieces}\}$.

We prove that this Morse relation, if it occurs at the time t of the deformation, should also occur at the time zero of the deformation (Lemma 2.14, Sect. 2.5). If c_{2k-1} is a cycle for ∂_{per} at time zero for which $(*)$ does not hold, then c_{2k-1} survives all the isotopy. This conclusion alone, without any further result, allows us to derive, in dimension 3 and under some restrictive assumption on $\beta \wedge \mathrm{d}\beta$ (which can be removed), Rabinowitz's result [29] for the standard contact structure on S^3.

The proof goes as follows: we first establish that $y^{(\infty)}_{2k}$ in $(*)$ cannot be a "shadow critical point at infinity" equal to a $(x_{2k-1} + \delta)^\infty$, i.e. built through Fredholm violation, see [3,4], with the use of a periodic orbit x_{2k-1} of index $(2k-1)$ and the addition of a "positive" or a "negative" "Dirac mass" on it , see [3,4] for more precisions.

This holds true if the Fredholm assumption is violated only along certain portions of the periodic orbit y_{2k-1} and not along all of y_{2k-1}.

Then having ruled out that $y_{2k}^{(\infty)}$ could be a $(x_{2k-1}+\delta)^{\infty}$, we examine $(*)$ at time zero for the standard contact structure $\alpha_0 = x_1dx_2 - x_2dx_1 + x_3dx_4 - x_4dx_3$ on S^3 and we find that the index of the critical points at infinity above periodic orbits of index $(2k+1/2k-1)$ (these come in pairs of such indexes after a slight perturbation of the standard contact structure on S^3) cannot be $2k$ and is higher than $(2k+1)$. Paul Rabinowitz's 35-year-old result follows, under restrictive assumptions and for the three-dimensional case (Lemma 2.14, Sect. 2.5 of this paper).

Before continuing with the existence issues, we explore more the relation $\partial_{\text{per}} \circ \partial_{\text{per}} = 0$ and how it is violated, we address the formation of triangles of dominations $x_m/x_{m-1}^{\infty}/x_{m-2}$. x_i designates here a periodic orbit of index i and x_i^{∞} designates a critical point at infinity of index i as well.

Such a triangle involves in fact a complicated Morse relation x_m/x_{m-1}^{∞}, especially when m is odd, equal to $(2k+1)$. Then x_{m-1}^{∞} is a collection (not a single!) of critical points at infinity y_{2k}^{∞} which is in a "point to circle", see [7], Morse relation with x_{2k+1}. This is explained in more detail in Sect. 2.3 of this paper.

We had observed in [7] that the unstable manifold of each simple elliptic orbit x_{2k-1} contained a copy of the complex projective space $\mathbb{PC}^{k-1}$ and that the classifying map for the S^1-action (for the effective action of S^1 on $C_\beta^* = C_\beta \setminus \{\text{periodic orbits}\}$) was provided by the map b where b is the v-component of the tangent vectors to the curves $x(t)$. Combining these two observations, we derive a map from $W_u(x_{2k-1})/\partial W_u(x_{2k-1})$ into $\mathbb{PC}^{k-1} \times [-1, 1]/(\mathbb{PC}^{k-1} \times \{-1, 1\})$.

Using then the Fadell–Rabinowitz index [13] and this map, we are able to prove that the relation

$$\partial y_{2k}^{(\infty)} = c_{2k-1} + x_{2k-1}^{\infty,t}(*)$$

cannot hold if $\partial_{\text{per}} c_{2k-1} = 0$ and c_{2k-1} is not empty and minimal, see above. It follows that rhombi violating the relation $\partial_{\text{per}} \circ \partial_{\text{per}} = 0$ at the odd indexes do not exist with minimal cycles.

We conjecture that the result $\partial_{\text{per}} \circ \partial_{\text{per}} = 0$ holds in almost full generality at the odd indexes to the least. The proof of this conjecture would involve in a first step the study, at the time zero of the deformation, that is e.g. for the first exotic contact form of Gonzalo and Varela [14], of the Morse relation $\partial y_{2k}^{(\infty)} = c_{2k-1} + x_{2k-1}^{\infty,t}(*)$ described above.

We proceed now with the proof of our claims. This paper is organized as follows:

Let us first describe some of the tools that we are assuming to be known from our previous papers:

We will constantly use Proposition 1, p 469 of [5] that modelizes the unstable manifold of a simple periodic orbit of index m with $m \pm v$- jumps, initially single jumps that can turn into "families" (sequence of consecutive $\pm v$-jumps with the same orientation as the deformation proceeds). The decreasing deformation takes place in $\cup \Gamma_{2k}$, where $\Gamma_{2k} = \{$curves made of k-pieces of ξ-orbits, alternating with k-pieces of $\pm v$-orbits$\}$. The process of "pushing away" a small $\pm v$-jump form the next and the process of "widening an oscillation" is described in [5], Proposition 20, p 519. This process is assumed to be understood in this paper. The Non-Fredholm analysis for the variational problem (J, C_β) near a periodic orbit is summarized in [4], Lemma 16, p 161; see also Sect. 12 of the present paper, an erratum covering minor modifications to the Proof of Lemma 1 of [3], p 26, also to the proof of Lemma 16 of [4], cited above and to Theorem 1.1' of [5], p 567–568. This erratum is inserted here for the convenience of the reader.

More results are assumed to be known from [4,5], e.g. Lemma 3, p 80 of [4]; but we will be recalling and referring to these results as their use becomes necessary.

This paper contains four parts that are independent almost completely. The second part is concerned with the proof of Theorem 1.3 (Sect. 2.1 of Part II). It uses the results of [7] and the Fadell–Rabinowitz index [13]. The third part is very short; it contains two observation. The first one is about S^1-equivariant maps and barycenter spaces on S^3. The second one indicates formal analogies between the contact form framework and the study of the Riemannian Einstein equations. The fourth part is a short erratum correcting minor points in [3–5] that can be of use in this paper. This should help a reader interested also in the details.

The first part is concerned with the proof of Theorems 1.1 and 1.2 and the construction of the family of pseudo-gradients Z_t. It contains 11 sections which we describe now.

In the first section, we will be considering a curve of Γ_{2m+2}, having $(m+1) \pm v$-jumps, near a periodic orbit w of index m. We assume that the $\pm v$-jumps of this curve have been re-ordered along the J_∞-process of "widening", "pushing away" of [5], cited above, so that these $\pm v$-jumps are equally spaced in terms of the v-rotation separating any two consecutive $\pm v$-jumps of this configuration. We then define a decomposition $F^+ \oplus F^-$ at this configuration for $J_\infty''(w)$ restricted to the space of curves or configurations having their $\pm v$-jumps at the same location than the curve we are considering. F^+ is a positive space (of dimension 1) for $J_\infty''(w)$, F^- is a negative space, of dimension m.

The second section is devoted to the proof that, along the decreasing flow-lines of the deformation that start at a periodic orbit, as we "bypass" critical points at infinity to achieve transversality, see Appendix 4 of [5], pp 562–566 using "companions" [5] and transforming the single $\pm v$-jumps of [5], Proposition 1, p 519, into families, we can spare one $\pm v$-jump, one initial $*$ to remain a single $\pm v$-jump. The re-ordering that is performed prior to the use of the decomposition $F^+ \oplus F^-$ of Sect. 2.1 acquires then several properties, see Sect. 2.3 for this.

The third section is devoted to the proof that some sequences of variational dominations, e.g. a periodic orbit of index w_{2k+1} dominates a critical point at infinity of index $2k$, w_{2k}^∞, which in turn dominates a periodic orbit of index $(2k-1)$, w_{2k-1}, are impossible (Lemma 2.8, Sect. 2.3). Other, similar and different flow-lines are ruled out in Propositions 2.2 and 2.3 and Lemmas 2.4 through 2.7 of this section. The general idea is that, along flow-lines coming from a periodic orbit of index m to a critical point at infinity, and from there going to a periodic orbit of index $(m-2)$, will inherit, because of the "steady edges" of the critical point at infinity, some rigidity, including some repetitions in the distribution of their $\pm v$-jumps (which are $(m+1) \pm v$-jumps or families of $\pm v$-jumps provided by the description of the unstable manifold of the periodic orbit of index m, Proposition 2.2 of [5]). This "rigidity" and these repetitions will imply that the resulting configurations are not along the F^+ defined in Sect. 2.1. The conclusion then follows. The results of this section are subtle as they assume that, over the intermediate w_{2k}^∞, that, e.g. a definite choice of a $\pm v$-jump γ can be made amongst the $(2k+1) \pm v$-jumps of $W_u(w_{2k+1})$ so that a repetition in the orientations of the $(2k)$ remaining $\pm v$-jumps exists. This choice can be made if w_{2k}^∞ is a single critical point at infinity. More generally, a coherent choice of γs can be completed unless the relation between w_{2k}^∞ and w_{2k-1} is a Morse relation of the type "point to circle"(Morse-Bott), so that a coherent choice of a family of γs cannot be completed over the domination, see Sect. 2.3 and Sect. 2.11 for more details.

Sections 2.4 is devoted, using the results of Sects. 2.1, 2.2 and 2.3, to the study of the Morse relations of this variational problem, starting from the time zero of the deformation and assuming that no creation or cancellations of **periodic orbits** occur along this path . It follows from the detailed study that these sections contain that a homology involving only periodic orbits can be defined on this special path. Assuming then that a deformation of contact forms can be followed by a "symplectic" or "Fredholm" deformation of pseudo-gradients, Definition 2.9, Sect. 2.4—(see [8] for the removal of this assumption)—we prove that this homology is invariant along this deformation. This homology has a generator in each odd dimension larger than or equal to 3 for the standard contact structure of S^3; it is not zero for a sequence of odd indexes tending to ∞ for the first exotic contact structure of Gonzalo and Varela [14]. Section 2.5 is a detailed study of the process of formation of "rhombi" $x_m/x_{m-1}/x_{m-1}^\infty/x_{m-2}$ that might arise as a creation or cancellation x_m/x_{m-1} of periodic orbits occurs. Over this process, the unstable manifolds of x_m and x_{m-1} change to take the normal form described in Proposition 1 of [5]; in their normal form, they contain as many $\pm v$-jumps as the index of the periodic orbit, m for x_m, $(m-1)$ for x_{m-1}.

Several "rules" (Lemmas 2.13 and 2.14 of Sect. 2.5) hold through the formation of these rhombi. Ultimately, they do not form when $m = (2k+1)$ is odd (Theorem 1.3 of part II). A proof of Rabinowitz result 35-year-old result [29], in dimension 3—the restriction $\frac{\beta \wedge d\beta}{\alpha \wedge d\alpha} \gneq 0$ can be removed—is derived in this section (without the use of Theorem 1.3).

Section 2.6 is devoted to the proof of an abstract deformation lemma needed for the results of Sect. 2.1.

Section 2.7 is devoted to the proof that the cobordisms changing the Γ_{2k}s along the deformation do not change the intersection operator; hence the homology.

Section 2.8, Appendix 1, is devoted to a further, more detailed and technical study of the violation of the Fredholm condition near a periodic orbit. Models are given for curves of the Γ_{2k}s, with large "oscillations"; the curves are in a weak neighbourhood of a periodic orbit. Computations of the variations, including the second-order variations are completed, for further reference and use also.

Section 2.9, Appendix 2, is devoted to the verification of the Palais–Smale condition for the functional at infinity J_∞ (see [2–4], it is the "natural" extension of J that is defined on C_β to $\cup\Gamma_{2s}$) on each Γ_{2k} for a suitable pseudo-gradient. The proof covers the case of the standard contact structure kerα_0 on S^3, with v a vector field of its kernel defining a Hopf fibration of S^3 over S^2 and the case of kerα_1, the first exotic contact structure of Gonzalo and Varela [14], with v the vector field of Vittorio Martino [25].

Section 2.10 provides the arguments for the proof of (ii) of Theorem 1.2. It is extracted from [6], Sect. 9 and it has been included here for the sake of completeness of the present paper.

Section 2.11 provides some detailed arguments required in the proofs of Sect. 2.3, covering the case of multiple dominations and non-transversal dominations of intermediate w_r^∞ acting as a w_{2k}^∞, by $W_u(w_{2k+1})$. The flow-lines of the pseudo-gradients are studied in detail near periodic orbits of elliptic and hyperbolic type. This is based, on the one hand, on the understanding of the H_0^1-flow in the vicinity of these periodic orbits and, on the other hand, on the definition of the attractive and repulsive (variational, ie Morse Lemma type) directions along a periodic orbit for a single $\pm v$-jump. A family of conditions have to be met by a curve supporting s small $\pm v$-jumps along a periodic orbit of Morse index m for this curve to be attracted by this periodic orbit. This provides, therefore, a "normal form" for the stable manifold of these periodic orbits in any slice $\Gamma_{2s}, \forall s$, with a precise count and description of the conditions to be met for the flow-line to reach the periodic orbit. Accordingly, we derive that tangencies between periodic orbits and critical points at infinity do not affect the operator ∂_{per}. Theorems 1.1 and 1.2 follow.

This section allows also to understand that, for "triangles" of dominations $x_{2k+1}/x_{2k}^\infty/x_{2k-1}$ to arise, x_{2k}^∞ must be built with a combination of critical points at infinity—a single one will not "work"—and the Morse relation between x_{2k+1} and x_{2k}^∞ must be of "point to circle" type.

2 Part I: The ∂_{per} and ∂_∞ operators

We prove on this first part Theorems 1.1 and 1.2 stated above and we compute the value of the homology of ∂_{per} starting from the standard contact form and the first exotic contact form of Gonzalo and Varela [14] on S^3. The statements about ∂_{per} and ∂_∞ are general, in between creations and cancellations of **periodic orbits**. The computation of the homology is completed on a path of contact forms that starts from the standard values of these contact forms for these two distinct contact structures. The assumption is that no creation or cancellation of periodic orbits occurs along this deformation.

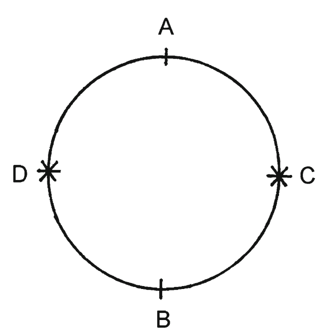
Through such creation or cancellations, the value of the homology is unchanged at the odd indexes (Part II, Theorem 1.3). the proof of Theorem 1.3 is independent of the statements and proofs of Part I.

2.1 Choice of $F^+ \oplus F^-$-representation near a simple periodic orbit

2.1.1 The case of a hyperbolic orbit of index 2, z_2

We take the example of a hyperbolic periodic orbit of index 2, z_2, to understand, given three $\pm v$-jumps located at equal v-rotation one from the next one along this periodic orbit, how we can choose continuously a positive direction in the three-dimensional space/manifold spanned by the curves close t z_2 made of three ξ-pieces and three $\pm v$-jumps located at these precise points.

A hyperbolic periodic orbit of index 2, z_2, comes with four nodes, coupled into two pairs. Indeed, since its Poincare return map reads $\begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}$, $\lambda \gneq 1$, there are two positions A and B along z_2 where v corresponds to the first eigenvector and two positions C and D where v corresponds to the second eigenvector.



From A to A and from C to C, the v-rotation is 2π. From A to B and from C to D, it is π.

Let us consider three "equidistant" locations for three $\pm v$-jumps along z_2; "equidistant" here refers to the equality of the v-rotation in between any two consecutive $\pm v$-jumps in the $+\xi$-direction.

We denote these three locations μ, γ, θ following $+\xi$.

The intervals $[A, C]$, $[C, B]$, $[B, D]$ and $[D, A]$ can be divided into two groups: those starting with a node corresponding to the eigenvalue $\lambda \gtrneqq 1$, namely $[A, C]$ and $[B, D]$, and those corresponding to the eigenvalue $\frac{1}{\lambda} \lneqq 1$, namely $[C, B]$ and $[D, A]$.

We claim that two locations among three for our three $\pm v$-jumps are in two distinct intervals of a given group, whereas the third one is in an interval of the other group. This claim can be easily derived from the recognition of these locations when one of the $\pm v$-jumps is at a node, e.g. at A. Indeed, since the v-rotation from A to C can be considered to be $\frac{\pi}{2}$, as well as the v-rotation from A to D $(\frac{-\pi}{2})$, whereas the v-rotation along $+\xi$ from A to B and from B to A is π, we find that the two other $\pm v$-jumps are located in (C, B) and (B, D) respectively. The claim follows.

Assume that one of the $\pm v$-jumps is located in (A, C), at a point that we denote u and let us build the curve near z_2 having a small positive v-jump of length $c \gtrneqq 0$ at u. Such a curve corresponds at first order to a variation $\delta z_2 = \lambda\xi + \mu v + \eta w$ of z_2. The w-component η of this variation satisfies at first order the equation:

$$\ddot{\eta} + a^2\eta\tau = c\delta_u$$
$$\eta(0) = \eta(1)$$

Thus, the vector $\dot{\eta}v + a\eta w$ is ξ-transported around z_2, from u to u, at first order and we also have

$$(*)\dot{\eta}(0) - \dot{\eta}(1) = ac;\ \eta(0) = \eta(1)$$

The position u corresponds to the times 0 and 1 along the curve. We then claim that $\eta(0) = \eta(1)$ is negative, whereas $\dot{\eta}(0)$ is also negative when u is close to A in (A, C).

These conclusions hold also true when u is in (B, D). These signs reverse when u is in (C, B) or when u is in (D, A). Accordingly, the second derivative of J_∞ along z_2

$$\int_0^1 (\dot{\eta}^2 - a^2\eta^2\tau)\mathrm{d}t = -\int_0^1 (\ddot{\eta} + a^2\eta\tau)\eta\mathrm{d}t = -c\eta(0)$$

is positive when u is in (A, C) or when u is in (B, D); it is negative when u is in (C, B) or when u is in (D, A).

This follows from the computations of [5], p 472. Along these computations, $(*)$ is solved with $c = -1$ at a node and near a node. Replacing ν by λ and η of [5] by $-\eta$, observing that $\lambda \gtrneqq 1$, the claim follows near A in (A, C).

In addition, we know that $\eta(0)$ is never zero when u varies in (A, C) because u is never a node. Therefore, the claim follows for (A, C). A similar argument works for (C, D), etc.

Next, we claim that the v-rotation around z_2, starting from u, is more than 2π in (A, C) and (B, D), whereas it is less than 2π in the two other intervals.

Indeed, assume that it is more than 2π when u is in (D, A). The H_0^1-index of z_2, based at u, is then 2 and the two H_0^1-directions spanning the two-dimensional negative eigenspace are, by construction, $J_\infty''(z_2)$-orthogonal to the third (negative) direction defined by η solution of $(*)$ at u. The negative eigenspace of $J_\infty''(z_2)$ is then of dimension at least three, a contradiction.

The v-rotation around z_2 is thus less than 2π when u is in (D, A). It then becomes more than 2π as u crosses A and the claim follows.

We now claim that, if we solve $(*)$ with u_1 in (A, C) and if we also solve $(*)$ with u_2 in (B, D), then, assuming (without loss of generality, see below) that the total v-rotation around z_2 is always close to 2π, we have

$$-\int_0^1 (\ddot{\eta}_1 + a^2\eta_1\tau)\eta_2\mathrm{d}t \lneqq 0$$

It follows that, for every $s \in [0, 1]$,

$$\int_0^1 (\dot{\eta}_s^2 - a^2\eta_s^2\tau)\mathrm{d}t \gtrneqq 0$$

with $\eta_s = (1 - s)\eta_1 - s\eta_2$.

Indeed, we know that when u_1 is close to A, $\eta_1(0) \lneq 0$ so that $\eta_1(0)$ is negative for every u_1 in $[A, C]$.

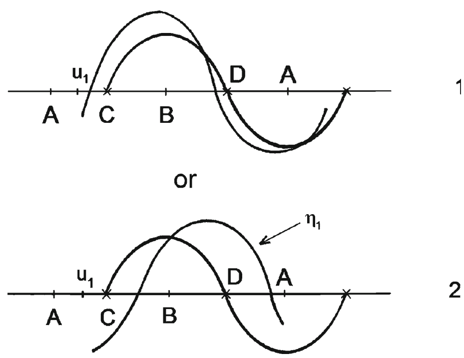
Let t_2 be the time corresponding to u_2, with 0 corresponding to u_1. Then,

$$-\int_0^1 (\ddot{\eta}_1 + a^2\eta_1\tau)\eta_2 \mathrm{d}t = -\int_0^1 (\ddot{\eta}_2 + a^2\eta_2\tau)\eta_1 \mathrm{d}t = -c\eta_1(t_2)$$

This will be negative if $\eta_1(t_2)$ is positive.

We need to understand how the solution η_1 of $(*)$ behaves for u_1 in (A, C).

Let us take u_1 in (A, C), close to C. We use the continuity of η_1 in function of the position of u_1, we use the fact that η_1 is negative at u_1 and we use the fact (derived from the computations of [5], p 472; η should be changed into $-\eta$) that $\dot{\eta}_1$ is positive when u_1 is near C. η_1 behaves as follows:



1- is in fact impossible. The only possible case is 2-, with the first zero of η_1 moving from C to B as u_1 moves from C to A.

It follows that η_1 is positive on the interval (B, D), hence at t_2; hence the claim.

Let us see that 1- is impossible and that the first zero of η_1 moves from C to B as u_1 moves from C to A.

Considering 1-, we claim that the zero of η_1 in (A, C)—it is non-degenerate—should move monotonically from C to A as u_1 moves from C to A. However, $\dot{\eta}_1$ at this zero is positive, whereas $\dot{\eta}_1$ at A is negative (this occurs when u_1 is at A); a contradiction.

Assume now that this zero does not move monotonically. We then have two positions $u_1 \lneq u_1'$ in (A, C), with two distinct solutions η_1 on $[0, 1]$ and η_1' on $[\delta, 1 + \delta]$. $\eta_1 = \eta_1'$ on $[\delta, 1]$. On $[0, \delta]$, η_1 is negative (its first zero is also the first zero of η_1', after u_1' or δ). η_1' and η_1 coincide at $0 = 1$ and at $\delta = 1 + \delta$. They both solve $\ddot{\eta} + a^2\eta\tau = 0$ and η_1 is negative on $[0, \delta]$; η_1 and η_1' (after a backwards shift of time equal to 1 for the latter) have then to coincide on $[0, \delta]$. η_1 does not produce a Dirac mass at δ; therefore, η_1', which is equal to η_1 on $[\delta, 1]$ and is equal to $\eta_1(t - 1)$ on $[1, 1 + \delta]$ does not have a Dirac mass at $(1 + \delta)$; a contradiction.

The claim follows: 1- is impossible.

The same argument implies that the first zero of η_1 moves from C to B as u_1 moves from C to A. $\eta_1(t_2)$ is, therefore, positive.

2.1.2 The case of a hyperbolic orbit of index $2k$, z_{2k}

When z_2 is replaced by z_{2k}, a simple hyperbolic periodic orbit of index $2k$, the above argument extends:

First, there is no loss[1] of generality if we assume that the v-rotation based at any point in z_{2k} is always equal to $2k\pi + o(e^{-k})$, where $o(e^{-k})$ is always a tiny number, positive or negative or zero (if we are at a node).

When we arrange the $(2k + 1) \pm v$-jumps so that the v-rotation on each interval is the same, we then find that this rotation should be $\pi - \frac{\pi}{2k+1} + o(e^{-k})$.

In order to understand how these $(2k + 1)*$s/$\pm v$-jumps distribute among intervals of the type $[A, C]$ or $[C, B]$, we assume that one of these $\pm v$-jumps is located at a node.

[1] What we are really doing here is modifying v along the periodic orbits so that v rotates monotonically of an amount close to $2k\pi$ in between the $4k$ nodes of this hyperbolic orbit This can be completed, with condition (A) satisfied throughout.

Iterating positively, we find that at least k consecutive $\pm v$-jumps are in the same kind of intervals. This follows from the fact that

$$\frac{k\pi}{2k+1} + o(e^{-k}) \lneq \frac{\pi}{2}$$

The same conclusion can be drawn for the negative iterations; but the intervals are of the other kind. We thus conclude that, in general, k consecutive $\pm v$-jumps are in the same type of intervals, e.g. $[A, C]$, whereas $k+1$ other consecutive $\pm v$-jumps are in the other type of intervals such as $[C, B]$.

Starting from the first u_1 in an (A, C) and tracking the next/previous k or $(k-1) \pm v$-jumps at $t_2, \ldots, t_m$, $m = k+1$ or k in the same type of intervals, we find that the sign of $\eta_1(t_j)$ alternates, $\eta_1(t_2)$ being positive, $\eta_1(t_3)$ being negative and so on.

We now build, given a configuration σ of $(2k+1) \pm v$-jumps distributed "equidistantly" along z_{2k} a decomposition $F_\sigma^+ \oplus F_\sigma^-$, with $\dim F_\sigma^+ = 1$, $\dim F_\sigma^- = 2k$. This decomposition should vary continuously with σ. In all the configurations σ, we assume that one $\pm v$-jump, which we denote γ, can be tracked. The $\pm v$-jump that precedes γ in the order defined by ξ is denoted μ.

We choose F_σ^+ so that it never contains a forced repetition (forced by the orientation of the $\pm v$-jumps) in $(\gamma, \mu]$.

We know that we can build a positive direction for $J_\infty''(z_{2k})$ by taking a sequence of alternating k or $(k+1)$ $\pm v$-jumps, depending on σ. The sizes of these $\pm v$-jumps are arbitrary; their orientations are given once the orientation of one of them is given. Observe that the negative eigenspace is always of dimension $2k$: it suffices for this to recognize that one of the $\pm v$-jumps is located at a base point for z_{2k} where the v-rotation across z_{2k} is more than $2k\pi$.

We now take F_σ^+ as we please along this sequence of k or $(k+1) \pm v$-jumps if it does not contain γ in its "middle"; that is this sequence might contain γ, but it is then one of its ends. The jumps of F_σ^+ are of various sizes, but their orientations agree with the orientations of the $\pm v$-jumps of the sequence. They can be taken to be zero and they are always zero at the locations where the $\pm v$-jumps of the sequence are zero.

Whenever γ starts to be included in the sequence—it is then its head—we gradually diminish the sizes of the $\pm v$-jumps that do not correspond to γ, eventually reducing F_σ^+ to γ. Then, F_σ^+ remains equal to $\mathbb{R}\gamma$ as long as the "head" of the sequence (oriented along $+\xi$) has not crossed γ.

Once this head has crossed over, we decrease the sizes of some of the $\pm v$-jumps and, in this way, we gradually retain from the sequence the subsequence that runs from its head to γ; then we suppress γ and so on.

In this way, there never is a forced repetition for F_σ^+ in the interval $[\mu, \gamma)$.

2.1.3 Choice of $F^+ \oplus F^-$ along an elliptic orbit

We will assume here, without loss of generality, that the v-rotation around the simple elliptic periodic orbit of index $(2k-1)$, w_{2k-1}, is $(2k\pi - \theta)$, $\theta \in (0, \pi)$.

Under this assumption, see the proof of Lemma 2.1 below, w_{2k-1} is an absolute minimum in Γ_2. Indeed, at any solution of

$$\ddot{\eta} + a^2\eta\tau = \delta_{\bar{t}}, \eta(\bar{t}) = \eta(\bar{t}+1)$$

we find that

$$J_\infty''(w_{2k-1}).\eta.\eta = -\int_0^1 (\ddot{\eta} + a^2\eta\tau)\eta = -\eta(\bar{t}) \gneq 0$$

If we change $\delta_{\bar{t}}$ into $-\delta_{\bar{t}}$, η changes into $-\eta$, with the same conclusion.

Therefore, with $2k \pm v$-jumps given along w_{2k-1} and equally spaced v-rotation-wise (after the use of the widening flow, see [5], Proposition 20 and Sect. 2.3, below), any location of one of the $\pm v$-jumps may be used as a positive direction F^+.

Once this $\pm v$-jump is chosen, assume that it is located at $\bar{t}$ because the v-rotation at $\bar{t}$ is $2k\pi - \theta$, $\theta \in (0, \pi)$, the H_0^1-index of $J_\infty''(w_{2k-1}).\eta.\eta$ in $H_0^1[\bar{t}, \bar{t}+1]$ is $(2k-1)$ and is represented by the H_0^1-subspace of

$$\bigoplus_{i=1}^{2k} \mathbb{R}\eta_i,$$

where η_i solves

$$\ddot{\eta}_i + a^2\eta_i\tau = -\delta_{\bar{t}_i}, \eta_i(\bar{t}_i) = \eta_i(\bar{t}_i + 1)$$

$(\bar{t}_1, \ldots, \bar{t}_{2k})$ are the locations of the $2k \pm v$-jumps. Denoting $p_-(u)$ the H_0^1-projection onto this H_0^1-negative space, given a configuration $u = \Sigma_{i=1}^{2k} \gamma_i\eta_i$, with $F^+ = \mathbb{R}\eta_1$, we find that

$$p_-(u) = u + \theta\eta_1, \theta \in \mathbb{R}$$

This defines our choice of $F^+ \oplus F^-$ at a given configuration (the choice of E^+ has to be continuous as u varies).

In addition to this choice, which we made explicit, we state now an interesting technical lemma, which we do not use in the present paper, but which may be useful for the construction of other global flows in the future.

Lemma 2.1 *Assume that an ordered configuration u of $2k$ $\pm v$-jumps near a simple elliptic periodic orbit supporting a v-rotation of $2k\pi - \theta$, $\theta \in (0, \pi)$, contains a forced repetition. Then given one of these $\pm v$-jumps S_0, which we choose to represent the positive eigenspace F^+, all the J_∞-decreasing flow-line from u to $p_-(u)$ contains at least two non-zero $\pm v$-jumps.*

Proof of Lemma 2.1 Since u contains a forced repetition, u must contain at least two non-zero $\pm v$-jumps. Denoting w_0 the configuration that yields a v-jump at S_0 we may assume that u reads

$$u = \bar{\alpha} w_0 + \sum_{i=1}^{2k-1} c_i\gamma_i$$

We can assume, without loss of generality that $\bar{\alpha} \geq 0$, for example.

The γ_is yield $\pm v$-jumps scattered at the other $(2k-1)$ locations. The functions $\eta(\gamma_i)$ are zero at S_0; they span the space $H_0^1[0, 1]$, based at S_0.

By construction, see above, $\Sigma_{i=1}^{2k-1} c_i\gamma_i$ is $p_-(u)$.

If $p_-(u)$ contains two non-zero $\pm v$-jumps besides S_0 (maybe), Lemma 2.1 holds.

We, therefore, may assume that $\Sigma_{i=1}^{2k-1} c_i\gamma_i$ contains exactly one $\pm v$-jump besides S_0, so that u reads

$$u = \alpha_1 w_0 + \beta_1 w_1$$

w_1 has a $\pm v$-jump at another location S_1, not at S_0.

We are assuming that u contains a forced repetition so that α_1 and β_1 are non-zero. We can take them to be positive. Then, going along $+\xi$ and assuming without loss of generality (at the expense of changing u into $-u$ if needed) that w_0 corresponds to a positive v-jump, w_1 is a positive v-jump if, strictly in between S_0 and S_1, there is an even number of other locations or $*s$. w_1 is a negative $(-v)$-jump otherwise.

With $F^+ = \mathbb{R}w_0$, $p_-(u)$ then reads $\beta_1 w_1 + \theta_1 w_0$, where θ_1 is computed so that the following holds:

$$\eta(\beta_1 w_1 + \theta_1 w_0)(S_0) = \beta_1\eta(w_1)(S_0) + \theta_1\eta(w_0)(S_0) = 0$$

$\eta(w_0)(S_0)$ is negative, but our argument is independent of this fact. Assuming that $\eta(w_0)(S_0)$ is negative, $\eta(w_1)(S_1)$ is negative if w_1 corresponds to a positive v-jump at S_1; it is positive otherwise.

Going along $+\xi$, we can locate the various $*s$, once S_0 is tracked and recognized, at a v-rotation equal to $\pi - \epsilon$ one from the next one, starting from S_0 and S_{2k-1}. With ϵ small enough, the sign of $\eta(w_1)$ will alternate from S_j to S_{j+1}, starting at S_0 and ending at S_{2k-1}.

It follows (from the fact that there is a forced repetition between w_0 and w_1) that $\eta(w_1)(S_0)$ is positive. Therefore, θ_1 is positive; Lemma 2.1 follows.

In the proof of Lemma 2.1 above, we have left out the proof of the fact that the sign of $\eta(w_0)$ or $\eta(w_1)$ alternates from S_j to S_{j+1}. This follows from the following:

We may assume that the ξ-transport is pure rotation along the elliptic orbit, the total rotation being $2k\pi - \epsilon_1$, ϵ_1 positive small. The ξ-transport matrix in the $(v, -[\xi, v])$-frame around the periodic orbit then reads:

$$\begin{pmatrix} \cos\epsilon_1 & \sin\epsilon_1 \\ -\sin\epsilon_1 & \cos\epsilon_1 \end{pmatrix}$$

w_0 corresponds then to a function η that solves $\ddot{\eta} + a^2\eta\tau = 0$ on $(0, 1)$ and such that

$$\begin{pmatrix} \cos\epsilon_1 & \sin\epsilon_1 \\ -\sin\epsilon_1 & \cos\epsilon_1 \end{pmatrix} \begin{pmatrix} \frac{\dot{\eta}(0)}{a} \\ \eta(0) \end{pmatrix} = \begin{pmatrix} \frac{\dot{\eta}(1)}{a} \\ \eta(1) \end{pmatrix} = \begin{pmatrix} \frac{\dot{\eta}(0)}{a} - 1 \\ \eta(0) \end{pmatrix}$$

It follows that $\eta(0)$ is negative and that $\dot{\eta}(0)$ is positive. With ϵ_1 small positive, η has a first zero at t_1 positive small, then at $t_1 + \pi$, etc. Let us compare with the function $\bar{\eta}$ solving $\ddot{\bar{\eta}} + a^2\bar{\eta}\tau = 0$ in $(0, 1)$ with $\bar{\eta}(0) = 0$, $\dot{\eta}(0) \lesssim 0$.

At the first zero of $\bar{\eta}$, η is positive; at the second zero of $\bar{\eta}$, η is negative and so on.

Our claim then follows from the fact that the locations of the $(2k - 1) \pm v$-jumps distinct from S_0 can be taken to be as close as we please to the zeros of $\bar{\eta}$ (to their left, after orienting the x-axis along $+\xi$).

2.2 Sparing a $*$ as a single $\pm v$-jump

We now establish that, whenever we have m recognizable $*s$, $m \geq 2$, we can arrange so that one of them, which we denote γ and which is a single $\pm v$-jump, remains a single $\pm v$-jump and never grows into a family as we bypass critical points at infinity or periodic orbits of index larger than or equal to m (there is a lack of transversality in the variational problem (J, C_β)).

The transversality issues in the variational problem (J, C_β) have been solved in [5], Appendix 4, pp 562–563 with the use of the New Hole Flow and the introduction of companions to given steady $*s$. The arguments have been detailed on characteristic pieces, but they can be adapted to non-degenerate ξ-pieces as well.

We need here to refine these arguments so that we can "bypass" any critical point at infinity w^∞ of index larger than or equal to m without ever creating companions to γ.

Since transversality can be assumed to hold on each Γ_{2s}, the issue is with the H_0^1-index of w^∞.

If γ corresponds to an edge of w^∞ and there is a "hole", see Appendix 4 of [5], on a ξ-piece of which γ is an edge, we can use the New Hole Flow which is defined starting with the **other** edge of this ξ-piece. This New Hole Flow will never introduce a companion to γ if w^∞ does not have a single edge represented by γ.

If now γ is a $*$ lying within a given characteristic ξ-piece containing a "hole", see [5], Appendix 4, also [6], section 15.1, then the New Hole Flow built starting from a given edge might introduce a companion to γ. If this happens, however, we know that γ has a steady orientation.

Indeed, otherwise, if γ is tiny and faces a "hole" (defined and found after the definition of a starting edge for the New Hole Flow), γ is moved into this "hole", thereby creating a "hole" "behind it" (between the edge and γ) and a companion to another $*$, the $*$ "behind γ", can be introduced in the new "hole" that has been created, provided of course this other $*$ is steady.

If it is not, an induction can be started. It eventually leads to the introduction of another $*$ than γ, "behind γ" and the definition of a decreasing deformation.

If γ is not tiny, then it cannot be moved "inside the hole" and this "hole", therefore, stays on the other side of γ. But then, we can use the **other** New Hole Flow that has the other edge as starting edge. This New Hole Flow will identify the hole in front of γ to be a hole that lies between this other edge and γ. Border-line configurations can be resolved with the introduction of companions to other $*s$ than γ on both sides of γ. The claim about γ thereby holds in this framework.

The argument above breaks down if w^∞ has only one edge, represented by γ. But we claim that this will happen only if the domination of w^∞ is occurring through Γ_2.

Indeed, if the domination is not occurring trough Γ_2, one of the $*s$ inside the ξ-piece of w^∞ is non-zero. We denote this $*$ $*_1$. $*_1$ is not tiny.

Starting from the large edge γ to the left (γ lies both at right and at left), we can "push away" the $*s$ one from the next one so that the v-rotation between them is larger than equal than a number smaller than π, but close to π. We complete this process until we reach $*_1$. This will never reverse the orientation of $*_1$.

Once this widening, under this form, is completed, we can use, if there is an "open hole" to the left of $*_1$, the New Hole Flow starting from $*_1$, which we take to be the starting edge and we can decrease the configuration.

If there is no "open hole" to the left of $*_1$, then there must be one, without any use of the widening flow [5], Proposition 20, to the right of $*_1$.

Between $*_1$ and this open hole, there might be several $*s$. If they are tiny or zero, we can move them into the "open holes' that they are facing (see [5], p 563) and we can fill the hole. Another hole then opens and it remains open.

Exhausting the $*s$ to the right of $*_1$ one after the other one, we find that there is "an open hole" with a $*$ immediately to its left that has a steady orientation and is not tiny. We can introduce a companion to this $*$ inside the hole and decrease the configuration.

We thus see that all these configurations can be decreased without ever introducing a companion to γ[2].

[2] This holds true as γ exits a ξ-piece.

Similar arguments may be used for transversality of flow-lines between periodic orbits. Finally, as we bypass periodic orbits of index m, starting from periodic orbits or critical points at infinity of index $m+1$ to which $(m+1)*$s may be associated, families can be rebuilt through the process and γ can be kept to be a single $\pm v$-jump.

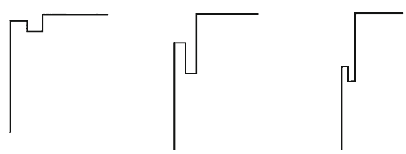
2.2.1 As γ exits a characteristic ξ-piece

The choices which we have made above for γ might have to be changed if a $*$ exits a ξ-piece. We will assume that this ξ-piece is characteristic for the sake of simplicity.

We claim that over this process we have to cross configurations of $*s$ on $W_u(x_m)$ that will "bypass" the critical point at infinity without the use of any companion.

For these configurations, there is no need to choose a γ yet; they are on direct flow-lines originating in w_m, and they have never reached a critical point at infinity. The choice of γ can be completed across these configurations and can be made to fit the new distribution of $*s$ over the various ξ-pieces of the critical points at infinity.

Similar phenomena, albeit more complicated on the surface, can occur as the $*$ corresponding to γ, keeping its orientation that might be opposite to the orientation of the edge, exits through this edge with the "help" of a third $*$. This third $*$ would then have the orientation of the edge and would become larger and larger.



We can build our deformation so that this never happens on the flow-lines originating in w^{∞}_{2k+1}. Indeed, as long as the $*s$ are represented by small $\pm v$-jumps (maybe multiple), we may assume that they are "spread", with a v-rotation separating them bounded away from zero.

Also γ, if it has the opposite orientation to the edge and if it has all the other $*s$ "far" rotation-wise from it, cannot exit from the edge and can be assumed to be "far" from this edge.

It must, therefore, be that some of the inside $*s$ must grow (in a given level set of the functional; all this process is happening in the space of configurations, in a level surface close to the level surface of w^{∞}_{2k+1})if γ, with the opposite orientation to the edge, is to exit through this edge.

The size to which they must grow does not depend on the level surface or on how close the configuration must be to w^{∞}_{2k+1}. It depends only on the process of "pushing away" $\pm v$-jumps one from the other. This process is embedded in the decreasing deformation.

It follows that the $*s$ that grow can be assumed to be "small but sizable", that is, they are small enough so that the "pushing away" can still take place; but they have become sizable so that if their sizes double, this very process cannot take place or γ would then reverse orientation through it.

Under such circumstances, these other $*s$ can become close to each other and to γ and γ can exit.

But, if this is to happen, J'_{∞} is not small at this configuration and some easy additional arguments prove that we have now left a neighbourhood of w^{∞}_{2k+1} and the configuration is then below the level of w^{∞}_{2k+1} by a large amount. We are not anymore in the prescribed level surface, close if below the level of w^{∞}_{2k+1}. The claim follows.

A cycle w^{∞}_{2k+1} could be created though in the reverse way, that is around an edge that would be formed by two consecutive $*s$ (an identifiable repetition) and this cycle would "end" on each side of the edge, if the two ξ-pieces are, e.g. characteristic because one of the $*s$ defining the edge would become a decreasing normal, see [5], pp 482–484. In such a case, either a second repetition is available as the edge is formed by two $*s$ and more. The transition between the two large chains of the cycle is achieved by configurations with $(2k-2)$ or less sign-changes in the orientations of their $\pm v$-jumps. This will be an "easier" case to handle. Or, this is a second case, the only repetition is this repetition and we have to find an "outside" γ. This γ is not difficult to find if the critical point at infinity has more than two characteristic ξ-pieces. The other cases are discussed in Sect. 2.11. We will assume for the time being that γ is defined without ambiguity over the cycle defined by w^{∞}_r, unless the number of sign-changes drops below $(2k-2)$.

2.3 Bypassing periodic orbits coming from infinity

The advantage to have γ to be a single $\pm v$-jump can be seen when the re-ordering is completed near a periodic orbit. Typically, if this periodic orbit is of index m, then we want to re-order along this periodic orbit $(m+1)*$s that might have turned into families.

These $(m+1)*$s come typically, directly or indirectly, see below, from the $(m+1)*$s associated with the unstable manifold of a periodic orbit of index $(m+1)$. When the periodic orbit is hyperbolic, then its unstable manifold, see [5], Proposition 1, p 469, is parametrized by $(m+1)*$s. When it is elliptic, then we can parametrize its unstable manifold with $(m = 2k)$ with $(2k+1) \pm v$-jumps or $(2k+2) \pm v$-jumps (not $*s$: for $m+1 = 2k+1$, the maximal number of sign-changes between these $\pm v$-jumps is $2k$), depending on whether we take the total v-rotation around this, simple, elliptic periodic orbit to be $(2k+1)\pi - \theta$ or $(2k+1)\pi + \theta$, $\theta \in (0, \pi)$.

Using the arguments of [5], p 473 and [4], p 85 and p 87, slightly extended to change also a simple hyperbolic orbit of odd index into an elliptic orbit (same argument than in [5], p 473, with a further adjustment of the ξ-transport map along the periodic orbit to turn it into pure rotation as the nature of the orbit changes; the orbit does not degenerate, but its iterate of order two does), we may assume that all simple periodic orbits of odd index are elliptic and that all hyperbolic orbits have even index.

Elliptic periodic orbits might have iterates that have an even index. However, for a given iterate, we can adjust the rotation of the prime elliptic orbit, without ever crossing $s\pi$, $s \in \mathbb{N}$, so that this iterate degenerates and is replaced in the arguments by a simple periodic orbit of even index. We may then assume, by the arguments above, that this periodic orbit is hyperbolic. These "adjustments/modifications" are assumed to have been completed in all our arguments below.

2.3.1 The unstable manifold of a dominating simple elliptic periodic orbit

We can choose, see [5] p 473 and [4] Proposition 15 p 85 and Lemma 7 p 87 , any of these values as we please, the contact form is changed, but not the homology. The change in the amount of rotation is completed, whereas the periodic orbit does not degenerate. Therefore, no critical point at infinity is created over this process.

In the sequel, we will take the v-rotation around a simple elliptic periodic orbit of index $(2k+1)$ that is **dominating** to be $(2k+1)\pi - \theta$, thereby creating a model for its unstable manifold with $(2k+1)*$s, whereas we will take this rotation to be $(2k+1)\pi + \theta$ if this simple elliptic periodic orbit is **dominated**.

Of course, an elliptic periodic orbit can be at the same time dominating and **dominated**. However, if we seek to understand the homology at some precise index, we will encounter one single type, dominating or dominated, not both.

2.3.2 Re-ordering $*s$ around a periodic orbit with the use of γ

We are now given $(m+1)*$s around a periodic orbit of index m. The v-rotation around this periodic orbit does not exceed $2m\pi + \theta$, $\theta \in (0, \pi)$.

There is barely enough room to re-order these $(m+1)*$s so that the v-rotation between two consecutive $*s$ is given, some number in $(\frac{\pi}{2}, \pi)$.

Since γ is a single $\pm v$-jump, we can start the re-ordering process by "pushing away" all the other $*s$, away from γ, so that the v-rotation from γ to its next neighbour is larger than or equal to a number less than π, but close to π. This "pushing away" never reverses the orientations of the next neighbours or the other $*s$, besides γ, so that any forced repetition taking place in the complement of γ is preserved. The next step, until the final step, in the re-ordering process does not involve γ. It involves only the other $*s$. Through such a process, a forced repetition cannot disappear: if one $*$ changes orientation through this process, then the next $*$ has the **reverse** orientation, and it is non-zero, so that the forced orientation maintains, with its edge $*s$ maybe changed (γ is never such an edge).

The arguments for re-ordering/rearrangement can be more involved as we consider the most general form of critical point at infinity w_r^∞ that could be dominated by $W_u(w_{2k+1})$; also multiple dominations may occur. This is discussed in Sect. 2.11 of this paper. We will see that, over a sequence $w_r - w_s^\infty - w_{r-1}$, involving one domination (with difference of Morse indexes equal to 1) and one tangency, w_s^∞ being a critical point at infinity, the intersection number $i(w_r, w_{r-1})$ does not change over a decreasing deformation that is globally defined, whereas for some sequences of dominations, $w_{2k+1} - w_{2k}^\infty - w_{2k-1}$, such a conclusion does not hold for certain configurations. The repetition in the orientations of the $\pm v$-jumps of the configurations and the value of the exterior γ cannot be assigned to specific sub-intervals of the full circle of the $(2k+1) \pm v$-jumps over these configurations.

In the next section, Sect. 2.4, we will be assuming that the value of γ is defined without ambiguity over the deformation classes; alternatively, we could also assume that the value for γ might change, but that over these changes re-arrangement of the $(2k+1) \pm v$-jumps can be completed as γ changes value, whereas the repetition(s) in the complement of these value(s) of γ are spared. Under such an assumption, we derive that the intersection numbers do not change. This conclusion, as stated above, holds in full generality for sequences involving a domination and a tangency, see Sect. 2.11.

2.3.3 Bypassing simple elliptic orbits coming from infinity

Also, and this is useful in the re-ordering process around a simple elliptic orbit of index $(2k-1)$ (dominated so that the v-rotation around it is $(2k-1)\pi+\theta, \theta \in (0,\pi)$), if a $\pm v$-jump or $*$ which is not γ is steady, then, after this re-ordering process is completed as above, one of the $\pm v$-jumps besides γ (which might or might not be steady) is steady.

Let us now imagine that the unstable manifold of a critical point at infinity w_{2k}^{∞} is parametrized by $2k*$s, one of them-γ-being reduced to a single $\pm v$-jump. Let us assume that w_{2k}^{∞} has more than one edge; or, if w_{2k}^{∞} is in Γ_2, that γ does not represent its edge.

We then claim that

Proposition 2.2 *Assume that the unstable manifold of w_{2k}^{∞} is parametrized with the use of $2k \pm v$-jumps. Then, the intersection number of w_{2k}^{∞} with the simple periodic orbits of index $(2k-1)$ is zero*

Proof We first recall,see above, that we may assume, see [5], p 473, also [4], p 85 and p 87, that all these simple periodic orbits of index $(2k-1)$ are elliptic. Being then considered as dominated, the v-rotation around them is $(2k-1)\pi+\theta, \theta \in (0,\pi)$.

Considering a configuration σ on $W_u(w_{2k}^{\infty})$, we know that it has $2k*$s. One of them is γ; it is a simple $\pm v$-jump. There is at least another $*$ that is steady.

After completing the re-ordering process as above, the $2k*$s are distributed over $2k \pm v$-jumps equidistant from each other. γ is one of them. There is another $\pm v$-jump, γ_1 that is non-zero.

We can take $\mathbb{R}\gamma$ to be E^+ for this configuration σ_1 (σ re-ordered). Since there another non-zero $\pm v$-jump, $p_-(\sigma_1)$ is non-zero. All the configurations of $W_u(w_{2k}^{\infty})$ can be moved away downwards and the claim about the intersection number follows. □

Considering now the case when w_{2k}^{∞} belongs to Γ_2 and dominated by a w_{2k} so that its unstable manifold is parametrized by $2k*$s (this will occur over a tangency $w_{2k} - w_{2k}^{\infty}$, along a deformation; an additional parameter is needed), we observe that the index at infinity of w_{2k}^{∞} should then be zero since $\dim(W_u(w_{2k}) \cap \Gamma_2 = 1$. Its H_0^1-index is then $2k$. This yields $(2k+1) \pm v$-jumps in the representation of its unstable manifold: an additional $\pm v$-jump is required for the large edge.

Because the index at infinity is zero, any flow-line from w_{2k}^{∞} to w_{2k-1} will involve a non-zero H_0^1 direction. Choosing γ to be the edge, the previous argument extends since there is a non-zero $\pm v$-jump outside of γ. The claim follows.

Let us now turn to the case when we have not $2k$, but $(2k+1)*$s near a simple periodic orbit of index $2k$ or $(2k-1)$:

2.3.4 Bypassing simple periodic orbits coming from infinity: the three-edges rule

Let us now assume that we are considering flow-lines supporting $(2k+1)*$s and that these $*s$ are $\pm v$-jumps. Let us assume that these flow-lines dominate w_m^{∞}. They can then be considered, over the lack of transversality, to dominate some disk D^r, of dimension $2k$ or $(2k+1)$ in the unstable disk of w_m^{∞} (the index m of w_m^{∞} might be larger than $2k$ or $(2k+1)$).

Let us assume that w_m^{∞} has three edges and that these three edges are represented by three distinct $*s$. We then claim

Proposition 2.3 *γ can then be chosen so that a forced repetition takes place in its complement. The intersection number of D^r with any simple periodic orbit of index $r-1$, w_{r-1} is zero.*

Proof Indeed, let us label $*_1$, $*_2$ and $*_3$ these three distinct $*s$, in the $+\xi$-order of w_m^{∞}, starting from $*_1$. Any two of these $*s$ define two intervals among the $(2k+1)*$s, which may be viewed on a circle. Given the orientations of the corresponding edges, one of these intervals harbors a forced repetition. γ should be chosen in the other interval. Therefore, $*_1$ and $*_2$ should be consecutive $*s$ among the $(2k+1)*$s; the corresponding edges should have reverse orientations. Otherwise, a γ can be found.

Using the same arguments, $*_2$ and $*_3$ should be consecutive, as well as $*_3$ and $*_1$. Then $k = 1$. But $*_3$ and $*_1$ cannot have reverse orientations and the claim follows.

Once γ is chosen and the forced repetition is identified, γ is spared as a single $\pm v$-jump and the forced repetition survives. After re-ordering around a simple periodic orbit of index $2k$ w_{2k} and using the choice for F_{σ}^{+} completed above, see Sect. 2.1, we derive that the intersection number with w_{2k} of D^r $(r = 2k+1)$ or w_{2k+1}^{∞} is zero.

The argument is more subtle because the projection onto F^- (see the choice of $F^+(\sigma) \oplus F^-(\sigma)$ in the first section) does not necessarily respect the existence of a repetition outside of γ in the case of a simple hyperbolic periodic orbit.

Therefore, whereas the conclusion about the intersection number holds for the **first** hyperbolic periodic orbit of index $2k$ w_{2k} encountered along these flow-lines, it is harder to prove that it holds for the other w'_{2k} below. The precise argument is made in section 6, below. We will proceed with the claim for now.

In the case of an elliptic orbit w_{2k-1}, of index $(2k-1)$, the forced repetition in the complement of γ implies that, once the re-ordering process (completed first by bringing all other $*s$ away from γ) is completed and the $*s$ are reduced to $2k*$s, one of the other $*s$ besides γ is non-zero. Indeed, after a first re-ordering, we find that two of the $*s$ that are not γ are very close and all these $*s$, with the pair confounded (ie considered to be at one location) are at the positions where they should be for $(2k-1)*$s (pair confounded, $*s$ outside of γ; the pair is a pair of consecutive $*s$ distinct from γ, chosen as we please). The orientations of the $2k*$s outside of γ imply a forced repetition. If the pair of $*s$ that are close have the same orientation or if one or both are zero, we can make the pair collapse into one single $*$. This $*$ is either a non-zero $*$ and we are done; or it is a zero $*$ and, under our assumptions (forced repetition), there must be another non-zero $*$ that is not γ. Again, we are done.

If the pair has opposite orientations, then the immediate neighbours must build with this pair an alternating sequence; otherwise, we can use a neighbour to collapse the pair and have a non-zero $*$, which can be found then among the immediate neighbours. But, then, since there is a forced repetition, there must be another $*$ that is not γ, not in the pair and not an immediate neighbour of the pair that is non-zero. The pair can be collapsed and one non-zero $*$ besides γ still exists. The claim about the intersection number then follows from the fact that E^+ is of dimension 1, see Sect. 2.1.

However, the same issue that we have encountered for hyperbolic orbits w_{2k} holds for the intersection number with the simple elliptic periodic orbits w_{2k-1}. Whereas the conclusion holds obviously for the first one encountered w_{2k-1}, it is harder to establish for the next one w'_{2k-1}.

We are helped here by the almost explicit form of the projection p_- on F^-. As stated in Sect. 2.1 (choice of $F^+ \oplus F^-$ along an elliptic orbit), given a configuration u represented by $\Sigma_{i=1}^{i=2k}\gamma_i\eta_i$ and choosing F^+ to be $\mathbb{R}\eta_\gamma$, η_γ representing the function η for the $*\gamma$ of choice to be a single $\pm v$-jump, then

$$p_-(u) = \eta(u) + \theta\eta_\gamma, \theta \in R$$

It follows that, from u to $p_-(u)$, the sequence of $\pm v$-jumps outside of γ is unchanged.

The re-ordering defined above and "centred" around γ starts with $(2k+1) \pm v$-jumps and a repetition outside of γ. As explained above, in the vicinity of a simple elliptic periodic orbit w_{2k-1}, the $(2k+1)*$s reduce to $2k*$s, or rather stay $(2k+1)*$s, with two $*s$ being very close, quasi-confounded and the repetition outside of γ subsiding.

We can define p_- for such a configuration u-with $(2k+1)*$s, 2 different from γ being very close, as an H_0^1-projection, that is

$$p_-(u) = \eta(u) + \theta\eta_\gamma$$

so that $\eta(u + \theta\gamma)$ is zero at the time $\bar{t}$ corresponding to γ.

Clearly, when two of these $2k*$s, distinct from γ, are very close, the deformation (as t increases from 0 to 1), $(1-t)u + p_-(u)$ decreases J_∞ and preserves the forced repetition outside of γ.

The induction from w_{2k-1} to w'_{2k-1} can then proceed as the forced repetition outside of γ is not destroyed by the use of the pseudo-gradient related to the $E^+ \oplus E^-$ decomposition near w_{2k-1}. □

2.3.5 Restrictions on flow-lines of Γ_2 and of Γ_4

Let us consider a generic deformation of the variational problem (J_t, C^t_β). Let w^t_{2k+1} be a simple dominating periodic orbit of odd index $(2k+1)$ that we track over this deformation. It is then, according to our assumptions above, an elliptic periodic orbit and its unstable manifold is described by $(2k+1)*$s.

Assume that, over a time t_0 of the deformation, w^t_{2k+1} dominates a critical point at infinity w^∞_m of index m, that is

$$W_u(w^\infty_m) \subset \overline{\underset{t\in[t_0-\epsilon,t_0+\epsilon]}{\cup} W_u(w^t_{2k+1})}$$

The notations w^∞_m and $W_u(w^\infty_m)$ cover here the case when the index of w^∞_m is more than m, but we would be only considering a subspace of dimension m (of decreasing flow-lines) in its unstable directions.

We then claim that

Lemma 2.4 *Assume that t_0 is a generic time for the deformation.*

(i) *If w^∞_m has one large $\pm v$-jump and the domination $w^{t_0}_{2k+1} - w^\infty_m$ is through Γ_2, then $m \lneq 2k+1$.*

(ii) *If w^∞_m has two large $\pm v$-jumps of opposite orientations, if the domination $w^{t_0}_{2k+1} - w^\infty_m$ is through Γ_4 and if there is one ξ-piece of w^∞_m that does not support any $*$ of w^t_{2k+1}, then $m \lneq 2k+1$.*

Proof $W_u(w^{t_0}_{2k+1}) \cap \Gamma_2$ is of dimension 1, so that the index at infinity i_∞ of w^∞_m must be zero in case (i). It follows that, if we argue by contradiction, the H^1_0-index of w^∞_m must be $m \geq 2k+1$. This implies that the maximal number of zeros for b ([3,4]) on $W_u(w^\infty_m)$ is $(2k+2)$ to the least.

On the other hand, the maximal number of zeros for b on $W_u(w^t_{2k+1})$ is $2k$. Domination cannot take place and (i) follows.

Turning now to (ii), we see that the index at infinity i_∞ of w^∞_m can be either 1 or 0.

If it is zero, the H^1_0-index of w^∞_m is $2k+1$ (if we argue by contradiction, again). Since we are assuming that there is no $*$ on one of the ξ-pieces separating the two large $\pm v$-jumps of w^∞_m, all this H^1_0-index must be supported by the other ξ-piece and the maximal number of zeros of b on $W_u(x^\infty_m)$ must be $2k+2$.

If i_∞ is one, we reach the same conclusion. Again, the maximal number of zeros of b on $W_u(w^{t_0}_{2k+1})$ is $2k$ and (ii) follows. □

Observe that either the assumptions of (i) or (ii) of Lemma 2.4 above hold, or w^∞_m has three distinct edges. Therefore,

Lemma 2.5 *Assuming we have a tangency $w_{2k+1} - w^\infty_{2k+1}$, the intersection number $i(w^\infty_{2k+1}, w_{2k})$ is zero.*

Lemma 2.6 *Let w_{2k+1} be a simple elliptic periodic orbit of index $(2k+1)$ and let w_{2k-1} be a simple elliptic periodic orbit of index $(2k-1)$. Let w^∞_{2k} be a critical point at infinity of index $2k$ (in the generalized sense described above) and assume that w_{2k+1} dominates w^∞_{2k} and that w^∞_{2k} dominates w_{2k-1}.*

(i) *If the domination of w^∞_{2k} by w_{2k+1} is through Γ_2, then the index at infinity i_∞ of w^∞_{2k} is zero.*

(ii) *If the domination of w^∞_{2k} by w_{2k+1} is through Γ_4, if the two large $\pm v$-jumps of w^∞_{2k} have opposite orientation and if no $*$ of w_{2k+1} separates them along the flow-lines of the domination, then the index at infinity i_∞ of w^∞_{2k} is 1.*

Proof (i) is straightforward. For (ii), i_∞ is one or zero since $\dim W_u(w_{2k+1}) \cap \Gamma_2 = 2$. If i_∞ is zero, then the H^1_0-index of w^∞_{2k} is $2k$. Since no $*$ of w_{2k+1} separates the two large $\pm v$-jumps of w^∞_{2k}, there is one ξ-piece of w^∞_{2k} with zero H^1_0-index (the actual H^1_0-index of this ξ-piece might not be zero, but it is not used in the representation of $W_u(w^\infty_{2k})$ under the domination by w_{2k+1} since no $*$ of w_{2k+1} lives on this ξ-piece). Thus, the other ξ-piece has an H^1_0-index equal to $2k$. Then the maximal number of zeros of b on $W_u(x^\infty_{2k})$ is $(2k+2)$, whereas it is only $2k$ on $W_u(w_{2k+1})$, a contradiction. (ii) follows

□

Let us now consider a w^∞_{2k} of index $2k$, of index at infinity $i_\infty = 1$. Let us assume that the curve supporting w^∞_{2k} is a curve of Γ_4, having two $\pm v$-jumps of opposite orientations.

Let us assume that a simple elliptic periodic orbit of index $(2k+1)$ w_{2k+1} dominates w_{2k}^{∞} and that, along the flow-lines of this domination that we will be considering, the two $\pm v$-jumps of w_{2k}^{∞} are represented by two **consecutive** $*s$ of w_{2k+1}.

Let w_{2k-1} be a dominated simple periodic orbit of index $(2k-1)$.

We then claim that

Lemma 2.7 (i) *Assuming that either the Fredholm assumption is violated along* w_{2k-1} *or that* $k \geq 2$, *we may define our pseudo-gradient so that no flow-line at infinity (ie involving the unstable direction at infinity) will connect* w_{2k}^{∞} *and* w_{2k-1}.

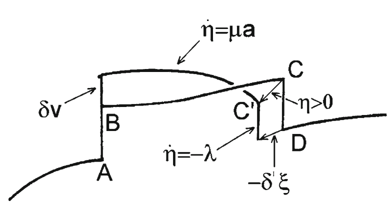
(ii) *Along the other flow-lines originating at* w_{2k}^{∞}, *the choice of a* $*\gamma$ *that is a simple* $\pm v$*-jump with an additional outside repetition (ie not involving* γ*) can be made so that no such flow-line will connect* w_{2k}^{∞} *and* w_{2k-1}. *Discontinuities in the choice of* γ *correspond to a drop in the number of sign-changes between the* $(2k+1)*s$ *below* $2k$, *to* $(2k-2)$ *or lower.*

Proof For (ii), we observe that, once we exclude the flow-lines at infinity originating at w_{2k}^{∞}, we find flow-lines that involve H_0^1-directions, that is flow-lines that involve at least a third non-zero $*$. Each additional non-zero $*$ defines two distinct intervals among the original $(2k+1)*$s once we take into account the two consecutive $*s$ representing the edges of w_{2k}^{∞}. Unless the number of sign-changes for b drops to $(2k-2)$ or less, only one of these intervals contains a forced repetition. Furthermore, if two or more $*s$ besides the two consecutive $*s$ of the edges are non-zero, the intervals of forced repetitions must be nested one in the other one, unless again the number of sign-changes drops to $(2k-2)$ or less. Therefore, the choice of a $*$ to be γ, not involved in the forced repetition, can be made in a continuous manner over all the configurations supporting a number of sign-changes equal to $2k$. This $*$ will be taken to be the $*$ of one of the edges.

For (i), we will use two arguments:

The first one uses the Fredholm violation along w_{2k-1}, which therefore will be assumed to occur along w_{2k-1}. By the results of [6], we know that this holds always true for the first exotic contact structure of Gonzalo and Varela [14].

We observe that $i_\infty = 1$, so that we are considering at most two flow-lines, originating at w_{2k}^{∞}, in Γ_4, ending at w_{2k-1}. We want to use the Fredholm violation along w_{2k-1} and modify these flow-lines so that they do not reach w_{2k-1} anymore. We may assume that the ξ-transport near w_{2k-1}, which is an elliptic periodic orbit, is pure rotation, so that J_∞ is invariant under such a rotation. Using this rotation, we locate the small positive v-jump of such a flow-line near w_{2k-1}, where we please along w_{2k-1}. Assume that the ξ-piece of w_{2k}^{∞} and of these curves that does not support inside $*s$ runs from the positive v-jump of x_{2k}^{∞} to the negative one. Under this assumption, we locate the positive large v-jump of w_{2k}^{∞} on a portion of the periodic orbit w_{2k-1} where the Fredholm violation is occurring along "positive Dirac masses" along v, that is along forth and back runs along v. Under the other symmetric assumption, we locate this positive v-jump on a portion of the periodic orbit w_{2k-1} where the Fredholm assumption is occurring along "negative Dirac masses" along v, or back and forth runs along v. We now use the "exterior" (exterior with respect to the ξ-piece supporting no $*$) $(2k-1)*$s and we push the small negative $(-v)$-jump towards the positive one (along this very ξ-piece) so that the v-rotation separating them is less than π. As we complete this process, we might exit Γ_4, since some of the $(2k-1)*$s that are zero might become non-zero. This might happen, but the number of sign-changes remains 2 and we are still considering two (at most) single flow-lines that we want to modify using the Fredholm violation. Assume now that the ξ-piece supporting no $*$ runs from the positive v-jump to the negative one. Expanding then the positive v-jump, we can decrease J_∞ and bring the negative $-v$-jump closer to the positive one:



that is, picking up δv, $\delta \gneq 0$ at B and ξ-transporting it to C, we find at C a vector having its $(-[\xi, v])$-component η to be positive. This $(-[\xi, v])$-component can be compensated by taking $-\delta'\xi$ at D, $\delta' \gneq 0$ and v-transporting it to C.

This process is J_∞-decreasing.

Over this process, either the negative $(-v)$-jump becomes very close to the positive v-jump. We can use the Fredholm violation, see Sect. 2.4 below and Appendix 1 in Sect. 2.8 to this paper and derive (i) under the assumption that Fredholm violation is occurring along w_{2k-1}. This is done as follows: the positive and the negative $\pm v$-jumps, are close, separated by a ξ-piece of small length δ. This is the beginning of a positive "Dirac mass" that we can expand, raising its size to s, s an increasing parameter. The curve is denoted x_δ. Using the expansion in Appendix 1, Sect. 2.8, (the Fredholm violation is occurring at $x_\delta(t_0)$):

$$J(x_\delta) = J(w_{2k-1}) + \delta(1 - \alpha_{x_\delta(t_0)}(D\phi_{-s}(\xi))) + O(\delta^2) = J(w_{2k-1}) + \delta\gamma(s) + O(\delta^2)$$

The expansion can be differentiated, see Appendix 1. We then choose δ as a function s, so that

$$\dot{\delta}\gamma + \delta\dot{\gamma} \le -c\delta$$

$c \gneq 0$ is a given, fixed and appropriate constant. When s is small, γ increases with s (see Sect. 2.9, Appendix 1). We then decrease δ appropriately so that the above inequality holds. This decrease continues as long as γ is positive, bounded away from zero. Eventually, because we are assuming that Fredholm violation is occurring at $x_\delta(t_0)$, γ starts to decrease and crosses zero. Then, $\dot{\gamma}$ is negative, bounded away from zero and the above inequality holds, with $\dot{\delta} = 0$. When γ crosses zero and Fredholm violation occurs, $J(x_\delta)$ becomes less than $J(w_{2k-1})$ and (i) follows.

Resuming our argument, it might also happen that the negative $(-v)$-jump becomes tiny. Decreasing a bit the positive v-jump decreases J_∞ and allows to move also the tiny $(-v)$-jump near the positive one. (i) again follows, under the same assumption.

If the Fredholm assumption is not violated along w_{2k-1}, we will assume that $k \ge 2$, so that w_{2k-1} is of index 3 to the least. Coming back to our argument above, we have now two $\pm v$-jumps, one positive and the other one very close, negative, tiny. We can move the negative v-jump from the positive one, scaling the positive one so that the process is J_∞-decreasing, reaching then a position such that the v-rotation along the ξ-piece between the two $\pm v$-jumps is less than 2π, but close to 2π. Since w_{2k-1} is at least of index 3, this is possible. It is not difficult to see then that the two $\pm v$-jumps can grow, whereas J_∞ decreases. (i) follows.

The construction above might seem to be not very well defined since the flow-lines originating at the curve supporting w_{2k}^∞ depend on the distribution of $*s$ of w_{2k+1} relative to the large $\pm v$-jumps of w_{2k}^∞.

However, the w_{2k}^∞s involved here should be considered as different depending on this distribution of $*s$, since they correspond to different H_0^1-subspaces of the unstable manifold of the critical point(s) at infinity defined by the underlying curve. The various flow-lines can then be combined (see the Sect. 2.2 about $*s$ travelling across large edges and the related drop in the number of zeros for b) into a global flow. □

If w_{2k}^∞ is in Γ_2 and is dominated by a periodic orbit w_{2k+1}, with a $w_{2k+1} - w_{2k}^\infty$ domination in Γ_2 (otherwise, the previous argument applies), then the index at infinity of w_{2k}^∞ is zero since $\dim W_u(w_m) \cap \Gamma_2 = 1$, see the proof of Proposition 1, p 469 in [5] (there could be exceptional times, where domination of critical points at infinity w_{2k}^∞ of index at infinity equal to 1 would occur, in principle. However, the domination $w_{2k+1} - w_{2k}^\infty$ remains a transversal domination at those times, because it does not occur in Γ_2 , but in a larger $\bigcup_{s=1}^{2k+1} \Gamma_{2s}$. Therefore, domination does occur before and after those times; the index at infinity of w_{2k}^∞ does not change at these special times)

Therefore, if w_{2k}^∞ dominates w_{2k-1}, this domination involves a non-zero H_0^1-direction. We have $(2k+1)*$s, one of the $*s$ is the $*$ of the edge, another $*$ is the $*$ of the H_0^1-direction that is non-zero. If these two $*s$ are not consecutive, then we can proceed as in the case of the three edges rule, choose a γ and a forced repetition outside of it. The various choices can be made coherent, unless there is a drop in the number of zeros. γ will be one of the $*s$ neighbouring the $*$ of the edge.

If the $*$ of the H_0^1-direction is a neighbour to the $*$ of the edge and if it does not have its orientation, then we may use again the Fredholm violation along w_{2k-1} and "cancel" the flow-line $w_{2k}^\infty - w_{2k-1}$. This settles all the cases: the last one involves a repetition between the edge and a neighbour. The choice of γ can be made as any of the other remaining $*s$.

Using Lemma 2.6 and Lemma 2.7 above and the results of the previous sub-section ("three edges rule"), we thus have established the following:

Lemma 2.8 *Assume that w_{2k+1} dominates w_{2k}^∞ with a non-zero intersection number, then the intersection number $i(w_{2k}^\infty, w_{2k-1})$ is zero.*

The above lemma holds under the assumption, as stated above, that γ is chosen uniformly over all the configurations, re-arrangement sparing the forced repetition being completed in the complement. This holds over sequences of Morse relations involving a domination (with a difference of Morse indexes equal to 1) and a tangency, see Sect. 2.11.

In the next section, we assume that Lemma 2.8 holds true in full generality over a continuous path α_t of contact forms verifying $\beta_t \wedge d\beta_t \gtrsim 0$, $\beta_t = d\alpha_t(v_t, .)$. The path is assumed to start at time zero from the contact structures with their standard contact forms (the standard contact form on S^3 and the first exotic contact form of Gonzalo and Varela [14]). With these forms, one can read directly that $\partial_{\text{per}} \circ \partial_{\text{per}} = 0$. It follows that Lemma 2.8 above holds at time zero, with w_{2k}^∞ intended as the collection of critical points at infinity dominated by w_{2k+1}. Using tangencies, we may assume that such a collection is made at time zero with a single w_{2k}^∞ and, therefore, Lemma 2.8 holds at time zero. This is a result that we need for the proof of Theorems 1.1 and 1.2 in the next section.

We will also be assuming—creations and cancellations are discussed in Sect. 2.5—that either no creation/cancellation occurs over this path; or if a creation/cancellation $x_m x_{m-1}$ occurs, no "rhombus" $x_m/x_{m-1}^\infty/x_{m-1}/x_{m-2}$ violating $\partial_{\text{per}} \circ \partial_{\text{per}} = 0$ will occur.

2.4 Morse relations

We will be considering in the sequel Morse relations of the type

$$\partial w_m = w_{m-1} + w_{m-1}^\infty$$

w_m is one or a collection of periodic orbits of index m; the same for w_{m-1} (with the index $m-1$).

w_{m-1}^∞ is one or a collection of critical points at infinity of index $(m-1)$.

We will be studying such Morse relations as a contact form α^t is deformed along a homotopy of contact forms, $t \in [0, 1]$. We will be assuming, in this section, except for Proposition 2.10, that no creation or cancellation of **periodic orbits** occurs along this path and we will be assuming that the contact forms are **at time zero** either the standard contact form on S^3 or the first exotic contact form of Gonzalo and Varela [14] on S^3. The arguments extend to all the other contact structures on S^3 for deformations of the same type.

Along such a deformation, creations/cancellations of critical points at infinity, singularities in $\cup\Gamma_{2k}$, tangencies between stable and unstable manifolds of critical points and critical points at infinity of the same index may occur. The issue of creations/cancellations of periodic orbits is addressed at the next section, Sect. 2.5, as these creations/cancellations might allow the formation of "rhombi" $x_m/x_{m-1}/x_{m-1}^\infty$ violating the relation $\partial_{\text{per}} \circ \partial_{\text{per}} = 0$.

We want to study the contents and the changes of these Morse relations as t varies in [0, 1].

For the sake of simplicity, the index t is dropped from these Morse relations; but it will be re-introduced when it is convenient and important to track the time t of the deformation along these Morse relations.

We first describe a key feature of the variational problem (J, C_β) and we discuss, before entering into the details of the study of the Morse relations, the impact of this key feature on these Morse relations and the related homology and intersection operator (restricted to periodic orbits):

Morse relations are related, for a given variational problem, to the definition of the intersection operator. This intersection operator is in turn related to the definition of a decreasing pseudo-gradient for this variational problem; to a certain extent, the expression "Morse relations" can be misleading if it is interpreted as "Morse relations of a given variational problem". The datum of the pseudo-gradient is essential for the definition of these Morse relations.

The pseudo-gradient that we will be using to define these Morse relations is a combination of the (semi)-flow defined on $C_\beta/\cup\Gamma_{2k}$ in [3,4]—see Sect. 2.9, Appendix 2 for the construction of a global flow, satisfying the Palais–Smale condition on each Γ_{2k}—with the local flow defined above in the previous sections. Once this definition is given, there is no possible confusion about what intersection operator we are referring to.

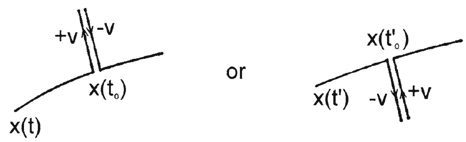
As we have pointed out above, the variational problem (J, C_β) misses one important property, usually satisfied in classical and even extended variational theory, namely it misses the verification of the Fredholm assumption.

This feature of (J, C_β) has been discussed in great extent in our previous works [1,3–5].

For the sake of self-consistency and for the reader, we summarize here the main features and consequences of this phenomenon described and studied in the references above:

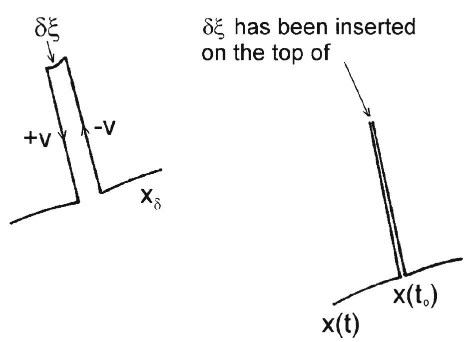
2.4.1 Violation of the Fredholm assumption: how it occurs

Given a piece of curve on M, $x(t)$, $t \in [0, 1]$, with $\beta(\dot{x}) = 0$, $\alpha(\dot{x}) = a$, a a positive constant, we can compute $J(x) = \int_0^1 \alpha(\dot{x})$. Since $\alpha(v) = 0$, the value of this functional does not change if we modify x into a new curve y derived by adding to x a piece of v-orbit at $x(t_0$ which we incorporate into $x(t)$ as a back and forth or forth and back run along v:



The curve y does not satisfy anymore the constraint $\alpha(\dot{x}) = a$, a a positive constant. If x were closed and were therefore a curve of C_β, y would not anymore be a curve of C_β. y does satisfy, however, the constraint $\beta(\dot{y}) = 0$ and y, with its piece of v-orbit thought of as a back and forth or forth and back run along v, can be approximated by a curve z_ϵ having $\alpha(\dot{z}_\epsilon) \gneq 0$; hence, after re-parametrization, having $\alpha(\dot{z}_\epsilon) = c_\epsilon$, c_ϵ a positive constant. $J(z_\epsilon)$ is very close to $J(x)$, so that we can think of all these curves z_ϵ as some sort of set that is associated with x, over which J does not change. This set is contractible (even after the insertion of several of these "runs" along v).

As we try to define decreasing variations of the curve x, we can "slide" x to any of the associated z_ϵ. Once the curve is located at z_ϵ, new variations can be defined. Indeed, see [3], pp 236–239, the back and forth or forth and back run along v, the so-called "Dirac mass" that we introduced along x, can be opened "up" at its "top" or "bottom" depending on the specific case and a small piece of orbit of ξ-orbit, of length δ, $\delta \gneq 0$ small, can be inserted. With appropriate adjustments, see [3], also Appendix 1 to this paper, Sect. 2.8, a new curve x_δ is thereby defined:



Computing $J(x_\delta)$-the piece of v-orbit, which we will take, for example, to be positively oriented, is of length s, ϕ_s is the one-parameter group of v-, we find, see [3], pp 236–239:

$$J(x_\delta) = J(x) + \delta(1 - \alpha_{x(t_0)}(D\phi_{-s}(\xi))) + O(\delta^2)$$

It might happen that, for certain $x(t_0)$s along $x(t)$ and for certain values of s-these cannot be small-$\alpha_{x(t_0)}(D\phi_{-s}(\xi))$ is larger than 1 so that $J(x_\delta) \lneq J(x)$. We then say that the Fredholm assumption is violated at the curve $x(t)$, for the value t_0. We can be more specific and state whether this is happening for $s \gneq 0$ ("positive Dirac masses") or for $s \lneq 0$ ("negative Dirac masses"). These violations might happen over disconnected intervals for s as well, so that the Fredholm assumption is then violated at multiple locations along the v-orbit across $x(t_0)$.

The terminology "violation" is here the right terminology because one can see that the linearized operator for J' is not Fredholm. This can be precisely traced back, see [3], 236–239 also, [4], pp 28–30, to the fact that the functional J is "insensitive" to the addition of these "Dirac masses".

2.4.2 How this violation interferes with the intersection operator: the topological "cancelation" of every periodic orbit

The Fredholm violation for (J, C_β) is a strong feature of this kind of variational problem. In the case of the standard contact structure of S^3, for a given contact form, this violation probably occurs at every periodic orbit of its contact vector-field. This, however, requires some more specific study: it is not, at this point, a theorem.

On the other hand, see [6], Propositions 2.16 and 2.17, section 8.3, in the case of the first exotic contact structure of Gonzalo and Varela [14], the Fredholm assumption is violated at every point of $S^3 \setminus \Sigma$, where Σ is a codimension hyper-surface in S^3.

This implies that this assumption is violated, for a generic contact form of this contact structure, at every periodic orbit of its Reeb contact vector-field.

Considering then a periodic orbit w of this contact form, a small unstable disk $D^r(w)$ for w, all of $D^r(w)$ (relative to its boundary S^{r-1} that will not move along this deformation) can be "moved down", below $J(w)$. This is accomplished by building appropriate "Dirac masses", at the appropriate location where the Fredholm assumption is violated—if this is happening at $w(t_0)$ for the periodic orbit, we will build these "Dirac masses" at $x(t_0)$ for the curves $x(t)$ of $D^r(w)$—along the curves of small unstable disk. These "Dirac masses" will grow in size, from non-existing to being of the appropriate size, as the curves "move up" in $D^r(w)$, from its boundary $S^{r-1}(w)$ to its "top", which is w.

Then, these "Dirac masses" are "opened up", a tiny ξ-piece is inserted, the curves are approximated very closely by curves of C_β and the whole process is J/J_∞-decreasing.

The difference of topology due to the periodic orbit w has been cancelled in the topological sense by a "shadow" critical point at infinity, built with the addition of a "Dirac mass" δ along w.

The study of this phenomenon is completed thoroughly, up to minor misprints and mistakes in [4], pp 151–178.

This additional "shadow critical point at infinity" that cancels w will be denoted in the sequel $(\delta + w)^\infty$. We have

$$\text{Morse index}((\delta + w)^\infty) = 1 + \text{Morse index}(w)$$

It is clear that the intersection number of $(\delta + w)^\infty$ with w is 1, so that for the intersection operator ∂-here, it will be for every "pseudo-gradient": $J_\infty(\delta + w)^\infty)$ is equal to $J(w)$-:

$$\partial(\delta + w)^\infty = w + \cdots$$

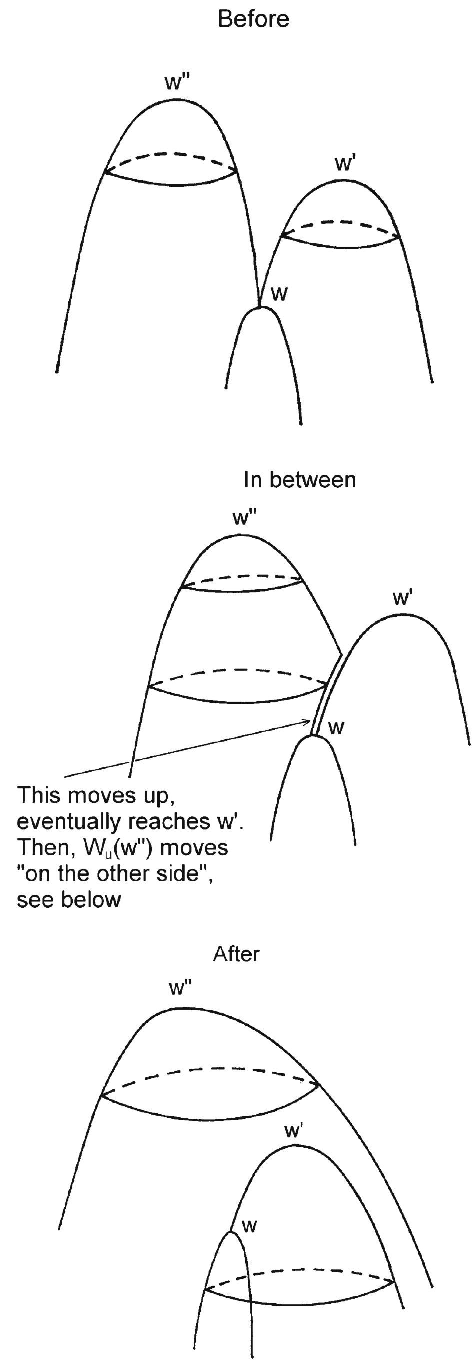
In addition to $(\delta + w)^\infty$, there are more complicated phenomena, where several "Dirac masses" are added along w and "moved around", giving rise to higher index "shadow critical points at infinity". The phenomenon can be studied completely, see [4], pp 151–178; again some re-writing and minor modifications are required, see Sect. 5.0.1, the erratum.

2.4.3 Impact on the homology

When we study the Morse relations for a functional f involving a given critical point w of index m as a **dominated** term, we seek all the critical points w', of index $(m + 1)$ that have flow-lines connecting them to w. Assuming general position arguments, we have isolated flow-lines that carry a local degree equal to 1 or to -1. Adding these degrees for a given w', we find the intersection number $i(w', w)$ of w' with w.

Let us assume that we have two critical points w' and w'', both of index $(m + 1)$, and let us assume that $f(w'') \gneq f(w)$. We compute for a given pseudo-gradient for f the quantities $i(w', w)$ and $i(w'', w)$, whatever we find them to be.

We claim now that we can modify $i(w'', w)$-modifying the pseudo-gradient flow-and replace it, for another pseudo-gradient by $i(w'', w) + ki(w', w)$, where k is any relative integer we please. This claim is not difficult to establish: it suffices for this to create a tangency of order k between the unstable manifold of w'' and the stable manifold of w'. This tangency of order k may be seen as k tangencies of order 1, one after the other one-along a deformation of pseudo-gradient flows. A tangency of order 1 is not hard to create: one "engineers" a deformation of the flow so that, over this deformation, the unstable manifold of w'' "swipes" over the unstable manifold of w' once. Perhaps the following drawing best illustrates this change of pseudo-gradient:



In the specific case of (J, C_β), for a periodic orbit w, we can take for w' the "shadow critical point at infinity" $(\delta + w)^\infty$. Since $i(w', w)$ is here 1, since $J(w') \lneq J(w'')$ for every w" of index index $(w) + 1$, we can modify the pseudo-gradient flow so that the intersection number $i(w'', w)$ for this flow is 0.

Considering then a homology where the dominant terms are periodic orbits, we find that w is not in the image of any other periodic orbit of index index $(w) + 1$. Completing this operation for every w of a certain index, restricting the boundary operators to be valued in periodic orbits, we find that its square is zero. Therefore, as we start from periodic orbits, all cycles survive in the corresponding homology, that is they are not boundaries.

We need to take our pseudo-gradient so that it is a "local" flow, a "Fredholm flow" near the periodic orbits, that is in an L^∞-neighbourhood of their graphs. If this neighbourhood is small enough, independently from the periodic orbit, once its critical value is bounded-then we know that the "Dirac masses" described above cannot be opened up in order to "engineer" a decrease $((1-\alpha(d\phi_{-s}(\xi)))$ is positive for s small). Therefore, the creation of this "Dirac mass" is not a natural phenomenon in a small L^∞-neighbourhood of the periodic orbit. However, as this neighbourhood expands, Fredholm violation can occur. There is no Morse Lemma to impede it.

We will be imposing the following rule along the deformation of our pseudo-gradients as the contact form is deformed:

2.4.4 Basic rule for the definition of the homology

Definition 2.9 The (continuous) deformation of the pseudo-gradients of the variational problems (J^t, C^t_β) and $(J^t_\infty, \cup\Gamma^t_{2k})$ along the deformation of the contact forms is **symplectic**: along this deformation, the union of the unstable manifolds of a periodic orbit w^t_m never dominate the unstable manifold of the "shadow critical points at infinity" $(\delta + w^t_{m-1})^\infty$, that is[3], $W_u((\delta + w^t_{m-1})^\infty) \not\subseteq \overline{\underset{t\in[0,1]}{\cup} W_u(w^t_{m-1})}$.

Another way to phrase the rule described above, the "symplectic" rule, is to state that there never is, at any time t along the deformation, a flow-line connecting w^t_m and $(\delta + w^t_{m-1})^\infty$.

We will be assuming that this rule holds for the deformation of pseudo-gradients Z_t here. We get rid of this assumption in [8].

2.4.5 The case of $(\delta^m + w')^\infty$, $m \geq 2$

We consider now the specific case of the "shadow critical points at infinity" $(\delta^m + w_1)^\infty$, $m \geq 2$, taken as w'' as we follow the framework and the arguments of the last section. We claim that for these w''s

Proposition 2.10 *For these w''s, the use of $(\delta + w)^\infty$ to make $i(w'', w)$ equal to 0 does not change the value of the homology. This holds true over cancellations of periodic orbits. In addition, the derived pseudo-gradient can be assumed to satisfy the property that the number of zeros of b does not increase along its decreasing flow-lines.*

Proof We need to prove that cancellations of periodic orbits and also of periodic orbits and critical points at infinity (see [4], pp 103–106) can proceed "normally" even as these intersection numbers are set to be zero. Observe that, with $w'' = (\delta^m + w_1)^\infty$, $m \geq 2$, assuming that, before any modification, $i(w'', w)$ was non-zero, the index of w_1 satisfied:

$$\text{index}w_1 + m = \text{index}w + 1$$

Therefore, the index of w_1 is always less than the index of w. Even along deformations of flows (involving one parameter), we may assume, by general position arguments, that w_1 never dominates w, that is there is no decreasing flow-line from w_1 to w. Nevertheless, since the intersection number was non-zero, $J(w_1)$ was more than $J(w)$. Thus, no cancellation between w_1 and w can take place since w_1, which has the lesser index, is **above** w, which has the higher index. This settles the first part of the claim.

For the second part, we observe that the domination by $(\delta^m + w_1)^\infty$, $m \geq 2$ of w cannot be "direct", through the domination by $(\delta^m + w_1)^\infty$ of w_1 (this one involves the "removal" of the "Dirac masses"). It takes place through flow-lines where the "Dirac masses" have been built and "opened up", since all the other ones

[3] See [4], pp 151–178, also Sect. 2.8, Appendix 1, which provides the framework for a parametrization of the phenomenon described in [4], in order to understand $(\delta + w^t_{m-1})^\infty$. Its unstable manifold is as follows: Proposition 2.2 of [5] gives a model for the unstable manifold of a periodic orbit of index $(m-1)$ using $(m-1)$ small $\pm v$-jumps located at $(m-1)$ (not arbitrary, they can be several choices) points along this periodic orbit. The "Dirac mass" can the be built by inserting a back and forth or forth and back run along v at any of those locations. Its size varies from 0 to s_0; s_0 is the size beyond which this "Dirac mass" can be opened and J_∞ then decreases (if this happens at some location. The rotation of v along the periodic orbit can be re-scaled so that, if there is Fredholm violation along the periodic orbit, then it will take place at one or all of these locations, see [4], p 162). Increasing the size from 0 to s_0 and then "opening up" the "Dirac mass" to decrease J_∞ provides the additional decreasing direction, to be added to the disk of dimension $(m-1)$ provided by the (small) $(m-1) \pm v$-jumps that modelize the unstable manifold of the periodic orbit in order to obtain the unstable manifold of $(\delta + w^t_{m-1})^\infty$. As s increases from 0 to s_0, J_∞ does not change if the "Dirac mass" is completely "closed", has no ξ-piece inserted in it. But, if a tiny ξ-piece is inserted in it and its size does not change, then J_∞ increases because the function $\gamma(s) = 1 - \alpha(d\phi_{-s}(\xi))$ (see above) is initially positive. s_0 is the size at which $\gamma(s)$ becomes zero and the "Dirac mass" can be opened. In this way, we derive a genuine additional unstable direction.

can go only to w_1 or to lower-index critical points (at infinity). The unstable manifold of $(\delta + w)^\infty$ carries no more zeros than the unstable manifold of w if the index of w is $2k$ ($2k$ zeros). It carries two more zeros than the unstable manifold of w if the index of w is $(2k - 1)$ (again it carries $2k$ zeros); see [4], pp 151–178, for these claims (Lemma 16, (i) and (iv), in particular).

On the other hand, by Lemma 2.5 , p 80 of [4] and by the property that the number of zeros of b never increases along the decreasing flow-lines of the pseudo-gradient of [3,4], the maximal number of zeros of b on the unstable manifold of $(\delta^m + w_1)^\infty$ is then at least $2k$ since $i(w'', w)$ is non-zero. The claim follows then (it is not difficult to adjust the intersection of $W_u((\delta^m + w_1)^\infty)$ with $W_s((\delta + w)^\infty)$ so that the property for the number of zeros of b holds on each single flow-line; this is not needed though in our framework. What is needed is the weaker property that if $w^{(\infty)}$ dominates $w_1^{(\infty)}$, the maximal number of zeros of b on the unstable manifold of $w^{(\infty)}$ is larger than or equal to the maximal number on the unstable manifold of $w_1^{(\infty)}$. This obviously holds). □

We now turn to the more specific issues of the changes affecting these Morse relations as we proceed with a generic deformation of contact forms that does not contain creations or cancellations of periodic orbits. We will be focusing on three main issues, which are the following: we need to understand these Morse relations at the time $t = 0$ of the deformation. Here, the example of the first exotic contact structure of Gonzalo and Varela [14] on S^3 will be considered. The arguments apply as well to the standard contact structure on S^3 and to a variety of other contact structures. All deformations start at time zero at the standard contact forms (see [14] for the exotic contact structures of Gonzalo and Varela on S^3) of these contact structures.

We also will study $\partial_{\mathrm{per}} w_m^\infty$; that is we will study the intersection numbers between critical points at infinity of index m and periodic orbits of index $(m - 1)$. The critical points at infinity w_m^∞ will either be in ∂w_{m+1}, that is they will be dominated by periodic orbits of index $(m + 1)$ through the flow of [3,4]. Or they will have undergone tangencies $w_m - w_m^\infty$, that is $W_u(w_m^{t_0}) \cap W_s(w_m^{t_0,\infty})$ will not be empty. It will typically be made of a single flow-line at some time t_0 along the deformation.

In addition, we will need to understand why we may assume if

$$\partial w_m = w_{m-1} + w_{m-1}^\infty$$

then, w_{m-1}^∞ does not contain critical points at infinity which we denote $(\delta + z_{m-2})^\infty$, built with the addition of a "Dirac mass" (a back and forth or forth and back run along v) to a periodic orbit of index $(m - 2)$, z_{m-2}.

All these issues are addressed in the sequel. Creations and cancellations of periodic orbits are addressed in the next section.

Let us first observe, as we pointed out earlier, that we will, each time we are considering a dominating w_m, view its unstable manifold in Γ_{2m}. This is possible, see [5], Proposition 2.2, p 469.

We will also be assuming—-without loss of generality—that all simple periodic orbits of odd index are elliptic.

2.4.6 Changes in Morse relations

A Morse relation such as

$$\partial w_m = w_{m-1} + w_{m-1}^\infty$$

can be changed in various ways along the deformation.

Indeed, ∂w_m can be changed by a tangency $w_m - w_m^{(\infty)}$. This will affect the Morse relation with the addition of a cycle $\sigma = \partial c$; that is, we will now have

$$\partial w_m = w_{m-1} + w_{m-1}^\infty + \partial c$$

This Morse relation could also be changed by tangencies $w_{m-1} - w_{m-1}^\infty$; that is a periodic orbit of index $(m - 1)$ in ∂w_m dominated another periodic orbit or another critical point at infinity of index $(m - 1)$. Observe that it then follows that the dominated w_{m-1}^∞ will inherit in the parametrization of its unstable manifold the parametrization of $W_u(w_{m-1})$ into $(m - 1)*$s.

We could also have tangencies $w_{m-1}^\infty - w_{m-1}^{(\infty)}$. The dominated $w_{m-1}^{(\infty)}$ will inherit the parametrization of $W_u(w_{m-1}^\infty)$ into $m*$s, derived from the fact that $w_{m-1}^\infty \subset \partial w_m$.

Finally, we could have singularities in $\cup\Gamma_{2k}$. We will have to study how these singularities might affect these Morse relations.

Routes As one may have noted from the arguments above, the w^∞_{m-1}s that appear in the Morse relations may either rise from ∂w_m or from tangencies $w_{m-1} - w^\infty_{m-1}$. These two different routes yield slightly different outcomes for w^∞_{m-1}, since in the first case its unstable manifold is parametrized by $m*$s, whereas in the second case $(m-1)*$s suffice.

Let us first consider the Morse relation

$$\partial w_{2k+2} = w_{2k+1} + w^\infty_{2k+1}$$

First Route We are interested in the first case, that is in w^∞_{2k+1} dominated by w_{2k+2}.

Let us assume that this Morse relation, with $\partial_{\text{per}} w^\infty_{2k+1} \neq 0$, is not happening at the time 0 of the deformation.

Then w^∞_{2k+1} arises later over the deformation and it would be part of a ∂c, where c has undergone a dominated tangency with w_{2k+2}. Assume then that c contains a w^∞_{2k+2} and that ∂w^∞_{2k+2} contains a periodic orbit w'_{2k+1}. We would then find a sequence $w_{2k+2} - w^\infty_{2k+2} - w'_{2k+1}$, or after a shift of indexes, a sequence $w_{2k} - w^\infty_{2k} - w'_{2k-1}$. We have ruled out in Sect. 2.3 (see Appendix 11 for the proof in full generality) that this may occur (observe that, by our rule above, w^∞_{2k} cannot be $(\delta + w_{2k-1})^\infty$). Therefore, once all w'_{2k+2} are removed from c (these contributions to ∂w_{2k+2} are of the same type that the Morse relation we started with in the first place; they yield only re-combinations of the same type of Morse relations), we find only w^∞_{2k+1}s in ∂c. Taken together, they have no boundary since $\partial^2 = 0$. We thus see that this route for changing this Morse relations does not affect the intersection numbers between w_{2k+2} and w_{2k+1}.

Let us now assume additionally that we are considering either a contact form in the standard contact structure of S^3 or a contact form in the first exotic contact structure of Gonzalo and Varela [14].

We then cannot either start at time zero with a Morse relation as above (with $\partial_{\text{per}} w^\infty_{2k+1} \neq 0$). Indeed, at time $t = 0$, we have the S^1-action of the vector-field X_0, see [6]. w_{2k+2} is of even index. Its unstable manifold is invariant under the S^1-action generated by X_0.

If w^∞_{2k+1} (that is, its unstable manifold) is transverse to the action of X_0, then the intersection number $w_{2k+2} - w^\infty_{2k+1}$ is zero, contrary to the assumption.

If $W_u(w^\infty_{2k+1})$ is invariant under the action of X_0, then we have a sequence of three unstable manifolds $W_u(w_{2k+2})$, $W_u(w^\infty_{2k+1})$ and $W_u(w_{2k})$; the three are invariant under the action of X_0.

This sequence defines a Morse relation that unravels and yields a Morse relations $w_{2k+1} - w^\infty_{2k} - w_{2k-1}$ that is transverse to the X_0-action. This Morse relation is taking place at time zero and this is not possible, see above.

More generally, Lemma 2.8 of Sect. 2.3 and Appendix 11 rule out Morse relations for any time t if they do not involve a "complicated" Morse relation over which the value of γ completes a full circle amongst the $(2k+1) \pm v$-jumps parametrizing $W_u(w_{2k+1})$.

Therefore, we conclude from this X_0-action that the intersection numbers $i(w^\infty_{2k}, w_{2k-1})$ are zero if $i(w_{2k+1}, w^\infty_{2k})$ is non-zero. This holds true at time zero, remains true through tangencies. This feature can only change through creations/cancellations of periodic orbits and we are assuming that, if such creations/cancellations do occur, no rhombus violating $\partial_{\text{per}} \circ \partial_{\text{per}} = 0$ arises, so that all over the deformation of contact forms that we are considering $i(w^\infty_{2k}, w_{2k-1}) = 0$ if $w^\infty_{2k} \in \partial w_{2k+1}$.

Second Route Considering now the second route, we find a Morse relation of the type

$$w_{2k+1} - w^\infty_{2k+1} - w_{2k}$$

These are ruled out over the deformation.

It follows that

Lemma 2.11 *If* $\partial w_{2k+2} = w_{2k+1} + w^\infty_{2k+1}$, *then* $\partial_{\text{per}} w^\infty_{2k+1} = 0$

We now consider the Morse relations:

$$\partial w_{2k+1} = w_{2k} + w^\infty_{2k}$$

Again, using the results of the previous section, also the rule stating that w_{2k+1} does not dominate (along tangencies) $(\delta + w_{2k})^\infty$, we find

Lemma 2.12 *If* $\partial w_{2k+1} = w_{2k} + w^\infty_{2k}$, *then* $\partial_{\text{per}} w^\infty_{2k} = 0$

2.4.7 *Morse relations and the* $(\delta + w_{2k-2})^\infty s$

We prove now that we cannot have Morse relations of the type

$$\begin{aligned}\partial w_{2k} &= w_{2k-1} + w_{2k-1}^\infty \pm (\delta + w_{2k-2})^\infty \\ \partial w_{2k+1} &= w_{2k} + w_{2k}^\infty \pm (\delta + w_{2k-1})^\infty\end{aligned}$$

[4] $(\delta + w)^\infty$) stands here for the critical point at infinity defined by the addition to the periodic orbit w of a back and forth or forth and back run along v, see above and [4], pp 151-178. Instead of adding one single "Dirac mass" to w, we could add more, at various locations, with various cycles. As stated above, we denote these—they have also been studied, up to some minor misprints/modifications in [4], cited above—$(\delta^m + w)^\infty$, $m \geq 2$. By Proposition 2.10 above, $(\delta^m + w)^\infty$, $m \geq 2$ have no boundary relevant to our arguments. We will ignore them.

In the two Morse relations above, w_{2k-1}^∞ and w_{2k}^∞ do not contain terms of the type $(\delta + w_j)^\infty$; or if they contain them, they are part of cycles and yield no boundary. They could contain terms of the type $(\delta^2 + w_j)^\infty$, etc. Since these do not yield any boundary relevant to our arguments, we will ignore them, as we already stated above.

We also observe that we can split w_{2k-1}^∞ into $w_{2k-1}^{\infty,1} + w_{2k-1}^{\infty,2}$. $w_{2k-1}^{\infty,1}$ has undergone dominated tangencies with w_{2k-1} and has, therefore, inherited the $(2k-1)*$s structure; whereas $w_{2k-1}^{\infty,2}$ is a ∂w_{2k}^∞ ($w_{2k} - w_{2k}^\infty - w_{2k-1}$ is impossible as we have seen above). Thus, $\partial w_{2k-1}^{\infty,2} = 0$.

Tangencies $w_{2k} - w'_{2k}$ yield, using induction starting from the time 0 of the deformation, a similar decomposition.

Considering the first Morse relation, $\partial w_{2k} = w_{2k-1} + w_{2k-1}^{\infty,1} \pm (\delta + w_{2k-2})^\infty + \partial w_{2k}^\infty$, it implies, since $\partial_{\text{per}} w_{2k-1}^{\infty,1} = 0$ by the results above, that w_{2k-1} is non-zero since it has a non-zero intersection number with w_{2k-2} (compute $\partial(\partial w_{2k})$ using the formula above).

We know that, at the time $t = 0$, the intersection number $i(w_{2k}, w_{2k-1})$ is zero. Therefore, such a Morse relation comes from the creation of a pair $w_{2k} - w_{2k-1}$.

As this pair is created, we may assume that the Morse relation takes the simple form $\partial w_{2k} = w_{2k-1}$ (for $k \geq 1$. For $k = 0$, we assume that there is no w_0, that is there is no periodic orbit of index 0).

Let us first assume that the unstable manifolds of the pair that is created are in the normal form given by Proposition 1 of [5].

Under such an assumption, to create a $(\delta + w_{2k-2})^\infty$, we either need a tangency $w_{2k} - w_{2k}^{(\infty)}$. These either produce a ∂w_{2k}^∞ that does not interfere with our arguments, or they produce a $\partial w'_{2k}$ that does not contain (use induction starting from the time 0 of the deformation) a $(\delta + w_{2k-2})^\infty$.

The other tangencies $w_{2k-1} - (\delta + w_{2k-2})^\infty$ or $w_{2k-1}^{\infty,1} - (\delta + w_{2k-2})^\infty$ are not allowed or are not possible: the second one changes the intersection number $i(w_{2k-1}^{\infty,1}, w_{2k-2})$. This intersection number must, therefore, be non-zero at a certain time of the deformation, which is not possible.

The first one is not allowed by our basic rule, see above, Definition 2.9.

The claim follows for the first Morse relation.

The second Morse relation implies, because $\partial_{\text{per}} w_{2k}^\infty = 0$, see Sect. 2.3 above, that

$$\partial w_{2k} = w_{2k-1} + \text{other terms (maybe)}$$

We know that this does not hold at the time 0 of the deformation. It might appear later under the form (see above)

$$\partial w_{2k} = w_{2k-1} + w_{2k-1}^{\infty,1}$$

This happens after a creation $w_{2k} - w_{2k-1}$. Using a general position argument, this creation never takes place on the unstable manifold of a critical point (at infinity) of index $(2k+1)$ (this argument is general; it

[4] For a simple periodic orbit of even index, the maximal number of zeros of b, the v-component of $\dot{x}$ on the unstable manifold of $(\delta + w_{2k-2})^\infty$, changes as the location of the "Dirac mass" changes along the periodic orbit w_{2k-2}. This number changes from $2k$ to $(2k-2)$ or vice-versa at the crossing of a node. Under the restriction that the maximal number of zeros is $(2k-2)$, we could therefore have several non-homotopic $(\delta + w_{2k-2})^\infty$. By difference, they can then yield cycles that are not boundaries. However, in $W_u(w_{2k})$, the corresponding maximal number is $2k$. Under this constraint, these cycles are boundaries. They do not interfere with our argument. The same observation holds for simple elliptic periodic orbit, only that the argument is then easier because then all $(\delta + w)^\infty$ are isotopic.

does not require any preliminary assumption: the stable manifolds of the critical points in the created pair can be assumed not to change through the process of creation).

Thus, the only way to insert w_{2k} in this Morse relation is through tangencies between a periodic orbit w'_{2k} or a critical point at infinity w^{∞}_{2k} in ∂w_{2k+1} with w_{2k}. The second case is impossible by the arguments above (it would imply a sequence $w_{2k+1} - w^{\infty}_{2k} - w_{2k-1}$).

Even after a tangency $w'_{2k} - w_{2k}$, we are left short of the term $(\delta + w_{2k-1})^{\infty}$ in ∂w_{2k+1}, which has to appear after the $w_{2k} - w_{2k-1}$ creation (using a general position argument, it cannot be inserted at the time of the creation).

Tangencies $w'_{2k} - (\delta + w_{2k-1})^{\infty}$ and $w^{\infty}_{2k} - (\delta + w_{2k-1})^{\infty}$ are not possible or not allowed again (same than above).

Thus, $(\delta + w_{2k-1})^{\infty}$ could only appear after a tangency $w_{2k+1} - w^{(\infty)}_{2k+1}$.

Over such a tangency, $w^{(\infty)}_{2k+1}$ would inherit the $(2k+1)*$s structure of w_{2k+1}. It can be thought, in all the arguments above, as w_{2k+1}.

$\partial w^{(\infty)}_{2k+1}$ would have to contain $(\delta + w_{2k-1})^{\infty}$ at a time of the deformation prior to ∂w_{2k+1}. An induction can be started; the claim follows.

Let us now remove, under some other much weaker assumption, our initial assumption about the unstable manifolds of the created pair.

Since we ask that the additional Morse relations introduced by the pair of periodic orbits reduce to the sole $\partial x_m = x_{m-1}$ as a creation or cancellation x_m/x_{m-1} occurs, it follows that the unstable manifolds of these periodic orbits are not in the normal form provided by Proposition 1 of [5]: The unstable manifold of x_{2k} in an x_{2k+1}/x_{2k} creation/elimination has one additional $\pm v$-jump ($(2k+1)$ in total). The unstable manifold of x_{2k-1} in an x_{2k}/x_{2k-1} creation/elimination has one additional $\pm v$-jump ($(2k)$ in total). This additional $\pm v$-jump does not count as an additional dimension of the unstable manifold. It is rather the additional dimension due to the deformation, as the unstable manifold takes its final normal form, starting from the form that warrants the Morse relations as we wish them to be through the creation/cancellation, that is that they should take the form $\partial x_m = x_{m-1}$ through these processes.

Tangencies with infinity become then possible and "rhombi" violating the relation $\partial_{\text{per}} \circ \partial_{\text{per}} = 0$ may arise, see Sect. 2.5 below for this. Outside of the creations/cancellations, these tangencies are impossible, the intersection operators $\partial_{per]}$ and ∂_{∞} do not mix by the results of Sect. 2.3 above and Appendix 11, below.

We, therefore, either need to assume that the relation $\partial_{\text{per}} \circ \partial_{\text{per}} = 0$ is not violated over the process of creations/cancellations of periodic orbits over our path; or, for the more special tangencies that we are considering here, where the critical point at infinity is a $(\delta + x_{m-2})^{\infty}$, through a creation/cancellation x_m/x_{m-1}, we may introduce another assumption, verified over a very large number of cases, if not in full generality. This assumption reads as follows: the existence of the critical point at infinity $(\delta + x_{m-2})^{\infty}$ assumes that there is Fredholm violation, see above and Sect. 2.9 along x_{m-2}. We assume that this Fredholm violation, in the direction of δ (δ could be a back and forth or a forth and back run along v), is not occurring at all points of w_{m-2}; there must be some points on w_{m-2} where the Fredholm violation does not occur in the direction of δ.

Under this assumption, see Appendix 1, Sect. 2.8 and see [4,8], see also Sect. 2.5 below, we can think of $(\delta + x_{m-2})^{\infty}$ as critical point at infinity of index at least 2 in Γ_4. One index direction is for the Fredholm violation, the other one is for the fact that this Fredholm violation is not occurring at every point of x_{m-2}, see [8] for more details.

Outside of the two large $\pm v$-jumps of $(\delta + x_{m-2})^{\infty}$, every small $\pm v$-jump counts for an H^1_0-index direction, see Sect. 2.11, the sub-section about the H^1_0-index based at γ, as long as the number of these small $\pm v$-jumps does not exceed the H^1_0-index of w_{m-2} based at γ.

Along a periodic orbit, there can be three types of positions for γ, see Sect. 2.2: γ can be at E^+ or E^- or E_0-type positions. At an E^+-position , a small $\pm v$-jump at γ is attractive for the second derivative; at E^-, it is repulsive and degenerate at E_0.

We may arrange, re-scaling the v-rotation along w_{m-2} as in [4], pp85-102, that the Fredholm violation—since it is not occurring uniformly along w_{m-2}—is occurring at positions that are not of E_0-type.

Then, if the "Dirac mass" is occurring at an E^+-position, $(\delta + x_{m-2})^{\infty}$ is of index 2 in Γ_4 and the H^1_0-index based at γ is $(m-2)$. With $s \pm v$-jumps, $s \leq m$, we find that $(\delta + x_{m-2})^{\infty}$ is of index s in Γ_{2s}, whereas $W_u(x_{m-1})$ is of dimension $(s-1)$ in Γ_{2s}, of dimension s if we add the additional parameter of deformation. The result is that no tangency is possible over the process of creation/cancellation of x_m/x_{m-1} between x_{m-1} and $(\delta + x_{m-2})^{\infty}$.

If the "Dirac mass" is occurring at an E^--position, $(\delta + x_{m-2})^\infty$ is of index 3 in Γ_4 and the H_0^1-index based at γ is $(m-3)$. The argument is slightly modified when $s = m$, but the conclusion is the same, see a variant of this argument in Sect. 2.5 below for the case when $m = (2k+1)$.

More general configurations are discussed in [8].

2.5 Rhombi violating $\partial_{\text{per}} \circ \partial_{\text{per}} = 0$ form when periodic orbits are created or cancel each other along a deformation

The deformations that we have defined above (see also Sect. 2.11) obey very stringent conditions that forbid tangencies such as $x_{2k} - x_{2k}^\infty - x_{2k-1}$, $x_{2k} - x_{2k-1}^\infty - x_{2k-1}$, $x_{2k+1} - x_{2k+1}^\infty - x_{2k-1}$, $x_{2k} - x_{2k-1}^\infty - x_{2k-1}$. This implies that no rhombus $x_m/x_{m-1}/x_{m-1}^\infty/x_{m-2}$ violating the homology can be created through such tangencies starting from time zero. Theorems 1.1 and 1.2 as well as the fact that the operator ∂_{per} does not change through these tangencies follow.

However, cancellations/creations of pairs of periodic orbits do happen. Along them, rhombi can be created: when, e.g. a pair x_{2k}/x_{2k-1} is created, the unstable manifold $W_u(x_{2k})$ of x_{2k} can be thought of as being along a single flow-line, thereby avoiding the lower y_{2k-1}s. We then have to give up the fact that $W_u(x_{2k-1})$ is built with the use of $(2k-1) \pm v$-jumps or the use of $2k \pm v$-jumps with $(2k-2)$ sign-changes between their orientations.

As we later deform this unstable manifold to its standard form given by Proposition 1 of [5], lower tangencies $x_{2k-1}/y2k-1$ might happen and these would change the intersection number $x_{2k} - y_{2k-1}$. Also tangencies/dominations $x_{2k-1} - y_{2k-1}^\infty - x_{2k-2}$ can happen and they induce "rhombi" $x_{2k}/x_{2k-1}/y_{2k-1}^\infty/x_{2k-2}$ violating the relation $\partial_{\text{per}} \circ \partial_{\text{per}} = 0$. We need to study this phenomenon in more detail:

Let us first observe that, since there is no need to adjust stable manifolds of periodic orbits over degeneracies and since tangencies involving critical points at infinity as a middle term are not allowed, the relation $\partial_{\text{per}} \circ \partial_{\text{per}} = 0$ cannot be violated "from top" along degeneracies. When x_{2k+1} is close to a degeneracy and it undergoes an $x_{2k+1} - x_{2k}$ degeneracy, its unstable manifold does not have more $\pm v$-jumps and a tangency $x_{2k+2} - x_{2k+1}^\infty - x_{2k}$ is not allowed. The same conclusion holds with x_{2k} in an $x_{2k} - x_{2k-1}$ degeneracy.

Therefore, "rhombi" that violate $\partial_{\text{per}} \circ \partial_{\text{per}} = 0$ because they involve a critical point at infinity cannot be created in this way (although they could pre-exist).

On the other hand, from "bottom", that is when unstable manifolds are involved and a degeneracy takes place, the phenomenon is different.

Typically, if we have an $x_m - x_{m-1}$ degeneracy, the unstable manifolds have to be adjusted to have $m \pm v$-jumps for the degeneracy to occur, if m is even for example. Along this process, rhombi can be created because tangencies become now possible.

Near a degeneracy, we can assume that we build $W_u(x_m)$ and $W_u(x_{m-1})$ with $m \pm v$-jumps in the vicinity of one flow-line so that $\partial W_u(x_m) = W_u(x_{m-1})$ globally; that is no other periodic orbit is dominated by x_m.

Assume that the Fredholm condition is violated at x_m. Then x_m is dominated by $\tilde{x}_m^\infty = (\delta + x_{m-1})^\infty$, a critical point at infinity built with a "Dirac mass" on top of x_{m-1}. Near the degeneracy, all $W_u(x_m) \cup W_u(\tilde{x}_m^\infty)$ can be assumed near the degeneracy to be achieved with $(m+1) \pm v$-jumps (one additional for $\tilde{x}_m^\infty$) in the vicinity of a stratified space of top dimension 2, one for the degeneracy, one for the "Dirac mass" and the Fredholm violation.

$W_u(x_m) \cup W_u(\tilde{x}_m^\infty)$ can be assumed, therefore, to be a cycle (for dimension reasons): it does not dominate critical points at infinity of index $(m-1)$ if m is large. It is eventually deformed, as the number of $\pm v$-jumps over $W_u(x_{m-1})$ is adjusted to $(m-1)$, so that it is achieved with $\pm v$-jumps.

Over this process, it can undergo tangencies. The tangencies will occur with critical points at infinity of index m, $y_m^{(\infty)}$. Assume that $\partial y_m^{(\infty)} = y_{m-1}^\infty + z_{m-1}^{(\infty)}$. Here y_{m-1}^∞ and $z_{m-1}^{(\infty)}$ are critical points or combinations of critical points, at infinity for y_{m-1}^∞, maybe at infinity for $z_{m-1}^{(\infty)}$. We also assume that each of y_{m-1}^∞ and $z_{m-1}^{(\infty)}$ dominate at least one periodic orbit x_{m-2} (and maybe more): this is needed for the creation of "rhombi".

Then, after the tangency takes place, $\tilde{x}_m^\infty$ has $y_{m-1}^\infty + z_{m-1}^{(\infty)}$ in its boundary. y_{m-1}^∞ can undergo a dominated tangency with x_{m-1} (since $W_u(x_{m-1})$ is achieved with $m \pm v$-jumps) and, then, a rhombus $x_m/x_{m-1}/y_{m-1}^\infty/x_{[m-2}$ is created, violating $\partial_{\text{per}} \circ \partial_{\text{per}} = 0$. The Morse relations below the degeneracy are unchanged; therefore, this is the sole phenomenon that occurs.

If we then modify the value of ∂_{per}, setting

$$\tilde{\partial}_{\text{per}} x_m = x_{m-1} + y_{m-1}^\infty$$

the homology is not modified and $\tilde{\partial}_{\text{per}} \circ \tilde{\partial}_{\text{per}} = 0$. Furthermore, one can prove, using the number of $\pm v$-jumps used for each unstable manifold of the critical points (at infinity) of such a rhombus, that y_{m-1}^{∞} can change for another critical point at infinity of the same index over a cancellation, but cannot be destroyed or replaced by a periodic orbit of the same index (over a cancellation of two critical points at infinity, the number of $\pm v$-jumps of the corresponding **cycles** must be the same because the cancellation occurs at infinity, not along H_0^1-directions).

However, the definition of $\tilde{\partial}_{\text{per}}$ is not "sturdy" and at even indexes, we cannot rule out that this homology might change through cancellations or creations of periodic orbits. At odd indexes, this does not happen as stated in Theorem 1.3. Theorem 1.3 is proved in Part II of this paper.

We claim that three key results hold over this process:

First, over creations/cancellations $x_{2k} - x_{2k-1}$, no tangency $x_{2k-1} - y_{[2k-1}$ is possible despite the fact that the unstable manifold of x_{2k-1} is undergoing a rapid change as described above.

Next, if a rhombus is created, a special relation $\partial c_{2k}^{\infty} = c_{2k-1} + h_{2k-1,\infty}^{t}$ must hold, where $h_{2k-1,\infty}^{t}$ is a collection of critical points at infinity of index $(2k-1)$ whose unstable manifold is parametrized with the use of $(2k)$ trackable $\pm v$-jumps (for all critical points of $h_{2k-1,\infty}^{t}$ together; this relation should hold from the time zero of the deformation. This implies, by direct checking that it does not hold for the standard contact form on S^3, the proof of Rabinowitz theorem [29] on S^3 for convex Hamiltonians. The proof can be extended, see section 6 of [8], to general star-shaped Hamiltonians.

Finally , as a rhombus $x_{2k+1}/c_{2k}^{\infty}/y_{2k}/c_{2k-1}$ is formed, we establish below, under minimal conditions that are always verified, e.g. for the first exotic contact structure of Gonzalo and Varela on S^3 [14] that c_{2k}^{∞} is never equal to $(\delta + x_{2k-1})^{\infty}$, where x_{2k-1} is a periodic orbit of index $(2k-1)$ in c_{2k-1}. $(\delta + x_{2k-1})^{\infty}$ is a critical point at infinity associated with the Fredholm violation along x_{2k-1}. These results are established in what follows:

Lemma 2.13 *As $W_u(x_{2k-1})$, along a x_{2k}/x_{2k-1} creation, is isotopically deformed to its standard form, no tangency $x_{2k-1} - y_{2k-1}$ occurs, so that the intersection number $x_{2k} - y_{2k-1}$ remains zero, for y_{2k-1} different from x_{2k-1}, if y_{2k-1} is not dominated by another y_{2k} with a non-zero intersection number*

Next, we claim that

Lemma 2.14 *Given $y_{2k-1} = c_{2k-1}$ a periodic orbit or a combination of periodic orbits of index $(2k-1)$ that has an intersection number equal to zero with every periodic orbit of index $2k$, y_{2k}, assume that c_{2k-1} cannot be written as $c_{2k-1} = \partial y_{2k}^{\infty} + h_{2k-1}^{\infty}$ where y_{2k}^{∞} is a critical point at infinity or a combination of critical points at infinity of index $2k$ whose unstable manifold is achieved with the help of $(2k+1)$ trackable $\pm v$-jumps and where h_{2k-1}^{∞} is a critical point at infinity or a combination of critical points at infinity of index $2k$ whose unstable manifold is achieved with the help of $2k$ (not $(2k+1)$) trackable $\pm v$-jumps. Then, no rhombus $x_{2k+1}/x_{2k}/x_{2k}^{\infty}/c_{2k-1}$ is formed along the isotopy.*

Proof of Lemma 2.13 Analyzing cancellations/creations x_{2k}/x_{2k-1}, we view the direction of degeneracy as provided with a line of curves having a single $\pm v$-jump. This line of curves connects x_{2k} and x_{2k-1} on one side, dropping off x_{2k} on the other side.

This direction of degeneracy is located at a point along x_{2k} where the v-rotation around the periodic orbit is a bit larger than $2k\pi$ (whereas, on most of x_{2k}, this v-rotation is less than $2k\pi$ because of the degeneracy with the elliptic x_{2k-1}). A single $\pm v$-jump located at such a point defines a decreasing direction at x_{2k} and an increasing direction at x_{2k-1}).

Let $\bar{x}$ be this point or rather this collection of points, which correspond to each other as we move from x_{2k} to x_{2k-1}. At x_{2k-1}, we may achieve the unstable manifold with functions η in the space $H_0^1[\bar{x}, \bar{x}+1]$. These functions are defined by $(2k-1)$ jump conditions $\dot{\eta}(t_i^+-) - \dot{\eta}(t_i^-) = c_i$, taken at $(2k-1)$ precise times t_i in the interval $[\bar{,}x + 1]$ (whereas $\eta(\bar{x}) = 0$).

This unstable direction may be followed as the curves move ascending (along this single flow-line connecting x_{2k} to x_{2k-1}) from x_{2k-1} to x_{2k}. At x_{2k}, these define $(2k-1)$ unstable directions to which we should add the direction defined by the single $\pm v$-jump at $\bar{x}$.

It follows that the unstable manifold of x_{2k-1} may be achieved, all along the isotopy, with the use of $2k \pm v$-jumps, one of them at $\bar{x}$. But it also follows that, all along the isotopy, on the boundary of a small unstable disk in $W_u(x_{2k-1})$, we may assume that one $\pm v$-jump to the least, **which is distinct from the $\pm v$-jump at $\bar{x}$**, is non-zero. Indeed, at the degeneracy and nearby, the curves with a single $\pm v$-jump at $\bar{x}$ are in the unstable dimensions at x_{2k-1}.

Coming then down, with such configurations, in the vicinity of y_{2k-1} and tracking the $2k \pm v$-jumps of $W_u(x_{2k-1})$, we locate the attractive direction E^+ for y_{2k-1} for these configurations at the $\pm v$-jump that was tracked from $\bar{x}$. Since one $\pm v$-jump besides this E^+-direction is non-zero, $W_u(x_{2k-1})$ can never dominate y_{2k-1} and the conclusion follows. □

Proof of Lemma 2.14 Considering a periodic orbit or a collection of periodic orbits c_{2k-1}, which we assume not to be dominated by a periodic orbit of index $2k$ with a non-zero intersection number, this cannot change through tangencies since x_{2k-1} cannot be dominated by another y_{2k}—there are no z_{2k} to dominate it to start with and tangencies/dominations $z_{2k} - z_{2k}^{\infty} - c_{2k-1}$ are forbidden—and it cannot be dominated over the isotopy by another y_{2k-1} (Lemma 3 of [4]) nor with a y_{2k-1}^{∞} that is dominated by an x_{2k}.

It follows that, outside of cancellations/creations of periodic orbits, it cannot be dominated by an x_{2k}.

Through a creation $y_{2k} - y_{2k-1}$, this domination cannot occur by Lemma 2.13.

However, if a creation $y_{2k+2} - y_{2k}$ occurs above c_{2k-1} and if a domination $y_{2k}^{\infty} - c_{2k-1}$ already exists or has come to existence through a $y_{2k}^{\infty} - y_{2k-1}^{\infty} - c_{2k-1}$ domination/tangency—Observe that the domination/tangency described above implies that the unstable manifold of y_{2k-1}^{∞} is achieved with $(2k+1) \pm v$-jumps or more. Otherwise, by the results of Sect. 2.3, it cannot take place—then we can imagine that, as $W_u(y_{2k})$ is still evolving to its final form, a domination/tangency $y_{2k} - y_{2k}^{\infty} - c_{2k-1}$ occurs. This process gives rise to a rhombus $y_{2k+1}/y_{2k}/y_{2k}^{\infty}/c_{2k-1}$ that violates $\partial_{\text{per}} \circ \partial_{\text{per}} = 0$.

We claim that this must become undone when $W_u(y_{2k})$ is achieved with $2k \pm v$-jumps, that is at the end of this process of adjustment.

Indeed, observe that, over the tangency $y_{2k} - y_{2k}^{\infty} - c_{2k-1}$, all of ∂y_{2k}^{∞} is dominated by y_{2k} after the tangency is completed. Therefore, if the rhombus is still there at the end of the adjustment process, ∂y_{2k}^{∞} is still dominated then by y_{2k} and its unstable manifold may be achieved with these of $2k \pm v$-jumps, whereas the unstable manifold of y_{2k}^{∞} has to use $(2k+1) \pm v$-jumps, not less (one of these $\pm v$-jumps is a steady orientation $\pm v$-jump at $\bar{x}$); otherwise, the tangency cannot take place.

Thus y_{2k}^{∞} is a chain of Γ_{4k+2}, dominating c_{2k-1} with an intersection number equal to 1 and the remainder of its boundary is achieved with the use of $2k$-trackable $\pm v$-jumps. In addition, the $\pm v$-jump from $W_u(y_{2k}^{\infty})$ that is lost over the domination of h_{2k-1}^{∞} is the additional one, with a steady orientation, introduced at an appropriate point $\bar{x}$ of the degenerating pair denoting above x_{2k}/x_{2k-1}.

We write

$$\partial y_{2k}^{\infty} = c_{2k-1} + h^{\infty},$$

where $h^{\infty} \in \Gamma_{4k}^{\text{track}}$, with obvious meaning of the notations. This holds true before the creation/elimination has taken place, at a time t_0^- of the isotopy (as well as t_0^+, after the creation/elimination is complete). We come back now to the time zero of the isotopy, assuming, without loss of generality that no other degeneracy of periodic orbits occurs. y_{2k}^{∞} might undergo changes, but it will never disappear and its unstable manifold cannot be achieved, because it dominates c_{2k-1} with less than $(2k+1) \pm v$-jumps.

What about h^{∞}? If the unstable manifold of one of the critical points at infinity of h^{∞} was achieved earlier with $2k \pm v$-jumps, can this change along the isotopy?

Let us analyze what can happen to h^{∞} over the isotopy: h^{∞} can be modified through tangencies between critical points at infinity of index $(2k-1)$, but its unstable manifold will still be achieved with the use of $2k$ trackable $\pm v$-jumps through such a process. Eliminations at infinity cannot change this feature since the unstable manifolds of the eliminating critical points at infinity are then achieved with the same number of $\pm v$-jumps and these are $2k$ at most when we start from t_0^-, moving backwards in time. It cannot happen also through the more subtle phenomenon of "collisions" and "transmutations", as in [4], pp 126–136, of a critical point at infinity of a kind with a critical point at infinity of another kind (typically having a different number of characteristic pieces): it is not very difficult to see, we refer to our detailed analysis of collisions in [4], that the unstable manifold at infinity of the one dominated by y_{2k}^{∞} is continuous through the "collision" and, therefore, the index at infinity $(s-1)$ (which drops by 1 with respect to the domination by y_{2k}^{∞} in the appropriate Γ_{2m} with a dominating index s)of this critical point is continuous through the collision, so is its H_0^1-index and, therefore, so is the number of the $\pm v$-jumps needed to describe its unstable manifold at infinity.

Since y_{2k}^{∞} has no boundary besides h^{∞}, which does not change nature, and c_{2k-1}, the boundary of y_{2k}^{∞} can change only through tangencies with $z_{2k}^{(\infty)}$. This only recomposes the chain into a new chain with the same boundary. We, therefore, find that the following relation should hold at time 0 of the deformation:

$$y_{2k}^{\infty}(t_0^-) - (y_{2k}^{\infty}(0) + \Sigma W_u(z_{2k}^{(\infty)})) = \phi^{\infty} + \partial w,$$

where $\phi^{\infty} \in \Gamma_{4k}^{\text{track}}$.

∂w, being a boundary, has an intersection number equal to zero with c_{2k-1}. ϕ^{∞}, being in $\Gamma_{4k}^{\text{track}}$, cannot dominate c_{2k-1}. The intersection number of $y_{2k}^{\infty}(t_o^-)$ with c_{2k-1} is equal to the intersection number of $y_{2k}^{\infty}(0) + \Sigma W_u(z_{2k}^{(\infty)})$ with c_{2k-1} and the relation $\partial y_{2k}^{\infty}(t_0^-) = c_{2k-1} + h^{\infty}$ is essentially unchanged, with y_{2k}^{∞} replaced by $y_{2k}^{\infty}(0) + \Sigma W_u(z_{2k}^{(\infty)})$.

Clearly, if c_{2k-1} is a periodic orbit that has been created "from bottom", that is c_{2k-1} has been created through a degeneracy with a periodic orbit x_{2k-2}, then we may assume that at time zero no y_{2k}^{∞} dominates c_{2k-1} since all of its stable manifold is in the neighbourhood then of a flow-line (no need for later adjustment on this side) and, therefore, such a relation is impossible. A periodic orbit x_{2k-1} that has been given birth to through a creation with an x_{2k-2}, if it later undergoes a cancellation, must undergo a cancellation "from bottom", with a y_{2k-2}. More generally, if c_{2k-1} is a collection of periodic orbits that has no intersection number with any y_{2k} and if the Morse relation $\partial y_{2k}^{\infty} = c_{2k-1} + h^{\infty}$, with $h^{\infty} \in \Gamma_{4k}^{\text{track}}$, does not hold, then no rhombus can be formed with c_{2k-1}. As we move back along the time of the isotopy, we track the relation:

$$\partial y_{2k}^{\infty} = c_{2k-1} + h_{2k-1,\infty}^{t}$$

and we check that this relation cannot disappear. Observe that the parametrization of $W_u(y_{2k}^{\infty})$, because it dominates c_{2k-1} requires the use of at least $(2k+1) \pm v$-jumps and observe that such a y_{2k}^{∞} cannot therefore cancel with $h_{2k-1,\infty}^{t}$ whose unstable manifold requires $2k \pm v$-jumps at most. Therefore, this relation cannot disappear through a $y_{2k}^{\infty} - h_{2k-1,\infty}^{t}$ cancellation, neither can it disappear through a $y_{2k}^{\infty} - c_{2k-1}$ cancellation: c_{2k-1} is made of periodic orbits and periodic orbits cancel only among themselves.

It cannot disappear also through a $y_{2k+1}^{\infty} - y_{2k}^{\infty}$ cancellation: then, y_{2k}^{∞} would have to be replaced by $z_{2k}^{(\infty)}$. $z_{2k}^{(\infty)}$ cannot be a periodic orbit z_{2k} since we are assuming that the intersection number $i(z_{2k}, c_{2k-1})$ is zero for all z_{2k}s: we are before any creation $z_{2k+1} - z_{2k}$ that might create a non-zero intersection number $z_{2k} - c_{2k-1}$. Being then a z_{2k}^{∞} and dominating $c_{[2k-1}$ that is not empty, its unstable manifold requires at least $(2k+1) \pm v$-jumps.

We now discuss the phenomenon of collisions, see [4], pp 126-136, when two critical points at infinity, one with s characteristic ξ-pieces and the other one with $(s-1)$ characteristic ξ-pieces may collide. Over the "collision", both have to survive; we might then create a critical point at infinity with a higher H_0^1-index, requiring more $\pm v$-jumps to describe its unstable manifold.

Typically, we then have two critical points "colliding", one of index at infinity i_{∞} and the other one with index at infinity $(i_{\infty} - 1)$. One, labeled C_1, has more characteristic ξ- pieces than the other one, labeled C_2, so that they both survive the collision. However, their "roles" are exchanged over the collision. We study in detail the case when C_1 has one more characteristic ξ-piece than C_2. Since we are deforming our variational problem over a path, we may assume by general position that only one ξ-piece at a time changes nature.

The only possibility then for the H_0^1-index to change and increase is an increase for C_2 from i_0 to $(i_0 + 1)$. Observe than C_2 does not degenerate in the full space of variations.

We then find that, over the "collision", we had a configuration $(C_2, i_{\infty}, i_0)/(C_1, (i_{\infty} - 1), i_0$ that became $(C_1, i_{\infty}, i_0)/(C_2, (i_{\infty} - 1), i_0$

All the H_0^1-indexes are the strict H_0^1-indexes, the only ones that matter are the ones related to the ξ-piece that changes nature. On all the other ones, whether the domination occurs with the H_0^1-flow or whether it involves the New Hole Flow [4], the "collision" does not change the nature of the domination, neither does it change the number of $\pm v$-jumps involved.

The only process through which the number of $\pm v$-jumps might increase stems from the fact that the description of $W_u(y_{2k}^{\infty})$ might require more than $2k \pm v$-jumps. Therefore, over the domination $y_2k^{\infty} - h_{2k-1,\infty}^{t}$, some $\pm v$-jump might be "lost". If this happens over the ξ-piece that changes nature and its H_0^1-index increases, then maybe the "lost" $\pm v$-jump might, after the collision, find "room" to occupy some H_0^1-position. Then $h_{2k-1,\infty}^{t}$ might over the collision disappear as a critical point at infinity in our Morse relation, or it might change nature.

Of course, this requires then that all the strict H_0^1-index positions over the changing nature ξ-piece are "filled" over the domination. The additional one(s) have to be "lost" over the domination.

These "collisions" could even involve y_{2k}^{∞} and some critical point at infinity $h_{2k-1,\infty}^{t}$ or they could involve some critical point at infinity of $h_{2k-1,\infty}^{t}$ and some other critical point at infinity.

If the collision involves y_{2k}^{∞}, then y_{2k}^{∞} must be above the corresponding critical point at infinity in $h_{2k-1,\infty}^{t}$; therefore, it is the one having the higher index at infinity. We then recognize that it is (C_2, i_∞, i_0) before the collision, dominating $(C_1, (i_\infty - 1), i_0)$. After the collision, y_{2k}^{∞} survives as (C_1, i_∞, i_0). Observe that, before the collision, the additional $\pm v$-jump of C_2 was already "lost" over C_2, if there were an additional one. This does not change as C_2 turns into C_1 (by continuity of the chain defined by z_{2k}^{∞}) and therefore no additional "lost" $\pm v$-jump will appear in $(C_2, (i_\infty - 1), i_0 + 1)$ since it is already "lost" as we reach C_1, before the collision.

The other case that we need to consider is the case when $h_{2k-1,\infty}^{t}$ has a collision **below** z_{2k}^{∞} with some other critical point at infinity. Again, the only "dangerous" scheme is the scheme when $h_{2k-1,\infty}^{t}$ becomes $(C_2, (i_\infty - 1), i_0 + 1)$. Before the collision, $h_{2k-1,\infty}^{t}$ could have been $(C_1, (i_\infty - 1), i_0)$, with one "lost" $\pm v$-jump. There could be a discontinuity in the chain through the collision.

However, we observe that we can modify the chain y_{2k}^{∞} with the addition, before the collision, of $W_u(C_2)$. C_2, being of index at infinity i_∞ and of H_0^1-index i_0 (before the collision) dominates $h_{2k-1,\infty}^{t}$ which is then $(C_1, (i_\infty - 1), i_0)$, with the same number of $\pm v$-jumps, $2k$ of them at most that can be tracked exactly as the ones of $h_{2k-1,\infty}^{t}$ are tracked. Indeed, the domination $(C_2, i_\infty, i_0) - (C_1, (i_\infty - 1), i_0)$ is at infinity and the $\pm v$-jumps, therefore, are in one to one correspondence. Observe that $W_u(C_2)$ cannot dominate any new x_{2k-1}; neither can it dominate c_{2k-1} since it is built with $2k \pm v$-jumps and C_2 is at infinity. It follows that the new chain $W_u(z_{2k}^{\infty}) \cup W_u(C_2)$ has the same features than $W_u(z_{2k}^{\infty})$. Furthermore, after the collision, it is continued by $W_u(z_{2k}^{\infty}) \cup W_u(C_1)$., with the same features. The claim follows. □

2.5.1 y_{2k}^{∞} in the above relation cannot be $(x_{2k-1} + \delta)^{\infty}$

This sub-section as well as the next one assume the results of Sect. 2.11 below:

Assume that y_{2k}^{∞} in the rhombus $y_{2k+1}/c_{2k}^{\infty}/y_{2k}/c_{2k-1}$ is a critical point at infinity of the type $(x_{2k-1}+\delta)^{\infty}$, that is it is a critical point at infinity associated with a Fredholm violation at x_{2k-1}, which is typically part of the cycle c_{2k-1} for ∂_{per}.

This rhombus has been formed through the creation of the pair y_{2k+1}/y_{2k} (for simplicity) through a tangency $y_{2k} - (x_{2k-1} + \delta)^{\infty} - c_{2k-1}$.

$W_u(y_{2k})$ is, through this creation, formed with $(2k + 1) \pm v$-jumps, instead of $2k \pm v$-jumps; this why the tangency can occur, whereas, if $W_u(y_{2k})$ were built only with $2k \pm v$-jumps; this tangency would be impossible.

Assume that the tangency $y_{2k} - (x_{2k-1} + \delta)^{\infty}$ occurs in Γ_{2s}, ie in the space of curves made of $s\xi$-pieces alternating with $s \pm v$-jumps.

In Γ_{2s}, the index of y_{2k} is $(s - 1)$ (in this transition period, when $W_u(y_{2k})$ is formed with the help of $(2k + 1) \pm v$-jumps).

If $s = 2$, then the index of y_{2k} in Γ_4 is 1, whereas, if we assume that the Fredholm assumption is violated along x_{2k-1} only **on part** of x_{2k-1}, we may assume that the index of $(x_{2k-1} + \delta)^{\infty}$ in γ_4 is 2: one for the "deconcentration" process, that is 1 for the ability of decreasing J_∞ by "opening up" its "Dirac mass" once it is large enough; the other one because the "Dirac mass" δ cannot be located at an arbitrary point on x_{2k-1}. At some points, the Fredholm assumption is verified in the direction of δ (positive or negative).

Therefore, tangency of y_{2k} and $(x_{2k-1} + \delta)^{\infty}$ cannot be achieved in Γ_4. some third $\pm v$-jump (non-zero) is involved. Therefore, a repetition can be identified and used as in the case of "simple dominations" (Sect. 2.3) to "bypass" c_{2k-1} out of $(x_{2k-1} + \delta)^{\infty}$.

However, this pseudo-gradient is related to a choice of γ (observe here that $W_u((x_{2k-1} + \delta)^{\infty})$ cannot dominate another y_{2k-1} because it can be described with the help of $2k \pm v$-jumps, two of them to the least steady), see Sect. 2.3 and Sect. 2.11, below, and the choice of γ varies with the repetition, so that we need to be able to decrease the whole set of configurations as we switch the value of γ. This is not always possible in the general framework since it is not always possible to perform a "re-arrangement" that preserves a given repetition in the complement of two neighbouring values of γ, γ_0 and γ_1.

In our present case, here, γ_0 and γ_1 can be taken to be one of the $\pm v$-jumps of the "Dirac mass" δ. These are two very close $\pm v$-jumps and it follows that the re-arrangement of the whole set of configurations as in Sect. 2.11 can then be performed whereas a given repetition in their complement is preserved. It follows that triangles $y_{2k} - y_{2k}^{\infty} - c_{2k-1}$ are not possible when $y_{2k}^{\infty} = (x_{2k-1} + \delta)^{\infty}$ and the claim holds.

Finally, we observe the following about this type of Morse relation:

2.5.2 "Point to circle" Morse relations $x_{2k+1} - y_{2k}^{\infty}$ in "triangles" $x_{2k+1}/y_{2k}^{\infty}/c_{2k-1}$

The arguments of Sect. 2.3 show that y_{2k}^{∞} cannot be a single critical point at infinity; it must be made of a collection of such critical points of index $2k$.

The arguments of Sect. 2.11, below, show more: namely, if x_{2k}^{∞} and $\tilde{x}_{2k}^{\infty}$ are two "neighbouring" critical points at infinity in the collection y_{2k}^{∞} of critical points at infinity (they then have some boundary in common), it is not always possible to switch the value of the $\pm v$-jumps γ associated with each of them by the above argument and keep the repetition in the complement of these γs. Over the transitions, typically over some critical point at infinity x_{2k-1}^{∞}, this switch should not be possible. This implies in particular that the number of sign-changes over the $\pm v$-jumps of the unstable manifold $W_u(x_{2k-1}^{\infty})$ should be $2k$ not less; otherwise, the switch is possible.

Some more arguments show, see Sect. 2.11, that, over these transitions, the repetition in the complement of γ has to "travel" along the $(2k+1) \pm v$-jumps and complete a full circle. It cannot be recognized to live within a strict subset of these $(2k+1)$-trackable $\pm v$-jumps. It must, therefore, be that the Morse relation y_{2k+1}/y_{2k}^{∞} as well as the Morse relation y_{2k}/y_{2k}^{∞} (which is a tangency occurring over the deformation, y_{2k}^{∞} is described using $(2k+1)$, not v$2k$ $\pm v$-jumps over a creation/cancellation (y_{2k+1}/y_{2k}))are "point to circle " Morse relations.

2.5.3 Morse relations and lack of transversality

We want here to track the effect of the lack of transversality on the Morse relations that we have been studying. We will focus on tangencies or Morse relations of the type

$$w_{2k+1} - w_{2k}^{\infty},\, w_{2k+1} - w_{2k+1}^{\infty},\, w_{2k} - w_{2k}^{\infty}$$

Considering $w_{2k+1} - w_{2k}^{\infty}$ and $w_{2k} - w_{2k}^{\infty}$, the third term in the Morse relations is then w_{2k-1}.

From our analysis above, if we reach a neighbourhood of w_{2k-1} with $2k{*}$s, one of them being a single $\pm v$-jump γ and another one being non-zero, then the flow-lines can be made to avoid w_{2k-1}.

In the case of $w_{2k} - w_{2k}^{\infty}$, we have $2k{*}$s starting from w_{2k}. Any "first" critical point at infinity w^{∞} encountered along the flow-lines $w_{2k} - w_{2k}^{\infty}$ (with $w^{\infty} = w_{2k}^{\infty}$ if needed) will provide a γ and another $*$ (using an edge of w^{∞}). Therefore, all these flow-lines will avoid w_{2k-1}.

Considering $w_{2k+1} - w_{2k}^{\infty}$ or $w_{2k+1} - w_{2k+1}^{\infty}$, we start with $(2k+1){*}$s.

Considering the "first" critical point at infinity w^{∞} encountered along these flow-lines, we know that if the covering of the edges of w^{∞} requires at least three $*s$, we can then designate a single $\pm v$-jump to spare as a γ and we have a forced repetition among the other $*s$ once γ is taken away.

These flow-lines cannot then reach w_{2k}, a simple hyperbolic orbit.

The choice of γ and the additional repetition can be thought of also—we used this above and detailed the argument—as having a simple $\pm v$-jump γ and $(2k-1)$ other $*s$, after relabeling, re-ordering, creating families, one of them to the least non-zero.

Thereafter, these can be tracked, even after w_{2k}^{∞} in the $w_{2k+1} - w_{2k}^{\infty}$ relation . Again, these flow-lines cannot reach w_{2k-1} (with a suitable flow).

If the edges of w^{∞} contain a repetition (no $*$ between them, same orientation), the choice of γ and of another non-zero $*$ extends.

Therefore, w^{∞} can only have one or two edges. The results of Sect. 2.3 above take care of these w^{∞}s and, therefore, the matter is settled here.

However, we want here to explore another direction where we try to overcome the lack of transversality with the use of companions. This will not lead to a framework where the homology is well defined, but it will lead to an interesting configuration that is worth describing since it can be useful in later studies.

In case w^{∞} has two edges, they cannot be separated by $*s$ on both sides, see above; the corresponding $*s$ must be immediate neighbours and they must have the reverse orientation.

In both cases, whether w^{∞} has one or has two edges, if there is lack of transversality, then the H_0^1-index of w^{∞} (the strict one) must be larger than $2k$ for one edge, larger than $(2k-1)$ for two edges. Otherwise, we find enough $*s$ to cover the H_0^1-index. On the other hand, transversality is achieved in $\cup\Gamma_{2m}$.

Since the strict H_0^1-index is larger than the number of $*s$ available, since these $*s$, besides the $*s$ of the edges, reach w^{∞} as zero $\pm v$-jumps (otherwise, we could choose a γ, etc), we can arrange so that they cover various index positions on the ξ-pieces, but leave one position of H_0^1-index near an edge not filled.

We can then "bypass" w^{∞} coming from w_{2k+1} by filling this position with a companion to this edge.

We might then reach another w^∞, w_1^∞, for which we can repeat the same arguments, only that one of the $*s$ is now a family. This family has to represent two edges of w_1^∞, having the same orientation. No $*$ of w_{2k+1} separates these two edges. Either there is a non-zero H_0^1 index in between these two edges, we can fill an H_0^1-position with a companion and decrease J; or there is none, none also separating these two edges from the other edge representing the steady other $*$ of w^∞—otherwise, we introduce a companion to the $*$ that is already a family in the unfilled position—and we can resume the argument used for w^∞ on the ξ-piece separating these two steady $*s$.

It might also happen that, after w_1^∞, we reach a w_2^∞ that has an additional edge represented by a third $*$.

Then, this third $*$ has to be an immediate neighbour to the two other ones, with a given orientation implying that there is a forced repetition; that is now we have three steady $*s$, $*_1, *_2, *_3$ which are immediate neighbours. $*_2$ is a family and $*_1$ and $*_3$ have the same orientation.

Otherwise, if, for example, $*_2$ is not a family but is a single $\pm v$-jump, we can take it to be γ, etc. The orientations of $*_1$ and of $*_3$ must be the same to force the repetition on the outside interval. Otherwise, $*_1$ and $*_2$ or $*_2$ and $*_3$ have the same orientation. $*_3$ and $*_1$ can be taken for γ, depending on the cases, etc.

An induction can be started. Neighbouring steady $*s$ with alternating signs are given birth to. All the intermediate $*s$ among these are families, the two extreme $*s$ can be single $\pm v$-jumps. They have the same orientation if the number of these steady neighbouring $*s$ is odd, opposite orientations otherwise.

With this type of configurations, transversality is overcome and we reach w_{2k}^∞. As we pointed above, this is not what we have been doing here; but this can nevertheless be an interesting observation.

2.6 An abstract deformation argument

We present now an abstract deformation argument that allows us to conclude, from our constructions and from our arguments above, that along the sequence $w_{2k+1} - w_{2k+1}^\infty - w_{2k}$, the intersection number $w_{2k+1}^\infty . w_{2k}$ is zero.

Here, w_{2k+1}^∞ is such that a $*$ denoted γ can be spared as a single $\pm v$-jump, with an outside repetition taking place, see above.

It is clear from the arguments used above that, along such a sequence, assuming that w_{2k} is the first simple hyperbolic periodic orbit reached by the flow-lines originating at w_{2k+1}^∞, this intersection number is zero. However, the argument used to reach this claim uses the decomposition $F^+(\sigma) \oplus F^-(\sigma)$ of section above.

The related flow does not respect the repetition of σ outside of γ. Thus, the flow-lines originating from a neighbourhood of w_{2k}—which can be taken to be as small as we please—even if they are in $W_u(w_{2k+1}^\infty)$, might not have anymore the property that a repetition takes place outside of γ.

The argument about the intersection number cannot be repeated then.

Of course, we expect this set of flow-lines (they come from a small neighbourhood of w_{2k+1}^∞) to be of dimension $2k$ or to be in a small neighbourhood of a set of dimension $2k$. Therefore, we might expect, using a general position argument, that these flow-lines do not actually go to some w'_{2k}, where w'_{2k} is another simple hyperbolic periodic orbit of index $2k$.

We need to turn this into a rigorous argument.

The idea for this is to create a deformation argument near w_{2k} so that all the flow-lines coming from w_{2k+1}^∞ either will continue, past w_{2k}, with γ and the repetition outside of γ spared; or, if not, they will be part of an exit set of dimension $2k$ at most.

The local decomposition $F^+(\sigma) \oplus F^-(\sigma)$, with the related pseudo-gradient for J_∞, allows then to conclude that w_{2k+1}^∞ covers w_{2k} with zero degree, whereas the use of the flow as described above allows us to proceed with the induction.

Indeed, all flow-lines of w_{2k+1}^∞ exiting then a neighbourhood of w_{2k} either are part of a set of dimension $2k$. Using a general position argument, these flow-lines will not go and reach a w'_{2k}.

The other remaining flow-lines of w_{2k+1}^∞ that will reach w'_{2k}, even if they are coming from a neighbourhood of w_{2k}, will have γ and an outside repetition spared, so that the argument can be repeated inductively.

We proceed now with the construction of the flow near w_{2k}:

The $(2k+1)*$s, one of them γ, are equally spaced near w_{2k} and there is a repetition outside of γ. Their algebraic sizes are $c_1, c_2, .., c_m$.

In order to spare γ and the repetition, we build our pseudo-gradient so that each constraint $c_j = 0$ is respected, that is if the jth $\pm v$-jump (we could count starting from γ, along $+\xi$) of a configuration σ is a zero $\pm v$-jump, then c_j is also zero for the deformed configurations originating at σ along the decreasing pseudo-gradient.

Under these constraints, several additional rest points are created. We need to prove that the related exit sets are of dimension $2k$ at most.

These exit sets are not the unstable manifolds of these additional rest points only. They also contain the decreasing normals—which violate the constraint $c_j = 0$—along these rest points. The claim is that all the exit set, unstable manifold and its span under the decreasing normals, all of it taken together, is of dimension $2k$ at most.

Let us consider the second derivative of J_∞ at w_{2k} on Γ_{4k+2} and, more particularly, on the equally spaced configuration σ which we find on $W_u(w^\infty_{2k+1})$ near w_{2k}.

The index of the second derivative at such a σ is $2k$. Indeed, it cannot be more; otherwise, the second derivative of J on $T_{w_{2k}}C_\beta$ would have a negativity larger than $2k$; a contradiction.

On the other hand, there is at least one $\pm v$-jump of σ that is located at a point $\bar{t}_{i_0}$ where the v-rotation around w_{2k} is more than $2k\pi$. Considering then the configurations $\Sigma_{i=1}^{2k+1} d_i \delta_{\bar{t}_i}$ of $(2k+1) \pm v$-jumps of sizes $d_1, \ldots, d_{2k+1}$ located exactly where the $\pm v$-jumps of σ are located and requiring that

$$\sum_{i=1}^{2k+1} d_i \eta_i(\bar{t}_{i_0}) = 0$$

(η_i solves $-(\ddot{\eta}_i + \eta_i \tau) = \delta_{t_i}$, η_i 1-periodic),

we span the so-called H_0^1 (based at $\bar{t}_{i_0}$) subspace of $\oplus_{i=1}^{2k+1} \mathbb{R}\eta_i$. We know that the second derivative is negative along this space, which is of dimension $2k$. This follows from the fact that the v-rotation at $\bar{t}_{i_0}$ is more than $2k\pi$ and from the fact that the $\bar{t}_i$s are equally spaced. For the same reason, η_{i_0} is a positive direction for the second derivative.

It follows that, under the constraint $\Sigma_{i=1}^{2k+1} c_i^2 = \theta$, θ a small positive constant and at any positive critical value of the second derivative $D^2 J_\infty(w_{2k})$ on the spheres around zero of $\oplus_{i=1}^{2k+1} \mathbb{R}\eta_i$—a positive critical value ia a positive eigenvalue—the index of $D^2 J_\infty(w_{2k}).\eta.\eta$ is exactly $2k$.

If none of the c_is is zero, we can use the flow provided on these spheres by $D^2 J_\infty(w_{2k}).\eta.\eta$ as a pseudo-gradient for J_∞ on the space of curves having their $\pm v$-jumps located at the $\bar{t}_i$s; the two functionals are very close. a Morse Lemma holds.

This pseudo-gradient will spare γ and the repetitions.

Because the negativity of $D^2 J_\infty(w_{2k})$ on this space is $2k$ and its positivity is 1, there is no zero eigenvalue. All the eigenvalues of $D^2 J_\infty(w_{2k})$ are non-zero. The positive one is entirely disconnected from the set of negative ones.

At any of these negative eigenvalues, $D^2 J_\infty(w_{2k}).\eta.\eta$ can be decreased by a dilation of η, $\eta \to c\eta$, $c \geq 1$. J_∞ decreases in this way and the c_is are expanded. They remain non-zero if they were non-zero to start with.

Along the positive eigenvalue, $D^2 J_\infty(w_{2k}).\eta.\eta$ increases radially. Therefore, if we want to decrease $D^2 J_\infty(w_{2k}).\eta.\eta$, we are led to decrease η and we reach w_{2k}. Transversally to this radial direction, the index is $2k$ all over these configurations.

There is a subtle phenomenon to be understood here: the equally spaced positions $(\bar{t}_1, \ldots, \bar{t}_{2k+1})$ form a circle. At each of these $(2k+1)$-positions, $D^2 J_\infty(w_{2k}).\eta.\eta$ has a unique positive eigenvalue and a unique associated normalized eigenvector on the sphere $\Sigma_{i=1}^{2k+1} c_i^2 = \theta, \theta \gneq 0$. We can track this eigenvector as we rotate along the circle of $(2k+1)$ equally spaced positions. The index along the $2k$-dimensional sphere $\Sigma_{i=1}^{2k+1} c_i^2 = \theta, \theta \gneq 0$ of $D^2 J_\infty(w_{2k}).\eta.\eta$ is unchanged, equal to $2k$.

Of course, some of the components c_i of the positive eigenvector might cross zero. We would not use a non-radial flow at such points because repetitions would not be spared. This is discussed below.

Overstepping this issue, this positive eigenvalue has at least one maximum and one minimum along this circle.

The circle direction $\frac{\partial}{\partial R}$ is a negative direction at a maximum. The $2k$-other negative directions are along the sphere $\Sigma_{i=1}^{2k+1} c_i^2 = \theta$. Putting them together, we would get a $(2k+1)$-dimensional negative space, a contradiction.

This in fact does not happen because this additional direction is not orthogonal to the $2k$-dimensional space tangent to the sphere $\Sigma_{i=1}^{2k+1} c_i^2 = \theta$. Some direct computations show that orthogonality (for $D^2 J_\infty(w_{2k})$) of this circle direction $\frac{\partial}{\partial R}$ to this $2k$-dimensional space requires additional conditions that are not met at a maximum or at a minimum of the positive eigenvalue on the circle, unless w_{2k} is degenerate, which we are not assuming.

This would settle our argument if we could change the c_is freely on the sphere $\Sigma_{i=1}^{2k+1} c_i^2 = \theta$.

However, this is not possible for us if one of the c_is is zero. We might then destroy our spared repetition outside of γ.

Therefore, if one or several c_is are zero, we need to build our flow so that it keeps these values to be zero; this introduces additional rest points.

Assume that s of these c_is are zero. Relabel them, for the sake of simplicity:

$$c_1 = c_2 = \cdots = c_s = 0$$

They have associated times $\bar{t}_1, \ldots, \bar{t}_s$.

Two possibilities arise then:

Assume first that the v-rotation around w_{2k} at one of these $\bar{t}_i$s for $i \in [s+1, 2k+1]$, $\bar{t}_{s+1}$ for example, is more than $2k\pi$. Then η_{s+1} is a positive direction for $D^2 J_\infty(w_{2k}).\eta.\eta$ and this direction is in the tangent space to the slice $\{c_1 = c_2 = \cdots = c_s = 0\}$ that we are considering. Since the positivity of $D^2 J_\infty(w_{2k})$ on the full space of ηs is 1, it cannot be more than 1 on the tangent space to the slice. The index in restriction to the slice is then $(2k - s)$. Up to now, the deformation spares the repetitions. But it has additional rest points that have unstable manifolds, in restriction to their slices, of dimension $(2k - s)$ at most.

Bypassing these rest points, we have to flow their unstable manifolds along the set of decreasing normals at these rest points. This is a set of dimension s at most and, therefore, we derive exit sets of dimension $2k$ at most in this case as claimed.

This argument works as well if the restriction of $D^2 J_\infty(w_{2k})$ to the tangent space to the slice has a positive eigenvalue.

If all the eigenvalues are non-zero negative (in restriction to the slice), we can decrease $D^2 J_\infty(w_{2k}).\eta.\eta$ and J_∞ by radial expansion. The exit set spares the repetitions in this case.

We are left with zero eigenvalues. For these, we can use the additional $\frac{\partial}{\partial R}$ direction to move the configuration:

at a critical point for the R-variable along the circle of evenly spaced configurations (for the restricted variational problems $D^2 J_\infty(w_{2k}).\eta.\eta$ and J_∞ along the slice), we may assume-by a general position argument-that the restriction of $D^2 J_\infty(w_{2k}).\eta.\eta$ to $\{c_1 = c_2 = \cdots = c_s = 0\}$ is non-degenerate. Thus, if we use also the R-direction, we will never encounter the zero eigenvalue case.

The only other cases are the cases discussed above.

The claim follows.

2.7 Singularities

Along a deformation of contact forms, the sets Γ_{2k} might undergo cobordisms and, therefore, singularities. We prove in this section that these singularities do not change the intersection numbers and, therefore, the homology.

We start with

Lemma 2.15 *Let* $x_m^{(\infty)}$ *and* $x_{m-1}^{(\infty)}$ *be two critical points (*at infinity*) of indexes* m *and* $(m-1)$ *in* Γ_{2k}.

If a singularity of Γ_{2k} *changes their intersection number and is generic, then this singularity must involve two critical points at infinity of indexes* 0 *and* $2k$.

Proof We first present the idea of the proof, then we expand the details of this proof until we reach a point where a normal form for this singularity (that changes the intersection number $x_m.x_{m-1}$) can be written. We then conclude invoking the initial argument.

Let x_r^∞ and x_s^∞ be the two critical points at infinity involved in the singularity, see [4], pp 127–130, for a preliminary study of these singularities and the behavior of x_r^∞ and x_s^∞[5] . r and s are their respective indexes. $J_\infty(x_r^\infty)$ is larger than $J_\infty(x_s^\infty)$.

$W_u(x_m^{(\infty)}) \cap W_s(x_r^\infty)$ is of dimension $(m-r)$, of dimension $(m-r-1)$ transversally (to a pseudo-gradient flow). $W_u(x_s^\infty) \cap W_s(x_{m-1}^{(\infty)})$ is of dimension $(s-m+1)$, of dimension $(s-m)$ transversally to a pseudo-gradient flow.

[5] All the ξ-pieces of these two critical points at infinity are "characteristic", [4], p 6. From one edge to the other edge of each their $\pm v$-jumps, ξ is mapped into $\theta\xi$ in the v-transport. The difference of their Morse indexes is even, [4], Proposition 26, p 127.

The singularity involves a collapse of x_r^∞ and x_s^∞ at a time t_0 along a deformation of Γ_{2k}.

At the time $t_0+\epsilon$, $\epsilon \gneq 0$, x_r^∞ and x_s^∞ have disappeared.

However, denoting c a level close to $J_\infty(x_r^\infty)$ (larger) and considering $E = W_u(x_m^{(\infty)}) \cap W_s(x_r^\infty) \cap J_\infty^{-1}(c)$, we can track E even after x_r^∞ is gone since the level $J_\infty^{-1}(c)$ deforms isotopically through t_0.

Similarly, denoting d a level close to $J_\infty(x_s^\infty)$ (below) and considering $F = W_u(x_s^\infty) \cap W_s(x_{m-1}^{(\infty)}) \cap J_\infty^{-1}(d)$, we can track F even after x_s^∞ is gone since the level $J_\infty^{-1}(d)$ deforms isotopically through t_0.

If the intersection number $x_m^{(\infty)}.x_{m-1}^{(\infty)}$ changes, then flowing down E to $J_\infty^{-1}(d)$, we find a set E_1 and $E_1\cap F$ must not be empty. Using a general position argument, this implies that

$$\dim E_1 + \dim F \geq 2k-1,$$

that is

$$\begin{aligned}(m-r-1)+(s-m) &\geq 2k-1\\ s-r &\geq 2k\end{aligned}$$

This implies that $s=2k$ and $r=0$, as claimed.

The scheme developed above is turned below into a rigorous argument.

In a first step, we observe that we must have $r \lneq s$. Indeed, the statement that the intersection number $x_m.x_{m-1}$ changes through the singularity implies that $r \lneq m$ and that $s \geq m$; these inequalities imply in turn that $r \lneq s$. Observe that, in addition, we already know that $r-s$ is even, see [4], Proposition 26, p 127.

In a second step, we claim that we must have $s = 2k - r$. The proof of this equality is somewhat involved. We provide the details below:

Let $d \lneq e \lneq c$ be three values such that

$$d \lneq J_\infty(x_s^\infty) \lneq e \lneq J_\infty(x_r^\infty) \lneq c$$

These values are all very close to each other. d and c are fixed values. The singularity of Γ_{2k} occurs through the collapse of x_r^∞ and x_s^∞ in $J_\infty^{-1}[d,e]$. No other phenomenon is occurring in this energy slice for the functional J_∞.

Define

$$A = J_\infty^{-1}(e)\cap\Gamma_{2k},\ B = J_\infty^{-1}(d)\cap\Gamma_{2k},\ B_1 = J_\infty^{-1}(c)\cap\Gamma_{2k}$$

A does not survive the singularity, but B and B_1 do. We might as well assume, without any loss of generality that they do not change over the deformation. After x_r^∞ and x_s^∞ have collapsed and are gone, B and B_1 are diffeomorphic.

Observe that we have the following homotopy equivalences (they are easily derived from the classical deformation lemma of Morse Theory, see, e.g. [16]):

$$\begin{aligned}A\cup D^r &\cong B\cup D^{2k-r}\\ A\cup D^{2k-s} &\cong B_1\cup D^s\end{aligned}$$

D^r and D^s are the unstable disks of x_r^∞ and x_s^∞ respectively, D^{2k-r} and D^{2k-s} are their stable disks.

The first homotopy equivalence implies readily that $H_\ell(A)$ and $H_\ell(B)$—we will take the coefficients to be real, for simplicity—are isomorphic for $\ell \neq r, r-1, 2k-r, 2k-r-1$.

The second one implies, in a similar way, that $H_\ell(A)$ and $H_\ell(B_1)$ are isomorphic for $\ell \neq s, s-1, 2k-s, 2k-s-1$.

Combining the two results, observing that $H_\ell(B) = H_\ell(B_1)$ and that $r \neq s$, $r-s = 2p$, we derive that either $r = 2k-s$ or that $H_\ell(A) = H_\ell(B)$ for every ℓ.

We now prove in a third step that we cannot have $H_\ell(A) = H_\ell(B)$ for every ℓ.

One of r or s is not k. Let us assume that it is, e.g. r.

Observe that, under this assumption, $2k-r$ is not r. Observe also that $2k-r\pm 1$ is not r as well.

We have the two long exact sequences:

$$\begin{aligned}&H_{2k-r+1}(D^r,S^{r-1}) = 0 \to H_{2k-r}(A) \to H_{2k-r}(A\cup D^r) \to H_{2k-r}(D^r,S^{r-1}) = 0\\ &\quad\to H_{2k-r-1}(A) \to H_{2k-r-1}(A\cup D^r) \to H_{2k-r-1}(D^r,S^{r-1}) = 0\end{aligned}$$

$$H_{2k-r+1}(D^{2k-r}, S^{2k-r-1}) = 0 \to H_{2k-r}(B) \to H_{2k-r}(B \cup D^{2k-r}) \to H_{2k-r}(D^{2k-r}, S^{2k-r-1}) = \mathbb{R}$$
$$\to H_{2k-r-1}(B) \to H_{2k-r-1}(B \cup D^{2k-r}) \to H_{2k-r-1}(D^{2k-r}, S^{2k-r-1}) = 0$$

Since $H_\ell(A) = H_\ell(B)$ for every ℓ and since $A \cup D^r \cong B \cup D^{2k-r}$, we find a contradiction. The equality $r = 2k - s$ follows.

In the last step, we consider B and B_1. They are each on one side of the slice where the singularity is taking place. They are diffeomorphic, identified by the variational flow after the collapse has taken place and x_r^∞ and x_s^∞ have disappeared.

Before the collapse, B contains a stable sphere S^{2k-r-1} of x_r^∞ and a stable sphere S^{r-1} of $x_s^\infty = x_{2k-r}^\infty$ (because $r \precsim s = 2k - r$, the unstable sphere of x_r^∞ does not intersect the stable sphere of x_{2k-r}^∞, which flows up to B, past x_r^∞).

Similarly, before the collapse, B_1 contains an unstable sphere S_1^{r-1} of x_r^∞ and an unstable sphere S_1^{2k-r-1} of $x_s^\infty = x_{2k-r}^\infty$ (again, because $r \precsim s = 2k - r$, the unstable sphere of x_r^∞ flows down to B_1, past x_{2k-r}^∞).

The variational flow before the collapse identifies $B \setminus (S^{2k-r-1} \cup S^{r-1})$ and $B_1 \setminus (S_1^{2k-r-1} \cup S_1^{r-1})$. After the collapse, it identifies B and B_1. Removing small disk neighbourhoods of the involved spheres to B and flowing down what is left, before the collapse, we find B_1 deprived of neighbourhoods of the corresponding spheres. The variational flow can be assumed not to change on these sets, over the singularity. Using connectedness, we find then (observe that $2k - r - 1 \neq r - 1$) that, after the collapse, a disk neighbourhood of S^{r-1} in B must flow to a disk neighbourhood of S_1^{r-1} in B_1. A similar claim holds for S^{2k-r-1} and S_1^{2k-r-1}.

Once this is understood and established, the normal form for the collapse and the singularity can be written explicitly: skipping details, the spheres of equal dimension in B and B_1 are identified by the flow after the collapse. At the collapse, they are flown to the singularity , which, therefore, "contains" deformations of large sets. However, these large sets can be coupled in pairs and the singularity resolves in a very natural way by flowing the two terms of each couple one into the other.

If the intersection number $x_m.x_{m-1}$ is to change, then the set $W_u(x_m) \cap W_s(x_r^\infty) \cap B$—observe that this set now can be tracked over the singularity—once flown down to B_1 (observe that it is part of S^{2k-r-1}), must intersect $W_s(x_{m-1}) \cap W_u(x_{2k-r}^\infty) \cap B_1$ (it is part of $S_1 2k - r - 1$, observe that this set also can be tracked now over the singularity).

Using a general position argument, we must have

$$(m - r - 1) + (2k - r - (m - 1) - 1) \leq 2k - 1$$

Thus, $r = 0$ and $s = 2k$ as claimed.

Let us consider a critical point at infinity x^∞ having all its ξ-pieces characteristic. We assume that $x^\infty \in \Gamma_{2k}$ so that all the k ξ-pieces of x^∞ are characteristic.

The tangent space $T_{x^\infty}\Gamma_{2k}$ can be split into a direct summand of two k-dimensional spaces $E_1 \oplus E_2$ that are $J''_\infty(x^\infty)$-orthogonal, see [3], pp 213–222 and [4], pp 120–126, Proposition 23, p 120 in particular.

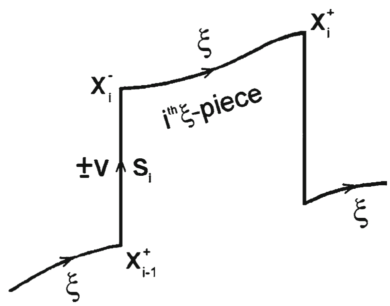
E_1 is spanned $\oplus_{i=1}^k \mathbb{R}\tilde{v}_i$. $\tilde{v}_i$ is related to the ith characteristic ξ-piece , whereas E_2 is spanned by k vectors $X_1, \ldots, X_k$ that are $J''_\infty(x^\infty)$-orthogonal to $\tilde{v}_1, \ldots, \tilde{v}_k$.

This construction extends to all curves nearby x^∞ in Γ_{2k}. However, the decomposition is obviously not J''_∞-orthogonal since it is anyway not intrinsic to define a second derivative at points which are not critical points of J_∞.

We are going in the sequel to construct $X_1, \ldots, X_k$ along x^∞ so that the following result will be obvious:

Proposition 2.16 *Over a singularity involving the collapse of two critical points at infinity x^∞ and y^∞ of Γ_{2k}, the difference of Morse indexes between x^∞ and y^∞ is at most k.*

Proof We come back to the computations of [3], p 213–222 and [4], p 120–126 to understand the behavior of J''_∞ on E_1. From these computations, we know that the $\tilde{v}_i$s are J''_∞-orthogonal. Let us draw the ith characteristic ξ-piece of x^∞. This ξ-piece runs from x_i^- to x_i^+. The $\pm v$-jump of x^∞ that precedes the ithξ-piece, runs from x_{i-1}^+ to x_i^- and is of length s_i. Let ϕ_s denote the one-parameter group of v:



We write

$$\mathrm{d}\phi_{s_i}(\xi) = (1 + A_i)\xi + B_i[\xi, v] + \gamma_i v$$

Assume that, at x^∞, v is mapped onto $\theta_i v$ from x_i^- to x_i^+, with a rotation in the ξ-transport equal to $k_i\pi$ in $\ker\alpha$.

We then have, [4], p 121:

$$J_\infty^{"}(x^\infty)\tilde{v}_i.\tilde{v}_i = \frac{A_i}{B_i}\theta_i \mathrm{d}a_i^c(v(x_i^-))$$

a_i^c is the length of the ξ-piece originating at ξ, defined so that v rotates k_i-times in the ξ-transport (in $\ker\alpha$) from one edge to the other edge of this ξ-piece.

At a singularity, B_i is zero, A_i is not zero as well as θ_i and $\mathrm{d}a_i^c(v)$. In the vicinity of a singularity, B_i and $\tilde{B}_i$ have opposite signs on x^∞ and y^∞ (the two critical points at infinity that are collapsing).

Therefore, any index-direction $\tilde{v}_i$ for x^∞ yields a co-index direction for y^∞ and vice-versa. $J_\infty^{"}(x^\infty)_{|E_1}$ is essentially equal to $-J_\infty^{"}(y^\infty_{|E_1})$.

We are going to see now that, on E_2, $J_\infty^{"}(x^\infty)$ and $J_\infty^{"}(y^\infty)$ are almost equal and have the same index. Proposition 2.16 will follow.

The claim follows from a construction of the vectors X_i, which we carry out below in some detail, that is slightly different from the one completed in [4], pp 123–126 and a construction that emphasizes the fact that $J^{"}_{\infty_{|E_2}}$ does not vary much from x^∞ and y^∞.

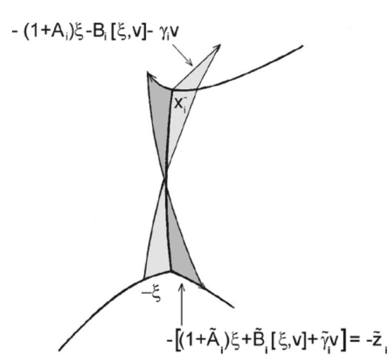
We consider again the ith $\pm\, v$-jump of x^∞. From x_{i-1}^+ to x_i^-, we have set

$$\mathrm{d}\phi_{s_i}(\xi) = (1 + A_i)\xi + B_i[\xi, v] + \gamma_i v$$

Similarly, from x_i^- to x_{i-1}^+, we set

$$\mathrm{d}\phi_{-s_i}(\xi) = (1 + \tilde{A}_i)\xi + \tilde{B}_i[\xi, v] + \tilde{\gamma}_i v = \tilde{Z}_i$$

These definitions can be extended to any piece of $\pm v$-orbit of length s running from an x^+ and x^-. We find then functions $B(s, x^+)$ and $\tilde{B}(s, x^+)$. Observe that B is zero if and only if $\tilde{B}$ is zero (β is a contact form, v is in its kernel) and the function $\frac{\tilde{B}}{B}$ is a C^∞-function of the base point x^+ and of the length s.



Considering the v-transport $\tilde{Z}_i$ of ξ from x_i^- to x_{i-1}^+, we can add to ξ at x_i^- a v-component equal to $-\tilde{\gamma}_i$ and remove from $\tilde{Z}_i$ its v-component. We can also change the length of the ξ-piece from x_{i-1}^- to x_{i-1}^+ so that $\tilde{Z}_i$ becomes $\hat{Z}_i = \frac{B_i[\tilde{\xi},v]}{\theta_{i-1}}$ at x_{i-1}^-, after backwards ξ-transport from x_{i-1}^+ to x_{i-1}^- and appropriate change of the length of the $(i-1)$thξ-piece as described above. θ_{i-1} is the analog , for the $(i-1)$th, of what θ is for the ithξ-piece. If there is only one ξ-piece, we do not need to perform this operation since ξ will be transported on itself from x_{i-1}^+ to x_i^- in the backwards ξ-transport.

We then take $\hat{Z}_i$ and transport it from x_{i-1}^- to x_i^- using the "history" of the curve x of Γ_{2k} (close to x^∞) at which this construction is completed. The curve x reads from x_{i-1}^- to x_i^- as a composition ℓ_i of ξ-transport maps, along ξ-pieces of lengths a_j with v-transport maps, along $\pm v$-pieces of lengths s_j. We use the differential of this map $\mathrm{d}\ell_i$ from x_{i-1}^- to x_i^- to transport $\hat{Z}_i$ to x_i^-.

The vector $\frac{\mathrm{d}\ell([\xi,v])}{\theta_{i-1}}$ has a non-zero component on $[\xi, v]$ because the ξ-pieces of x are nearly characteristic and its $\pm v$-jumps nearly map ξ onto $\lambda\xi$.

$\mathrm{d}\ell_i(\hat{Z}_i)$ then reads

$$\mathrm{d}\ell_i(\hat{Z}_i) = \tilde{B}_i(M_i\xi + c_i(x_{i-1})[\xi, v] + N_i v)$$

Therefore, $\frac{B_i\hat{Z}_i}{\tilde{B}_i c_i(x_{i-1})} = T_i$ has the $[\xi, v]$-component of $\mathrm{d}\phi_{s_i}(\xi)$.

$\mathrm{d}\phi_{s_i}(\frac{B_i\tilde{Z}_i}{\tilde{B}_i c_i(x_{i-1})} + \xi)$ now differs from T_i by a vector that splits on ξ and v since $\mathrm{d}\phi_{s_i}(\tilde{Z}_i)$ is along ξ and v and since $\mathrm{d}\phi_{s_i}(\xi)$ has the $[\xi, v]$-component of T_i. We may, therefore, scale the ithξ-piece length and the length s_i of the $\pm v$-jump abutting to it and match the two vectors. The value of T_i is not changed through this process because the addition of ξ at x_{i-1} is not changed through this process because the addition of ξ at x_{i-1}^+ can be compensated at x_{i-1}^- by scaling the ξ-length of the $(i-1)$thξ-piece. Again, this is not needed if there is only one ξ-piece.

We find a tangent vector at x^∞ that we denote $\hat{X}_i$. $\hat{X}_i$ goes not belong to E_1 and substraction from it the appropriate components on the $\tilde{v}_j$s, we find

$$X_i = \hat{X}_i - \sum_{i=1}^{k} \omega_i \tilde{v}_i$$

with

$$\mathrm{d}a_j(\hat{X}_i) - \mathrm{d}a_j^c(\hat{X}_i) = -\omega_j \mathrm{d}a_j^c(v(x_i^-))$$

at a curve having all its ξ-pieces being characteristic so that $(\mathrm{d}a_j - \mathrm{d}a_j^c)(X_i) = 0$ for $i = 1, \ldots, k$.

We recall that a_j is the ξ-length of the jthξ-piece, a_j^c is the corresponding characteristic length function. It follows that, when x^∞ and y^∞ are replaced by nearby curves of Γ_{2k} that have their ξ-pieces nearly characteristic, or when x^∞ collapses with y^∞ at the singularity, the computation of the ω_js can be carried out: this computation is semi-local in nature. It does not involve the whole curve. For the index j and to compute ω_j, the contributions that are not zero come from $\tilde{v}_j$ and $\tilde{v}_{j-1}$. The contribution of $\tilde{v}_{j-1}$ yields a vector at x_j^- of the order of $(a_{j-1} - a_{j-1}^c)$ in magnitude, see [4], p 121. These are small quantities and, therefore, the computation of the ω_js from the above system of equations is only a perturbation from he computation when all the ξ-pieces of x are characteristic, which is straightforward.

Skipping the details—they are not difficult; they are a generalization of the computations of [4], which were carried for Γ_2)—we build in this way k vectors $X_1, \ldots, X_k$ and they are independent.

When $x^\infty = y^\infty$ and here is a singularity, $X_1, \ldots, X_k$ exist and are well defined in a neighbourhood of $x^\infty = y^\infty$.

We first compute $J'_\infty(x).X_i$ since X_i is defined for every x close to x^∞. Using these computations, we find that we can compute

$$(X_j.(J'_\infty(x).X_i)_{|x=x^\infty} = J''_\infty(x^\infty).X_i.X_j$$

This computation can be carried out precisely; it depends only on the value that X_i takes at each x_j^+, x_j^- of x:

$$J'_\infty(x).X_i = \sum_{j=1}^{k}(\alpha(X_i(x_j^-)) - \alpha(X_i(x_j^+)))$$

The value of $(X_k.(J'_\infty(x).X_i)_{|x=x^\infty}$ follows then. We derive, as we track these computations that they can be carried out when $x^\infty = y^\infty$ and they imply that $J''_\infty(x^\infty)_{|E_2}$ and $J''_\infty(y^\infty))_{|E_2}$ are very close to the same (non-degenerate after a general position argument)matrix.

The claim follows. □

2.8 Appendix 1

We need to develop in this Appendix a precise understanding of the process of formation of a "Dirac mass", why and how it is created, why and how it evolves. This is what we describe in what follows:

2.8.1 Formation of "Dirac masses"

Let us consider a point x_0 and the v-orbit through x_0. For the sake of the simplicity, we will be considering only "positive Dirac masses", that is "Dirac masses" that correspond to a forth and back run along v.

Considering the positive v-orbit $\phi_s(x_0)$, $s \geq 0$, we study the function

$$\alpha_{x_0}(D\phi_{-s}(\xi(\phi_s(x_0)))) = \gamma(s)$$

Observe that

$$\alpha_{x_0}(D\phi_{-s}([v, \xi](\phi_s(x_0))) = \gamma'(s)$$

so that the extrema of γ occur when

$$D\phi_{-s}([v, \xi]) = \lambda[v, \xi] + \mu v$$

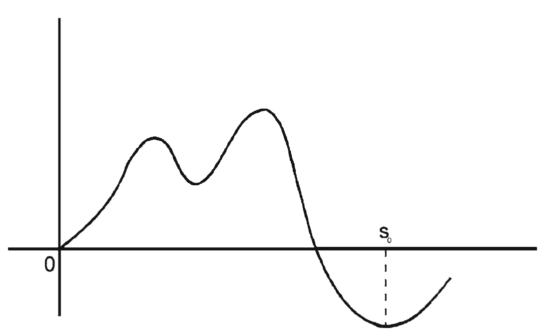
that is when $[v, \xi]$ is mapped onto itself by the one parameter group of v, ϕ_s. These are the **coincidence** points of x_0. Computing, we find that

$$\alpha_{x_0}(D\phi_{-s}([v, [v, \xi]])) = \gamma''(s)$$

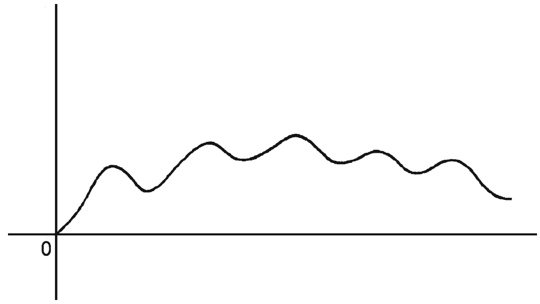
At $s = 0$,

$$\gamma''(s) = \alpha_{x_0}([v, [v, \xi]]) = d\alpha_{x_0}(v, [\xi, v]) = -1$$

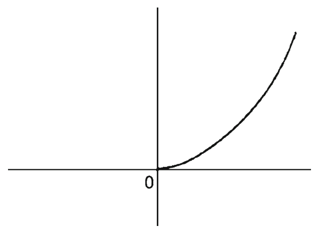
The function $\theta = 1 - \gamma$ thereby behaves as follows:



It might happen that values s_0 such that θ is negative never occur:

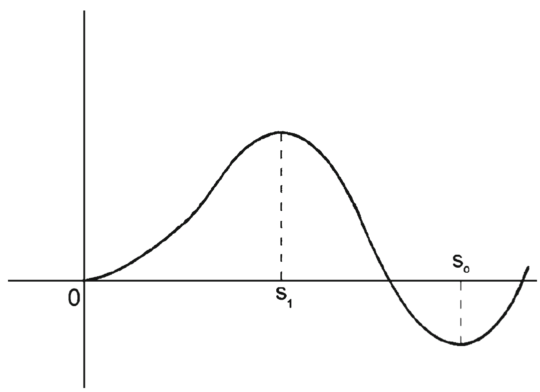


Then, the Fredholm assumption is not violated in this direction. We might as well consider that θ would behave as

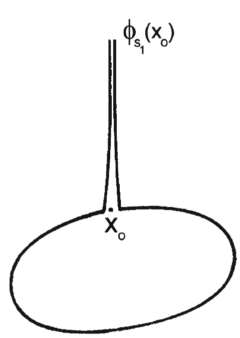


That is it behaves as if zero were a unique minimum. This is at least what happens topologically.

On the other hand, if s_0 does exist with $\theta \lesssim 0$, the various oscillations of θ before s_0 cancel each other to leave room only to zero and to a unique maximum s_1, **before** s_0; that is, topologically (i.e after cancellation of additional maxima and minima), θ behaves as

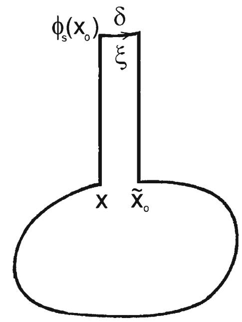


The pattern might repeat thereafter, but it belongs to the same phenomenon and is solved by the same techniques. Because θ is negative at s_0, the point s_1 acts as a critical point for the functional J_∞ on γ_4. The actual curve is x_1 with a forth and back run at the point x_0 of x_1, from x_0 to $\phi_{s_1}(x_0)$ and back:



If the "Dirac mass" on this curve is "opened up" after the "Dirac mass" has been stretched to reach $\phi_{s_0}(x_0)$, then the functional J_∞ decreases. The graph indicates that this "curve +" "Dirac mass" acts as a critical point in Γ_4 (with one ξ-piece collapsed) and its index is at least 1. Our arguments below will not use this latest observation. They will rely on a careful analysis of the behaviour of the various tangent directions to Γ_4 in the vicinity of such curves.

Let us imagine x_1 with an additional "Dirac mass" from x_0 to $\phi_s(x_0)$ and let us "open it", inserting in it a ξ-piece of length $\delta \gneq 0$:

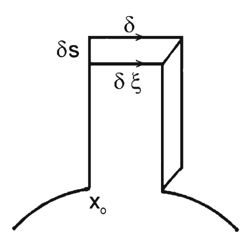


2.8.2 Remarkable tangent vectors

There are four basic tangent vectors along such a curve. We describe in what follows in great detail three of them:

The two first ones belong to the same family; they do not increase the size of the ξ-piece δ. Let us describe in detail one of these directions:

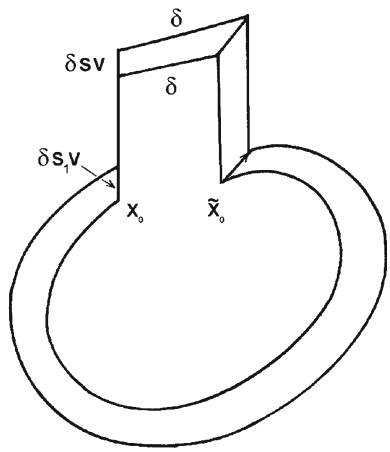
Starting from x_0, we change the size of the positive v-jump from x_0 to $\phi_s(x_0)$. This size was s, it becomes $s + \delta s$:



We then transport δs along the ξ-piece of length δ. We derive at the other edge of this ξ-piece a vector in $\ker\alpha$ that reads $\delta s[(v + \delta[\xi, v]) + O(\delta^2)]$, which we re-transport down using the one parameter group of v over the time $-t$. We derive at $\tilde{x_0}$ a vector that reads

$$\delta s D\phi_{-t}((v + \delta[v, \xi]) + O(\delta^2))$$

Using a $\delta s_1 v$ at x_0 that we transport backwards along the large ξ-piece from x_0 to $\tilde{x}_0$, we compensate the $[\xi, v]$ component of this vector and we create a tangent vector, which we denote z_0:



The variation of J_∞ along this vector z_0 is

$$\partial J_\infty.z_0 = -\delta(1 + O(\delta))\alpha(D\phi_{-t}([v, \xi]))$$

We thus see that this variation is zero if and only if

$$\alpha(D\phi_{-t}([v, \xi])) = 0$$

Another tangent vector z_1, belonging to the same family, is defined following the same construction, but with t in lieu of s that is using the negative $(-v)$-jump of the "Dirac mass" instead of its positive v-jump. Using the same computations, we derive that $\partial J_\infty.z_1$ is zero if and only if

$$\alpha(D\phi_s([v, \xi])) = 0$$

The two conditions above define at each $\bar{x}_0$ of the periodic orbit x_1 and for each $\delta \gneq 0$ small enough a unique curve $C(\bar{x}_0, \delta)$ that we find as follows:

Considering $\bar{x}_0$ on x_1, we build a section σ_0 to ξ at $\bar{x}_0$, tangent to kerα at $bar x_0$ and tangent to v.

From a point $x_0 \in \sigma_0$, we build a positive v-jump through x_0 of length s. s is defined to be the first positive time such that

$$\alpha(D\phi_{-s}([v, \xi])) = 0$$

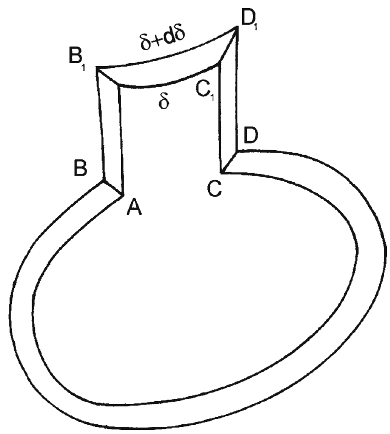
We then follow ξ during the time δ from $\phi_s(x_0)$ to a new point x_δ^+. From x_δ^+, we follow the $(-v)$-orbit during the time t, until we reach a point (the first such point) with

$$\alpha_{\phi_{-t}(x_\delta^+)}(D\phi_{-t}([\xi, v])) = 0$$

We reach then a point $\tilde{x}_0 = \phi_{-t}(x_\delta^+)$. Starting from $\tilde{x}_0$, we follow the ξ-orbit until we hit σ_0 again.

This defines a map $f_\delta : \sigma_0 \to \sigma_0$. It is easy to see that f_δ is continuously differentiable. Continuity of the differential holds also in terms of δ. df_0 is the differential of the Poincare-return map of x_1. Existence, continuity, etc. of $C(\bar{x}_0, \delta)$ follow. The tangent direction that corresponds to increasing δ is denoted z_3.

Let us come back to our earlier curves, as we "opened up" the oscillation and we inserted a ξ-piece of length δ and let us vary δ now:



Let Ψ_C be the ξ-time advance map from C to A. We then have

$$D\Psi_C(\overrightarrow{CD}) = \overrightarrow{AB} + \delta c\xi + o(|\overrightarrow{AB}|)$$

Furthermore,

$$\frac{\overrightarrow{AB}}{|\overrightarrow{AB}|} = h + o(1)$$

with $h \in \ker\alpha$.

By construction, $\ker\alpha$ is v-transported from A_1 to A and from C_1 to C. It follows that

$$\overrightarrow{CD} = \overrightarrow{AB} + O(\delta)|\overrightarrow{AB}| + O(\mathrm{d}\delta) + o(|\overrightarrow{AB}|)$$

Thus,

$$D\Psi_C(\overrightarrow{CD}) = \overrightarrow{CD} + O(\delta)|\overrightarrow{CD}| + O(\mathrm{d}\delta) + o(|\overrightarrow{CD}|) + \delta c\xi$$

Since $D\Psi_C - Id$ is invertible on $\ker\alpha$ (δ is small), the above equations imply that $|\overrightarrow{CD}|$ and $|\overrightarrow{AB}|$ are $O(\mathrm{d}\delta)$.

Next, we compute the variation of J between the two curves. We find that it is (at the differential level)

$$\alpha(\overrightarrow{AB}) - \alpha(\overrightarrow{A_1B_1}) + \alpha(\overrightarrow{C_1D_1}) - \alpha(\overrightarrow{CD})$$

This is

$$\begin{aligned}&(1-\alpha_A(D\Phi_s(\xi(A_1))))\mathrm{d}\delta + (\alpha_A(D\Phi_s(\xi(A_1))) - \alpha_C(D\Phi_{\tilde{s}}(\xi(C_1)))\alpha(\overrightarrow{C_1D_1})\\ &\quad = (1-\alpha_A(D\Phi_s(\xi(A_1))))\mathrm{d}\delta + O(\delta)|\overrightarrow{C_1D_1}| = (1-\alpha_A(D\Phi_s(\xi(A_1))))\mathrm{d}\delta + O(\delta\mathrm{d}\delta)\end{aligned}$$

This defines a third tangent vector z_2

2.8.3 Fredholm violation on curves of Γ_4

We now combine z_1 and z_2, assuming that the function $\theta(t)$ becomes zero for a first, e.g. positive value $t = t_0$, then negative after crossing t_0. For $t \lneq t_0$, $\theta(t)$ is positive. If t is not close to t_0, we use z_2 above. $(1-\alpha_A(D\Phi_s(\xi(A_1))))$ is not $O(\delta)$ and we can decrease δ to decrease J_∞. We want at the same time to increase s and t, that is the size of the "Dirac mass". This involves the use of z_1. We thus add to z_1 Mz_2, where M is a suitable constant so that $z_1 + Mz_2$ decreases J_∞. Using the expansions above for $J'_\infty.z_i$, the existence of M is straightforward. δ can be assumed to decrease exponentially over this combination, whereas s and t increase at a speed close to 1 (exactly 1 for s).

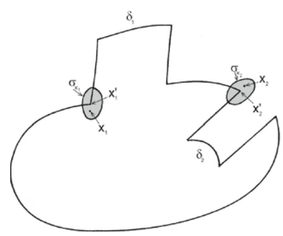
Now, as t increases and becomes close to t_0, the function $\theta(t)$ decreases and its derivative becomes negative, bounded away from zero. This derivative is $-\alpha(D\phi_{-t}([v, \xi])$ and, therefore, the use of z_0 or the use of z_1 to increase the size of the "Dirac mass" is J_∞ decreasing.

Combining the two processes, the one involving the combination of z_1 and the use of z_2 when t is smaller, not close to t_0, with the later process when t becomes close to t_0, we decrease J_∞, increase the size of the "Dirac mass" and keep δ positive. Once we cross over t_0 and go sizably beyond this value, θ becomes negative, bounded away from zero. Increasing then δ decreases J_∞. The decrease in unhindered and J_∞ can be decreased below the critical value of the elliptic orbit, quite sizably.

Fredholm violation, taking place in Γ_4 already, is enacted.

2.8.4 More "Dirac masses"

Given p points $(x_1, \ldots, x_p)$ on the periodic orbit, we can build at each x_i a section σ_{x_i} to ξ tangent at x_i to $\ker\alpha$ at each of these points. We then prescribe $\delta_1, \ldots, \delta_p \gneq 0$ and we seek a curve $C(x_1, \ldots, x_p, \delta_1, \ldots, \delta_p)$,



Such curves are derived, just as in the case of a single (x_1, δ_1) through a fixed point problem. Then, $\frac{\overrightarrow{x_1x_1'}}{|\overrightarrow{x_1x_1'}|} \in \ker\alpha_{x_1} + o(1)$, $\frac{\overrightarrow{x_2x_2'}}{|\overrightarrow{x_2x_2'}|} \in \ker\alpha_{x_2} + o(1)$. All the arguments and claims developed above generalize to this framework.

2.9 Appendix 2: The verification of the Palais–Smale condition on each Γ_{2k}

We study in what follows the verification of the Palais–Smale condition for J_∞ on each Γ_{2k}. The present study is focused on the two specific cases of the standard contact structure α_0 on S^3, with v a vector-field defining a Hopf fibration in its kernel and the case of the first exotic contact structure of Gonzalo and Varela [14], with v the vector-field of Martino [25] and its kernel.

2.9.1 The case of α_0

In the case of the standard contact structure on S^3, we can use, for convex Hamiltonians, a vector-field v in $\ker\alpha_0$ defining a Hopf-fibration of S^3 over S^2. All the orbits of v are closed, and therefore, there is an intrinsic periodicity in the behavior of J_∞ on Γ_{2k}: given a curve x of Γ_{2k} and one of its $\pm v$-jumps, the value of $|\partial J_\infty(x)|$ will not change as we add or substract a closed orbit of v to this $\pm v$-jump.

Therefore, $|\partial J_\infty(x)|$ can be computed on a compact set of curves of Γ_{2k} (given an a priori bound on J_∞) and the Palais–Smale condition follows because $\partial J_\infty(x)$ as well has this periodicity built in, so that every addition of a closed orbit to a $\pm v$-orbit of x under deformation will yield a decrease in J_∞ lower-bounded by a fixed positive constant, provided x is not in a small neighbourhood of a critical point (at infinity). As the positive constant is made smaller and smaller, the neighbourhood can also be made smaller and smaller. The claim follows.

2.9.2 The case of α_1

We consider now the case of the first contact structure α_1 of Gonzalo and Varela [14] on S^3 along the vector field v of Martino [25]. If we are considering the more specific $\gamma_1\alpha_1$, γ_1 is a C^2 positive function on S^3, we assume that

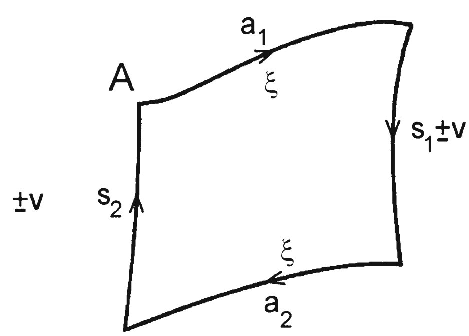
$$d(\gamma_1\alpha_1)(v, .)\text{is a contact form with the same orientation than } \alpha_1$$

In [25], it is proved that this assumption holds for perturbations of α_1.

Let x be a curve of the related space Γ_{2k}. Let $A_1, A_2, \dots, A_k$ be the sizes in absolute value of its $\pm v$-jumps. Let:

$$T = \text{Sup}(A_1, \dots, A_k)$$

Let ℓ be the Poincare-return map of x; ℓ follows the history of x, never changing the size of a ξ-piece or the size of a $\pm v$-jump of x. We can base ℓ at any of the corners or at any of the edges of x



that is, if x is defined by $\gamma_{s_2} \circ \phi_{a_2} \circ \gamma_{s_1} \circ \phi_{a_1}(A) = A$- ϕ_s is the one-parameter group of ξ, γ_s is the one-parameter of v, then $\ell = \gamma_{s_2} \circ \phi_{a_2} \circ \gamma_{s_1} \circ \phi_{a_1}$ and $d\ell = d\gamma_{s_2} \circ d\phi_{a_2} \circ d\gamma_{s_1} \circ d\phi_{a_1}$.

The ξ-pieces of x are ordered with indexes ranging from 1 to k, the first one starting at A. We then define ξ_i to be the transport of γ from the ithξ-piece of x to A along $d\gamma_{s_k} \circ \cdots \circ d\gamma_{s_{i+1}} \circ d\phi_{a_{i+1}} \circ d\gamma_{s_i}$, that is, along the history of the curve x, from the ithξ-piece to A.

We then consider at A the $(k-1)$ vectors:

$$\xi_1 - \xi_i, i = 2, \dots, k$$

and the vector space:

$$E = Span\{(d\ell - Id)(\mathbb{R}^3), \xi_1 - \xi_i; i = 2, \dots, k\}$$

We then have

Proposition 2.17 (i) *If E is not $\mathbb{R}^3$, k independent conditions must be verified by the curve x.*
(ii) *If E is $\mathbb{R}^3$ and some $\pm v$-jump of x does not take place between conjugate points, then a decreasing direction for J_∞ z can be defined at x and this decreasing direction verifies*

$$z.T = \frac{\partial T}{\partial s} = 0$$

Clearly, under (ii), these directions z taken at various xs can be convex-combined using a partition of unity so that the differential inequality

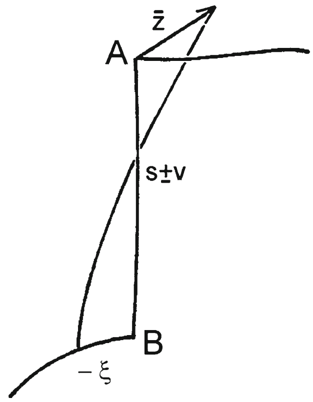
$$z.T = \frac{\partial T}{\partial s} \leq -z.J_\infty = -\frac{\partial J_\infty}{\partial s}$$

is verified (assuming that (ii) holds and the curves x are not critical points at infinity).

A decreasing pseudo-gradient for J_∞ that satisfies the above differential inequality is a flow that verifies the Palais–Smale condition. Therefore, the above Proposition leads us to study the curves x for which the assumption of (ii) are not verified. At these curves, (i) of Proposition 2.17 tells us that k independent conditions must be verified at x.

Proof of Proposition 2.17 (i) follows from general position arguments on $(\ker\alpha_1, v)$, see Proposition 29 , p198 of [3].

For (ii), we consider a $\pm v$-jump of x; let s be its length along $\pm v$:



We denote A,the base point, one of its edges. We pick up $-\xi$ at the other edge of this $\pm v$-jump and we $\pm v$-transport at A, defining $\bar{z} = d\phi_s(-\xi)$.
Since E is of dimension 3, we can write

$$\bar{z} = d\ell(h) - h + \sum_2^k \delta a_i(\xi_1 - \xi_i)$$

This defines a tangent vector at the curve x. Along this tangent vector, $\frac{\partial T}{\partial s}$ is zero and $\frac{\partial J_\infty}{\partial s} = -1 + \alpha_A(\bar{z})$. Assuming that $\alpha_A(\bar{z})$ is not 1, we can change $-\xi$ into ξ at B if needed so that $\frac{\partial J_\infty}{\partial s} \lneq 0$. The definition of the direction z of (ii) follows.
Then, if such a direction z cannot be defined, $\alpha_a(\bar{z}) = 1$. This equation must then be verified at the two edges of this $\pm v$-jump of x. Therefore, this $\pm v$-jump must take place between two conjugate points. (ii) follows. □

We now have

Proposition 2.18 (i) *Assume that the size of any $\pm v$-jump A_i is strictly less than $T = \mathrm{Sup}(A_1, \ldots, A_k)$. Then, either two additional independent conditions are verified at x or a J_∞-decreasing direction z can be found at x verifying*

$$z.T = \frac{\partial T}{\partial s} = 0$$

(ii) *Under the same assumption than (i), the curves x of Γ_{2k} for which x cannot be defined verify $2k$-independent conditions and thereby form an isolated set.*

We will denote in the sequel a_c^i the characteristic length ([4], p120) (for a given number of rotations of v along ξ) taken at the left edge of the ithξ-piece of x.
Let $\bar{z}_i = \lambda_i\xi + \mu_i v + \eta_i w$ be the v-transport of ξ along the ithξ-piece, from one edge to the other one (in a given direction).
Combining Proposition 2.18 with (ii) of Proposition 2.17, we derive

Proposition 2.19 (i) *If a J_∞-decreasing direction z verifying $z.T = \frac{\partial T}{\partial s} = 0$ cannot be found at x of Γ_{2k}, then either x is part of an isolated set, or in addition to the k conditions of (i) of Proposition 2.17, x verifies the $(k-1)$ additional independent conditions:*

$$A_i = A_j, i \neq j \in 1, \ldots, k$$

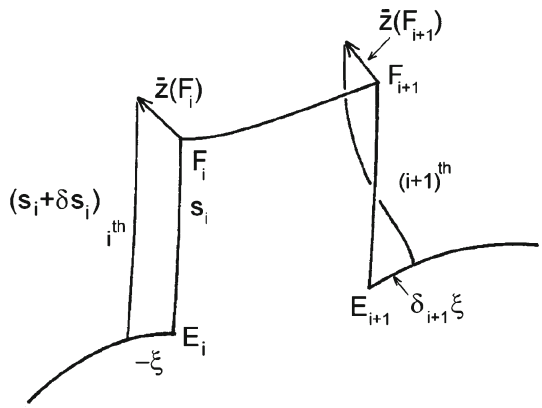
(ii) *If the condition $z.T = \frac{\partial T}{\partial s} = 0$ is relaxed into the condition $z.T = \frac{\partial T}{\partial s} \leq CT\ln T \times (-z.J_\infty) = -CT\ln T\frac{\partial J_\infty}{\partial s}$, then, in addition to the $(2k-1)$ conditions of (i), we must have:*

$$\left(\left|\frac{\lambda_i - 1}{\eta_i}\right| + \left|\frac{\lambda_{i+1} - 1}{\eta_{i+1}}\right|\right)|a_i - a_c^i| \leq \frac{\bar{C}}{T\ln T}$$

$\bar{C}$ is a fixed constant depending on C above; the number of v-rotations along the ithξ-piece of x, once T is large, can be derived from the value of a_i and the knowledge of an edge of the ξ-piece.

Proof of Propositions 2.18 and 2.19 Proof of (i) of Proposition 2.18:

Let us consider the ith and the $(i+1)$th $\pm v$-jumps. We are assuming that A_i is strictly less than T:



We pick up $-\xi$ at E_i and v-transport it to F_i into:

$$\bar{z}(F_i) = -\mathrm{d}\gamma_{s_i}(\xi) + \delta s_i v$$

We take $\delta\xi$ at E_{i+1} and we v-transport it at F_{i+1} into:

$$\bar{z}(F_{i+1}) = \delta_{i+1}\mathrm{d}\gamma_{s_{i+1}}(\xi)$$

Observe that we have not used $\delta s v$ at F_{i+1} since A_{i+1} could be T.

We then ξ-transport $\bar{z}(F_i)$ to F_{i+1} into $\mathrm{d}\phi_{a_i}(\bar{z}(F_i)) + \delta a_i \xi$ and we seek δs_i and δa_i so that

$$\bar{z}(F_{i+1}) = -\mathrm{d}\phi_{a_i} \circ \mathrm{d}\gamma_{s_i}(\xi) + \delta s_i \mathrm{d}\phi_{a_i}(v) + \delta a_i \xi$$

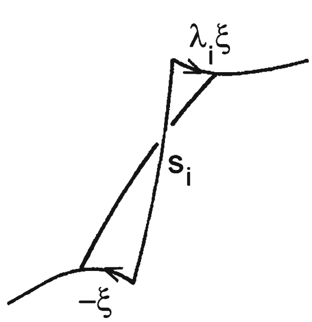
Assuming that (ii) of Proposition 2.19 holds—its proof is independent—and T is large or C is small, δs_i and δa_i can be used to adjust the ξ and v-components of this identity. $\mathrm{d}\phi_{a_i}(v)$ is then almost $\pm\theta_i v$, $\theta_i \gtrapprox 0$ (bounded away from zero). Thus, the verification of the above identity, with the additional use of δ_{i+1} relies on the simple fact that $\mathrm{d}\phi_{a_i} \circ \mathrm{d}\gamma_{s_i}(\xi)$ has a non-zero component on $[\xi, v](F_i)$. If indeed this happens, then we have built a tangent vector z at x.

If, in addition, $z.J_\infty(x) = \partial J_\infty(x).z$ is non-zero, we may assume that it is negative (changing ξ into $-\xi$ if needed). Clearly, $z.T = \frac{\partial T}{\partial s} = 0$, so that z has been found.

If $\partial J_\infty(x).z = 0$, we find one condition at x.

It might also happen that $\mathrm{d}\phi_{a_i} \circ \mathrm{d}\gamma_{s_i}(\xi)$ is equal to $\lambda_i \xi + \gamma_i v$. Then, one condition is again verified at x. If $-\lambda_i - 1$ is non-zero, taking $\delta s_i = -B_i$, we find a vector z at x such that (after having possibly changed z into $-z$):

$$\partial J_\infty(x).z \neq 0; \; z.T = \frac{\partial T}{\partial s} = 0.$$



If $1 + \lambda_i = 0$, we find an additional condition.

Summarizing, using the $(i+1)$th, the ith and the $(i-1)$th $\pm v$-jump at x and the construction above (repeated with $(i-1, i)$ in lieu of $(i, i+1)$), we find two additional conditions related to the ith $\pm\, v$-jump and the fact that its length A_i is strictly less than T. The independency is addressed below. It is based on Proposition 29, p 198 of [3].

Proof of (ii) of Proposition 2.18:

Assume now that $m \pm v$-jumps of x have a size less than T.

We then find at least $(m+1)$ additional conditions related to these $m \pm v$-jumps. The remaining $(k-m) \pm v$-jumps verify the equation

$$A_i = T$$

This gives us $(k-m-1)$ additional conditions, which, when combined with the other k conditions of (i) of Proposition 2.17, yield $2k$ conditions. The independence of these conditions follows from Proposition 29 , p198 of [3] that states that, along each $\pm v$-jump of a specific curve, e.g. our curves here, the differential of the v-transported map can be perturbed freely (subject to the condition $d\gamma_s(v) = v$).

Using this result, the independency of all our conditions follows. □

Proof of Proposition 2.19

After Proposition 2.18, the proof of (i) of Proposition 2.19 is straightforward. The independency of the $(2k-1)$ conditions follows again from the use of Proposition 29 of [3].

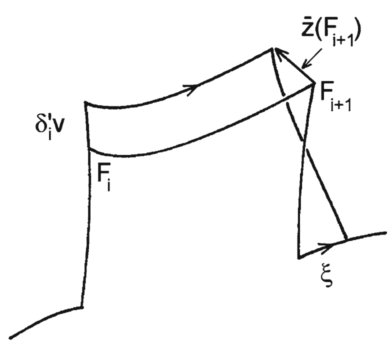
For (ii), let us consider the ith ξ-piece and let us assume that

$$|\frac{\lambda_{i+1}-1}{\eta_{i+1}}||a_i - a_c^i| \geq \frac{\bar{c}}{T \ln T}$$

Then, picking up $\delta_i' v$ at F_i, see figure above, and ξ-transporting it to F_{i+1}, we find that the transported vector at F_{i+1} has a $[\xi, v]$-component that is of the order of $|(a_i - a_c^i)\delta_i'|$. It follows that the $[\xi, v]$-component of $\bar{z}(F_{i+1})$—which we denoted η_{i+1}—taken with $\delta_{i+1} = 1$, see above, can be "compensated", see Figure below, with the use of a $\delta_i' v$, δ_i' being of the order of $\frac{|\eta_{i+1}|}{|a_i - a_c^i|}$.

Let λ_{i+1} be the ξ-component of $\bar{z}(F_{i+1})$ at F_{i+1}.

We have built a vector z:



We know that

$$z.J_\infty = \lambda_{i+1} - 1$$

and we also know (a_i is a priori bounded) that the v-component of $\bar{z}(F_{i+1}) = d\phi_{a_i}(\delta_i' v) + \delta a_i \xi$ is bounded by $C(a_i)|\delta_i| \leq C_1|\delta_i'|$.

Therefore,

$$\frac{\partial T}{\partial s} \leq (1 + C_1)|\delta_i'| \leq \frac{(1+|C_1|)|\eta_{i+1}|}{\bar{C}'|a_i - a_c^i|} \leq \frac{(1+|C_1|)|\eta_{i+1}\frac{\partial J_\infty}{\partial s}|}{\bar{C}'|(a_i - a_c^i)(\lambda_{i+1}-1)|}$$

The claim of (ii) of Proposition 2.19 follows, after adjusting z into $\pm v$ so that $\frac{-\partial J_\infty}{\partial s} \gtrsim 0$.

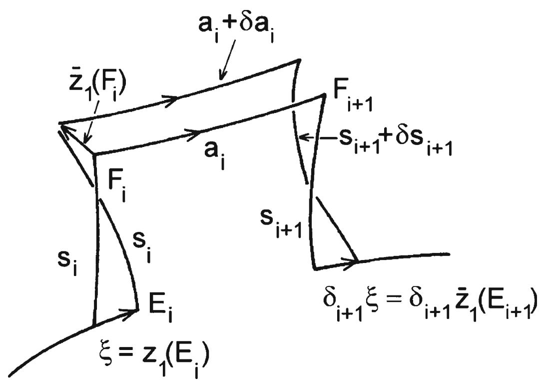
Although Propositions 2.18 and 2.19 seem to cover all the possible cases of tangent vector-fields (two per ξ-piece), these vector-fields are built so that no compensation can take place in the changes of the $\pm v$-jumps along the deformation.

As we will see now, we will build yet another tangent vector z_1 at x, around the ithξ-piece. This tangent vector z_1 will induce changes in the sizes of the ith and the $(i+1)$th $\pm v$-jumps; but these changes are built so that these sizes stay bounded away from infinity as $(a_i - a_c^i)$ tends to zero.

This allows to derive k other conditions to be "almost satisfied" (up to $O(\frac{1}{\ln T})$ or up to $O(\frac{1}{T \ln T})$, depending on how we scale them) as T tends to ∞. z_1 is constructed as follows: we take ξ at the base E_i of the ith$\pm v$-jump of x and we v-transport it to F_i, thereby deriving a vector $\bar{z}_1(F_i) = d\gamma_{s_i}(\xi)$.

We complete the same construction with the $(i+1)$th $\pm v$-jump.

We then scale $z_1(E_{i+1})$ into $\delta_{i+1}\bar{z}_1(E_{i+1})$ and $d\phi_{a_i}(\bar{z}(E_i))$ match. Using ξ and using v at E_{i+1}, we can then build a tangent vector z_1:



Using the results of [6], Lemma 10, see also the derived computations below (the function $R(\bar{y})$ is introduced in [6], Lemmas 2.6 and 2.15):

$$|\bar{z}_1(E_i)| = O\left(T|\frac{\partial R}{\partial \xi}(E_i)| + 1\right)$$

$|\frac{\partial R}{\partial \xi}|(E_i)$ is in fact $\frac{\partial}{\partial \xi}(R(\bar{y}))$ at the v-orbit through E_i.

It follows that $d\gamma_{a_i}(\bar{z}_1(F_i))$ is also $O(1 + T|\frac{\partial R}{\partial \xi}(E_i)|)$. Assuming that $|a_i - a_c^i| \leq \delta$, δ a fixed positive constant, then $|\delta_{i+1}|$ is bounded above and below by fixed positive constants.

Then,

$$|\bar{z}_1(F_{i+1})| = O\left(1 + T|\frac{\partial R}{\partial \xi}(E_{i+1})|\right)$$

Therefore,

$$\delta s_{i+1} = O\left(1 + T\left(\left|\frac{\partial R}{\partial \xi}(E_i)\right|\right) + \left|\frac{\partial R}{\partial \xi}(E_{i+1})\right|\right)$$

and

$$z_1.T = \frac{\partial T}{\partial s} = O\left(1 + T\left(\left|\frac{\partial R}{\partial \xi}(E_i)\right|\right) + \left|\frac{\partial R}{\partial \xi}(E_{i+1})\right|\right)$$

On the other hand,

$$z_1.J_\infty = \frac{\partial J_\infty}{\partial s} = 1 - \lambda_i + \delta_{i+1}(\lambda_{i+1} - 1)$$

Observe that λ_i and λ_{i+1} are $O(T)$ ([6], lemma 10, maybe generalized to include the case $\bar{y} \lesssim \bar{y}_0$ as well), so that this expression can tend to infinity with T. In fact, these expressions are more precisely $O(1 + T(|\frac{\partial R}{\partial \xi}(E_i)|) + |\frac{\partial R}{\partial \xi}(E_{i+1})|))$.

We thus derive that

$$(*)|1-\lambda_i+\delta_{i+1}(\lambda_{i+1}-1)| \leq \frac{c|\frac{\partial T}{\partial s}|}{T\ln T} \leq \frac{cO(1+T(|\frac{\partial R}{\partial \xi}(E_i)|)+|\frac{\partial R}{\partial \xi}(E_{i+1})|))}{T\ln T}$$

Otherwise, after choosing z_1 or $-z_1$ so that $\frac{-\partial J_\infty}{\partial s}$ is positive:

$$|\frac{\partial T}{\partial s}| \leq \frac{T\ln T}{c} \times \frac{-\partial J_\infty}{\partial s}$$

and we have built a direction z_1 satisfying the relaxed condition.
$(*)$ should be read as

$$(**)\frac{|1-\lambda_i+\delta_{i+1}(\lambda_{i+1}-1)|}{1+T(|\frac{\partial R}{\partial \xi}(E_i)|)+|\frac{\partial R}{\partial \xi}(E_{i+1})|)} \leq \frac{c}{T\ln T}$$

In particular, if $|\frac{\partial R}{\partial \xi}(E_i)|)+|\frac{\partial R}{\partial \xi}(E_{i+1})|$ is not $o(1)$, $(**)$ becomes

$$(***))\frac{|1-\lambda_i+\delta_{i+1}(\lambda_{i+1}-1)|}{1+T} \leq \frac{c'}{T\ln T}$$

□

This can be pushed further and Proposition 2.19 can be refined; but this is not our purpose here. We proceed now and we prove that the Palais–Smale condition is satisfied for (J_∞, Γ_{2k}) with the use of a suitable flow, in the case of $(\ker\alpha_1, v)$, $\ker\alpha$ the first contact structure of Gonzalo and Varela [14] and v the vector-field of Vittorio Martino [25]. We are assuming here that $\beta = d(\gamma_1\alpha_1)(v, .)$ is a contact form having the same orientation than α_1 (γ_1 is a positive C^2-function from S^3 to the positive reals. The proof is based on Propositions 2.17, 2.18 and 2.19, with some additional arguments:

Proposition 2.19 allows us, under suitable conditions, to build a pseudo-gradient that verifies the differential inequality:

$$|\frac{\partial T}{\partial s}| \leq \frac{C|\eta_i| \times \frac{-\partial a}{\partial s}}{|\lambda_i-1||a_i-a_c^i|}$$

$\frac{|\eta_i|}{|\lambda_i-1|}$ can be replaced above by $\eta_{i+1}||\lambda_{i+1}-1|$ or a combination of both. Depending on the construction of the pseudo-gradient and the partition of unity used, we can get such combinations.

At a specific x, where $A_i = A_j$, we are going to focus on the ith $\pm\, v$-jump to build our deformation, at least in a first step. However, later, we will generalize the argument and include combinations of decreasing directions defined with the ith$\pm v$-jump and other decreasing deformations defined with the $(i+1)$th$\pm v$-jump.

We now claim that, as T increases and as the $\pm v$-jumps cross the torus T_0 of S^3, defined by $T_0 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4; x_1^2+x_2^2 = x_3^2+x_4^2 = \frac{1}{2}\}$ see [6], we may arrange over each additional crossing or over each additional 2π-rotation of $\ker\alpha_1$ along v, so that we find an interval $[T_-, T_+]$ over the ith$\pm\, v$-jump so that

$$\frac{|\eta_i|}{|\lambda_i-1|} \leq C$$

C is a given fixed constant.

The claim derives from two observations and the use of Lemma 4.6 of [6].

First, observe that $\ker\alpha_1$ "turns well" along v; that is starting from any point z of S^3, the rotation of $\ker\alpha_1$ along v in a v-transported frame is infinite. This is clear if the v-orbit through z is not in T_0. Indeed, in this case, $\ker\alpha_1$ turns π from one crossing of T_0 to the next one, see [6], Lemma 4.6 and the conclusion follows then using the monotonicity of the rotation of $\ker\alpha_1$ along v.

If the v-orbit is in T_0, the same conclusion holds; however, it follows this time from the fact that the time $T(z)$ needed to cross T_0 twice starting from $z \in T_0$ is bounded above independently of z (z is a point in T_0 such that the v-orbit through z is not in T_0). Using a continuity argument, the fact that $\ker\alpha_1$ "turns well" also when the v-orbit is in T_0 follows.

The claim in $T(z)$ stems from the formula for $T(z)$ in Lemma 3.2 of [6]:

$$T(z) = 4\int_{\frac{1}{2}}^{\bar{y}} \frac{dy}{\sqrt{k(y) - k(\bar{y})}}$$

It is not difficult to see that, see [6], the formula for $k'(y)$ in the proof of Lemma 3.2 of [6] ,$g(y)$ is defined in Sect. 3 of [6], the dynamics of v, of [6], that $k'(c)$, for $c \in [y, \bar{y}]$, y and $\bar{y}$ larger than $\frac{1}{2}$, is larger than $c(2y-1)$, c a fixed positive constant. Then,

$$T(z) \leq c' \int_{\frac{1}{2}}^{\bar{y}} \frac{dy}{(2y-1)(\bar{y}-y)}$$

and the conclusion follows.

We now know for a fact that kerα_1 "turns well" along v. We then use Lemma 3.2 of [6]. Let N be the number of "cycles", i.e. the number of consecutive crossings of T_0 that a given $\pm v$-orbit that we will be considering, completes. One can also see N, when N large, as the number of π-rotations of kerα_1 along v.

Either $|\eta_i|+|\lambda_i|$ tends to ∞ as T tends to ∞; that is either $|\eta_i|+|\lambda_i|$ is very large (this corresponds to having $N\delta^2$ very large in Lemma 3.2 of [6]). Since the transported vector $\lambda\xi_i + \eta_i[\xi, v]$ rotates in the $(\xi, [\xi, v])$-frame, there are certainly, over each cycle, as kerα_1 adds more and more rotations along this ith $\pm$ v-jump, intervals $[T_-, T_+]$ over which $|\eta_i|$ and $|\lambda_i|$ are both large and the ratio $\frac{|\eta_i|}{|\lambda_i|}$ is bounded above, so that $\frac{|\eta_i|}{|\lambda_i - 1|}$ is again bounded above.

$(T_+ - T_-)$ is bounded above and below by fixed constants; this follows from our first observation above and the estimate on $T(z)$ derived above.

Following again Lemma 3.2 of [6], if now $N\delta^2$, is bounded above, then $|\lambda_i| + |\eta_i|$ is bounded. Again,kerα_1 rotates along v and the v-intervals for a full rotation of kerα_1 are of lengths bounded above and below by fixed constants, so that we can find a position where $\lambda_i\xi + \eta_i[\xi, v]$ is along $[\xi, v] + \bar{c}\xi$ and stays close to $[\xi, v] + \bar{c}\xi$ in direction for a certain portion of the $\pm v$-jump. This portion also has a size bounded above and below by fixed positive constants. Indeed, the speed of rotation (measured along v) of kerα_1 in a transported frame is bounded above and below by fixed positive constants: the estimate derived above on $T(z)$ above implies this result.

If now $\bar{c}$ is chosen appropriately (assuming $N\delta^2$ of Lemma 10 of [6] is bounded; otherwise, the argument is straightforward), the existence of the interval $[T_-, T_+]$, with the required properties, is warranted.[6]

Over such an interval $[T_-, T_+]$, we have

$$\left|\frac{\partial T}{\partial s}\right| \leq \frac{C \times \frac{-\partial a}{\partial s}}{|a_i - a_c^i|}$$

On the other hand, since $\frac{|\eta_i|}{|\lambda_i|}$ is bounded below by a fixed constant on such an interval

$$\left|\frac{\partial T}{\partial s}\right| \geq C|\delta_i'| \geq \frac{C|\eta_i|}{|a_i - a_c^i|} \geq \frac{C_1}{|a_i - a_c^i|}$$

, see the estimate on δ_i' in the proof of Proposition 2.18.

The pseudo-gradient here is then derived with the use of δ_i' or δ_{i+1}', as in the proof of Proposition 2.19 above, that is, it is derived after transporting $\pm\xi$ from the edge of a neighbouring ξ-piece, to the left or to the right of the ithξ-piece, and "compensating" $\eta_i[\xi, v]$ or $\eta_{i+1}[\xi, v]$ thereby derived with $\delta_i' v$ or $\delta_{i+1}' v$, see Figure above.

We then have along this pseudo-gradient:

$$\left|\frac{\partial}{\partial s}(a_i - a_c^i)\right| = O(|\lambda_i| + |\delta_i'||\mathrm{d}a_c^i(v)|)$$

if we are using the ith $\pm$ v-jump.

[6] The above arguments use Lemma 3.2 of [6]. This Lemma considers orbits of v that are transverse to T_0. There are two orbits of v that are periodic and are in T_0, see the definition of v and its dynamics, Sect. 3 of [6]. From these two periodic orbits, the above arguments can be repeated with very little modification; the basic phenomenon is that kerα_1 "turns well" along v.

We may assume in our construction of $[T_-, T_+]$ that

$$|\lambda_i| \leq C_1|\eta_i| = o(\delta'_i)$$

if $(a_i - a_c^i)$ is small; so that $|\frac{\partial}{\partial s}(a_i - a_c^i)|$ is then of the order of $|\frac{\partial T}{\partial s}|$, assuming that $\mathrm{d}a_c^i(v)$ is bounded away from zero at least on a sizable portion of the interval $[T_-, T_+]$.

We will address the issue of having $\mathrm{d}a_c^i(v)$ bounded away from zero later, as we "finalize" our argument.

However, we might have to use the ith $\pm$ v-jump and the $(i+1)$th $\pm$ v-jump in combination. Then δ'_i is replaced by a combination of δ'_i and δ'_{i+1}. The previous estimate does not hold anymore then.

Then, upper-bounding from above $|\lambda_i|$ by $|\lambda_i - 1| + 1$, same for $|\lambda_{i+1}|$, we derive

$$\left|\frac{\partial}{\partial s}(a_i - a_c^i)\right| = O\left(\frac{-\partial a}{\partial s}\right) + O(1) + O\left(\left|\mathrm{d}a_c^i(v)\right|\left|\frac{\partial T}{\partial s}\right|\right.$$

In $[T_-, T_+]$, we may assume—using again the fact that $\ker\alpha_1$ turns well along v—that $\frac{-\partial a}{\partial s}$ is bounded below by a fixed constant $c \gneq 0$. Therefore, if the time interval in s is bounded below, $J_\infty(x)$ will have decreased by a sizable amount. Iterating the argument as T tends to ∞, we then conclude that, under such circumstances, the Palais–Smale condition is satisfied.

We cannot exclude the possibility that the time interval in s becomes very small. This happens if $|\frac{\partial T}{\partial s}|$ becomes very large, assuming existence of the flow, as defined above, over $[T_-, T_+]$. Indeed, under this assumption, if $|\frac{\partial T}{\partial s}|$ is bounded, the corresponding s-interval has a size bounded below by a fixed positive constant.

Assuming that $|\frac{\partial T}{\partial s}|$ stays large, we derive that there is a sizable time-interval in $[T_-, T_+]$, $[T_-^0, T_+^0]$ over which $\mathrm{d}a_c^i(v)$ is bounded away from zero. Indeed, using general position arguments, we may assume that, if $\mathrm{d}a_c^i(v)$ is zero, $|\mathrm{d}^2a_c^i(v)| + |\mathrm{d}^3a_c^i(v)|$ is bounded below by a fixed positive constant, so that any v-orbit will eventually move away, after some short time on the v-orbit, from the hyper-surface $\{\mathrm{d}a_c^i(v) = 0\}$.

We need to make this argument more general, as it is possible that $|\frac{\partial T}{\partial s}|$ does not stay uniformly large over $[T_-, T_+]$ or a sizable connected sub-interval of this interval.

Observe that we may assume that the total decrease in $J_\infty = a$ over this interval is $o(1)$. Since we may assume that $\frac{-\partial a}{\partial s}$ is bounded below by a fixed constant $c \gneq 0$ over the whole $[T_-, T_+]$ interval, we conclude that $O(\frac{-\partial a}{\partial s}) + O(1)$ has a total contribution, after integration, equal to $o(1)$.

Under the construction of (ii) of Proposition 2.19, the contributions of the ith and of the $(i+1)$th $\pm$ v-jumps to $\frac{-\partial a}{\partial s}$ add up, they have the same sign. Therefore, both $|\lambda_i - 1|$ and $|\lambda_{i+1} - 1|$ both contribute $o(1)$ after integration; thus, all ξ-displacements $\lambda_i\xi$, λ_{i+1} and $O(1)$ contribute little to the displacements of the edges of the ith and the $(i+1)$th $\pm$ v-jumps. On the other hand, on $[T_-, T_+]$, $\frac{|\eta_i|}{|\lambda_i - 1|}$ and $\frac{|\eta_{i+1}|}{|\lambda_{i+1} - 1|}$ are bounded above. Thus, the displacements along $[\xi, v]$ are also, after integration, $o(1)$.

Since some of the $\pm v$-jumps have to increase and gain a "cycle", the displacement has to be essentially along a v-orbit and our argument about $\mathrm{d}a_c^i(v)$ and the interval $[T_-^0, T_+^0]$ extends.

Then, either we have existence of the flow over all of $[T_-^0, T_+^0]$. Over this interval, $|\frac{\partial}{\partial s}(a_i - a_c^i)|$ is of the same order than $|\mathrm{d}a_c^i(v)||\frac{\partial T}{\partial s}|$. Since $\mathrm{d}a_c^i(v)$ is bounded away from zero, if T increases from T_-^0 to T_+^0, $(a_i - a_c^i)$, which was small, becomes large and the decrease in $a = J_\infty$ becomes sizable, implying the verification of the Palais–Smale condition over these sequences.

Otherwise, T does not increase from T_-^0 to T_+^0 and this is just because $(a_i - a_c^i)$ tends to zero.

We then find $2k$ conditions at this limit curve to the least, since we also have that $(a_i - a_c^i)$ is zero, an additional condition; unless we can define a decreasing deformation in its vicinity, with the appropriate bounds. These $2k$ conditions can be seen again to be independent with the use of Proposition 29, p 198 of [3].

The conclusion is that, outside of a discrete isolated set, we can define a decreasing deformation with the appropriate bounds and this deformation will have its limit points either at critical points of J_∞—-a welcome conclusion—or at this discrete set.

At any point of this discrete set, there is a decreasing deformation, but it does not verify the appropriate bounds. However, we can use it for a tiny time, so that the bounds will hold depending on a previous decrease in J_∞; that is , we allow an increase in T under this deformation, but this increase will be, e.g. dominated by a previous decrease of J_∞, $-\Delta_j a$ that has occurred between the previous point in this discrete set on our flow-line and this new point of the same discrete set, in a tiny neighbourhood of this point where the flow-line has ended. $-\Delta_j a$ will be a priori bounded below for all flow-lines starting from all flow-lines that are very close to this flow-line and abutting in this tiny neighbourhood.

The construction of the flow can proceed. Because this is done by induction on the points of the discrete set as they are encountered, the flow might be a non-autonomous flow; but the decreasing deformation will proceed and the bounds on the curves hold on the flow-lines of this flow. The Palais–Smale condition follows.

If we are working on Γ_{2k} with $k \geq 2$, it is not difficult to repeat our previous construction on another ξ-piece. We have now to be careful and check that the condition $a_i = a_c^i$ derived on the ithξ-piece will be respected by our new deformation; this can be done with the use of the vector $\tilde{v}_i$ related to this ξ-piece, see [4], p 124. We may assume that $da_c^i(v)$ is non-zero; otherwise, we derive $(2k+2)$ conditions that are independent. These curves do not exist by general positions arguments. The use of $\tilde{v}_i$ as in [4], p124, to verify over the deformation the condition $a_i = a_c^i$ is then warranted. Bounds still check also. Therefore, after working o this additional ξ-piece, without destroying our previous work, we conclude that we must have now $(2k+1)$ conditions that are independent (the additional condition reads $a_j = a_c^j$ on the jthξ-piece on which we would have worked now); hence, again these curves do not exist and the Palais–Smale condition now holds with an autonomous flow. □

Our proof is now complete.

2.10 Appendix 3: The value of the homology for the first contact form/structure of Gonzalo and Varela

For the sake of completeness of the present paper, we include the argument in [6], Sect. 9 which establishes (ii) of Theorem 1.2. (i) of Theorem 1.2 follows from the precise knowledge of the periodic orbits and their indexes for the standard contact form of S^3.

Let us consider the two simple periodic orbits O_0 and O_1 corresponding to $r_1 = 0$ and $r_2 = 0$, respectively.

We first claim that

Proposition 2.20 *The v-rotation on the simple orbits corresponding to $r_1 = 0$ or $r_2 = 0$ is at least 7π. Therefore, the index of the iterate of order $\bar{p}$, $i_{\bar{p}}$, is at least $7\bar{p}$.*

Proof of Proposition 2.20

We consider neighbouring periodic orbits to the simple periodic orbit O_0 corresponding to $r_1 = 0$. ++It is not very difficult to see that as O_0 is elliptic, this involves the computation of the linearized operator at O_0; it is a long, but straightforward computation of the quantity τ, see [1], p 2, [4], p21, involved in the formula of the linearized operator $\ddot{\eta} + \eta\tau$.

The neighbouring periodic orbits have associated numbers (p, q), see section 7 of [6], that tend both to $-\infty$ as r_1 tends to zero: the ratio $\frac{\tilde{A}}{\tilde{B}}$ is irrational at $r_1 = 0$.

p is the number of counter-clockwise rotations in the "surviving" (x_3, x_4)-plane. We thus may consider our neighbouring periodic orbits as made of p distinct pieces of nearly closed ξ_0-pieces of orbits. Each of this distinct piece converges to the periodic orbit O_0 as r_1 tends to zero.

We consider some base point x_0 on O_0. We pick up v at x_0, equal, therefore, to $v(x_0)$ and we ξ_0-transport it around the periodic orbit O_0 over p-revolutions. This transported vector is denoted $u = u(s)$, where s is the running parameter over the periodic orbit O_0, based at x_0 and iterated an infinite number of times. Over each of these $p\xi$-pieces, $u(s)$ will coincide with v a certain number of times. This number of times can be n or $n-1$, where n is the H_0^1-index of O_0, with no base point assigned, that is , starting from any point of O_0, v turns more than $n\pi$ and less than $(n+1)\pi$ over O_0.

On the approaching ξ_0-orbits, we can take a base point close to x_0 and define a ξ_0 transported vector $\hat{u}(s)$, equal to v at the base point. Using continuity, v will coincide, on each of the p-pieces of ξ-orbit with $\hat{u}(s)$ at most n-times. It follows that on the whole approaching ξ_0 periodic orbit, v will coincide with the transported vector $\hat{u}(s)$ at most pn-times. The index of this periodic orbit is then less than or equal to $pn+1$, since it is less than or equal to pn under the constraint that the variation of the curve is along v at the base point.

Thus, the ratio of the index i_p to p is less than or equal to $\frac{pn+1}{p}$. Its limitsup, as p tends to infinity, is, therefore, less than or equal to n. The ratio $\frac{i_p}{p}$ is equal to $\frac{-2(\tilde{A}-\tilde{B})}{\tilde{A}}$ (section 7 of [6]) at the periodic orbit. This ratio is 2π at O_0. It follows that n is larger than 6. The claim follows. □

The other claim needed for the proof of (ii) of Theorem 1.2 is about the number of hyperbolic orbits of ξ_0 of index $2k$. It reads

Proposition 2.21 *Let H_{2k} be the set of periodic orbits of index $2k$ of ξ_0 , with $0 \lneq r_1 \lneq 1$ and let n_k be its cardinal. Then, $n_{k-1} + 4 \geq n_k \geq n_{k-1} + 2$ as k tends to infinity.*

Proof of Proposition 2.21 For $r_2 \geq \frac{1}{2}$, we consider the ratio of the index i to the number q. This ratio is equal to $\frac{-2(\tilde{A}-\tilde{B})}{\tilde{A}}$ (section 7 of [6]). The minimum m of this function on $[\frac{1}{2}, 1]$ is strictly larger than 1 and strictly less than 2.

It follows that if $\frac{i}{q+1} \lneq m \leq \frac{i}{q}$, then $\frac{i+2}{q+3} \lneq \frac{i}{q+1} \lneq m \leq \frac{i}{q} \leq \frac{i+2}{q+1}$ for p or q large enough. There is at least one more hyperbolic orbit in H_{i+2} with respect to H_i in the r_2-interval $[\frac{1}{2}, 1]$, maybe 2. The claim follows using the symmetry between r_1 and r_2. □

We are now ready to prove (ii) of Theorem 1.2; we will provide two proofs.

Proof of (ii) of Theorem 1.2

We consider the periodic orbits of prescribed index i. This set is denoted C_i. C_i is made of two subsets. To see this, we first consider the odd index $(2k-1)$. Then C_{2k-1} is made of the periodic orbits of index $(2k-1)$ having $0 \lneq r_1 \lneq 1$ and of the iterates of the elliptic orbits O_0 and O_1 (corresponding to $r_1 = 0$ and $r_2 = 0$, respectively). This latter set is denoted K_{2k-1}. The set of periodic orbits of index $(2k-1)$ having $0 \lneq r_1 \lneq 1$ is in one to one correspondence with the set H_{2k} of periodic orbits of index $2k$ introduced earlier. The iterates of O_0 and O_1 have a strictly increasing index since the v-rotation on each of them is larger than 3π, so that their index is at least three. Therefore, there are either two iterates contributing to the index i or none.

Thus, C_{2k-1} is made of H_{2k-1} that has as many elements as H_{2k} and of K_{2k-1}, that is empty or has two elements which are iterates of O_0 and O_1.

The same conclusion applies to C_{2k}.

By Corollary 7.2 of [6], the intersection operator from H_{2k} to H_{2k-1} is zero. Furthermore, by Proposition 2.20, there must be an infinite number of intervals of iterations $[p_m, p_m + 5]$ where the $K_j = \emptyset$ for $j \in [p_m, p_m + 5]$. Considering an odd index $(2l-1)$ in this interval, such that $2l$ and $(2l-2)$ are also in this interval, the claim of (ii) of Theorem 1.2 follows now from Proposition 2.21 and the fact that the intersection operator from $C_{2l} = H_{2l}$ into $C_{2l-1} = H_{2l-1}$ is zero.

2.11 Addendum for section 3

The following Addendum discusses how to extend the proof of Sect. 2.3 to cover all the cycles defined by the critical points at infinity dominated by a w_{2k+1} or a w_{2k} and the Morse relations that they define.

Special attention is given to the fact that the repetitions R singled out in Sect. 2.3 may change over a cycle, as well as the values that the "spared" $\pm v$-jump γ may take (see Sect. 2.3).

We complete in what follows detailed study of the distribution of the repetitions and the various choices for γ over the space of configurations as well as the study of the H_0^1-flow near a periodic orbit. This leads to an understanding of the stable and unstable manifolds of a hyperbolic as well as an elliptic periodic orbit in the Γ_{2m}s. This is useful in the study of the Fredholm properties of the various pseudo-gradients of the variational problem $(J, C_\beta)/(J_\infty, \cup\Gamma_{2m})$, [8].

Coming back to the framework of Sect. 2.3 and considering flow-lines coming from a w_{2k+1} and reaching a w_r^∞ that has three distinct edges, a (forced) repetition is singled out among the $(2k+1) \pm v$-jumps of the unstable manifold of w_{2k+1} near w_r^∞ on one of its ξ-pieces, whereas a single $\pm v$-jump γ_0 is chosen on another ξ-piece. Once these choices are completed and the flow-lines reach a periodic orbit of index $2k$ or $(2k-1)$, a decreasing, "bypassing" deformation has been sketched in Sect. 2.3; several technical points have been left aside about this deformation. They are as follows:

First, we need to define and track, in the vicinity of the cycle defining a critical point at infinity, on the flow-lines coming from $W_u(w_{2k+1})$ (see Sect. 2.3), a repetition in the presentation of the $*s$ or the $\pm v$-jumps unambiguously.

Second, we need the definition of a v-jump, again a $*$, denoted γ, outside of this repetition that we can track in a coherent manner. We will allow for switches in the definition of these $*s$, outside of a given repetition.

Third, given a periodic orbit, hyperbolic or elliptic, of index $2k$ or $(2k-1)$, respectively, or higher, we need to define a decreasing deformation that will "bypass" this periodic orbit. The proof was sketched in Sect. 2.3. We introduce here a process of re-arrangement of the $\pm v$-jumps near a periodic orbit that will move these configurations below the level of the periodic orbit.

We start with the definition of the repetitions on a given cycle:

2.11.1 Local analysis

Given $(2k+1)*$s to be distributed between the large $\pm v$-jumps of the cycle associated with a critical point at infinity w_r^∞, let us first assume that this critical point at infinity has at least two large $\pm v$-jumps. One, two or more $*s$ could contribute to form an edge.

If only one $*$ defines an edge, then one of the ξ-pieces of w_r^∞ supports a **forced** repetition. If there are more than one repetition, then the number of possible sign-changes between the $*s$ of the configuration drops below $2k$ to $(2k-2)$ or less.

If several $*s$ form an edge, we can slightly modify the deformation lines so that these configurations contain a forced repetition due to the fact that all the $*s$ forming the same edge can be brought to have the same orientation.

Assuming that a single $*$ defines an edge, we can order by "pushing away" and "widening" [5] all the $*s$ inside a ξ-piece of w_r^∞. When the $*s$ are not enough to cover the H_0^1-index of a ξ-piece, we also use the New Normal Flow of Appendix 4 of [5], which is discussed below.

All ξ-pieces of non-zero strict H_0^1-index can be assumed to be characteristic [4], i.e. the H_0^1-problem can be assumed to be degenerate.

If a ξ-piece of strict H_0^1-index equal to zero is non-degenerate, then either its edges are of the same orientation and w_r^∞ contains a **forced** fixed repetition. Or they have opposite orientation. The first case is simple as all approaching configurations must have this precise repetition, whereas the second case is more complicated.

Considering then a ξ-piece that is characteristic, of strict H_0^1-index i_0^j, we will say in the sequel that it is super-filled if it supports $(i_0^j+1)*$s or more.

There are two types of such pieces: with exactly $(i_0^j+1)*$s over such a ξ-piece, we can find that there must be a forced repetition between the $(i_0^j+1)*$s together with the edges. We will say then that this characteristic ξ-piece is of type (I). Otherwise, we will say that it is of type (II).

Characteristic pieces of type (I), supporting $(i_0^j+1)*$s [7] cannot be perturbed easily. That is assuming that one additional or more $*$ enters the characteristic ξ-piece or assuming that one or more $*s$ exits the characteristic ξ-piece, we find (all $*s$ in an edge can be assumed to have the same orientation) a configuration with at least two repetitions (in the edge and within the characteristic ξ-piece with its edge), hence with $(2k-2)$ zeros or less.

These configurations are thereby isolated; we can choose γ to be a $*$ inside this characteristic ξ-piece with the repetition defined by the sequence of $*s$ exterior to this ξ-piece (edges included).

Thus, we may assume for the remainder of our arguments that all super-filled characteristic ξ-pieces are of type (II), thereby bearing a repetition (any additional $*$ beyond the $(i_0^j+1)*$s may be assumed to be part of the edge, building a repetition with another $*$ of the edge).

There can be only one such super-filled characteristic ξ-piece and all other ξ-pieces are not super-filled; otherwise, the number of zeros drops again below $(2k-2)$. We claim that such a ξ-piece has a decreasing normal, see [5], pp 482–484, that has the orientation of the neighbouring edge. Indeed, from one edge to the next one, we find, since there is a forced repetition that i_0^j is even if the orientations of the edges are opposite and i_0^j is odd otherwise.

It follows that if the decreasing normal near one edge requires a $\pm v$-jump with an orientation opposite to the orientation of the edge, then the decreasing normal near the other edge requires a $\pm v$-jump with the orientation of the neighbouring edge. Therefore, if a $*$ exits or enters through this edge, it yields a decrease below the corresponding critical level of the critical point at infinity since it then has the orientation of the edge. Thus, the local cells that associate to build a cycle dominated by w_{2k+1} or x^∞ have repetitions (assuming that the number of zeros does not drop below $2k$) that are localized near such a super-filled characteristic piece and its neighbour.

Indeed, either this neighbour is also super-filled once an additional $*$ enters into it and the above argument holds. Or it is not super-filled with this additional $*$. As it enters, the configuration moves down.

This happens also whenever a $*$ on a characteristic ξ-piece that is not super-filled moves out of its nodal position. Again, a decreasing normal can be used on this $*$ and the configurations are then moved down.

When the H_0^1-index of a characteristic ξ-piece is not achieved by the number of descending $*s$ jailed between the two large edges of a critical points at infinity and when not all these $*s$ are zero $\pm v$-jumps on a given flow-line, we use the New Normal flow of [5], Appendix 4. The configurations can then be thought of

[7] The case when there are more than $(i_0^j+1)*$s on this ξ-piece can be reduced to this one

as having to the least two non-zero large $\pm v$-jumps, one coming from the critical point at infinity, the other one being this non-zero $*$, on this unfilled characteristic ξ-piece, subject to the New Normal Flow. If there is another non-zero $\pm v$-jump, typically if the underlying critical point at infinity has two large $\pm v$-jumps; or if there is another non-zero $\pm v$-jump on an unfilled characteristic ξ-piece, the arguments developed above apply without change since the New Normal Flow does not reverse the orientation of the given $*s$. As the distribution of $*s$ between these given three non-zero $*s$ changes, repetition develops. Over these forced repetitions, the cycle "ends", i.e. the critical point at infinity is bypassed as some non-zero $*$, subject to the New Normal Flow is moved out of the attractive New Normal line associated with this $*$ [5].

If there are only two non-zero $*s$ and they are consecutive $*s$, then we can argue as in Sect. 2.3, Lemmas 2.4-2.8, with identical conclusions.

It follows that, over all possible cycles, the choice and the tracking of repetitions can be completed.

2.11.2 Choice of γ

The choice of γ is required up to the start of the above process of decrease. In between, the configurations have moved down, past the level of the critical point at infinity. We do not need γ anymore.

γ can be constant, in between these processes of decrease. γ can also be forced to travel, e.g. if the support curve w of the critical point at infinity w_s^∞ has exactly two characteristic pieces: as a $*$ travels from one ξ-piece to the next one, the repetition follows and γ has to travel.

Checking through all arguments above over the choice of γ, we can assert that, in between the processes of decrease, over a single sequence (of at most two cells usually) of cells over which the choice of γ is completed, γ never completes a full turn over the whole sequence of $*s$.

Trivialization at w, the base curve, follows, that is, for our choice of γs to be completed, we need now to define γ at w. The definition of w, the support curve, might involve locally one less $*$ than the definition of the cycles that it supports and of those that we encounter over our processes of deformation, starting from the w_{2k+1}s, see above.

If it does not, γ is defined without ambiguity, there is no additional $*$. If it does, then γ varies over one cycle and in between cycles. We need to bring it back to a single a priori chosen insertion/position in the other remaining $2k*$s defining w.

Because the sequence of γs over a given cycle in between the processes of decrease never accomplishes a full turn over all $*s$, see the discussion above about the decreasing normals, this trivialization is accomplished through a contraction of the sequence of γs for one cycle to a constant position γ_0.

Over w, these γs may be viewed as zero $\pm v$-jumps. Zero $\pm v$-jumps can be made to "travel", the roles played by various $*s$ being switched over this travel, until they reach the position γ_0. The claim follows.

2.11.3 Critical points at infinity having a ξ-piece of H_0^1-index zero and critical points at infinity of the type $(\delta + w^\infty)^\infty$; choice of γ

We know that there are also critical points at infinity built with "Dirac masses" (back and forth or forth and back runs along v) along a critical point at infinity.

We need to choose γ over the associated configurations that come close to such a critical point at infinity.

Since w^∞ has at least one large $\pm v$-jump, we can choose γ to be a $\pm v$-jump in the complement of the two $\pm v$-jumps of a given "Dirac mass". If a $\pm v$-jump of a neighbouring configuration crosses the "Dirac mass", then a repetition occurs in the $\pm v$-jumps (they are then at least three of them) building the "Dirac mass". We can then switch the value of γ so that it remains in a different interval, far from the repetition. All arguments used above, when there were no "Dirac masses" involved, extend immediately.

We must also extend the choice of γ as one of the $\pm v$-jumps of the "Dirac mass" or both $\pm v$-jumps disappear. As long as one of them is not zero, the choice defined above, with the switches added, works. If both $\pm v$-jumps disappear and w^∞ has more than one large $\pm v$-jump, the switch is again possible. If w^∞ has exactly one $\pm v$-jump and it is involved in our proofs, dominated by some configuration out of $W_u(x_{2k+1})$, then it is of index at infinity equal to zero. Any flow-line coming out of w^∞ and reaching out to some w must involve some non-zero H_0^1-index and the choice of γ, with switches, becomes again possible.

The arguments and the choice for γ above works as well, it is easier, when the critical point at infinity has a non-degenerate ξ-piece of H_0^1-index zero, with edges having opposite orientation. This is a case that we left open above.

2.11.4 *Compatible choices for γ over the deformation*

Starting from the local construction as above, we flow down our configurations, with the related choices of the $*s$ γ outside of the repetitions. The flow that we use here is the "natural flow", inside each Γ_{2k}, which never reverses the orientation of a given $*$ (and therefore keeps a zero $*$ equal to zero).

Using this flow, we can reach a periodic orbit. We will discuss this later, in the next sub-sections.

We can also reach another critical point at infinity w_s^∞, to which the local analysis above applies. We then find a new local construction. On the other hand, we have the descending local construction, with its repetitions and γs. We need to "glue" these two local constructions.

For this, we first observe that, given the new local (forced) repetitions in the sequence of $(2k+1)$ descending $*s$, these local repetitions have a support, between two large consecutive edges with appropriate orientations, depending on the number of $*s$ jailed between them.

On the other hand, the "natural flow" carries with it the repetitions that the configurations inherited from the first critical point at infinity encountered.

These two or more repetitions might have different supports. These supports cannot be disjoint though and, near the new local set of configurations, we may define a dividing set where a repetition must occur over the intersection of supports, unless the number of sign changes drops below $2k$ and glueing is straightforward.

We thus reach this dividing set with several repetitions maybe. These repetitions include a smallest basic one.

2.11.5 *Various leaves or various $W_u(w_{2k+1})$ dominating the same w_r^∞*

We prove here that when several distinct $W_u(w_{2k+1})$ dominate the same w_r^∞, or when several leaves of the same $W_u(w_{2k+1})$ dominate the same w_r^∞, the choice of the repetitions and of the γs can be completed coherently, up to a process of switching of the γs, among choices exterior to the repetitions. We first show that we can assume that the repetitions are the same, whenever such an occurrence arises, but the γs may be different.

Given a characteristic ξ-piece of the (multiply) dominated w_r^∞, we observe that, over this multiple domination of the **same** cycle, the unstable directions at infinity of w_r^∞ must be covered by both (or more) sets of $(2k+1)*$s reaching to w_r^∞. In addition, on each ξ-piece, the same negative subspace must be covered by both sets for the problem with fixed edges. The issue of coherence arises only if , given a ξ-piece with an interior (strictly interior, distinct from the edges) $*$ for one leaf of the domination, no H_0^1-direction on this ξ-piece is left unfilled, that is, on each ξ-piece with an interior $*$ over the dominations (not for all of them; only one suffices) the spaces must be of the total H_0^1-index; otherwise, it is possible to split the cycle that we are considering in w_r^∞ in two distinct unstable manifolds, one for each domination and the glueing is not needed.

This observation implies that the issue of coherence in the choice of the repetitions arises only (see our discussion above on the cycles themselves) when the cycles are identical on the various leaves of the domination. The choice of the repetition does not vary then. However, over a sequence of multiple dominations, the choice for γ could vary among γs that are all exterior to the same repetition; or else, there are multiple distinct repetitions and the number of sign changes in the orientations of the $(2k+1) \pm v$-jumps drops below $2k$.

The argument above covers the case when a ξ-piece is unfilled and the New Normal flow of [5] is used on this ξ-piece.

2.11.6 *Non-characteristic ξ-pieces*

For the non-characteristic pieces, we observe that for any such ξ-piece with a non-zero H_0^1-index, we may assume using the techniques of [4], pp 77–102 that the number γ defined in [4], p78, Definition 4, is zero.

We then argue as follows: if the number of $*s$ on such a ξ-piece is at most its H_0^1-index and if a $*$ is then to leave this ξ-piece, then it must have the steady orientation of the edge through which it is leaving. As it starts moving towards the edge, we can assume that it is non-zero and still at a position of H_0^1-index. It follows that we may assume that the related cycle "ends" over such a transition.

On the other hand, if the number of $*s$ on such a ξ-piece. increases beyond this H_0^1-index, then because the number γ related to this ξ-piece is zero, the number of sign-changes drops below $2k$.

We are left with non-characteristic ξ-pieces of H_0^1-index zero, with reverse orientation of their edges. The only case to discuss is the case when there is a single characteristic ξ-piece and all the other ξ-pieces are non-characteristic of H_0^1-index zero, with reverse orientations of the edges. There is then a cycle that is obtained by making a $\pm v$-jump "travel" from the single characteristic ξ-piece across the non-characteristic ξ-pieces and back to this characteristic ξ-piece on the other side. Over this cycle, the characteristic ξ-piece is "super-filled", see Sect. 2.11.1 above—it has (one; if more, the number of sign-changes drops below $2k$)more $*s$ than its strict H_0^1-i at both "ends" of the cycle. Otherwise, a $\pm v$-jump γ outside of an identifiable repetition can be found and chosen. This fact will be used below to prove that we can then bypass, over flow-lines carrying such configurations, all the periodic orbits down.

If there are several non-characteristic pieces, then we can define dividing lines as the additional $\pm v$-jump crosses an edge of a non-characteristic ξ-piece. There is then a non-zero $\pm v$-jump outside of the repetition; again, we will define a downwards deformation that will move such configurations down, past the periodic orbits. We thus may assume that we are considering a critical point at infinity with a single non-characteristic piece of index 0, with reverse edge orientations and another single characteristic ξ-piece.

The total number of $*s$ is $(2k+1)$. The two edges of the characteristic ξ-piece have a reverse orientation; therefore, the number of $*s$ inside the characteristic ξ-piece when it is "super-filled" is odd. This forces the decreasing normals, [5], pp 482–484, related to the two edges to be "well-oriented" for one edge (the orientation of the decreasing normal for this edge is the reverse of the orientation of the edge) and "ill-oriented" for the other one. When the travelling $\pm v$-jump "enters" the characteristic ξ-piece through an edge, it has the orientation of the edge. Therefore, there is one side where it does not define a decreasing normal as it enters. For the cycle to be completed, the orientation of this $\pm v$-jump must reverse; or the repetition must "expand" and must involve the other $\pm v$-jumps living on this ξ-piece. We prove below that the curves of Γ_4 in $W_u(w_{2k+1})$ do not reach w_{2k}. It follows that we have at least three non-zero $\pm v$-jumps over the configurations that we are studying and that transitions involve at least four non-zero $\pm v$-jumps. The repetitions at both ends of the cycle are different: one occurs on the left side of the characteristic ξ-piece, starting from the left edge until some $*$ that is not the $*$ of the right edge is reached, whereas the other one is reached on the right side.

Since four $\pm v$-jumps to the least are non-zero over the transitions, a given repetition stays on one side, left or right, unless the number of zeros drops to $(2k-2)$. For the decreasing normal to be achieved, the repetition must be on a given side, eg the right side. Therefore, at the other end of the cycle, the cycle will not be complete unless we cross configurations such that the number of sign-changes drops to $(2k-2)$ or less. The switch of γs is straightforward over such configurations. The claim follows.

2.11.7 Reaching out to a hyperbolic orbit of index $2k$ or to an elliptic orbit of index $(2k-1)$. Choice of F^+ and preservation of the repetitions

In this sub-section, we extend, at the light of the multiple coherent repetitions singled out above, the arguments of Sect. 2.2 for the choice of the stable direction F^+ along a hyperbolic orbit of index $2k$. Of course, the use of such an $F^+ \oplus F^-$ decomposition is warranted only after a process of re-arrangement of the $(2k+1)*$s is completed with equal spacing between them, see Sect. 2.2. This will be addressed in the next sub-sections; a very precise process of re-arrangement will be defined, over a precise definition of the H_0^1 and the "pushing away" flows, until a given configuration is moved down, or equal spacing is achieved or a cycle of low dimension is left above. The definition of the homology does not depend on these low-dimension cells.

We thus assume that we are near a simple hyperbolic orbit w_{2k}, of Morse index $2k$, coming from the $w_{2k+1} - w_{2k+1}^\infty$-tangency. Re-arrangements, see below are enacted; the H_0^1-flow has been defined, see below.

Along the various local choices for γ, we need to find various (maybe) local choices for the space F^+, so that the configuration never finds itself along F^+.

Considering a set of such descending configurations coming from some w_r^∞, out of some w_{2k+1}, which have been close maybe to several other w_s^∞, we single out the various repetitions and we track the smallest one R_0 (this repetition does exist, otherwise the configuration allows only for $(2k-2)$ sign-changes and glueing is straightforward). All repetitions for approaching configurations R_j $(j \neq 0)$ contain R_0. For any given R_j, there is an exterior γ, a $*$ that is not included in the support of the repetition.

Given the R_js, a family of forced repetitions intersecting in a basic forced repetition R_0 (that might not be minimal at the configuration), we consider the sequence S of k or $(k+1)$ consecutive alternating $\pm v$-jumps modelizing the attractive directions for $J''(w_{2k})$, see Sect. 2.2.

Several different occurrences may arise:

Case 1 S covers R_j.

Case 2 S covers one extreme $*$ of R_j, but not the other one and intersects the exterior of R_j.

Case 3 S covers the exterior of R_j (possibly including a boundary $*$ of R_j).

We choose F^+ in a continuous way as the cases evolve (and the configurations change) with the following rule: in case 3, F^+ is chosen over $\pm v$-jumps completely exterior to R_j (not having any $\pm v$-jump in common with R_j).

As we convex-combine the various F_j^+s for the various R_js to build a continuously varying F^+, these F_j^+s and their combinations can never combine and build an alternating configuration that covers the two boundary $*s$ of R_0 **and** the exterior of R_0.

Indeed, if so, the sequence S must cover the exterior of R_0. We are in case 3. It then covers the exterior of R_j and, therefore, F_j^+ is reduced to a subset of the part of S completely exterior to R_j, hence to R_0 as well. The claim follows.

It follows that the configuration, after convex-combining the various rearrangements so as to preserve them and preserve R_0 as well, never adjusts along F^+ and can be moved down, past w_{2k}.

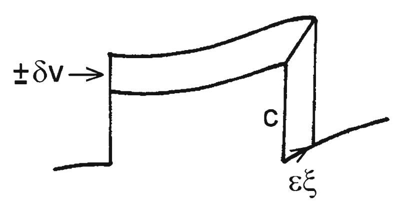
A similar, slightly easier argument, can be made for w_{2k-1}.

Before starting a detailed analysis of the H_0^1-problem based at γ—this includes the re-arrangement process—we make some technical observations that will support our arguments below, for the "pushing away" flow defined below in particular:

2.11.8 Technical observations

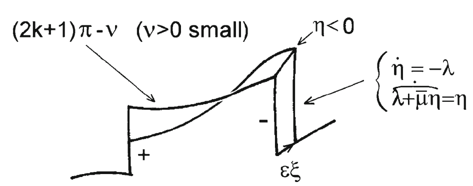
The observations are supported by a related sequence of drawings and computations embedded in these drawings:

Observation 1 *Given two consecutive small $\pm v$-jumps of a curve in Γ_{2s}, separated by a ξ-piece that is not characteristic, we can decrease the functional J_∞ on this curve by "pushing" these two $\pm v$-jumps further apart (see below). This can be completed while keeping anyone of the two $\pm v$-jumps's location unchanged whereas its size might change.*

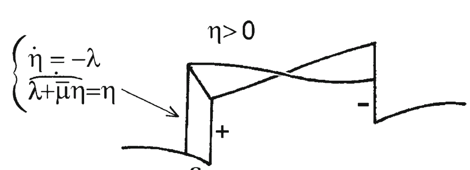


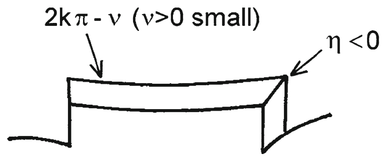
$$\Delta J_\infty = -\varepsilon + \varepsilon\left(1 - \frac{c^2}{2} + o(c^2)\right)$$

Observation 2 *Assuming that these two consecutive small $\pm v$-jumps have opposite orientations and assuming that the v-rotation on the ξ-piece separating them is $k\pi^-$, a bit less than $k\pi$, then through this "pushing away", either J_∞ will decrease substantially, or one of these $\pm v$-jumps will become tiny, whereas the other one increases in size. This latter one may be chosen as we please among these two $\pm v$-jumps. The configuration can then be brought down, below the level of the periodic orbit w by a simple application of the flow at infinity.*

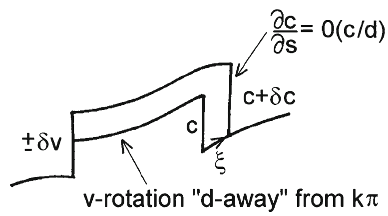


Here, we are decreasing the positive $\pm v$-jump. This increases the negative $\pm v$-jump.





We are now decreasing the negative $\pm v$-jump. This increases the positive $\pm v$-jump.
This, time, both $\pm v$-jumps increase in size.



A transported vector (along a $\pm v$-jump) $z = \lambda\xi + \mu v + \eta w$ satisfies $\dot{\eta} = O(\lambda)$, so that η is, in absolute value, close to c at the top of the right $\pm v$-jump, see [4], p 26. "Compensating" this $[\xi, v]$-component, we find that the size of the left $\pm v$-jump changes by an amount of the order of $\frac{c}{d}$, where d measures the difference of the v-rotation along the intermediate ξ-piece to the next larger $k\pi$. The size of the right $\pm v$-jump changes then of an amount of the same order (use the transport equations along ξ now, same reference than above). The decrease in the value of J_∞ 0ver a time interval of the above process $[0, s]$ is of the order of $-\int_0^s c^2(x)dx$ and $c(x)d(x) \cong c(0)d(0)$. $d(x)$ decreases from d_0 to 0 and can be taken, up to irrelevant constant, to be $d(x) = d(0) - x$. It follows that either J_∞ decreases substantially, or the $\pm v$-jump becomes very large and a simple use of the flow at infinity brings these curves below the level of the periodic orbit w.

2.11.9 Deforming the flow-lines of $W_u(w_m) \cap \Gamma_2$ and the flow-lines of $W_u(w) \cap \Gamma_4$ after a domination or a tangency

We prove here that we can deform the configurations of $W_u(w_{m+1}) \cap \Gamma_2$ and the configurations of $W_u(w_{m+1}) \cap \Gamma_4$ that went to a w_m^∞ or that are involved in a tangency $w_{m+1} - w_{m+1}^\infty$, without creating additional $\pm v$-jumps and without reversing the orientations of their $\pm v$-jumps, past a periodic orbit w_m.

We observe that, in the case of $W_u(w_{m+1}) \cap \Gamma_2$, this set is of dimension 1. After domination—this includes the case of the tangencies $w_{m+1} - w_{m+1}^\infty$, since the tangency doe not occur through Γ_2 by general position—this set is of dimension 0, that is empty since this is a set of flow-lines. For $W_u(w_{m+1}) \cap \Gamma_4$, the dimension after domination is 1 and, therefore, we are considering a single flow-line in Γ_4. The argument is not difficult.

The following considers the more general case when we deform **all** of $W_u(w_{m+1}) \cap \Gamma_4$, with $m = 2k$, past a simple hyperbolic orbit w_{2k}: in section to the flow, the configurations that we consider form a stratified set of top dimension 1. We single out in this set the configurations such that the v-rotation from the positive $\pm v$-jump to the negative one is $s\pi$, $s \in \mathbb{N}$. These define a finite number of isolated points in the same section to the flow. Using general position, we may assume that this positive $\pm v$-jump is not at the top E^+-position (as in Sect. 2.11.10). Therefore, we can manipulate these configurations down, moving the positive $\pm v$-jump to an E^--position; the v-rotation from + to − is unchanged. All these configurations are then moved down through expansion of the positive $\pm v$-jump. The initial manipulation involves an expansion of the functional where the η function $\eta = \eta_+$ corresponding to one $\pm v$-jump, e.g. the positive v-jump is projected onto the H_0^1-space defined by the other one. η_+ is modified after this projection into $\bar{\eta}_+$. Then, after the total η-function corresponding to the sum of η_+ and η_- is rewritten using $\bar{\eta}_+$ and a corresponding $\bar{\eta}_- = \eta_+ + \eta_- - \bar{\eta}_+$, we manipulate $\bar{\eta_-}$, increasing or decreasing this quantity so that the functional decreases. We may assume, using general position, that the position at which the $\pm v$-jumps are located and their seizes allow this manipulation with a strict decrease. We perform this manipulation, without moving the base points of the $\pm v$-jumps until either the configuration has been moved below the level of the hyperbolic orbit or the positive $\pm v$-jump has become tiny. The size of the negative $\pm v$-jump has not changed throughout this manipulation. We, then, are considering a configuration made essentially of a single negative $\pm v$-jump located at a point that is not a top

E^+-position. We can slide it down, as stated above, to an E^--direction over a decreasing deformation and then make this negative $\pm v$-jump grow so that the configuration moves below the level of the periodic orbit.

We are then left with configurations such that the v-rotation from the positive to the negative $\pm v$-jumps is never $s\pi$, $s \in \mathbb{N}$. If this v-rotation is between $(2\ell-1)\pi$ and $2\ell\pi$, $\ell \in \mathbb{N}$, we expand the interval between them, "pushing away" one of them from the other. If one of them becomes tiny with respect to the other one over this process, we may continue this process, moving away the tiny one from the other one whereas manipulating the size of the larger one so that the process is still J_∞-decreasing. Once the v-rotation between them is close to $2\ell\pi$, both expand and J_∞ decreases below the critical level (with the help of a pseudo-gradient if needed).

The configurations that are left are such that the v-rotation from the positive to the negative $\pm v$-jump is in $(2\ell\pi, (2\ell+1)\pi)$. All the other ones have been moved down. We perform the same "pushing away" and the negative $\pm v$-jump will eventually become tiny with respect to the positive one (if we "push away" the positive one from the negative one). We can then move it away across the $(2\ell+1)\pi)$-position (it is tiny) from the positive v-jump in a J_∞ decreasing process and we resume as above, with the same conclusion.

We cannot perform this when $(2\ell+1) = 2k-1$ since we would then end up having the negative $\pm v$-jump very close to the positive one. We observe now that all the configurations such that the v-rotation from $+$ to $-$ is less than or equal than $(2k-2)\pi$ have been moved down. For the other ones, we may assume that the complement v-rotation is less than 3π. We then resume the same argument, but we reverse the role played by $+$ and the role played by $-$. The conclusion follows.

2.11.10 H_0^1-index based at γ, flows

Let us consider a simple hyperbolic periodic orbit w_{2m}, of Morse index $2m$, with $m \geq k$. The case of elliptic orbits will follow from the study of sub-cases developed here.

We assume that configurations of $(2k+1) \pm v$-jumps out of a simple periodic orbit of index $(2k+1)$, w_{2k+1}, are reaching near w_{2m}, after having undergone a tangency with the stable manifold of a critical point at infinity of index $(2k+1)$ or having been attracted by a critical point at infinity w_s^∞.

Let us assume that a $\pm v$-jump γ has been selected over a given set of configurations, outside of a repetition R_j that must be spared. There might be switches in the choices for γ, between two neighbouring $\pm v$-jumps; this will be discussed later.

The $\pm v$-jumps of our configurations are re-ordered as indicated above, through the process of "pushing away", essentially away from γ.

Let us understand the H_0^1-problem based at γ, along w_{2m}. According to our analysis in Sects. 2.1.1, 2.1.2, γ can be located either in an interval of type E^+ or in an interval of type E^-, or at a node. The case of elliptic orbits can be included in the cases when γ is at an E^+ or at an E^--position, depending on the v-rotation around the elliptic orbit.

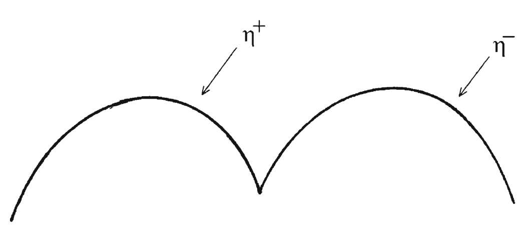
At a point of E^+, the v-rotation around w_{2m} is $2m\pi+\epsilon$, $\epsilon \gtrsim 0$; at a point of E^-, it is $2m\pi-\epsilon$. ϵ is small in both cases.

We introduce the solutions $\eta^\pm$ of the ordinary differential equation:

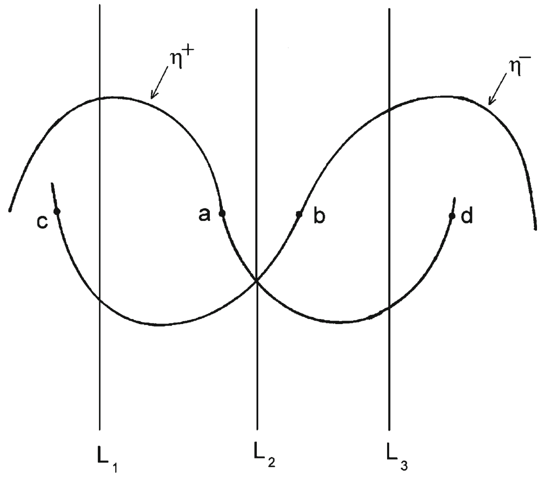
$$\ddot{\eta}^\pm + a^2\eta^\pm\tau = 0;\ \eta^\pm(0) = 0,\ \dot{\eta}^\pm(0) = 1$$

The only difference between η^+ and η^- is that η^+ is found after orienting w_{2m} along ξ, whereas η^- is found after orienting w_{2m} along $-\xi$.

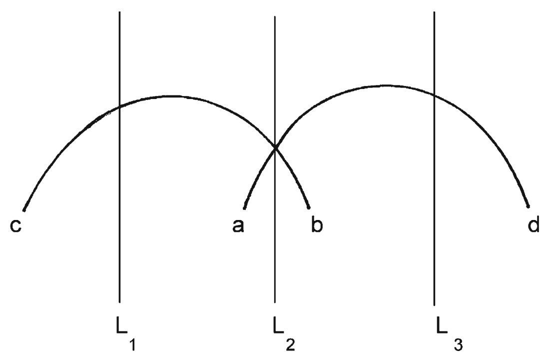
When γ is at a node, η^+ equals $-\eta^-$:



However, when γ is in E^+, η^+ and $-\eta^-$ are different. We find



Whereas, when γ is in E^-, η^+ and η^- behave as follows:



The drawings indicate three possible positions L_1, L_2, L_3 for a small $\pm v$-jump θ along the orbit w_{2m}. We assume here that there is "room" for this $\pm v$-jump, i.e. there are no other $\pm v$-jumps in $[c, d]$. Let us introduce the solution η_θ of the ordinary differential equation:

$$\ddot{\eta}_\theta - a^2\eta_\theta\tau = 0;\ \eta_\theta(L_i) = \eta_\theta(L_i + 1);\ \dot{\eta}_\theta(L_i^+) - \dot{\eta}_\theta(L_i^-) = 1$$

L_i can be any position among L_1, L_2, L_3.

When L_i is at a or when L_i is at b, η_θ becomes equal to $c\eta^+$ or $c\eta^-$ from γ to a along $+\xi$ or from γ to b along $-\xi$, depending on the cases, extended by 0 on the complement interval; c is an appropriate non-zero constant. Thus η_θ is in $H_0^1[0, 1]$ and $\int_0^1(\dot{\eta}_\theta^2 - a^2\eta_\theta^2\tau)\mathrm{d}t = 0$.

On the other hand, if L_i is neither a nor b, η_θ is not in $H_0^1[0, 1]$. It has an H_0^1-projection, which is a combination of η_θ and of η_γ, the function η for the $\pm v$-jump at γ, $\eta_\theta - \eta_\theta(\gamma)\eta_\gamma = p_{H_0^1}(\eta_\theta)$ $(\eta_\gamma(\gamma) = 1)$.

Arguing as in [4], pp 151, it is not difficult to see that $\tilde{\eta} = p_{H_0^1}(\eta_\theta)$ satisfies:

$$\int_0^1 (\dot{\tilde{\eta}}_\theta^2 - a^2\tilde{\eta}_\theta^2\tau)\mathrm{d}t \geq 0$$

if γ is in E^+ and $L_i = L_2$.

$$\int_0^1 (\dot{\tilde{\eta}}_\theta^2 - a^2\tilde{\eta}_\theta^2\tau)\mathrm{d}t \leq 0$$

if γ is in E^+ and $L_i = L_1$, $L_i = L_3$,
whereas $\tilde{\eta} = p_{H_0^1}(\eta_\theta)$ satisfies

$$\int_0^1 (\dot{\eta}_\theta^2 - a^2\tilde{\eta}_\theta^2\tau)\mathrm{d}t \le 0$$

if γ is in E^- and $L_i = L_2$.

$$\int_0^1 (\dot{\eta}_\theta^2 - a^2\tilde{\eta}_\theta^2\tau)\mathrm{d}t \ge 0$$

if γ is in E^- and $L_i = L_1$, $L_i = L_3$

These claims are derived from the counting of the H_0^1-indexes of the quadratic form $\int \dot{\eta}^2 - a^2\eta^2\tau$ on the complement intervals $[\gamma, \theta]$ and $[\theta, \gamma+1]$ (in $H_0^1[\gamma, \theta]$ and $H_0^1[\theta, \gamma+1]$, with obvious notations). If the addition of these independent indexes equals the total H_0^1-index at γ, $\tilde{\eta}_\theta$ is positive for $\int_0^1 \dot{\eta}^2 - a^2\eta^2\tau$. If this addition is less (by 1 then), $\tilde{\eta}_\theta$ is negative for $\int_0^1 \dot{\eta}^2 - a^2\eta^2\tau$. We can also combine various choices of positions for the L_i, over the various nodes of the functions $\eta_\gamma^\pm$. If chosen appropriately, one with respect to the other, then, all positions of type L_2 (taken a bit less than πv-rotation wise away from each other), combined together, will be positions of negative index, whereas all positions of type L_1, L_3 (taken in between the consecutive L_2s, at a v-rotation close to $\frac{\pi}{2}$ from the L_2s), combined together, will be positions of positive index in $H_0^1[\gamma, \gamma+1]$ if γ is in E^-. We can also mix the two types and, adjusting further the relative positions, the conclusion is the same. These roles reverse when γ is in E^+ (now the $L_2 s$ are positive positions that a tiny bit more than π apart, whereas the L_1, L_3 are negative positions close to $\frac{\pi}{2}$ away from the L_2s). In all the arguments above, the v-rotation around the hyperbolic orbit is always $2m\pi + o(1)$.

Observe that $\tilde{\eta}_\theta$ corresponds to two $\pm v$-jumps, one at θ and the other one at γ, of appropriate relative algebraic sizes (so that $\tilde{\eta}_\theta$ is in H_0^1, based at γ).

Let us define three fixed positions for $L_1, L_2, L_3, \bar{L}_1, \bar{L}_2, \bar{L}_3$, that evolve continuously in their respective regions as γ evolves from E^+ to E^-.

Assuming that a $\pm v$-jump located at θ is on $\bar{L}_2$ and γ is in E^+, θ defines a positive direction in the space $H_0^1[\gamma, \gamma+1]$ for $\int_0^1 \dot{\eta}^2 - a^2\eta^2\tau$. Let us assume now that γ moves continuously from E^+ to E^-. We claim that $\tilde{\eta}_\theta$ can be followed continuously, for θ on $\bar{L}_2$, as γ moves. When γ is in E^-, $\tilde{\eta}_\theta$ defines a negative direction in $H_0^1[\gamma, \gamma+1]$.

The formula for $\tilde{\eta}_\theta$ is $\tilde{\eta}_\theta = \eta_\theta - \frac{\eta_\theta(\gamma)}{\eta_\gamma(\gamma)}\eta_\gamma$. η_γ is the solution of

$$\ddot{\eta}_\gamma + a^2\eta_\gamma\tau = 0, \eta_\gamma(\gamma) = \eta_\gamma(\gamma+1), \dot{\eta}_\gamma(\gamma^+) - \dot{\eta}_\gamma(\gamma^-) = 1$$

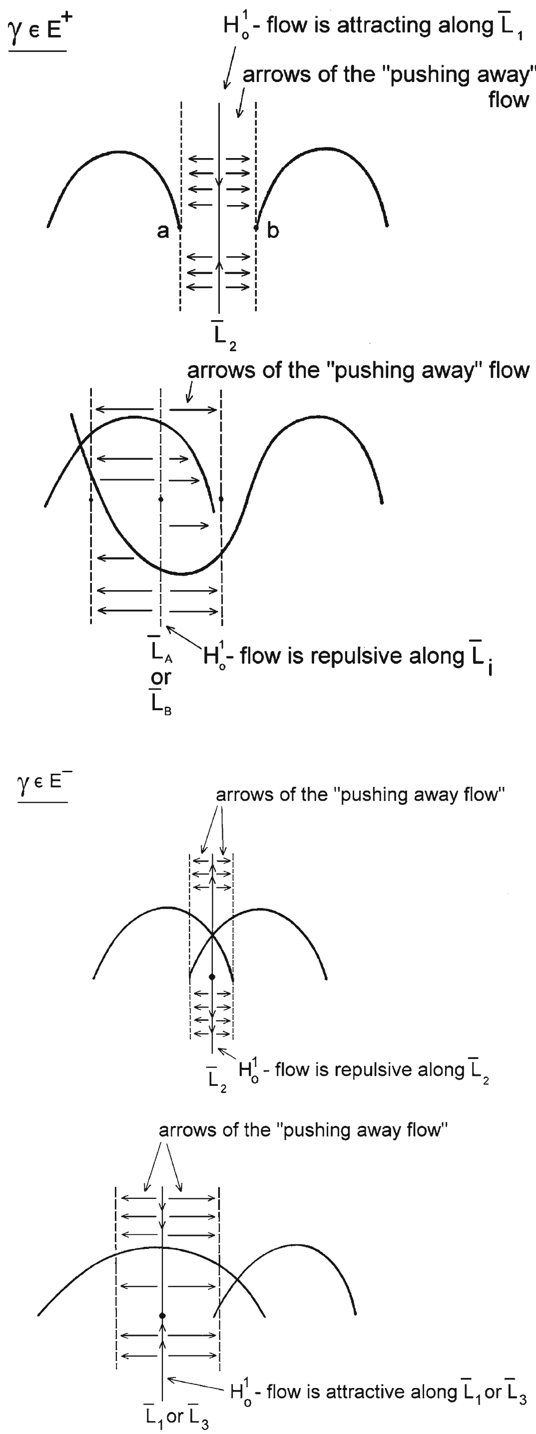
As γ evolves from E^+ into E^-, it crosses a node. Then $\eta_\gamma(\gamma) = 0$. If $\eta_\theta(\gamma)$ is not zero, that is if θ is not then at a node, $\tilde{\eta}_\theta$ does not converge and we cannot follow its value continuously. This is what happens when θ is at $\bar{L}_1$ or $\bar{L}_3$. However, when θ is on $\bar{L}_2$, then it goes to a node whenever γ goes to a node. The claim follows.

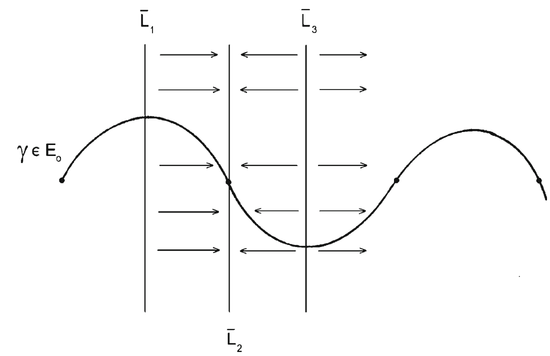
2.11.11 The "Pushing Away" flow when a $\pm v$-jump of the configuration is not located at any of the $\bar{L}_i$s

Having understood the behavior of the H_0^1-index based at γ, with the three types of positions L_1, L_2, L_3 and how we use them as γ evolves from E^+ to E^-, we add to the decreasing H_0^1-flow (based at $\bar{L}_2$ if γ is in E^-, based at $\bar{L}_1$, $\bar{L}_3$ if γ is in E^+) the "pushing away" flow, away from γ, also with the use of the $\pm v$-jumps of R_j (without ever changing their orientations) and the other $\pm v$-jumps over this process. The assumption is that the $\pm v$-jump that we are "pushing away" is not located at any of the $\bar{L}_i$s .

There are two ways of "pushing away" from γ, either along an intermediate nearly ξ-piece or along an intermediate $-\xi$-piece. The dividing lines are chosen to be the various $\bar{L}_i$ (there are several of these, for each choice of i) and if a $\pm v$-jump of the configuration is not located at one of the $\bar{L}_i$s, then "pushing it away" from γ will bring it close to the next point such that the v-variation is $j\pi$-away from γ, $j \in \mathbb{Z}$.

Typically, we find either


γ ϵ E+
H°1- flow is attracting along L̄1
arrows of the "pushing away" flow
a
b
L̄2
arrows of the "pushing away" flow
L̄A or L̄B
H°1- flow is repulsive along L̄i
γ ϵ E−
arrows of the "pushing away flow"
L̄2
H°1 - flow is repulsive along L̄2
arrows of the "pushing away flow"
H°1 - flow is attractive along L̄1 or L̄3
L̄1 or L̄3



2.11.12 Dimension of cells associated with these flows

It follows that each $\bar{L}_i$ defines a rest point for the flow. If we are considering $\bar{L}_2$ and γ is in E^- or we are considering $\bar{L}_1$ or $\bar{L}_3$ and γ is in E^+, these rest points have an unstable manifold of index 2, one for the position (the "pushing away" flow decreases J_∞ whereas the base points of the $\pm v$-jumps are moved further away from $\bar{L}_i$) and one for the H_0^1-index (a $\pm v$-jump along $\bar{L}_i$ is along an unstable H_0^1-direction).

On the other hand, for a $\pm v$-jump to be attracted to such a rest point, two conditions must be fulfilled: this $\pm v$-jump must lie on $\bar{L}_i$ and it must be a zero $\pm v$-jump. Otherwise, the flow will move it away from the rest point.

If we are considering $\bar{L}_2$ and γ is in E^+, or if we are considering $\bar{L}_1$ or $\bar{L}_2$ and γ is in E^-, this rest point has an unstable manifold of dimension 1 since the H_0^1-direction along $\bar{L}_i$ is a stable direction.

Let us observe also that if a collection of $\pm v$-jumps are zero on $W_u(w_{2k+1})$ and the curves are, therefore, in $\Gamma_{4k+2-2s}$, then their algebraic sizes $(c_1, \ldots, c_s)$ form a set of coordinates for $W_u(w_{2k+1})$ transversally to its intersection with $\Gamma_{4k+2-2s}$. This easily follows from the fact that the one-parameter group of decreasing deformation along $W_u(w_{2k+1})$ preserves each Γ_{2l} and its differential along curves of $\Gamma_{2l}, l \lneq (2k+1)$, maps the transverse directions to Γ_{2l} in Γ_{4k+2} near w_{2k+1} on transverse directions to Γ_{2l} as the decreasing deformation proceeds.

2.11.13 Never increasing the number of sign-changes

When we use the "pushing away" flow, we want the number of sign changes over the configurations never to increase. In view of our technical observations above, this requires to proceed as follows: nearly zero $\pm v$-jumps can be "pushed away", but to "push away" other neighbouring $\pm v$-jumps from them, we need the v-rotation separating them to be less than π. Therefore, as a $\pm v$-jump becomes tiny (and the configuration is then in the neighbourhood of configurations satisfying the condition that this $\pm v$-jump is zero), we perform a decreasing deformation, using the fact that some other $\pm v$-jump is not "tiny" when compared to this $\pm v$-jump (being "tiny" is a relative feature), and we make the "tiny" $\pm v$-jump travel so that the v-rotation separating it from the neighbouring $\pm v$-jump with respect to which the "pushing away" is performed becomes less than π. Then, by our observations above, "pushing away" can be performed safely and the number of sign-changes in the configurations do not increase.

The "tiny" $\pm v$-jumps travel back to their former positions as their sizes increase.

Observe also that, when a $\pm v$-jump is "tiny" or zero, we can "push away" the other $\pm v$-jumps across or over this tiny $\pm v$-jump. The deformations is J_∞-decreasing.

2.11.14 Crossing periodic orbits

We consider now an elliptic periodic orbit w_{2k+1} and we assume that it either dominates a w_{2k}^∞: we are then interested in the flow-lines from w_{2k}^∞ to periodic orbits, periodic orbits of index $2k$ or of index $(2k-1)$ in particular; or a tangency $w_{2k+1} - w_{2k+1}^\infty$ takes place and we are then interested in the flow-lines again from w_{2k+1}^∞ to periodic orbits, hyperbolic periodic orbits of index $2k$ in particular.

Although in both cases we will have to discuss periodic orbits that are elliptic as well as periodic orbits that are hyperbolic, we label the first case "elliptic periodic orbits" whereas we label the second case "hyperbolic periodic orbits", since in the first case we are interested to the effect of the critical points at infinity on the ∂-operator having elliptic orbits of index $(2k-1)$ as a target, whereas, in the second case, the target is hyperbolic orbits of index $2k$.

For elliptic periodic orbits, as the value of γ switches between two consecutive $\pm v$-jumps, we find that over tangencies/dominations $w_{2k+1} - w_{2k}^{\infty} - w_{2k}$ and $w_{2k+1} - w_{2k+1}^{\infty} - w_{2k}$, re-arrangement can be completed over such a switch with preservation of the repetition(s) in the complement: the argument for a decreasing deformation is complete. For Morse relations $w_{2k+1} - w_{2k}^{\infty} - w_{2k-1}$, the same argument does not work and a global decreasing deformation cannot always be defined. Indeed, if the forced (forced by w_{2k}^{∞}) repetition(s) complete a "full circle" amongst the $(2k+1) \pm v$-jumps instead of staying located within a sub-interval, we cannot define a global deformation.

For dominations/tangencies $w_{2k} - w_{2k}^{\infty} - w_{2k-1}$ or $w_{2k} - w_{2k-1}^{\infty} - w_{2k-1}$, the argument is straightforward, so that the conclusion is that ∂_{per} and ∂_{∞} do not mix over tangencies. The relation $\partial_{\text{per}} \circ \partial_{\text{per}} = 0$ can only be violated over creations/cancellations of periodic orbits x_m/x_{m-1}, "rhombi $x_m/x_{m-1}^{\infty}/x_{m-1}/x_{m-2}$" can form over the adjustment of the unstable manifold of x_{m-1} from the form that it has over the process of creation/cancellation to the normal form of Proposition 1 of [5].

2.11.15 *Elliptic periodic orbits*

We first observe that, given a γ and a repetition R_j to spare, we do not encounter any problem if we view R_j as R_{2k}, that is if we view R_j as a repetition occurring between **all** the $\pm v$-jumps of the configuration once γ is removed.

Therefore, we can move our configurations down, below the level of w, in this way. Only that γ and R_j are used over part of the configuration space, not over all of it. Along dividing lines of dimension $(2k-2)$, transversally to the flow (we are on $W_u(w_{2k}^{\infty})$, in a section to the flow; and we consider there dividing lines) we need to be able to switch γs.

Using one "given" γ that is outside of R_j, we can try to perform re-arrangement and bring the configuration down. As explained above, we find that this possible if we leave behind some cells of top dimension $(2k-1)$.

We would like to prove that this will not happen along this dividing line of dimension $(2k-2)$.

Using the L_is defined above, we see that the "top" of this cell that does not move down corresponds to the verification of $(2k-2)$ conditions at the configuration: $(2k-j)$ conditions for the $\pm v$-jumps not in R_j to be along the L_is and $(j-2)$ conditions for the $j \pm v$-jumps of R_j to have a zero projection on the strict H_0^1-index of the nearly ξ-piece that runs from predecessor of R_j to successor.

On a set of top dimension $(2k-2)$, this gives rise, by general position, to the occurrence of isolated configurations.

If γ is at E_0, we find an additional condition. With $(2k-1)$ conditions, on a set of top dimension $(2k-2)$, we may assume that we never encounter such configurations.

Also, if γ is in E^-, we find an additional condition see below, and these configurations are ruled out.

If γ is in E^+, it can be assumed, using the same general positions argument, to avoid a family of specific positions. The remaining isolated configurations may be "driven", all relative positions of the $\pm v$-jumps of the configurations unchanged, so that γ is in E^- (see below again). An additional condition at γ is then not met and these configurations are moved down, past the level of w.

The switch is thereby possible and the deformation is complete if w is a hyperbolic orbit of index $2s \geq 2k$ or if w is an elliptic orbit of index higher than $(2k-1)$:then, γ is in E^-.

We are left with the periodic orbits of index $(2k-1)$, when γ is forced to be in E^+. A "cell" is left behind. It is of dimension $(2k-1)$, see our count below: one for γ (γ is in E^+), $(2k-j)$ for the "free" $(2k-j) \pm v$-jumps that are not γ and are not involved in R_j, each one contributes 1 in the H_0^1-problem based at γ and $(j-2)$ for the "squeezed" repetition R_j.

This cell is actually spanned by cells of dimension $(2k-2)$, rotating—the transport equations around ξ may be considered to be a pure rotation in the vicinity of an elliptic periodic orbit—as some $\pm v$-jump, before or after the repetition R_j, runs around the periodic orbit.

The $j \pm v$-jumps of the repetition R_j, with their predecessor and their successor, span a nearly ξ-piece supporting a v-rotation less than $(j-1)\pi + \epsilon$, ϵ a small as we please. Beyond this amount of v-rotation, the $\pm v$-jumps of the repetition R_j are expanded over this ξ-piece and the configuration moves down, below the level of ${}_{2k-1}$.

From γ to $1+\gamma$, the v-rotation is $2k\pi - \epsilon_0$, where ϵ_0 is a small fixed quantity. Using the "pushing away flow", performed between consecutive $\pm v$-jumps starting from γ, we can shorten the ξ-piece: it now supports a v-rotation at most equal to $(j-1)\pi - \frac{\epsilon_0}{2}$. We use more specifically here the "pushing away flow" starting from the predecessor of R_j, on the interval exterior to R_j. In this way, the repetition R_j is spared.

Then, when this shortened ξ-piece supports a v-rotation equal to $(j-1)\pi - \frac{\epsilon_0}{4}$, it supported before "shortening" of the ξ-piece, a v-rotation larger than $(j-1)\pi + \frac{\epsilon}{2}$. It can be assumed to be below the level of w_{2k-1}.

The $j \pm v$-jumps of R_j can then be collapsed into $(j-2) \pm v$-jumps. This might involve a modification of the predecessor and the successor; however, over this (J_∞-decreasing) process, either the repetition R_j is spared, or the $j \pm v$-jumps collapse into $(j-2) \pm v$-jumps.

The "cell" of dimension $(2k-1)$ "left behind" has then only $(2k-1) \pm v$-jumps. These $(2k-1) \pm v$-jumps, as the energy level drops below the level of w_{2k-1}, recompose into $(2k+1) \pm v$-jumps through the addition of two zero $\pm v$-jumps that gradually assume the proper orientations and recompose R_j, once the configurations have move away (below) w_{2k-1}.

The "cell" supports also a large "hole", that is the curves composing the "cell" contain a ξ-piece bearing a v-rotation strictly more than π (by an amount lower-bounded away from zero) and where no inside $*$ lives.

2.11.16 "Bypassing" the periodic orbit

We need to define the deformation across the dividing lines when there is a switch in the value of γ. The repetition R_j may be assumed to be the same across these dividing lines, expanding or retracting or both on each of the domains thereby defined.

The dividing lines arise for two reasons: for the case of the elliptic orbits w_{2k-1}, they arise through the $w_{2k}^\infty - w_{2k-1}^\infty$-dominations in the $w_{2k+1} - w_{2k}^\infty - w_{2k-1}$ sequences of dominations. They are also due to the fact that the cycles associated with the w_{2k}^∞s in these sequences of dominations might be complicated cycles, made of several chains for which γ and R_j are well-defined for each chain, but change as the chains change through the dividing lines.

For the case of the hyperbolic orbits w_{2k}, they arise through the tangencies and dominations $w_{2k+1} - w_{2k+1}^\infty - w_{2k}^\infty$ in the $w_{2k+1} - w_{2k+1}^\infty - w_{2k}$ sequences; and they also arise for a second reason as above: the cycles associated with the w_{2k+1}^∞ might be complicated.

Let us first discuss the first case, that is the w_{2k}^∞- w_{2k-1}^∞-dominations. This w_{2k-1}^∞ might be dominated by an w'^∞_{2k} that is itself dominated by w_{2k+1}.

The cycle in w_{2k-1}^∞ that is dominated must be the same and this implies, after some work related to the understanding of the critical points at infinity of this variational problem, see [4,5] in particular, including Appendix 4 of [5] and the various flows, that the repetition R_j is identical across these dominations. However, the γs might be different.

Assume that, outside of R_j, the dominations $w_{2k}^\infty - w_{2k-1}^\infty$ and $w_{2k}^\infty - w_{2k-1}^\infty$ involve a "hole" as in Appendix 4 of [5] and a $\pm v$-jump with a steady orientation associated with this "hole". This means that this $\pm v$-jump with a steady orientation collapses along a nodal line of the New Hole Flow of Appendix 4 of [5] and expands on each side of this nodal line.

If this does not happen, then either some $\pm v$-jump outside of R_j collapses on some ξ-piece of w_{2k-1}^∞, the number of sign changes over the configurations of this dividing line drops to $(2k-2)$ and the switch of γs can be performed easily; or there is no collapse and the large $\pm v$-jumps of w_{2k}^∞ and $w'^{\,\infty}_{2k}$ must be the same and the dominations are H_0^1-dominations outside of R_j. Then, γ can be taken to be the same across w_{2k-1}^∞. These cases are, therefore, solved.

The case when γ is defined unambiguously and also the case when the dividing lines support configurations tolerating at most $(2k-2)$ sign-changes in the orientations of their $\pm v$-jumps also. This is the best result that we have reached in the case of elliptic orbits of index $(2k_1)$ over sequences of dominations $w_{2k+1}/w_{2k}^\infty/w_{2k-1}$; it assumes no "point to circle" domination between the three terms of the Morse relation (the first one or the second one). Bypassing an elliptic orbit w_{2k_1} over a tangency w_{2k}/w_{2k}^∞ or over a domination/tangency $w_{2k}/w_{2k-1}^\infty/w_{2k-1}$ is immediate.

For the case of hyperbolic orbits, we develop in the next sub-section another argument, based on the study of the H_0^1-flow near a periodic orbit and this allows us to bypass these hyperbolic orbits, without further condition on γ, contrary to the elliptic case. This argument allows us also to understand well the stable and unstable manifold of a hyperbolic periodic orbit of index $2k$ in the stratified spaces Γ_{2k} and Γ_{2k+2}. This is useful in the study of the verification of the Fredholm assumption along flow-lines (work in preparation).

2.11.17 Hyperbolic periodic orbits

We consider now the case of hyperbolic orbits and, therefore, we consider the case when we have a connection $w_{2k+1} - w^{\infty}_{2k+1} - w_r$, involving at some time t_0 of the deformation a tangency between the unstable manifold of w_{2k+1} and the stable manifold of w^{∞}_{2k+1}.

2.11.18 A crossing an elliptic orbit coming w^{∞}_{2k+1}

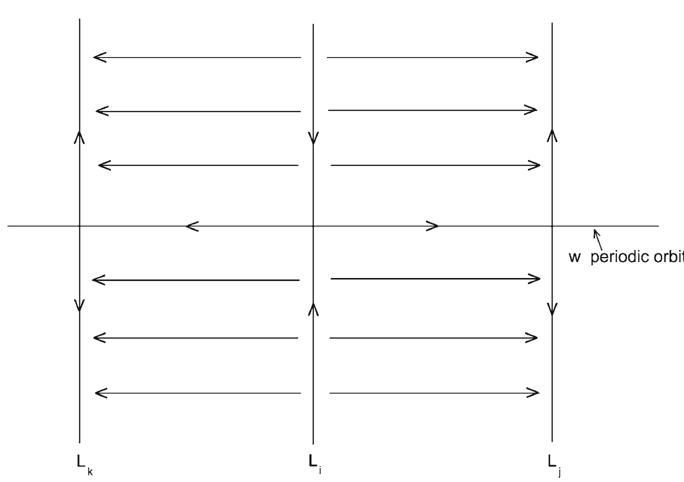
When we consider a $w_{2k+1} - w^{\infty}_{2k+1}$ tangency, we can avoid discussing dominations of an elliptic w_{2r+1} by $w^{\infty}_{2k+1}, r \geq k$ because a direct argument based on the counting of the number of conditions on the configurations transversally to the flow (on a set of dimension $2k$) yields $(2k+1)$ conditions (recall that, since $(2k+1)$ is more than $(2k-1)$, a single $\pm v$-jump along such an elliptic periodic orbit is in E^- and this provides one more condition.) for a domination of w_{2k+1} of w_{2r+1}, with $r \geq k$, to take place. This yields too many conditions with respect to the dimension of the stratified set defined by the unstable manifold of w_{2k+1} transversally to the flow and such dominations do not take place at any time and cannot be induced by tangencies $w_{2k+1} - w^{\infty}_{2k+1}$.

2.11.19 Crossing a hyperbolic orbit coming from w^{∞}_{2k+1}: the $(2k-1)$ dimensions count; more generally, the dimension count in the H^1_0-problem based at γ at an elliptic as well as at a hyperbolic orbit

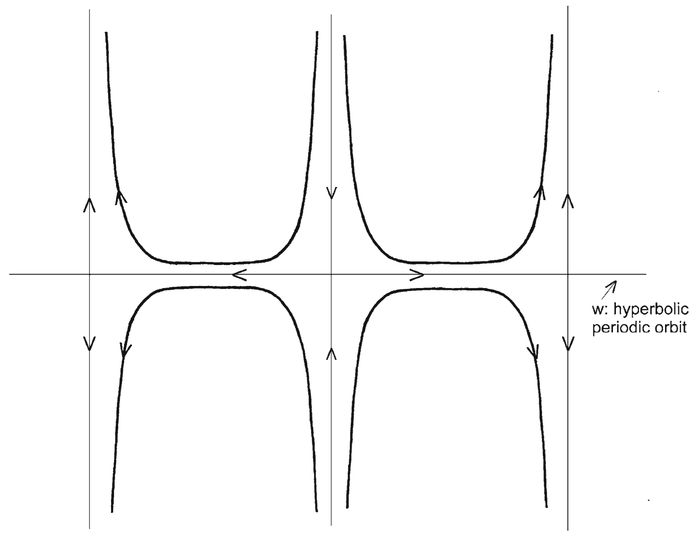
We now prove that, when coming from a similar tangency $w_{2k+1} - w^{\infty}_{2k+1}$, we try to cross a hyperbolic periodic orbit, we "leave behind" a cycle of dimension $(2k-1)$ at most. The discussion proceeds according to the position of γ along the hyperbolic orbit w and it includes a counting of the dimensions of the cells that cannot be deformed downwards, when a repetition R_j is to be spared and lives on a nearly ξ-piece supporting a v-rotation at most $(j-1)\pi$. This dimension count is performed for the H^1_0-problem based at γ; this argument works whether performed along a hyperbolic w or an elliptic w. This has been used above. An additional counting argument of the same type has to be performed at γ and gives 1 at a hyperbolic orbit w (see below) or at an elliptic orbit where the v-rotation is a bit less than $2s\pi$, $s \in N$. At an elliptic orbit with v rotation a bit less than $(2k-1)\pi$, the count can give 1 or 2, depending on whether γ over the set of configurations corresponding to the cell can span the whole periodic orbit or not.

We start with an outline of the proof:

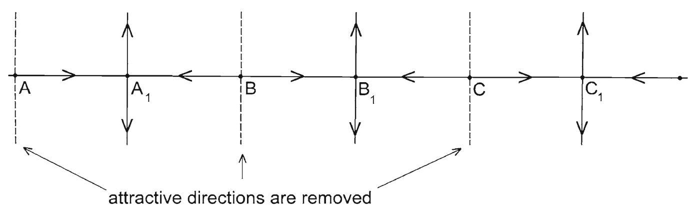
When γ is in E^+ or when γ is in E^-, we will revisit the H^1_0-problem based at γ and the related "pushing away flow" between the various $\bar{L}_i$. We will observe that this "pushing away flow" can be defined unhindered from any attractive line $\bar{L}_i$ to a neighbouring repulsive line $\bar{L}_j$; the node in between these two lines does not hinder the continuation of this flow because it is a node for the other, for the reverse direction.



It follows, see below, that the incoming $\pm v$-jumps, once γ is defined, can all be deformed onto the attractive directions, to which we add the periodic orbit w:



The set on which we retract by deformation is



A, B, C, A_1, B_1, C_1 are all rest points for this flow. Zero $\pm v$-jumps can be included in this retraction by deformation.

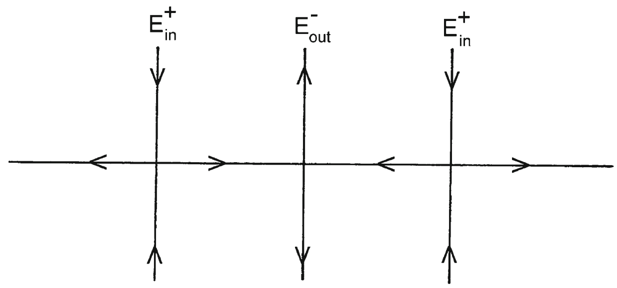
Since γ is in E^+ or in E^-, the total v-rotation at γ around the hyperbolic w is not $2s\pi$ and we may assume that the $j \pm v$-jumps of R_j have been collapsed into $(j-2) \pm v$-jumps.

Then, each of the $(2k-j) \pm v$-jumps of the configuration, when we exclude γ and the $j \pm v$-jumps of the repetition R_j, gives rise to one dimension. These dimensions are attached to the level set $J_{\infty,d-\epsilon}$, where d is the level of w. Indeed, as can be derived from the drawing above, this "structure" is attached along a set where one of the $\pm v$-jumps is not "small" along a repulsive line. The energy level is, therefore, less than the level for w by an amount to the least $\epsilon \gtrsim 0$. The remainder of the set is made of curves such that at least one of $\pm v$-jumps is not small and this set can thereby deformed below the level of w using classical arguments of deformation and Morse theory.

The $j \pm v$-jumps of R_j are collapsed into $(j-2) \pm v$-jumps between predecessor and successor of R_j and yield at most $(j-2)$ dimensions. This yields $(2k-2)$ dimensions, but we have to add the dimension spanned by the variations of γ.

At a hyperbolic orbit w, the count is the same than above for the $(2k-j) \pm v$-jumps and yields one additional dimension. Indeed, we now have to think about E^+ and E^-. Along a direction of E^+, γ decreases in size and along a direction of E^-, γ increases (in fact, this is not γ itself, but the function η derived at γ by adding to η_γ, see below, the contribution of the other $\pm v$-jumps of the configuration after H_0^1-projection on the H_0^1-problem based at γ, see below again).

We may then define, in each interval of type E^+, a special position E^+_{in}. When γ is at E^+_{in}, this function η (in absolute value) and the functional J_∞ decreases. We can also define, in each interval of type E^-, a special position E^-_{out}. When γ is at E^-_{out}, the function η defined above increases in absolute value and J_∞ decreases again. In between, we can "slide" the position of γ from an E^+_{in} into an E^-_{out}. Again, J_∞ decreases, see below.



We thus will find $(2k-1)$ dimensions, only that this count is completed when γ is in E^+ or when γ is in E^-, so that the H^1_0-problem at γ is non-degenerate. We can then use also the previous analysis and describe very precisely the sets along which these "structures" attach to the set when γ is in E_0. The attachment sets are of dimension $(2k-2)$ and are not difficult to describe.

Once we understood how we can deform our configurations on a set of top dimension $(2k-1)$ when γ runs in $E^+ \cup E^-$, we will move to understand what happens when γ is in E_0.

This dimension count is distinct from the arguments of Sect. 2.3: once the repetition R_j is spared and its span in terms of v-rotation along ξ is more than $(j-1)\pi$, re-arrangement can be completed and the arguments of Sect. 2.3 take over and allow to bypass the hyperbolic periodic orbit x_{2k}.

The outline completed above is made into a rigorous argument now:

In order to complete the adjustment of, e.g. the first $\pm v$-jump after γ along the "structure" defined above, we need to adjust the other $\pm v$-jumps along positions that are $s\pi$ or $s\pi + o(1)$ away from γ along $-\xi$.

When they are all $s\pi + o(1)$ away, the "attractive" and the "repulsive" directions in this "structure" become effectively attractive and repulsive, the nodes are as described and the deformation proceeds.

However, a given other $\pm v$-jump may occupy several of these $s\pi + o(1)$ positions, for different values of s. The deformation argument is not complete then since, in between such positions, we might not be able to complete this deformation. In particular, the "attractive" and the "repulsive" directions in this "structure" for the first $\pm v$-jump might not be effectively attractive or repulsive.

This can be overcome as follows:

Given another $\pm v$-jump and two consecutive positions P_1 and P_2 for this $\pm v$-jump that are $s\pi + o(1)$ and $(s+1)\pi + o(1)$ away from γ in the direction of $-\xi$, the deformation is not hindered by a $\pm v$-jump that is $o(\Sigma|c_i|)$.

Indeed, when it is a zero $\pm v$-jump, we can "ignore it" and change from the H^1_0-problem on $[\gamma, P_1]$ to the H^1_0-problem on $[\gamma, P_2]$ without problem (given that the other $\pm v$-jumps would also be at positions $s\pi + o(1)$ from γ, along $-\xi$, or would be zero).

If it is $o(\Sigma|c_i|)$ also, its effect is little and can be compensated by a manipulation of the size of a "sizable $\pm v$-jump"; in this way, the process is inserted in a decreasing deformation. In such a case, the process of "pushing away" can be completed across these tiny $\pm v$-jumps.

Now, deformation on this "structure" for this first $\pm v$-jump has been completed when all the other $\pm v$-jumps besides γ and this first $\pm v$-jump are either $o(1)$-close to positions that are $s\pi$-away from γ along $-\xi$ or are $o(\Sigma|c_i|)$.

Using the "pushing away" flow, we can move all the configurations such that some other $\pm v$-jumps, different from the first $\pm v$-jump, are not $o(1)$-close to positions that are $s\pi$-away from γ onto positions where that are of that type. Once these $\pm v$-jumps are locked in such positions, deformation for this first $\pm v$-jump on the required "structure" can be performed. The argument requires an induction if there are several such other $\pm v$-jumps. All configurations are deformed over this "pushing away" process. However, the new configurations that are not $o(1)$-close to positions that are $s\pi$-away from γ after deformation are configurations for which the first $\pm v$-jump is already deformed on the required "structure". Since this $\pm v$-jump is not deformed over this process, we do not need to perform any further deformation on these configurations and we can just continue our process on the new configurations for which the other $\pm v$-jumps are now within $o(1)$ of the special positions.

Deformation onto the "structure" is thereby performed for the first $\pm v$-jump.

This "structure" is made of $\pm v$-jumps that are either zero or almost zero, or of $\pm v$-jumps that grow along the unstable directions. "Pushing away" the other $\pm v$-jumps from these unstable directions, or "pushing across them" when they are tiny, starting from γ can be performed so as to achieve the deformation on the expected “structure” for the second $\pm v$-jump. Indeed, consecutive unstable directions, that is consecutive unstable directions of exit, in the “structure” defined above, are separated by an amount of v-rotation less than π, whereas consecutive stable directions are separated by an amount of v-rotation a bit more than π. Any two unstable directions are separated by an amount a bit less than $s\pi$ and any two stable directions are separated by an amount a bit more than $s\pi$, s taking an appropriate value. Unstable directions and stable directions in the structure, taken together between γ and $(1+\gamma)$, or between γ and P , where P is $s\pi$-away from γ, are effectively unstable and stable directions in the corresponding H_0^1-problems(s).

On the other side, along $-\xi$, the deformation argument through “pushing away” from $(1+\gamma)$ or from a P-type position is as above, for the first $\pm v$-jump.

The conclusion is that this deformation can be completed when γ is in E^+ or when γ is in E^-. It extends to the case when γ is in E_0 and the “structures” thereby derived for $\gamma \in E^{\pm}$ are of dimension $(2k-1)$ and are attached to the “structure” at E_0 along recognizable sets of dimension $(2k-2)$.

We are left with establishing that the “structure” derived when γ is at E_0 is also of dimension $(2k-1)$ at most.

The argument for $\gamma \in E_0$ is different; indeed, we do not know anymore that the directions to which we are “pushing” (these are also E_0-type positions; there are two kinds of positions of this type, those that are $s\pi$-away from γ and those that are $s\pi + \frac{\pi}{2}$-away from the same γ) are also directions of “exit” or “entry” and we cannot conclude as above:“pushing away” the $\pm v$-jumps between E_0-positions might induce some additional dimensions.

We, therefore, rely on two facts: first, the subset that we are trying to deform is, in the $(2k)$-dimensional set of configurations, (transversally to the flow) of dimension $(2k-1)$: γ is subject to one constraint, namely that γ is in E_0.

Second, there is a “good” variational theory when γ is in E_0. This variational theory relies on the one hand on the deformation completed above (similar to the ones completed for $\gamma \in E^{\pm}$) and on the other hand, on another deformation completed when all the $\pm v$-jumps are either zero (tiny) or on positions of type E_0 that are $s\pi$ away from γ. We describe this deformation in more detail below. The important fact is that these combined decreasing deformations move this $(2k-1)$ dimensional set onto a set of the same dimension Σ which can be written as $w \cup \mathcal{C}$, w the periodic orbit, with $\mathcal{C}$ of dimension $(2k-1)$ and below the level of w.

The other “structures’, derived when $\gamma \in E^{\pm}$, maybe deformed, are attached to Σ and the claim about the dimension of the cell ”left behind“ through this deformation (that it does not exceed $(2k-1)$ is complete.

We conclude with the arguments for the second part of the deformation when γ is in E_0. We consider now the variational problem defined by J_∞ when all $\pm v$-jumps are constrained to live on an E_0-position that is $s\pi$-away from γ. J_∞ is then a functional that is cubic in the algebraic sizes c_i of the various $\pm v$-jumps. There is an additional term that is $O(\Sigma c_i^4)$, but it can be neglected in the study.

Applying general position, we may assume that the critical points of this functional, under the constraint $\Sigma c_i^2 = \epsilon, \epsilon \gtrsim 0$ small, are non-degenerate. Also, we may assume, using general position, that all the corresponding critical values are non-zero.

Along such a positive critical value, 0 becomes a minimum radially, that is it is a critical point in the full space of variations, radial dilation included, of index equal to the index computed in restriction to the sphere of radius ϵ, to which 1 is added for the radial decrease.

A negative critical value can be avoided by radial expansion.

There is, therefore, a full variational theory and 0, viewed as all sizes $c_i = 0$, hence as w, is the only critical point.

It follows that the set of configurations corresponding to $\gamma \in E_0$ can be deformed, using the combination of the flow hereby defined for the configurations having all $\pm v$-jumps either tiny or at E_0-type positions as above with the flow that corresponds to “pushing away” to these E_0-positions as defined when we were deforming on the “structures”.

The claim follows.

Let us observe that our deformations are compatible with the preservation of R_j because, when the $\pm v$-jumps of R_j are tiny, we can “push away” across them. When γ is in E^+ or when γ is in E^-, the first step of the argument is to collapse the $j \pm v$-jumps of R_j into $(j-2) \pm v$-jumps. Deforming as above, we find $(2k-1)$ dimensions at most. Refining as γ reaches E_0, we reach the other deformations, attach along an explicit map, etc. The proof is now complete.

2.11.20 The conditions at w hyperbolic of index $2k$ or more

Let us "visualize" the conditions for a configuration not to move below a hyperbolic w of index $2k$ to the least; this is used in the argument about elliptic orbits: starting from γ, we "push away" the next $\pm v$-jump and we then "push away" the next $\pm v$-jump from the next $\pm v$-jump etc. We can also perform these operations in the other direction and combine the two movements. As long as we are "pushing away" between consecutive $\pm v$-jumps and "pushing away from γ, never "pushing away" γ, the number of sign-changes will not increase, a forced repetition outside of γ will be spared if it existed and at least two non-zero $\pm v$-jumps will survive if the configuration did harbor two non-zero $\pm v$-jumps or more before the process was started.

Over the whole process, using the technical observations, see above, the configuration will move below the level of w unless $2k$ conditions are met: the other $\pm v$-jumps should be either zero or $2s\pi$, $s \in \mathbb{N}$ away from γ.

This builds $2k$ conditions that might collapse into $(2k-2)$ conditions if two $\pm v$-jumps are already zero. One can easily convince himself that the other $(2k-2)$ conditions are here to stay. The argument proceeds.

One might ask, and we already commented about this above: how much of the deformation on the $(2k-1)$ dimensional "cell" defined above depends on γ. The answer relies on the observation that each of these "cells" is tied to the repetition R_j solely, γ being any of the $\pm v$-jumps outside the $j \pm v$-jumps of R_j besides the successor of R_j. At w_{2k-1}, the deformation downwards defined with the introduction of two additional $\pm v$-jumps on the ξ-piece of R_j having a hole, does not depend on γ. We can then choose one γ over all the cycle, different from the successor and, as we approach the boundary of the cycle, below w_{2k-1}, make it more specific in order to tie to the previous choices for γ. Our deformation may now be viewed as follows: given the R_js taken over all possible choices, deform downwards, outside of the cycles associated to each R_j. The choices of γs are given at the boundaries of these cycles, below w_{2k-1}. Then, choose one γ near each top etc.

2.11.21 The "mixing properties" of the "pushing away" flow

As explained above, the "pushing away" flow has to be completed carefully so that, when the orientation of a $\pm v$-jump reverses, the number of sign-changes reverses. For this, tiny, nearly zero $\pm v$-jumps can be "pushed away", but in order to "push away" other neighbouring $\pm v$-jumps from them, we need the v-rotation separating them to be less than π.

The "pushing away" flow, even when completed as above between consecutive $\pm v$-jumps and always away from γ, never the reverse way, has then some "mixing properties" that read as follows: if we start with a configuration of $2k \pm v$-jumps outside of γ, one of them non-zero to the least, living on or a near a periodic orbit w_{2k-1}, and we implement this "pushing away" flow amongst them and also away from γ various times, with various amounts of "pushing", we reach, given a set Σ of configurations, another set of configurations that is "generic" in that a preassigned set of $(2k-1)$ conditions on the sizes of the $\pm v$-jumps of Σ' will yield $(2k-1)$ independent constraints.

The proof of this fact goes through an explicit computation. Namely, we can assume that these $2k \pm v$-jumps are re-arranged, them and γ, so that they are separated by intervals of equal v-rotation.

Let us assume, for simplicity, that the sizes of these $2k \pm v$-jumps are denoted $c_1^0, c_2^0, \ldots, c_{2k}^0$ at time zero and that c_1^0 is non-zero. They are denoted, in that order, "the first jump", "the second jump", till the "$2k$th or last jump".

The other cases can be reduced to this one.

Let us choose $(4k-2)$ extremely small positive times $t_1, \ldots, t_{4k-2}$. We will "push away" the first jump from the second one over the time t_1, then after that, we "push away" the second jump from the third jump over the time t_2 and so forth, "pushing away" the $(2k-1)$th from the last jump over the time t_{2k-1}. We then resume the procedure a second time, starting with "pushing away" the first jump from the second jump over the time t_{2k} and so forth.

Our claim is that, after these procedures are applied, the new sizes of the $2k \pm v$-jumps will be "complicated" functions of the initial sizes $c_1^0, c_2^0, \ldots, c_{2k}^0$ and of the t_is. The combined space where Σ is multiplied by intervals $(0, \epsilon_i)$, $\epsilon_i \gneq 0$, $i = 1, \ldots, (4k-2)$-the product is mapped, the map is denoted f_t, onto the result derived by applying the "pushing away" flow to a configuration of Σ over $(4k-2)$ times $t_i, \ldots, t_{4k-2}$ to be taken in the intervals $(0, \epsilon_i)$-is a space where general position arguments can be applied. Denoting $\bar{t}$ the choice of $t_i = \frac{\epsilon_i}{2}$, $i = 1, \ldots, (4k-2)$, we can perturb $f_{\bar{t}}(\Sigma)$, by perturbing the value of $\bar{t}$, so that it becomes transverse to the $(2k-1)$ conditions at w_{2k-1} under which a configuration cannot be flown down. If needed, the procedures defined above can be iterated until general position is achieved.

We provide now the formulae, at first order when the t_is are tiny, for the evolution of the various sizes c_is of the $2k \pm v$-jumps through this process. The general position argument follows. Observe that, in addition, over this process, repetitions outside of γ are not destroyed, so that our arguments above and the present general position argument can be used together.

Given $c_1^j, \ldots, c_{2k}^j$ and tiny small positives times $t_{1+(2k-1)j}, \ldots, t_{2k-1+(2k-1)j}$, the sizes of the new $\pm v$-jumps, given at first order with respect to the t_is and obtained after the use of the "pushing away" flow with the t_is as above is (θ_k is a fixed positive constant, bounded away from zero and depending only on the ξ-transport equations along w_{2k-1}):

$$c_i^{j+1} = c_i^j + \theta_k c_{i-1}^j t_{i-1+(2k-1)j}$$

One can check easily, e.g. by considering the example of $k = 2$ that general position holds.

As a final remark, we observe that increasing the size of the nearly ξ-piece between predecessor and successor near these cycles that we are discussing can be turned, after appropriate perturbation of the functional, into a decreasing deformation bringing this cycle below the level of w_{2k-1}, since, once this nearly ξ-piece has increased in size to include a v-rotation larger than $(j-1)\pi$, the decreasing deformation proceeds through expansion of the sizes of the $\pm v$-jumps of the repetition R_j. This provides an alternative proof that these cycles can be bypassed.

2.11.22 Under the 2k conditions: additional condition when γ is in E^-; deformation of γ from E^+ into E^-

We assume now that we have $2k$-conditions on the configuration, and, therefore, isolated points.

γ is given. Using general position arguments, we may assume that γ is not at a node or very close to a node. Thus, γ is either in E^- or γ is in E^+; E^+ is the set of (m or $(m+1)$ if the index of the hyperbolic orbit is $2m$) intervals of positivity for the second derivative $J"_\infty$, E^- is the corresponding set of m or $(m+1)$ intervals of negativity defined in Sect. 2.2. These are not the positive and negative eigenspaces for the second derivative $J"_\infty$, F^+ and F^- defined in Sect. 2.2.

If γ is in E^-, we find an additional condition:

Indeed, we may then consider the H_0^1-problem based at $\gamma = 0, 1$. Denoting σ the configuration, with its various $\pm v$-jumps of size c_i at t_i, we find a family of functions η_i. $\eta(\sigma)$ reads as $\Sigma_{i=1}^{2k} c_i\eta_i + c\eta_\sigma$, where each η_i solves:

$$\ddot{\eta}_i + a^2\eta_i\tau = 0;\ \eta_i(t_i) = \eta_i(t_i+1);\ \dot{\eta}_i(t_i) - \dot{\eta}_i(t_i+1) = 1$$

Projecting on H_0^1 based at γ, we write

$$\eta(\sigma) = p_{H_0^1}\left(\sum_{i=1}^{2k} c_i\eta_i\right) + d(\gamma)\eta_\sigma$$

Accordingly,

$$J"_\infty(w).\eta(\sigma).\eta(\sigma) = J"_\infty(w).p_{H_0^1}\left(\sum_{i=1}^{2k} c_i\eta_i\right).p_{H_0^1}\left(\sum_{i=1}^{2k} c_i\eta_i\right) - a(\gamma)d(\sigma)^2$$

here $a(\gamma)$ is positive since γ is in E^-.

Accordingly, we can expand $d(\sigma)$ if is non-zero and this decreases the functional without changing the c_is. It is only the size of γ that changes. Thus, we can move the configurations below the level of w unless $d(\sigma) = 0$.

This yields an additional condition. We now have $(2k+1)$ constraints on a stratified space of dimension $2k$. General position rules out this possibility.

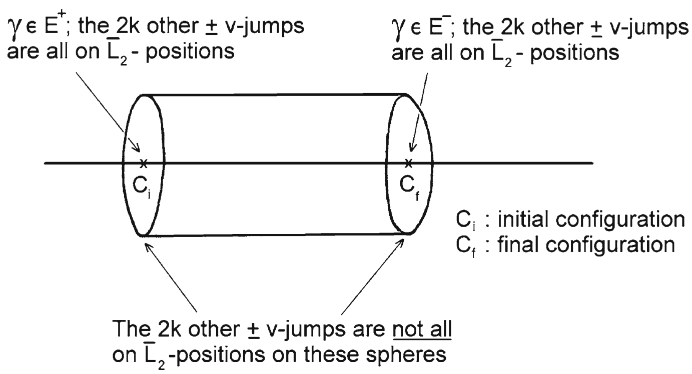
2.11.23 γ in E^+ and 2k conditions

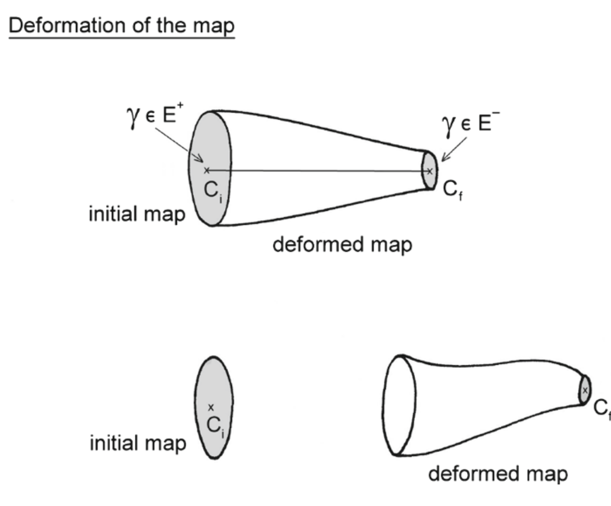
We are left with the case when γ is in E^+. Here, we are going to deform the map so that γ is in E^-. We need for this to single out the cell that is covered by our deformation process and to see that, along a path where γ is moved from E^+ to E^-, this cell can be continuously deformed whereas J_∞ decreases over the deformation. This now brings us back to the understanding of the H_0^1-problem based at γ, when γ is in E^+.

Let us then recognize the related cells: we may assume, see above, that our $2k \pm v$-jumps, besides γ, are on positions of type L_2 or are zero: the other positions, of type L_1 or L_3, are negative positions. The configurations have non-zero $\pm v$-jumps at such locations can be moved down.

Considering such cells, built with $\pm v$-jumps at the L_2s or zero, observing that we may assume, by general position, that γ is not at some preassigned positions, one in each interval of type E^+, we move γ from E^+ into E^- without crossing the preassigned positions.

The cells follow, in a decreasing deformation, because the $\pm v$-jumps are either zero or on positions of L_2-type. This fact can be checked easily. The deformation obeys the following drawings in the configurations's space:





The argument extends to cover the case singled out above, in the previous sub-section, when we reach $(2k-1)$ conditions coming from $W_u(w_{2k+1})$, after reaching $W_u(w_{2k}^\infty)$, also the E_0 case described above.

2.11.24 Switching γs when transition configurations have been moved down

We observe that the deformation arguments above can be convex-combined and the processes over various distinct γs can be glued: using the process for the first γ, we move the configuration below, up to a low-dimensional cycle. As we start a similar process around a new γ, the configurations that we manipulate have already been subjected to the deformation process around the first γ. Over the overlap region, those that are below the level of the periodic orbit w will go even further below, whereas the cells of low dimension that have not been deformed down will not change their dimension. Therefore, the two processes are compatible and can be glued together.

2.11.25 Additional observations about rearrangements and repetitions

We have completed above rearrangements along the periodic orbits of the $(2k+1)$ descending $\pm v$-jumps. We observe here that, as we consider various neighbouring configurations harboring various repetitions, we can complete them so that they preserve repetitions and allow for the construction of the direction F^+ as indicated above.

First, we observe that, over all configurations defined nearby a point, there is a partial order relation on repetitions, if we discard the configurations over which the configurations the number of sign changes is $(2k-2)$ or less.

Indeed, through the "natural flow" that preserves each Γ_{2s}, repetitions are preserved and therefore, going to local situations, the repetitions will become finer, with equal or smaller support over bordering configurations.

If a repetition R_1 dominates another repetition R_0 in that its support includes the support of R_0, we observe that the rearrangements that we have defined above that preserve R_0 can be deformed into the rearrangements that preserve R_1 **among rearrangements preserving** R_1.

Indeed, when is it that, along a rearrangement around R_0, we might destroy R_1?

Analyzing the rearrangement process, we find that this can be made to happen only over the process of H_0^1-decrease, as the $\pm v$-jumps of R_0 expand. Then, the orientations of some $\pm v$-jumps not in R_0 might reverse. Otherwise, all other processes can be made, even as they reverse the orientations of the boundary $\pm v$-jumps of R_1 (not included in R_1), to preserve an inside repetition.

As we now rearrange, trying only to preserve R_1 and not R_0, we use less and less the H_0^1-decrease through the increase of the $\pm v$-jumps of R_0, when they are in appropriate positions and we gradually replace it with a process of "pushing away", from the $\pm v$-jumps of R_0, reaching out to those of R_1.

Over this process, the $\pm v$-jumps of R_0 might reverse orientation; then, R_0 expands into R_0' etc. But it never expands beyond R_1 and, eventually, we find the R_1-process.

In this way, a rearrangement around R_0 can be replaced by a rearrangement around R_1 and, over the transition, a repetition inside R_1 will be preserved.

Using this process, we can define rearrangements in a continuous way over all rearrangements. Glueing follows.

The proofs of Theorems 1.1 and 1.2 are now complete.

3 Part II

3.1 Proof of Theorem 3

Proof of (i)(assuming (ii)): we first observe that the Morse relation $\partial\sigma = c_{2k-1} + h^t_{2k-1,\infty}$ implies that $\partial(c_{2k-1}+h^t_{2k-1,\infty}) = 0$. This will be used throughout our argument. We also assume that $\partial_{\text{per}} c_{2k-1} = 0$ and that c_{2k-1} cannot be decomposed in smaller cycles of ∂_{per}. This implies, see below, that the map $\tilde{b}$ of [7], restricted to a neighbourhood of the periodic orbits of c_{2k-1}, covers the generator of $\mathbb{PC}^{k-1}\times[-1,1]/\mathbb{PC}^{k-1}\times\{-1,1\}$ with non-zero degree. Indeed, either the number of periodic orbits of c_{2k-1} is odd and this conclusion is immediate using Propositions 3 and 4 of [7]. Or, it is even; we then use the fact that it is minimal and we write it as the sum of two chains of periodic orbits c_1+c_2, with an odd number of periodic orbits in each. We claim that the orientations coincide. Indeed, these two chains have a periodic orbit of index $(2k-2)$ in a common boundary. Below, we prove the orientations of c_1 and c_2 then coincide, see the end of the proof of Theorem 1.3.

Let us consider a collection of decreasing flow-lines Z that defines a stratified set of top dimension $(2k-1)$. Let $\Sigma w^{(\infty)}_{2k-3}$ be the collection of critical points (at infinity)of index $(2k-3)$ in Z. Let $L^\pm$ be the collection of curves in $\cup\Gamma_{2m}$ such that b is non-zero and contains no sign-change. Let $T = \overline{W_u(\Sigma w^{(\infty)}_{2k-3}) \setminus (L^+\cup L^-)}$. We assume in the sequel that T is connected. There is no loss of generality in this assumption. If T is not connected, we can restrict to one connected component of T and use the same arguments.

T has several boundaries that are related on the one hand to the fact that we have removed $L^\pm$ from $W_u(\Sigma w^{(\infty)}_{2k-3})$ and on the other hand to the fact that $\Sigma w^{(\infty)}_{2k-3}$ might dominate other $w^{(\infty)}_{2k-4}$s. Finally, T_1 has a trace on the connected components of **contractible** curves in $J_\infty^{-1}(\epsilon)$. Let $\mathbb{B}_\epsilon$ denote this trace. Let us assume that the critical points (at infinity) of index 1 of J_∞ do not disconnect this component of $J_\infty^{-1}(\epsilon)$. Then $\mathbb{B}_\epsilon$ is connected, of top dimension $(2k-4)$—viewed in the quotient $\cup\Gamma_{2m}/S^1$—and is part of the boundary of T. This boundary is then made of a part related to the removal of $L^\pm$. That part maps equivariantly into $L^\pm$. The target images are denoted in the sequel L_d^+, L_d^-, which we assume in a first step to be of low Fadell–Rabinowitz index γ_{FR} [13] compared to k, $\gamma_{FR}\le(k-2)$, see below for the removal of this assumption: we will discuss this issue in details. The other part is connected by assumption. It follows that the Fadell–Rabinowitz index of $\mathbb{B}_\epsilon$ is at most $(k-2)$ and the pull-back of the orientation class of $\mathbb{PC}^{k-2}$ through the classifying map on $\mathbb{B}_\epsilon$ is zero.

Using the Darboux reduction for a contact form, we can directly check that no critical point (at infinity) disconnects the connected component of $J_\infty^{-1}(\epsilon)$ made of contractible curves. Therefore, the above argument is general.

We proceed with our proof:

We use the Morse relation $\partial\sigma = c_{2k-1}+h^t_{2k-1,\infty}$. Since $\partial(c_{2k-1}+h^t_{2k-1,\infty})=0$, we can lift the top cells of $\bar{W}_u(c_{2k-1}+h^t_{2k-1,\infty})$ and their boundaries into a set Σ attached to a stratified set $\overline{W_u(\Sigma x^{(\infty)}_{2k-3})}$. $\Sigma\cup\overline{W_u(\Sigma x^{(\infty)}_{2k-3})}$ and $\overline{W_u(c_{2k-1}+h^t_{2k-1,\infty})}$ are cohomologous in the quotient of the loop space or C_β by the action of S^1-relative to their trace on the "bottom set" on the connected component of contractible curves in $J_\infty^{-1}(\epsilon)$ and relative to their "boundaries" with $L^\pm$.

Σ is assumed to have a constant classifying map. This happens, e.g. when the chain σ has undergone a dominated tangency with a periodic orbit y_{2k}: the top cell of $W_u(y_{2k})$ has a constant classifying map.

We then use the Morse relation $\partial\sigma = c_{2k-1}+h^t_{2k-1,\infty}$ and we deform Σ into $\tilde{\Sigma}$ that is attached to $\overline{W_u(\Sigma x^{(\infty)}_{2k-3})}$ with the same attaching map than Σ, but $\tilde{\Sigma}$ does not dominate $\Sigma x^{(\infty)}_{2k-3}$ anymore. This is obtained by cancelling the domination of $c_{2k-1}+h^t_{2k-1,\infty}$ by Σ "over the chain made of the combination of unstable manifolds of σ".

Then, we find a new stratified set $\tilde{\Sigma}\cup\overline{W_u(\Sigma w^{(\infty)}_{2k-3})}$ and deforming further, we find a $\tilde{\tilde{\Sigma}}\cup\overline{W_u}(\Sigma w^{(\infty)}_{2k-3})$ where $\tilde{\tilde{\Sigma}}$ has a constant classifying map and is attached to $\overline{W_u(\Sigma w^{(\infty)}_{2k-3})}$ along some $\mathbb{B}'_\epsilon$. $\mathbb{B}'_\epsilon$ has the same property that $\mathbb{B}_\epsilon$ above: it of Fadell–Rabinowitz index at most $(k-2)$. $\mathbb{B}_\epsilon$ is contained in $\mathbb{B}'_\epsilon$. The classifying map of $\mathbb{B}'_\epsilon$ extends the classifying map of $\mathbb{B}_\epsilon$. It is derived after extending the (prescribed) classifying map on $\overline{W_u(\Sigma w^{(\infty)}_{2k-3})}\cap J_\infty^{-1}(\epsilon)$ to a subset of the deformation of the top cell, which is of Fadell–Rabinowitz index 0 and is attached to $\overline{W_u(\Sigma w^{(\infty)}_{2k-3})}$ along $\mathbb{B}_\epsilon$. Arguing as above now, we find that our new set is of Fadell–Rabinowitz index at most $(k-1)$. The value of the classifying map on $\overline{W_u(\Sigma w^{(\infty)}_{2k-3})}\cup(W_u(c_{2k-1}+h^t_{2k-1,\infty})\cap J_\infty^{-1}(\epsilon)$ is prescribed, so that the conclusions of Proposition 3 and 4 of [7] apply. This yields a contradiction once we prove (i) and (ii).

In order to prove (ii) and (iii), we consider the collection of critical points at infinity Σw^∞_{2k-3} of critical points at infinity w^∞_{2k-3} dominated by $h^t_{2k-1,\infty}$. We add to these the periodic orbits of index $(2k-2)$ and of index $(2k-3)$ dominated by $h^t_{2k-1,\infty}$, with $L^\pm$ removed from their unstable manifolds. After deformation, the dimension of these strata drops to $(2k-3)$, $(2k-4)$. Because $\partial_{\text{per}}h^t_{2k-1,\infty}=0$, we can achieve the related strata of dimension $(2k-3)$ into subsets of $W_u(h^t_{2k-1,\infty})$ where b is non-zero. The definition of these new strata, in the vicinity of the strata of dimension $(2k-3)$ derived from the $W_u(x_{2k-2})$s once $L^\pm$ is removed from them and they have been further deformed, is straightforward.

.The extension problem:

Under this new set of assumptions, we need to solve the following extension problem:

Let Σ_{2k-4} be a $(2k-4)$-dimensional stratified set in $\cup\Gamma_{2m}/S^1$, which is part of the boundary of a stratified set of top dimension $(2k-3)$, W_{2k-3}, also in $\cup\Gamma_{2m}/S^1$. We assume that the Fadell–Rabinowitz index of Σ_{2k-4} is at most $(k-1)$ and the top two cells, of dimension $(2k-4)$ and of dimension $(2k-5)$ of this set build a connected set.

We consider in the sequel different homotopies of the classifying map on a subset of this space denoted S of top dimension $(2k-5)$, valued into $\mathbb{PC}^{k-2}$ and we want to build a homotopy between them that is still valued $\mathbb{PC}^{k-2}$.

The problem formulated as follows: we consider $S^1 * S \times [0, 1] \times [0, 1]$, over $S \times [0, 1] \times [0, 1]$, S^1 acting on the first factor, and we consider a part of its boundary $S \times \partial([0, 1] \times [0, 1])$. This set is clearly also of Fadell–Rabinowitz index $(k - 1)$ at most.

We assume that we have some map from this set into $\mathbb{P}\mathbb{C}^{k-2}$ a first classifying map—it will made of homotopies of other classifying maps—and also another classifying map from S into $\mathbb{P}\mathbb{C}^{k-2}$. Since $S \times ([0, 1] \times [0, 1])$ projects equivariantly over S, we may consider that this second equivariant map is also defined on $S \times \partial([0, 1] \times [0, 1])$, but is constant on the second factor.

We claim that these two maps are homotopic as maps valued in $\mathbb{P}\mathbb{C}^{k-2}$, that is that the first map extends to the disk, with value at the centre of the disk equal to the other map.

We know that the two maps are indeed homotopic. For dimension reasons, we may assume that the homotopy is valued into $\mathbb{P}\mathbb{C}^{k-2}$, since $(2k - 5) + 2 = 2k - 3$.

.The use for this extension argument:

In the framework where we use the argument below, we have a fibration over S of dimension 2 in the fiber. The fiber defines locally a cone C of dimension 2 over S, with vertex at a point of S. The homotopies are homotopies between values of the classifying maps at—assuming for simplicity that the fibration is trivial—S with the classifying map at $S \times \partial C$. This does not look like a homotopy. However, we may consider that our maps are independent of the ∂C position in C and that they depend only on the "radial" parameter of C. Indeed, the classifying map on $S \times C$ (S will be typically part of the "bottom set" of a $\Sigma w^{(\infty)}_{2k-3}$) is derived by "glueing" as in [13], the classifying map on S with the classifying map on the top cell of $h^t_{2k-1\infty} \setminus (L^+ \cup L^-)$. This classifying map can take two values: the value $\tilde{b}$ defined above or the value defined by restricting b to one sign change that one can track over the top cell of $h^t_{2k-1\infty} \setminus (L^+ \cup L^-)$. These two maps glue in a natural way; $\tilde{b}$ is the natural map to use all over the curves such that their v-component b has at least two zeros and at most $(2k - 2)$ zeros, and the other map is the one that gives a constant classifying map on the top cell of $h^t_{2k-1\infty} \setminus (L^+ \cup L^-)$. There are also two homotopic values for the map on S, see below.

.The classifying map on $h^t_{2k-1,\infty}$:

The solution of the extension problem defined above allows to specify the value, under additional assumptions, of the classifying map on the unstable manifold of $h^t_{2k-1,\infty}$. Indeed, let us assume that there are two "slices" of dimension $(2k - 2)$ where special assumptions hold allowing to define on these slices the classifying map more specifically and to solve the extension problem. The classifying map is then defined on these two slices and also on the top cells of $h^t_{2k-1,\infty}$ in between these two slices, with the centres of these cells removed, since these latter cells, without their centres will deform down onto the union of subsets of these slices, $\overline{W_u(\Sigma w^{(2k-3)}_{2k-3})}$ and Σ_1.

We want to extend it to these top cells, centres included and we find, therefore, a new extension problem, with a map from a sphere S^{2k-2} into $\mathbb{P}\mathbb{C}^{k-2}$ that lifts into an equivariant map from $S^{2k-2} \times S^1$ into S^{2k-3}.

Extending this map equivariantly amounts to extending the section of this map from $S^{2k-2} \times \{1\}$ into S^{2k-3} to the disk D^{2k-1}. Since $\pi_{2k-2}(S^{2k-1}) = \mathbb{Z}_2$, the square of this map extends.

It is possible to square an equivariant map into an equivariant map while keeping the S^1-action to be effective. The target values of the classifying maps are not modified in this way. We may, therefore, assume that the map has been extended then to these top cells, valued into $\mathbb{P}\mathbb{C}^{k-2}$.

Observe now that, since $\partial(c_{2k-1} + h^t_{2k-1,\infty})$ is zero and since $\partial_{\text{per}} c_{2k-1}$ is zero, $\partial_{\text{per}} h^t_{2k-1,\infty}$ is zero. We have discussed above how the periodic orbits of index $(2k-2)$ that $h^t_{2k-1,\infty}$ might dominate give rise to a set of dimension $(2k-3)$ after $L^\pm$ is removed and how their unstable manifolds after removal of $L^\pm$ may be achieved "above", with b non-zero and non-constant, with $2k$ zeros at most, so that they may be considered as part of $\overline{W_u(c_{2k-2} + h^t_{2k-1,\infty})}$. We are left with the boundary of infinity of $h^t_{2k-1,\infty}$. It is a shared boundary with c_{2k-1} and, therefore, the maximal number of zeros of b on the unstable manifolds of these critical points at infinity is at most $(2k - 2)$. This boundary has itself no boundary and, therefore, after appropriate adjustments (the adjustments leave the intersection with L^+ and L^- unchanged), it dominates only critical points (at infinity), not of L^+ or L^-, of index $(2k - 4)$ at most. Alternatively, we can argue that since $\partial\partial_\infty c_{2k-1} = 0$, the cells of dimension $(2k - 3)$ in $\partial_\infty c_{2k-1}$ build a coherent "puzzle" that can be lifted to be part of the top cell, with constant classifying map. The critical points of index 1 are again the ones disconnecting L^+ and L^- from each other and from the bottom set $J^{-1}_\infty(\epsilon)$.

Consider all the $w^{(\infty)}_{2k-3}$ such that they are dominated by $c_{2k-1} + h^t_{2k-1,\infty}$, but their unstable manifolds are not contained in $A_{2k-2} = \{x \in \cup\Gamma_{2m}; b\text{has at most } (2k - 2) \text{ zeros}\}$. Let us denote II the collection of these critical points (at infinity) of index $(2k-3)$ and let $T_1 = \overline{\underset{II}{\rightarrow} \cup W_u(w_{2k-3}^{(\infty)})} \cap A_{2k-2}^c$. The critical points of II are dominated by $h_{2k-1,\infty}^t$ and not by c_{2k-1} since their unstable manifold is not in A_{2k-2}. Assume that T_1 has a boundary in $L^+ \cup L^-$. This boundary is made of dominations of critical points (at infinity) of index $(2k-4)$, w_{2k-4}^{∞} by critical points (at infinity) from II.

These dominations can be cancelled after a modification of the pseudo-gradient over $\overline{W_u(h_{2k-1,\infty}^t)}$ that cancels the domination by $h_{2k-1,\infty}^t$ of the $w_{2k-3}^{(\infty)}$s of II that have such a $w_{2k-4}^{(\infty)}$ in their boundary. The modification of the pseudo-gradient involves "pushing" the flow-lines of $W_u(h_{2k-1,\infty}^t) \cap W_s(w_{2k-3}^{(\infty)})$ out of $W_s(w_{2k-3}^{(\infty)})$ in a level surface above the level of $w_{2k-3}^{(\infty)}$ through the boundary of $W_s(w_{2k-3}^{(\infty)})$ defined by $W_s(w_{2k-4}^{(\infty)})$. The deformation is an isotopy. It is unclear that we can preserve the fact that our sets are defined with $(2k)$ **trackable** $\pm v$-jumps. However, it is clear that the isotopy can be assumed to take place in Γ_{2k} so that the number of sign-changes of b will never exceed $2k$. This latter fact is not used in what follows. The flow-lines out of c_{2k-1} are not altered over this modification. In (i), we developed an argument to show that $\mathbb{B}_\epsilon$, the trace of T (the T of (i)) on the component of $J^{-1}(\epsilon)$ made of contractible curves, was of Fadell–Rabinowitz index $(k-2)$ at most. The argument contained two parts: on the one hand this trace was identified as part of the (connected) boundary in the contractible curves of a connected stratified set of dimension $(2k-3)$. On the other hand, the other possible pieces of boundary for this set, L_d^+ and L_d^- in $L^+ \cup L^-$, were assumed to be of low Fadell–Rabinowitz index $\leq (k-2)$. Here, we replace T in (i) by T_1 defined above. After our modification, T_1 has no boundary in $L^+ \cup L^-$. It might have a boundary $\mathbb{B}_1$ in A_{2k-2}. Once $L^+ \cup L^-$ is removed from this piece of boundary, the classifying map for $\mathbb{B}_1$ is given by $(b - \int_0^1 b)$ or its orthogonal projection onto the unit sphere S^{2k-3} of $Span\{\cos 2j\pi kt, \sin 2j\pi kt, j = 1, \ldots, (k-1)\}$.

$\mathbb{B}_1$ is part of the boundary ∂T_1 of T_1. This boundary is a stratified set of top dimension $(2k-4)$. It can be thought of as Σ_{2k-4} from our extension arguments above, with a value for the classifying map prescribed on the part $\mathbb{B}_1$. The extension problem is then formulated on ∂B_1. It allows to glue the two pieces of the classifying map: the one on $\overline{W_u(h_{2k-1,\infty}^t) \setminus (L^+ \cup L^-)} \cap A_{2k-2}$ defined by $\tilde{b}$ with the one defined on the trace of $\overline{W_u(h_{2k-1,\infty}^t \setminus (L^+ \cup L^-)}$ on $J_\infty^{-1}(\epsilon)$, viewed as the union of the trace of T_1 with the trace of the top cell of $h_{2k-1,\infty}^t$.

This gives us the value of the classifying map on one "slice" of $W_u(h_{2k-1,\infty}^t \setminus (L^+ \cup L^-))$. The value of the classifying map on the part of $\overline{\underset{I+II}{\cup} W_u(w_{2k-3}^{(\infty)})}$ in $L^+ \cup L^-$, where $I + II$ denotes all the $w_{2k-3}^{(\infty)}$s dominated by $h_{2k-1,\infty}^t$, is also assumed to be in a low $\mathbb{PC}^{k-2}$ (This assumption is also removed below). It can be extended, valued for dimension reasons in $\mathbb{PC}^{k-2}$, to all of $\overline{\underset{I+II}{\cup} W_u(w_{2k-3}^{(\infty)})}$. The part in $L^+ \cup L^-$ can be glued with the classifying map on the top cell and the resulting classifying map is valued in $\mathbb{PC}^{k-2}$.

It follows that, if we remove a suitable neighbourhood $\mathbb{V}$ of the top critical points (at infinity) of index $(2k-1)$ and $(2k-2)$ in $h_{2k-1,\infty}^t$ (without touching at the common boundary with the critical points of c_{2k-1}) and also a neighbourhood $\mathbb{W}$ of the $w_{2k-4}^{(\infty)}$ of $L^+ \cup L^-$ that $h_{2k-1,\infty}^t$ might dominate directly, not through T_1 , we find a classifying map valued into S^{2k-3} or $\mathbb{PC}^{k-2}$ for $\overline{W_u(h_{2k-1,\infty}^t)} \setminus (\mathbb{V} \cup W)$, if we assume that this set has two boundaries (of dimension $(2k-2)$), in L^+ or L^- or in both or one in one of them and the other one, in common with c_{2k-1} or through a "slice" of dimension $(2k-2)$, in A_{2k-2}.

In all these cases, the Faddell–Rabinowitz index of $\overline{W_u(h_{2k-1,\infty}^t \setminus L^+ \cup L^-)}$ is at most $(k-1)$. Adding now the set $\mathbb{W}$, we find that are adding a set of dimension $(2k-4)$ after deformation, along an intersection that has a classifying map valued into $\mathbb{PC}^{k-2}$. This will not modify substantially our concluding argument below.

The remaining case is when $h_{2k-1,\infty}^t$ has no such boundaries. Its classifying map may then be valued in $\mathbb{PC}^{k-1}$, but there is no additional factor $[-1, 1]$ added to it. It cannot cover the $(2k-1)$-dimensional generator of $(\mathbb{PC}^{k-1} \times [-1, 1], \mathbb{PC}^{k-2} \times [-1, 1] \cup \mathbb{PC}^{k-1} \times \{-1, 1\})$.

.Classifying maps and elliptic periodic orbits of Reeb vector-fields:

The above arguments yield a map i from the pair $\overline{(W_u(c_{2k-1} + h_{2k-1,\infty}^t) \setminus (L^+ \cup L^-)}$, $\overline{(W_u(c_{2k-1} + h_{2k-1,\infty}^t) \setminus (L^+ \cup L^-))} \cap [(\partial L^+ \cup \partial L^- \cup J_\infty^{-1}(\epsilon)) \cup \overline{(W_u(h_{2k-1,\infty}^t) \setminus (L^+ \cup L^-))}])$ into the pair $(\mathbb{PC}^{k-1}\times[-1,1],\mathbb{PC}^{k-2}\times[-1,1]\cup\mathbb{PC}^{k-1}\times\{-1,1\})$. This map yields a homomorphism:

$$\begin{aligned}i_*:&\ H_{2k-1}(\overline{W_u(c_{2k-1}+h^t_{2k-1,\infty})\setminus(L^+\cup L^-)},\overline{(W_u(c_{2k-1}+h^t_{2k-1,\infty})\setminus(L^+\cup L^-))}\\&\cap[(\partial L^+\cup\partial L^-\cup J_\infty^{-1}(\epsilon))\cup(\overline{W_u(h^t_{2k-1,\infty})\setminus(L^+\cup L^-)})])\\&\longrightarrow H_{2k-1}(\mathbb{PC}^{k-1}\times[-1,1],\mathbb{PC}^{k-2}\times[-1,1]\cup\mathbb{PC}^{k-1}\times\{-1,1\})\end{aligned}$$

i_* may be rewritten after excision of $\overline{W_u(h^t_{2k-1,\infty})\setminus(L^+\cup L^-)}$ as a composition of exc_* with j_*, where j maps $(\overline{W_u(c_{2k-1})\setminus(L^+\cup L^-)},\overline{(W_u(c_{2k-1})\setminus(L^+\cup L^-))}\cap[(\partial L^+\cup\partial L^-\cup A_{2k-2}\cup J_\infty^{-1}(\epsilon))])$ into $(\mathbb{PC}^{k-1}\times[-1,1],\mathbb{PC}^{k-2}\times[-1,1]\cup\mathbb{PC}^{k-1}\times\{-1,1\})$. A_{2k-2} is here derived by flowing down a set of curves such that the v-component b has at most $(2k-2)$ zeros. The construction of the map j follows the same arguments, simpler, than the map i.

$$\begin{aligned}exc_*:&H_{2k-1}(\overline{W_u(c_{2k-1}+h^t_{2k-1,\infty})\setminus(L^+\cup L^-)},\overline{(W_u(c_{2k-1}+h^t_{2k-1,\infty})\setminus(L^+\cup L^-))}\cap[(\partial L^+\\&\cup\partial L^-\cup J_\infty^{-1}(\epsilon))\cup(\overline{W_u(h^t_{2k-1,\infty})\setminus(L^+\cup L^-)})])\\&\longrightarrow H_{2k-1}(\overline{W_u(c_{2k-1})\setminus(L^+\cup L^-)},\overline{(W_u(c_{2k-1})\setminus(L^+\cup L^-))}\cap[(\partial L^+\cup\partial L^-\cup A_{2k-2}\cup J_\infty^{-1}(\epsilon))])\\j_*:&\ H_{2k-1}(\overline{W_u(c_{2k-1})\setminus(L^+\cup L^-)},\overline{(W_u(c_{2k-1})\setminus(L^+\cup L^-))}\cap[(\partial L^+\cup\partial L^-\cup A_{2k-2}\cup J_\infty^{-1}(\epsilon))])\longrightarrow\\&H_{2k-1}(\mathbb{PC}^{k-1}\times[-1,1],\mathbb{PC}^{k-2}\times[-1,1]\cup\mathbb{PC}^{k-1}\times\{-1,1\})\end{aligned}$$

The map l from $(\overline{W_u(c_{2k-1}+h^t_{2k-1,\infty})\setminus(L^+\cup L^-)},\overline{(W_u(c_{2k-1}+h^t_{2k-1,\infty})\setminus(L^+\cup L^-))}\cap[(\partial L^+\cup\partial L^-\cup J_\infty^{-1}(\epsilon))])$ into $(\overline{W_u(c_{2k-1}+h^t_{2k-1,\infty})\setminus(L^+\cup L^-)},\overline{(W_u(c_{2k-1}+h^t_{2k-1,\infty})\setminus(L^+\cup L^-))}\cap[(\partial L^+\cup\partial L^-\cup J_\infty^{-1}(\epsilon))\cup(\overline{W_u(h^t_{2k-1,\infty})\setminus(L^+\cup L^-)})])$ is the natural quotient map. It yields i_* in homology. i_* is onto in the top dimension $(2k-1)$ for these cycles.

$$\begin{aligned}l_*:&\ H_{2k-1}(\overline{W_u(c_{2k-1}+h^t_{2k-1,\infty})\setminus(L^+\cup L^-)},\overline{(W_u(c_{2k-1}+h^t_{2k-1,\infty})\setminus(L^+\cup L^-))}\cap[(\partial L^+\\&\cup\partial L^-\cup J_\infty^{-1}(\epsilon))])\longrightarrow\\&H_{2k-1}(\overline{W_u(c_{2k-1}+h^t_{2k-1,\infty})\setminus(L^+\cup L^-)},\overline{(W_u(c_{2k-1}+h^t_{2k-1,\infty})\setminus(L^+\cup L^-))}\cap[(\partial L^+\cup\partial L^-\\&\cup J_\infty^{-1}(\epsilon))\cup(\overline{W_u(h^t_{2k-1,\infty})\setminus(L^+\cup L^-)})])\end{aligned}$$

Finally, the map m from $(\overline{W_u(c_{2k-1}+h^t_{2k-1,\infty})\setminus(L^+\cup L^-)},\overline{(W_u(c_{2k-1}+h^t_{2k-1,\infty})\setminus(L^+\cup L^-))}\cap[(\partial L^+\cup\partial L^-\cup J_\infty^{-1}(\epsilon))])$ into $(\mathbb{PC}^{k-1}\times[-1,1],\mathbb{PC}^{k-2}\times[-1,1]\cup\mathbb{PC}^{k-1}\times\{-1,1\})$ is defined as above. It yields the map m_* in homology:

$$\begin{aligned}m_*:&\ H_{2k-1}(\overline{W_u(c_{2k-1}+h^t_{2k-1,\infty})\setminus(L^+\cup L^-)},\overline{(W_u(c_{2k-1}+h^t_{2k-1,\infty})\setminus(L^+\cup L^-))}\cap[(\partial L^+\\&\cup\partial L^-\cup J_\infty^{-1}(\epsilon))])\longrightarrow\\&H_{2k-1}(\mathbb{PC}^{k-1}\times[-1,1],\mathbb{PC}^{k-2}\times[-1,1]\cup\mathbb{PC}^{k-1}\times\{-1,1\})\end{aligned}$$

.Removal of S^1-equivariant pieces:

Observe that if $S^1*W_u(x^\infty_{2k-2})$ is attached to c_{2k-1}, it is attached along $S^1*W_u(x^\infty_{2k-3})$. Removing it will not affect the covering of $\frac{\mathbb{PC}^{k-1}\times[-1,1]}{\mathbb{PC}^{k-1}\times\{-1,1\}}$. The argument extends. The proof of (ii) of Theorem 1.3 is now complete.

.Proof of (iii):Removing the assumptions on L^+_d, L^-_d and the trace in L^+, L^- of the Morse complex of dimension $(2k-3)$ dominated by $W_u(h^t_{2k-1,\infty})$

The above argument works also under the weaker assumption $\gamma_{FR}((L^+\cup L^-)_d)\le(k-2)$; here $(L^+\cup L^-)_d$ stands for the Morse complex of $(L^+\cup L^-)$ dominated by $\Sigma w^{(\infty)}_{2k-3}$. Under this assumption, the trace $\mathbb{B}_\epsilon$ of $\Sigma w^{(\infty)}_{2k-3}$ on the bottom set $J_\infty^{-1}(\epsilon)$ has low Fadell–Rabinowitz index, at most $(k-2)$. This step is an essential part of the extension argument which allows, under the assumption that $c_{2k-1} + h^t_{2k-1,\infty} = \partial\sigma$, to find a contradiction with Proposition 3 of [7]. Summarizing this argument, the classifying map in the extension argument is prescribed on $\Sigma w^{(\infty)}_{2k-3} \cup (\overline{W_u(c_{2k-1} + h^t_{2k-1,\infty})} \cap J^{-1}_\infty(\epsilon))$. After cancellation of the domination of $\Sigma w^{(\infty)}_{2k-3}$ by $\overline{W_u(c_{2k-1} + h^t_{2k-1,\infty})}$ using the Morse relation $c_{2k-1} + h^t_{2k-1,\infty} = \partial\sigma$, we find that a set with constant classifying map is attached to $\Sigma w^{(\infty)}_{2k-3} \cup (\overline{W_u(c_{2k-1} + h^t_{2k-1,\infty})} \cap J^{-1}_\infty(\epsilon))$ along $\mathbb{B}_\epsilon$ and the contradiction follows since the Fadell–Rabinowitz index of $\mathbb{B}_\epsilon$ is at most $(k-2)$.

Let us consider the sets L^+_d and L^-_d dominated by the $\Sigma w^{(\infty)}_{2k-3}$ that are themselves dominated by periodic orbits of c_{2k-1}. We denote them as above $\sum_I w^{(\infty)}_{2k-3}$ and let us assume that the related L^+_d, L^-_d are of Fadell–Rabinowitz index $(k-1)$. Let us assume for simplicity that each of these stratified sets is connected and that the classifying map for each of them (they are of top dimension $(2k-4)$) is valued in $\mathbb{PC}^{k-2}$ with (non-zero) equal and opposite degrees. This is justified below, maybe with multiple components for each of L^+_d, L^-_d. Let us first argue when there is no $h^t_{2k-1,\infty}$. It then follows that all the boundaries of $\sum_I w^{(\infty)}_{2k-3}$ are in L^+, L^- and $J^{-1}_\infty(\epsilon)$; all other boundaries can be cancelled after moving the top cell of $c_{2k-1}(+h^t_{2k-1,\infty})$ out of the related stable manifold of the critical point (at infinity) of index $(2k-4)$ that $\sum_I w^{(\infty)}_{2k-3}$ dominates.

After this cancellation, $\mathbb{B}_\epsilon$ has a classifying map valued into $\mathbb{PC}^{k-2}$ of zero degree. If $\mathbb{B}_\epsilon$ is connected, then we may assume that a homotopy of this map, the homotopy still taking values into $\mathbb{PC}^{k-2}$, is valued into $\mathbb{PC}^{k-3}$. The argument of (i) then applies and we derive a contradiction as above.

Assume now that $\mathbb{B}_\epsilon$ is not connected. This would follow from the fact that $\sum_I w^{(\infty)}_{2k-3}$ would split into to separate $\Sigma h^{(\infty)}_{2k-3}$ and $\Sigma \ell^{(\infty)}_{2k-3}$, one having a boundary $\mathbb{D}^+_\epsilon$ in L^+_d and the other one having a boundary $\mathbb{D}^-_\epsilon$ in L^-_d, each of them with classifying map of equal, non-zero and opposite degree.

Either $\Sigma h^{(\infty)}_{2k-3}$ and $\Sigma \ell^{(\infty)}_{2k-3}$ can be seen as part of a boundary for a (collection of)critical point(s) (at infinity) $w^{(\infty)}_{2k-3}$ dominated by c_{2k-1}. Then, their cells of dimension $(2k-3)$ can be lifted as part of the top cell, with constant classifying map; and the argument proceeds.

Otherwise, they are not part of any such boundary. The domination $c_{2k-1} - \sum_I w^{(\infty)}_{2k-3}$ bypasses the dimension $(2k-2)$ and involves, therefore, compact manifolds (not a stratified space) of dimension 2, $\mathbb{C}^i_2$, that are equal to $W_u(c_{2k-1}) \cap W_s(w^{(\infty)i}_{2k-3})$ where w^i_{2k-1} is part of $\sum_I w^{(\infty)}_{2k-3}$. It then also follows (from the assumption not being part of a boundary) that the respective traces of $\overline{W_u(\Sigma h^{(\infty)}_{2k-3})}$ and $\overline{W_u(\Sigma \ell^{(\infty)}_{2k-3})}$ in the component of **contractible** curves of $J^{-1}_\infty(\epsilon)$, which we denote $\mathbb{B}^+_\epsilon$ and $\mathbb{B}^-_\epsilon$, can be deformed one onto the other one in $\overline{W_u(c_{2k-1} + h^t_{2k-1,\infty})} \cap J^{-1}_\infty(\epsilon)$. We then find, inserting this homotopy in between $\Sigma h^{(\infty)}_{2k-3}$ and $\Sigma \ell^{(\infty)}_{2k-3}$ a cycle of dimension $(2k-3)$ which we denote θ_{2k-3}; this cycle is relative to $L^+ \cup L^-$. The portion made of unstable manifolds of $w^{(\infty)}_{2k-3}$s can be assumed to have no boundary outside of L^+, L^- and $J^{-1}_\infty(\epsilon)$. This follows after cancelling any domination from the top cell $c_{2k-1} + h^t_{2k-1,\infty}$ of a $w^{(\infty)}_{2k-4}$ such that $W_u(w^{(\infty)}_{2k-4})$ contains sign-changes for b, the argument has been made for $h^t_{2k-1,\infty}$ above; (recall that we are assuming that there no $h^t_{2k-1,\infty}$ at this point, this removed below). Then $\sum_I w^{(\infty)}_{2k-3}$ is a stratified space of dimension $(2k-3)$, without boundary outside of L^+, L^-. It can be made into a manifold in these dimensions. In order to form θ_{2k-3}, we add a homotopy H, which we may consider to be valued into $W_u(c_{2k-1}) \setminus (L^+ \cup L^-)$. This homotopy exists for symmetry reason between v and $-v$, L^+ and L^-, L^+_d and L^-_d: there is, under the assumption that $\beta = d\alpha(v, .)$ "turns well" [1] along ξ an isotopy of vector-fields v_s in kerα, starting at v and ending at $-v$, with $d\alpha(v_s, .)$ a contact form. The functional $\int_0^1 \alpha(\dot{x})$ does not change along this path. Thus, L^+_d, L^-_d are isotopic and the Morse relations for the functional can be deformed isotopically one onto the other. Each of $\ell^{(\infty)}_{2k-1}$ and $h^{(\infty)}_{2k-1}$ therefore defines isotopic traces D^+_ϵ and D^-_ϵ which are homologous to the traces on the component of $J^{-1}_\infty(\epsilon)$ made of contractible curves. The homotopy between the traces follows from the fact that they can be contracted in $\overline{W_u(c_{2k-1})} \setminus (L^+ \cup L^-)$. The set $\overline{W_u(c_{2k-1})}$ is denoted w_{2k-1} in the sequel. It is a stratified space of top dimension $(2k-1)$, without boundary. Therefore, it can be turned into a manifold in dimension $(2k-1)$ and in dimension $(2k-2)$. H being defined from a set of dimension $(2k-3)$ into a stratified space of dimension $(2k-1)$, which is a manifold in dimensions $(2k-1)$, $(2k-2)$, can itself be considered a manifold in dimensions $(2k-3)$ and $(2k-4)$ since $(2k-3)+(2k-3)\lesssim(2k-1)+(2k-4)$. Therefore, θ_{2k-3} is a manifold—by general position arguments—in dimension $(2k-3)$ and $(2k-4)$.

We now write, using an appropriate Eilenberg–Zilber map (the Alexander-Whitney diagonal approximation) and with an abuse of notation, w_{2k-1} as $\mathbb{C}_2\otimes w_{2k-3}$. This follows from Morse Theory applied to $(W_u(w_{2k-1}), W_u(w_{2k-3}))$. Each of these is a collection of flow-lines relative to their "bottom sets". w_{2k-3} is a cycle relative to $L^+\cup L^-\cup J_\infty^{-1}(\epsilon)$ and we have that $w_{2k-1}=\Sigma C_2^i\otimes w_{2k-3}^i$ in the chain group of $C_\beta, L^+\cup L^-\cup J_\infty^{-1}(\epsilon))$, with $\Sigma w_{2k-3}^i=w_{2k-3}$. The notation $\mathbb{C}_2\otimes w_{2k-3}$ is thus an abuse of notation for a sum of terms of this type, where the first factors are all two-dimensional cycles C_2 and the second factors, when collected together rebuild w_{2k-3}.

Let f be the classifying map on $W_u(w_{2k-1})$. In the sequel, we will be writing w_{2k-1}, w_{2k-3} for $W_u(w_{2k-1})$, $W_u(w_{2k-3})$.

$$f_*(w_{2k-1})=\Sigma f_*(\mathbb{C}_2^i)\otimes f_*(w_{2k-3}^i)$$

w_{2k-1} is then viewed as a cycle of dimension $(2k-1)$ relative to $L^+\cup L^-\cup J_\infty^{-1}(\epsilon)$. $f_*(w_{2k-3})$ is a cycle of dimension $(2k-3)$ and each $f_*(C_2)^i$ is also a cycle of dimension 2. We now observe that the pair $(w_{2k-1}, w_{2k-1}\cap(L^+\cup L^-\cup J_\infty^{-1}(\epsilon))$ maps in fact into in $\mathbb{PC}^{k-1}\times[-1,1]/(\mathbb{PC}^{k-1}\times\{-1,1\}\cup\mathbb{PC}^{k-1}\times\{0\})$ because the bottom set $w_{2k-1}\cap(L^+\cup L^-\cup J_\infty^{-1}(\epsilon))$ splits in several connected components; the component corresponding to contractible curves is one of them and it maps into $\mathbb{PC}^{k-1}\times\{0\}$, whereas the other components in $L^\pm$ map into $\mathbb{PC}^{k-1}\times\{\pm1\}$. Using then Proposition 3 of [7], we derive that w_{2k-1} "covers" through this map the generator μ_{2k-1} of $\mathbb{PC}^{k-1}\times[-1,1]/(\mathbb{PC}^{k-1}\times\{-1,1\}$ Let ω^* be the generator of $H^2(\mathbb{PC}^{k-1}\times[-1,1])$. The cap-product $\omega^*\cap\mu_{2k-1}$ is non-zero, equal to the generator μ_{2k-3} of $H_{2k-3}(\mathbb{PC}^{k-2})\times[-1,1]/\mathbb{PC}^{k-2})\times\{-1,1\})$.

Denoting $x_i=\omega^*f_*(C_2^i)$, we have $\mu_{2k-3}=\Sigma x_i f_*(w_{2k-3}^i)$. We will denote in the sequel w'_{2k-3} the chain $\Sigma x_i f_*(w_{2k-3}^i)$. Starting with w_{2k-3}, which we had split into $\ell_{2k-3}^{(\infty)}$ and $h_{2k-3}^{(\infty)}$, we had built above a cycle θ_{2k-3}, relative to $L^+\cup L^-$. The construction of θ_{2k-3} used the fact that the boundary of w_{2k-3} split into two or more pieces $\mathbb{B}_\epsilon^\pm$ relative to the $\ell_{2k-3}^{(\infty)}/h_{2k-3}^{(\infty)}$ decomposition and that these pieces were homotopic in the component of contractible curves in $J_\infty^{-1}(\epsilon)$ of w_{2k-1}. The symmetry holds for $\Sigma x_i w_{2k-3}i$ as well, since all the Morse relations of the variational problem "rotate" as v is changed to $-v$ along the isotopic deformation v_s derived from the fact that $\beta=\mathrm{d}\alpha(v,.)$ "turns well" [1] along ξ. Therefore, we may assume that the θ_{2k-3} that we have built above is in fact associated to $\Sigma x_i w_{2k-3}i$ rather than w_{2k-3} as above. This is not needed for our argument here, but will be used in what follows, for the general case.

Observe, in addition, that the x_i are integers, because the C_2^i are unions of two-dimensional manifolds without boundary that may have critical points of c_{2k-1} and critical points (at infinity) of $w_{2k-3}i$ in common. Observe also that we may assume that the boundaries in L^+ and L^- of to $\Sigma x_i w_{2k-3}i$ and therefore of θ_{2k-3} are connected because L^+ and L^- are connected. Indeed, since the cycle of ∂_{per} is assumed to be minimal, any periodic orbit of index $(2k-1)$ of c_{2k-1} has a non-trivial boundary, a periodic orbit of index $(2k-2)$ with the other periodic orbits of c_{2k-1} (if there are any). Using this common domination, we find a path in c_{2k-1} transverse to this common boundary and connecting in L^+, respectively in L^-, the exit sets in L^+, respectively, in L^-, at the various periodic orbits: it suffices for this to consider the first eigenvalue and the first eigenspace of the "linearized" operator $\ddot{\eta}+a^2\eta\tau$ under periodic boundary conditions along the curves of the path.

After some reasoning, this implies indeed that the various boundaries of the various $w_{2k-3}i$ in L^+ and L^- are connected sets of dimension $(2k-4)$. The Fadell–Rabinowitz index of these two sets is assumed to be non-zero; this means that the classifying maps on these sets are valued in $\mathbb{PC}^{k-2}$, of non-zero degree. We may add the integers x_i in front of the $w_{2k-3}i$ in our previous arguments (those of (i) and (ii)); since this is ultimately a reasoning on degree, they are unchanged. Multiplicity in the domination is viewed through the x_is. These can be resolved in a sum of 1s or -1s by creating $|x_i|$ critical points (at infinity) $x_{2k-3}i^j$, $j=1,\ldots,|x_i|$s nearby each other compensated by $|x_i|-1$ critical points of index $(2k-2)$. Once the dominations are with multiplicity 1, the argument becomes a simple argument of degree; paths have to be created between the various components of the boundaries in L^+ and L^- and a connected manifold of dimension $(2k-4)$ is then built, whose classifying map covers the generator of $\mathbb{PC}^{k-2}$ with the required degree. The assumption on the Fadell–Rabinowitz index becomes an assumption on the degree of this classifying map.

Resuming our previous argument, it then follows that, denoting λ the generator of $H^2(\mathbb{PC}^\infty)$, the cap-product $f^*(\lambda)\cap w_{2k-1}$ is equal to θ_{2k-3}.

Here, $f^*(\lambda)$ is viewed in $H^2(C_\beta \setminus (L^+ \cup L^- \cup J_\infty^{-1}(\epsilon))$ and w_{2k-1} is viewed in $H_{2k-1}(C_\beta, L^+ \cup L^- \cup J_\infty^{-1}(\epsilon))$.

Assume now that $w_{2k-1} = \partial w_{2k}^{(\infty)}$ in $H_{2k-1}(C_\beta, L^+ \cup L^- \cup J_\infty^{-1}(\epsilon))$. Then,

$$\theta_{2k-3} = f^*(\lambda) \cap \partial w_{2k}^{(\infty)} = \partial(f^*(\lambda) \cap w_{2k}^{(\infty)})$$

. Thus θ_{2k-3} is a boundary in $H_{2k-3}(C_\beta, L^+ \cup L^- \cup J_\infty^{-1}(\epsilon))$.

Furthermore, $f_*(\theta_{2k-3})$ covers the generator of $\mathbb{PC}^{k-2} \times [-1, 1]/(\mathbb{PC}^{k-3} \times [-1, 1] \cup \mathbb{PC}^{k-2} \times \{-1, 1\})$ and $\partial\theta_{2k-3} \cap L^+$, $\partial s_{2k-3} \cap L^-$ cover the generator of $\mathbb{PC}^{k-2}$. This is exactly as w_{2k-1}, with a shift of 2 in the index. We can start a decreasing induction. In the end, we find that the generator of $H_1(C_\beta, L^+ \cup L^- \cup J_\infty^{-1}(\epsilon))$ is zero, a contradiction.

Let us now consider the more general case when there are $h^t_{2k-1,\infty}$s, but the Fadell–Rabinowitz index of L_d^+ an d of L_d^- is $(k-1)$.

Let us then replace in the above argument the bottom set $J_\infty^{-1}(\epsilon)$ with $J_\infty^{-1}(\epsilon) \cup \partial_\infty c_{2k-1}$, with ∂_∞ denoting the boundary operator at infinity. The reasoning is changed, but leads to the same contradiction: as noted above, $\partial_\infty c_{2k-1}$ has no algebraic boundary. Therefore, we can lift its top cells of dimension $(2k-2)$ together with the cells of dimension $(2k-3)$ into a top set S of constant classifying map. We can also use Morse theory to cancel through tangencies all the dominations of the critical points (at infinity) of index $(2k-3)$ by the critical points at infinity of $\partial_\infty c_{2k-1}$. The unstable manifolds of the various critical points at infinity of $\partial_\infty c_{2k-1}$ are recomposed. This might change L_d^+ and L_d^-, but it does not change, using the symmetry between b and $-b$, see below, the fact that the new L_d^+ and L_d^- are cobordant: these are the boundaries in $L^\pm$ of the $w_{2k-3}^{(\infty)}$s that are dominated by the periodic orbits of c_{2k-1}(and they are not dominated by the critical points at infinity of $\partial_\infty c_{2k-1}$). Observe that, even without this cancellation process, as the variational problems are deformed along the isotopy v_s from v to $-v$, see below, no tangency will occur, by Theorems 1.1 and 1.2 of Part I, between the critical points at infinity of $\partial_\infty c_{2k-1}$ and the periodic orbits dominated by c_{2k-1}.

The $w_{2k-3}^{(\infty)}$s that are dominated by the periodic orbits of c_{2k-1} undergo changes as the boundary $\partial_\infty c_{2k-1}$ changes through creations and cancellations of critical points at infinity. We want to follow their contribution in a "continuous" (through cobordisms of stratified sets) manner over the deformation. We need for this to introduce two additional observations about the Eilenberg–Zilber decomposition $w_{2k-1} = \Sigma C_2^i \otimes w_{2k-3}^i$ above: The boundary $\partial_\infty c_{2k-1}$ can dominate critical points (at infinity) of index $(2k-4)$, w_{2k-4}^j that are not dominated by a w_{2k-3}^i. Then, the intersection $\overline{W_u(\partial_\infty c_{2k-1}) \cap W_s(w_{2k-4}^j)}$ is a union of two-dimensional manifolds D_j that have the top critical points of $\partial_\infty c_{2k-1}$ and have w_{2k-4}^j in common. D_j is the boundary for a three-dimensional stratified space $F_j = \overline{W_u(c_{2k-1}) \cap W_s(w_{2k-4}^j)}$. This changes the decomposition above for w_{2k-1} with the addition of a sum of terms $F_j \otimes w_{2k-4}^j$ that might come from $\partial_\infty c_{2k-1}$.

Observe now that we may introduce the critical point (at infinity) of index $(2k-3)$ $S_*^1 w_{2k-4}^j$ derived after time translation on w_{2k-4}^j. Clearly, the domination $\partial_\infty c_{2k-1} - (S_*^1 w_{2k-4}^j)$ is made of a number of isolated flow-lines, so that the domination $\partial_\infty c_{2k-1} - w_{2k-4}^j$ is made of a corresponding number of disks whose boundary collapses to a point when we mod out by the S^1-action of time translation. This union is D_j and since D_j is the boundary of F_j, the algebraic number of these flow-lines is zero and after perturbation, we may view F_j as a cobordism of genuine, disjoint copies of $S^2 = \mathbb{PC}^1$. The number of copies is always the same since these are copies of $\mathbb{PC}^1$; we may assume, after perturbation, that these copies are disjoint and that F_j is a family of cylinders running from one copy of $\mathbb{PC}^1$ to another one having the reverse orientation. The [0, 1] factor of the cylinders can be resolved with the use of a Morse function having critical points of index 1 only. Combined with w_{2k-4}^j, we find critical points of index $(2k-3)$, $w\prime_{2k-3}^j$ and then, with these additional critical points, the formula

$$w_{2k-1} = \Sigma C_2^i \otimes w_{2k-3}^i$$

becomes general and, in this formula, all the critical points w_{2k-4}^j in $L^+ \cup L^- \cup \partial_\infty c_{2k-1}$ are (part of) boundaries for w_{2k-3}^is. Through deformations, the w_{2k-3}^is then build cobordisms of the w_{2k-4}^js. The terms coming from the boundary $\partial_\infty c_{2k-1}$ have a contribution equal to zero algebraically since the D_js are the boundaries of the F_js.

The deformation of the sets L_d^+ is now a deformation of stratified sets of top dimension $(2k-4)$; this deformation is a cobordism in dimension $(2k-4)$. Furthermore, taking the same pseudo-gradient flow for the times 0 and 1 of the deformation, we conclude that each of L_d^+ and of L_d^- does not change when compared at the time 0 and at the time 1 of the deformation. It follows that the L_d^+ and the L_d^- are isotopic in dimension $(2k-4)$. This is the starting point in the argument.

The other critical points (at infinity) of $\partial_\infty c_{2k-1}$ build a Morse complex $b_{2k-4}^{(\infty)}$. The critical points of index 1 are again the ones disconnecting L^+ and L^- from each other and from the bottom set $J_\infty^{-1}(\epsilon)$. The orientation class of w_{2k-1} is viewed in $H_{2k-1}(w_{2k-1}, (w_{2k-1}\cap(L^+\cup L^-\cup J_\infty^{-1}(\epsilon))\cup(S\cup b_{2k-4}^{(\infty)}))$. Under the classifying map f, this maps onto the generator of $H_{2k-1}(\mathbb{PC}^{k-1}\times[-1,1]/(\mathbb{PC}^{k-2}\times[-1,1]\cup\mathbb{PC}^{k-1}\times\{-1,1\}))$. ω^* is as above the generator of $H^2(\mathbb{PC}^{k-1}\times[-1,1])$. We may view its pull-back $f^*(\omega^*)$ in $H^2(\mathbb{PC}^{k-1}\times[-1,1],S)$. Then, the cap-product $f^*(\omega^*)\cap w_{2k-1}$ is in $H_{2k-3}(w_{2k-1},(w_{2k-1}\cap(L^+\cup L^-\cup J_\infty^{-1}(\epsilon))\cup(b_{2k-4}^{(\infty)}))$ and the result of this computation is then mapped into the homology group $H_{2k-3}(\mathbb{PC}^{k-1}\times[-1,1]/(\mathbb{PC}^{k-2}\times\{0\}\cup\mathbb{PC}^{k-1}\times\{-1,1\})\cup V_{2k-4})$. The computation for the pair $(\mathbb{PC}^{k-1}\times[-1,1]/(\mathbb{PC}^{k-1}\times\{-1,1\})$ yields the generator of the homology of dimension $(2k-3)$ of this pair and this maps non-zero into the homology of dimension $(2k-3)$ of the pair $(\mathbb{PC}^{k-1}\times[-1,1]/(\mathbb{PC}^{k-2}\times\{0\}\cup\mathbb{PC}^{k-1}\times\{-1,1\})\cup V_{2k-4})$. It follows that, if $f^*(\omega^*)\cap w_{2k-1}$ is non-zero in $H_{2k-3}(w_{2k-1},(w_{2k-1}\cap(L^+\cup L^-\cup J_\infty^{-1}(\epsilon))\cup(b_{2k-4}^{(\infty)}))$, then, since $b_{2k-4}^{(\infty)}$ is attached to $w_{2k-1}\cap(L^+\cup L^-\cup J_\infty^{-1}(\epsilon))$ along cells of dimension $(2k-5)$ and lower, the result comes from a cycle of $H_{2k-3}(w_{2k-1},(w_{2k-1}\cap(L^+\cup L^-\cup J_\infty^{-1}(\epsilon))$. We claim that this cycle is the θ_{2k-3} found above, due to the symmetry L^+/L^- (a consequence of the fact that $\beta=d\alpha(v,.)$ "turns well" [1] along ξ), properly modified. The boundary $\partial_\infty c_{2k-1}$ has no effect on the construction of θ_{2k-3} since $\sum_I w_{2k-3}$ does not, after manipulation, have a boundary outside of $L^+\cup L^-$. Since the boundaries of θ_{2k-3} in L^+ and in L^- cover, by assumption on the Fadell–Rabinowitz index, the generator of $\mathbb{PC}^{k-2}$ with equal and non-zero degree, we conclude that indeed $f^*(\omega^*)\cap w_{2k-1}$ is non-zero.

The decreasing induction and the conclusion follows under this assumption.

We thus need to prove that $f^*(\omega^*)\cap w_{2k-1}=\theta_{2k-3}$: since the critical points (at infinity) $w_{2k-3}i$ are not dominated at the order $(2k-2)$, we know that $W_u(w_{2k-1})\cap W_s(w_{2k-3})$ is a manifold of dimension 2 without boundary C_2^i. Each $w_{2k-3}i$ has no boundary in $\partial_\infty c_{2k-1}$ outside of $L^+\cup L^-$. We, therefore, build the chain, relative to $(L^+\cup L^-\cup J_\infty^{-1}(\epsilon)\cup\partial_\infty c_{2k-1})\cap w_{2k-1}$, $\Sigma(C_2^i)\otimes w_{2k-3}i$. It is viewed using Morse Theory, see above. It is a cycle d in $H_{2k-1}(w_{2k-1},(w_{2k-1}\cap(L^+\cup L^-\cup J_\infty^{-1}(\epsilon))\cup(S\cap b_{2k-4}^{(\infty)})$. Computing $f^*(\omega^*)\cap d$, we find that it is θ_{2k-3} in $H_{2k-3}(w_{2k-1},(w_{2k-1}\cap(L^+\cup L^-\cup J_\infty^{-1}(\epsilon))\cup(b_{2k-4}^{(\infty)}))$, which is non-zero. It follows that d must be w_{2k-1} and this concludes the argument.

As pointed out above, if $\beta=d\alpha(v,.)$ "turns well" [1] along ξ—a condition which is verified for the standard contact form on S^3 as well as for the first exotic contact form of Gonzalo and Varela [14] on S^3—then we find a path $v_s, s\in[0,1]$, $v_0=v$, $v_1=-v$, of non-singular vector-fields in kerα such that $d\alpha(v_s,.)$ is a contact form with the same orientation than α. This path is found as the result of the transport of v along ξ between a given point x_0, arbitrary in the manifold, and the next "coincidence point" along ξ (see [1], with (β,ξ) in lieu of (α,v) used in [1]. Therefore, we can follow the sets $L_d^+(s)$ and $L_d^-(s)$ defined above as s changes from 0 to 1; this defines an isotopic deformation from L_d^+ to L_d^-; the assumptions of symmetry between L_d^+ and L_d^- of our argument are verified and the minimal cycles of ∂_{per} survive a deformation of contact forms.

For the other exotic contact structures of S^3, we have models provided by Gonzalo and Varela [14]. $\beta=d\alpha(v,.)$ does not "turn well" [1] uniformly along ξ since the condition (A) is not verified. However, a path $v_s, s\in[0,1]$, $v_0=v$, $v_1=-v$, of non-singular vector-fields in kerα can be defined. The variational problem (J,C_β) can be tracked along this path. For the contact **forms** provided in [14], the periodic orbits of the Reeb vector-fields can be divided in three sets: those along which v "turns well" in the ξ-transport and (A) is verified; those along which, again, v "turns well" along ξ, but $\beta\wedge d\beta$ has the reverse orientation when compared to $\alpha\wedge d\alpha$; finally, a third subset of ξ-orbits on a surface, when v does not "turn well" along ξ.

Accordingly, we can define two Morse complexes, one related to the periodic orbits such that v "turns well" along ξ and (A) is verified in the vicinity of these periodic orbits. This Morse complex is as above; and another Morse complex such that v "turns well" along ξ and (A) is not verified in the vicinity of these periodic orbits; for the latter, the functional $J(x) = \int_0^1 \alpha(\dot{x})\mathrm{d}t$ is replaced with $-J(x) = -\int_0^1 \alpha(\dot{x})\mathrm{d}t$. One can prove that these two Morse complexes "ignore" each other. We can then define two homologies and two families of cycles as above. The fact that they survive a deformation of contact forms when they are minimal relates then to the symmetry between L_d^+ and L_d^-. The argument is not straightforward as above since the variational problems are now defined on a singular manifold C_β (β is not a contact form anymore) or on $\cup\Gamma_{2s}$, which can now also be singular. We need to prove that along this deformation, the sets L_d^+ and L_d^- are cobordant in their top dimension, that is that the singularities do not interfere with the cobordism and the related degree argument. This is of course conjectured to be true, but requires a detailed proof.

.Additional observations:

What about cycles that are not minimal? If we follow the arguments used above when c_{2k-1} is a minimal cycle and we try to extend them to the difference or to the sum of two minimal cycles d_1 and d_2, we find that these arguments would extend if the sets $L_d^i\pm$ corresponding to each of d_1 and of d_2 are not cobordant in $L^\pm$. Also, we find that the relation $\partial y_{2k}^\infty = d_1 - d_2 + h_{2k-1,\infty}$ must define a cobordism between d_1 and d_2 which can be described as a continuous deformation of cycles of dimension $(2k-1)$ that span (each of them) the generator of $\mathbb{PC}^{k-1} \times [0,1]/\mathbb{PC}^{k-1} \times \{0,1\}$. This gives more restrictions on the way y_{2k}^∞ dominates d_1 and d_2. We already know that these relations involve some sort of "point to circle" Morse relation. The flow-lines connecting y_{2k}^∞ to d_1 and to d_2 must be in the closure of the set in $W_u(y_{2k}^\infty)$ where the v-component b of $\dot{x}$ has $2k$ sign-changes. Because the chain y_{2k}^∞ must define a cobordism as described above, these flow-lines must be in the same connected component of this set. Since y_{2k}^∞ is a combination of critical points at infinity, this is a serious restriction on the location of these flow-lines relative to the H_0^1-unstable manifold of this combination of critical points at infinity.

Finally, we also have, with y_{2k} a periodic orbit, $\partial y_{2k} = d_1 - d_2 + h_{2k-1,\infty}$. Each of d_1 and of d_2 spans the generator, up to sign reversal, of $\mathbb{PC}^{k-1} \times [0,1]/\mathbb{PC}^{k-1} \times \{0,1\}$. Modding out this Morse relation by the set $A_{2k-2} = \{x$such that b has at most $(2k-2)$ sign-changes$\}$, we find a restriction on the relative orientation of d_1 and d_2.

Combining all the above arguments, it seems possible that there are examples of contact structures on S^3 for which several distinct odd cycles survive the deformation of contact forms. This result would involve a more detailed study which we have not been able to complete.

.Coherent orientations via the map $\tilde{b}$ of the unstable manifolds of elliptic periodic orbits of index $(2k-1)$ satisfying the equation $\partial_{\text{per}} = 0$:

We proved in [7] that the map j restricted to one simple periodic orbit of odd index $(2k-1)$ mapped onto the generator of $H_{2k-1}((\mathbb{PC}^{k-1} \times [-1,1], \mathbb{PC}^{k-2} \times [-1,1] \cup \mathbb{PC}^{k-1} \times \{-1,1\})$. The argument of [7] does not quite state this result, but it is actually only a reformulation. We now have a collection of periodic orbits of the same index $(2k-1)$. The claim is that they all yield the same generator, with the same orientation. Indeed, the map is essentially generated by b, the v-component of the tangent vector $\dot{x}$ to the curves x, maybe modified into $(b - \int_0^1 b, \psi(b))$, $\psi(x) = Min(1, |x|)sgn(x)$ has been defined above. We know, see [4,5], that b near a periodic orbit reads at first order as the linearized operator $\ddot{\eta} + \eta\tau$, under periodic boundary conditions. Because the orbits are elliptic, we may assume that the ξ-transport along them is pure transport, that is that τ is constant. Then, the unstable modes are represented by a fixed space of dimension $(2k-1)$, the addition of the space of constants, a copy of $\mathbb{R}$ of dimension 1, with a complex space of dimension $(k-1)$, $2(k-1)$ real.

In order to prove now that the orientation of all the unstable manifolds of the elliptic periodic orbits of c_{2k-1} are the same if $\partial_{\text{per}} = 0$, we derive this orientation over a coherent process, starting from dominated periodic orbits of index $(2k-2)$.

We first observe that these unstable manifolds can all be achieved as subsets of dimension $(2k-1)$ of Γ_{4k-2}, with the help of $(2k-1)$ trackable $\pm v$-jumps. At a curve x in $W_u(x_{2k-1})$, x_{2k-1} in c_{2k-1}, the tangent space to Γ_{4k-2} reads [3,4]

$$\mathrm{d}\ell(h) - h = \sum_{i=1}^{2k-1} (\delta s_i \delta_{t_i} + \delta a_i \mathrm{d}\ell_i(\xi))$$

ℓ is above the transport map along the curve x [3], $\mathrm{d}\ell$ is its differential. $(t_1, \ldots, t_{2k-1})$ are the times at which the large $\pm v$-jumps of the curve x of $W_u(x_{2k-1})$ occur. These are continuous functions of the time evolution s along $W_u(x_{2k-1})$. ℓ_i are partial transport maps along x from the ithξ-piece of x to the base point. δa_i is a variation of the length of the ithξ-piece.

The differential form $\mathrm{d}s_1 \wedge \mathrm{d}s_2 \wedge \cdots \wedge \mathrm{d}s_{2k-1}$ can be followed continuously over $W_u(x_{2k-1})$. It is, however, unclear whether it provides an orientation of the tangent space.

At a periodic orbit, which can be either x_{2k-1} or another periodic orbit x_{2k-2} that x_{2k-1} dominates—we can then define on $W_u(x_{2k-2})$, near x_{2k-2}, $(2k-1)$ times $\bar{t}_1, \ldots, \bar{t}_{2k-1}$ that extend the $(2k-1)$ functions of time evolution $t_1(s), \ldots, t_{2k-1}(s)$ on $W_u(x_{2k-1})$—the equation of the tangent space rereads:

$$\ddot{\eta} + a^2\eta\tau = \sum_{i=1}^{2k-1} \delta s_i \delta_{t_i};\, \eta 1 - periodic$$

Since b is at first order $\ddot{\eta} + a^2\eta\tau$, [3,4], we conclude that the orientation of the tangent space to $W_u(x_{2k-1})$ at a periodic orbit x_{2k-1} is derived by pull-back of the form $\mathrm{d}s_1 \wedge \mathrm{d}s_2 \wedge \cdots \wedge \mathrm{d}s_{2k-1}$ using the map b . We should be careful here because Γ_{4k-2} is not a manifold at x_{2k-1}. However, Γ_{4k-2} is a manifold at every curve nearby and also at x_{2k-1} once the times $\bar{t}_1, \ldots, \bar{t}_{2k-1}$ are given.

This pull-back can be completed also at x_{2k-2} and, in fact, it can be completed at any periodic orbit x_s dominated by x_{2k-1}, the only issue is to define $(2k-1) \pm v$-jumps that "fit" with the functions $t_1(s), \ldots, t_{2k-1}(s)$ on n$W_u(x_{2k-1})$.

More generally, this pull-back can be completed at any curve x of $W_u(x_{2k-1})$ using the equation of the tangent space and it thereby defines a continuous set of positive basis for the tangent space of $W_u(x_{2k-1})$, provided $\mathrm{d}\ell - Id$ is invertible.

Of course, $\mathrm{d}\ell - Id$ might not be invertible, this happens on a stratified set of dimension $(2k-2)$ in $W_u(x_{2k-1})$. The important observation however is that $det(\mathrm{d}\ell - Id)$ has the same (positive) sign at **all** simple elliptic periodic orbits and it has the same (negative) sign at **all** simple hyperbolic periodic orbits.

Therefore, given an orientation of $W_u(x_{2k-1})$ near x_{2k-2}, x_{2k-2} dominated by x_{2k-1}, which we define by pull-back of $\mathrm{d}s_1 \wedge \mathrm{d}s_2 \wedge \cdots \wedge \mathrm{d}s_{2k-1}$ using the linearized operator at x_{2k-2}, we find an orientation at x_{2k-1} that is opposite to this orientation (this established below).

Assuming that x_{2k-2} is dominated by x_{2k-1} and by y_{2k-1} from c_{2k-1} with intersection numbers equal to 1 and to -1, respectively, the orientations of $W_u(x_{2k-1})$ and of $W_u(y_{2k-1})$ near x_{2k-2} are then the same. Therefore, the orientations derived at x_{2k-1} and at y_{2k-1} by pull-back from $\mathrm{d}s_1 \wedge \mathrm{d}s_2 \wedge \cdots \wedge \mathrm{d}s_{2k-1}$ using the linearized operator are the same, since they are opposite to the orientation at x_{2k-2}. The two unstable manifolds, taken with this orientation, combine to form a cycle at x_{2k-2}. This orientation maps to $\mathrm{d}s_1 \wedge \mathrm{d}s_2 \wedge \cdots \wedge \mathrm{d}s_{2k-1}$ under the map defined by b. The claim follows.

We, therefore, need only to prove that the pull-back of $\mathrm{d}s_1 \wedge \mathrm{d}s_2 \wedge \cdots \wedge \mathrm{d}s_{2k-1}$ through the equation of the tangent space:

$$\mathrm{d}\ell(h) - h = \sum_{i=1}^{2k-1} \delta s_i \delta_{t_i}$$

changes orientation at the crossing x_s, $s \in [-\epsilon, \epsilon]$ of a simple zero of $det(\mathrm{d}\ell - id)$.

At such a crossing, the equation $\mathrm{d}\ell(u) - u = 0$ has a one-dimensional set of solutions $\mathbb{R}u_0$. The equation $\mathrm{d}\ell_{x_s}(u_s) - u_s = s\delta_{t_1(s)}$ can be continuously solved across the crossing.

The range of $\mathrm{d}\ell_{x_0} - Id$ is of dimension 1 and, therefore, with an appropriate choice of coefficients non-zero a_i, $i = 1, \ldots, 2k-1$, that might slightly depend on s, the equation

$$\mathrm{d}\ell_{x_s}(h) - h = \sum_{i=1}^{2k-1} \delta s_i \delta_{t_i}$$

can be continuously solved across the degeneracy, provided $\Sigma_{i=1}^{2k-1} a_i \delta s_i = 0$.

We thus find $(2k-2)$ vectors, $u_1, \ldots, u_{2k-2}$. With u_0, they build a basis whose orientation can be tracked continuously across the degeneracy. We claim that the orientation of this basis reverses with respect to the pull-back of the orientation of $\mathrm{d}s_1 \wedge \mathrm{d}s_2 \wedge \cdots \wedge \mathrm{d}s_{2k-1}$.

Indeed, observe that $\mathrm{d}s_1 \wedge \mathrm{d}s_2 \wedge \cdots \wedge \mathrm{d}s_{2k-1}(u_0, \ldots, u_{2k-2})$ equals $\mathrm{d}s_1(u_0)(\mathrm{d}s_2 \wedge \cdots \wedge \mathrm{d}s_{2k-1}(u_1, \ldots, u_{2k-2}))$ since $\mathrm{d}s_i(u_0) = 0$, $i = 2, \ldots, (2k-1)$. Clearly, we may assume that $\mathrm{d}s_2 \wedge \cdots \wedge \mathrm{d}s_{2k-1}(u_1, \ldots, u_{2k-2})$ does not vanish at the crossing and then, the conclusion follows.

4 Part III: additional remarks

In this short section, we make two main observations.

The first one is about the existence of an S^1-equivariant map from $\cup\Gamma_{2s}(M)$ into $\cup B_s(S^3)$, where $B_s(S^3)$ is the space of barycenters of order s on S^3. For this, we choose a degree 1 map f from M into S^3. Given a curve x in Γ_{2k} having $k \pm v$-jumps at points $x_1, \ldots, x_k$ of M, assume that these $\pm v$-jumps take place at times $t_1, \ldots, t_k$. Let $s_1, \ldots, s_k$ be the algebraic sizes of the $k \pm v$-jumps along v. we then define the map:

$$\Phi : \Gamma_{2k} \longrightarrow B_k(S^3)$$
$$\Phi(x) = \sum_{j=1}^{j=k} s_j f(x_j) \exp(-it_j)$$

It is easy to see that this map is S^1 equivariant. $\cup B_k(S^3)$ is a contractible space on which S^1 acts effectively. Rationally, we may view it as the total space of the classifying space for the S^1-action.

The second observation is about the analogy between the Einstein equations and Contact Form Geometry. S^1 is to be replaced by the group of diffeomorphisms of a manifold M and the v-component b of the tangent vectors to C_β is to be replaced by the Ricci tensor $Ric(g)$ at the metric g of M. Just as in Contact Geometry, the linearization of b represents the second variation at a periodic orbit [7], the linearization of the Ricci tensor, that is the Licnerowitz Laplacian, represents the second variation at an Einstein metric. Just as in Contact Geometry, b provides us with the classifying map for the S^1-action on the space $C_\beta \setminus$ {periodic orbits} [7], Ric provides us with the classifying map for the action of the diffeomorphism group.

If we solve the Yamabe part of the Einstein problem, we take away the conformal group from the diffeomorphism group and we can find then compact Lie groups. It is, therefore, natural to conjecture that Einstein metrics can be derived using this classifying map and that the contribution of the critical points at infinity or whatever variant does not exhaust the pull-back of the equivariant classes. There is in general no fixed point for the action of the diffeomorphism group, unlike what happens for the S^1-action on the loop space of a contact manifold M.

5 Part IV: erratum

5.0.1 Reference [3]

1-p26 of [3], the definitions of $t_1^{\pm}$ should read

$$t_1^+ = \text{Inf}\{t, t_1 \le t \le \bar{t}, b(s) \ge \mu_1 \text{for} s \in [t, \bar{t}]\}$$
$$t_1^- = \text{Sup}\{t, \bar{t} \le t \le t_2, b(s) \ge \mu_2 \text{for} s \in [\bar{t}, t]\}$$

The statement of Lemma 1 is unchanged. The proof is slightly changed in that t_0 in the proof is now larger than or equal to $t_1^+(\mu_1)$, not less than or equal to $t_1^+(\mu_1)$, see line 7, p27.

When $t_0 = t_1^+(\mu_1)$, the induction (for decreasing ts starts and the conclusion of Lemma 1 is reached.

2-pp 91–184 of [3] can be avoided in a first reading. After the construction of the flow, pp1-91-this construction requires minor modifications, mainly misprints, the reader can jump to the ν-stretched curves of p184.

The results and claims of pp 91–184 are interesting and contain part of the H_0^1-flow described in [2,6] (the latter in great detail).

5.0.2 Reference [2]

1-p139 of [2]: Line −15: Proof of Proposition 28 (in lieu of Proposition 27 Line −11: $\ell(i_0 + \gamma) + 2n$ in lieu of $m(i_0 + \gamma) + 2n'$ Line −8 and −7: the equations are now changed, but the conclusion stays the same. If $\ell = m$, then $n = n'$. Line −2: Proof of Proposition 29 (in lieu of Proposition 28)

2-p161 of [2]: Statement of Lemma 16, (iv): add "at the index $\bar{m}$ at the end of the statement.

3-pp171–173: Line 25 of p171 till line 14 of p173 can be removed from the proof.

4-p177: Line −6, after "and, in addition" till Line −3, till "to be zero" should be removed. The claim is wrong.

5-p178: (iv) has also to be corrected with the addition of "at the index $\bar{m}$ at the end, as in Lemma 16, p161.

The mistake stems from the fact that a simple hyperbolic orbit of index $2k$ has $4k$ nodes, not $2k$ nodes: over each 2π rotation of v in the ξ-transport along this periodic orbit, starting at a node, v coincides with the eigenvectors of the Poincare-return map at 4 positions, not at 2 positions.

6-p201 is to be removed. The argument is (partly) incorrect.

5.0.3 Reference [5], *Compactness*

Appendix 4, pp 562–566, till line −11, is correct and establishes that transversality holds for the (J, C_β), with the use of companions. The argument for Theorem 1.1', p566, after line −11, till p567 is incorrect. The statement of Theorem 1.1' p568 is also incorrect, see [6] for a detailed correction. The assumption in Theorem 1.1', (ii), is not about characteristic pieces as stated in [3]; it is about **non**-characteristic pieces.

Section 15.1 of [6], which uses the Fredholm violation for the first exotic contact structure of Gonzalo and Varela [14], completes the argument for Theorem 1.1' for the case of this contact structure.

References

1. Bahri, A.: Pseudo-orbits of contact forms. In: Pitman Research Notes in Mathematics Series No. 173. Scientific and Technical, London (1988)
2. Bahri, A.: Un probleme variationnel sans compacite en geometrie de contact. C. R. Acad. Sci. Paris **299**, Serie I, **15**, 757–760 (1984)
3. Bahri, A.: Classical and quantic periodic motions of multiply polarized spin-manifolds. In: Pitman Research Notes in Mathematics Series No. 378. Longman and Addison-Wesley, London and Reading (1998)
4. Bahri, A.: Flow-lines and Algebraic invariants in Contact Form Geometry PNLDE, vol. 53. Birkhauser, Boston (2003)
5. Bahri, A.: Compactness. Adv. Nonlinear Stud. **8**(3), 465–568 (2008)
6. Bahri, A.: On the contact homology of the first exotic contact form/structure of J. Gonzalo and F. Varela. Arab. J. Math. (2014, this issue). doi:10.1007/s40065-014-0097-2
7. Bahri, A.: Homology computation. Adv. Nonlinear Stud. **8**(1), 1–17 (2008)
8. Bahri, A.: Fredholm pseudo-gradients for the action functional on a sub-manifold of dual Legendrian curves of a three dimensional contact manifold (M^3, α). Arab. J. Math. (2014, this issue). doi:10.1007/s40065-013-0089-7
9. Bourgeois, F.: A Morse-Bott approach to contact homology. In: Eliashberg, Y., Khesin, B., Lalonde, F. (eds.) Symplectic and Contact Topology: Interactions and Perspective, Fields Institute Communications, vol. 35, pp 55–77. American Mathematical Society, USA (2003)
10. Bourgeois, F.; Oancea, A.: Symplectic homology, autonomous Hamiltonians, and Morse–Bott moduli spaces. Duke Math. J. **146**(1), 71–174 (2009)
11. Eliashberg, Y.: Contact 3-manifolds twenty years since J. Martinet's work. Ann. Inst. Fourier, Grenoble **42**(1–2), 165–192 (1992)
12. Eliashberg, Y.; Givental, A.; Hofer, H.: Introduction to symplectic field theory. Geom. Funct. Anal. Special volume, Part II, pp. 560–673 (2000)
13. Fadell, E.R.; Rabinowitz, P.H.: Generalized cohomological index theories for Lie group actions with an application to bifurcation questions for Hamiltonian systems. Invent. Math. **45**, 139–174 (1978)
14. Gonzalo, J.; Varela, F.: Modeles globaux des varietes de contact. In: Third Schnepfenried Geometry Conference, Asterisque No. 107–108, vol. 1, pp. 163–168. Soc. Math. France, Paris (1983)
15. Gromov, M.: Pseudo-holomorphic curves in symplectic manifolds. Invent. Math. **82**, 307–347 (1985)
16. Hirsch, M.W.: Differential Topology, 6th edn, vol. 33. Springer GTM, Berlin (1997)
17. Hofer, H.: Pseudoholomorphic curves in symplectizations with applications to the Weinstein conjecture in dimension three. Invent. Math. **114**, 515–563 (1993)
18. Hutchings, M.: An index inequality for embedded pseudoholomorphic curves in symplectizations. J. Eur. Math. Soc.**4**, 313–361 (2002)
19. Hutchings, M.: The embedded contact homology index revisited. New perspectives and challenges in symplectic field theory. In: CRM Proc. Lecture Notes, vol. 49, pp. 263–297. Amer. Math. Soc, Providence (2009)
20. Hutchings, M.: Embedded contact homology and its applications. In: Proc. 2010 ICM **II**, 1022–1041 (2010)
21. Hutchings, M.: Quantitative embedded contact homology. J. Diff. Geom. **88**, 231–266 (2011)
22. Hutchings, M.: TaubesÆs proof of the Weinstein conjecture in dimension three. Bull. AMS **47**, 73–125 (2010)
23. Hutchings, M.: Recent progress on symplectic embedding problems in four dimensions. PNAS **108**, 8093–8099 (2011)
24. Hutchings, M.: Embedded contact homology as a (symplectic) field theory, P.B. Kronheimer and T.S. Mrowka. In: Monopoles and Three-Manifolds. Cambridge University Press, Cambridge (2008)
25. Martino, V.: A Legendre transform on an exotic S^3. Adv. Nonlinear Stud. (2010)

26. McDuff, D.: The local behaviour of J-holomorphic curves in almost complex 4-manifolds. J. Diff. Geom. **34**, 143–164 (1991)
27. Mario, J.M., Brian, W.: The structure of branch points in minimal surfaces and in pseudoholomorphic curves. Ann. Math. Second Ser. **141**(1), 35–85 (1995)
28. Oancea, A.: These D'Habilitation: La suite de Leray–Serre en homologie de Floer des variétés symplectiques compactes a bord de type contact. Universite Paris 11, Orsay (2003)
29. Rabinowitz, P.H.: Periodic solutions of Hamiltonian Systems. Commun. Pure. Appl. Math. **31**, 157–184 (1978)
30. Salamon, D.: Lectures on Floer homology. In: Eliashberg, Y.; Traynor, L. (eds.) Symplectic Geometry and Topology, vol. 7, pp. 143–229. IAS/Park City Math. Series. AMS, Providence (1999)
31. Schwarz, M.: Cohomology Operations from S^1-cobordisms in Floer homology. Dissertation, Swiss Federal Institute of Technology, Zurich, (1995)
32. Taubes, C.H.: The Seiberg–Witten equations and the Weinstein conjecture. Geom. Topol. **11**, 2117–2202 (2007)
33. Taubes, C.H.: Embedded contact homology and Seiberg–Witten Floer homology I. Geom. Topol. **14**, 2497–2581 (2010)
34. Taubes, C.H.: Embedded contact homology and Seiberg–Witten Floer homology II. Geom. Topol. **14**, 2583–2720 (2010)
35. Taubes, C.H.: Embedded contact homology and Seiberg–Witten Floer homology III. Geom. Topol. **14**, 2721–2817 (2010)
36. Taubes, C.H.: Embedded contact homology and Seiberg–Witten Floer homology IV. Geom. Topol. **14**, 2819–2960 (2010)
37. Taubes, C.H.: Embedded contact homology and Seiberg–Witten Floer homology V. Geom. Topol. **14**, 2961–3000 (2010)
38. Weinstein, A.: On the hypotheses of Rabinowitz' periodic orbit theorems. J. Differ. Equ. **33**(3), 353–358 (1979)
39. Yau, M.: Vanishing of the contact homology of overtwisted contact three manifolds. Bull. Inst. Math. Academia Sinica (New Series) **1**(2), 211–229 (2006)