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Solvability of a fractional boundary value problem with fractional derivative condition

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Abstract In this paper, we investigate a boundary value problem for fractional differential equations with fractional derivative condition. Some new existence results are obtained using Banach contraction principle and Leray–Schauder nonlinear alternative.

Mathematics Subject Classification 26A33 · 34B15 · 34B27

ملخص

في هذه المقال نبحث عن حل لمسألة حدية للمعادلات التفاضلية الكسرية مع شرط المشتقة الكسرية بإستخدام مبدأ باناخ للإنكماش و البديل الغير الخطي للوري شودار.

1 Introduction

Differential equations of fractional order have recently been addressed by many researchers of various fields of science and engineering such as physics, chemistry, biology, economics, control theory, and biophysics. On the other hand, fractional differential equations also serve as an excellent tool for the description of memory and hereditary properties of various materials and processes. With these advantages, the models of fractional order become more and more practical and realistic than the classical models of integer order, such effects in the latter are not taken into account. As a result, the subject of fractional differential equations is gaining much attention and importance. For more details on this theory and on its applications, we refer to the recent monographs of Kilbas et al. [13], Oldham [17], Hilfer [11] and the researches of Engheta [5].

The existence of solutions to fractional boundary value problems is under strong research, see [10,19,21] and references therein. More recently, some papers have considered nonlocal boundary value problems for fractional differential equations, in particular Benchohra et al. [3] discussed the existence and uniqueness of solutions for the boundary value problems for differential equations with fractional order and nonlocal conditions of the form

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$$
\begin{aligned}
{}^{c}D^{\alpha}y(t) &= f(t, y(t)), 0 \le t \le T, 1 < \alpha \le 2, \\
y(0) &= g(y), y(T) = y_T,
\end{aligned}
$$

where ${}^{c}D^{\alpha}$ is the Caputo fractional derivative of order α, $f \in C([0, T] \times \mathbb{R}, \mathbb{R})$, $g \in C([0, T], \mathbb{R})$ and $y_T \in \mathbb{R}$.

Gorenflo et al. in [8] presented some general results for the fractional boundary value problems. They dealt with boundary value problems for pseudo-differential equations with the operator:

$$
\frac{\partial^2}{\partial y^2} + A(D),
$$

where $A(D)$ is an elliptic pseudo-differential operator and with boundary operators depending on a positive real parameter α.

Bai [2] considered the existence of positive solutions of the fractional boundary value problem:

$$
\begin{aligned}
D_0^{\alpha}u(t) + f(t, u(t)) = 0,\ 0 < t < 1,\ 1 < \alpha \le 2 \\
u(0) = 0,\ \beta u(\eta) = u(1),
\end{aligned}
$$

where D_{0+}^{α} denotes the Riemann–Liouville differentiation.

Goodrich studied in [6] a similar problem for fractional differential equation where the nonlinear term depends only on u and t, he considered the following problem:

$$
\begin{aligned}
-D_{0+}^{v}y(t) &= f(t, y(t)) \\
y^{i}(0) = 0,\ &\left[D_{0+}^{\alpha}y(t)\right]_{t=1} = 0,
\end{aligned}
$$

where $0 \le i \le n-2$, $1 \le \alpha \le n-2$, $v > 3$ satisfying $n-1 < v \le n$, $n \in \mathbb{N}$, is given, and D_{0+}^{v} is the standard Riemann–Liouville fractional derivative of order v. The author established the existence of positive solution using cone theoretic techniques, then in [7] he extended this study to systems of differential equations of fractional order.

Motivated by the results mentioned above, we are concerned with the existence and uniqueness of solutions of the fractional boundary value problem generated by a fractional differential equation and fractional derivative condition (FBVP)(P1):

$$
{}^{c}D_{0+}^{q}u(t) = f\left(t, u(t), {}^{c}D_{0+}^{\sigma}u(t)\right), 0 < t < 1, \tag{1.1}
$$

$$
u(0) = u''(0) = 0, u'(1) = {}^{c}D_{0+}^{\sigma}u(1), \tag{1.2}
$$

where $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, $2 < q < 3$, $0 < \sigma < 1$ and ${}^{c}D_{0+}^{q}$ represents the standard Caputo fractional derivative of order q. The case $q = 2$ is studied in [9], where the second-order equations are used to model various phenomena in physics, chemistry and epidemiology. It is shown that by introducing fractional derivatives and fractional integrals, we get an adequate mathematical modelling of real objects and processes. Moreover, the introduction of the Caputo's fractional derivative, allows the utilization of physically interpretable boundary conditions. For more details on the geometric and physical interpretation for fractional derivatives Caputo types, see [18].

By the use of nonlinear alternative of Leray–Schauder and the Banach fixed-point theorem, we show the existence and uniqueness of solutions for the above problem. Our results allow the derivative condition to depend on the fractional derivative ${}^{c}D_{0+}^{\sigma}u$, which leads to extra difficulties. No contributions exist, as far as we know, concerning the existence of solutions of the fractional differential Eq. (1.1) jointly with fractional derivative condition (1.2).

The rest of this paper is organized as follows. First, we list some notations, definitions and lemmas to be used later. In Sect. 3, we present and prove our main results which consist of uniqueness and existence theorems. Finally, we give some examples to illustrate our work.

2 Preliminaries and lemmas

In this section, we cite definitions and some fundamental facts from fractional calculus which can be found in [13].

Definition 2.1 If $g \in C([a,b])$ and $\alpha > 0$, then the Riemann–Liouville fractional integral is defined by

$$I_{a^+}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{g(s)}{(t-s)^{1-\alpha}} \mathrm{d}s.$$

Definition 2.2 Let $\alpha \geq 0$, $n = [\alpha] + 1$. If $g \in AC^n([a,b])$ then the Caputo fractional derivative of order α of f is defined by

$${}^c D_{a+}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{g^{(n)}(s)}{(t-s)^{\alpha-n+1}} \mathrm{d}s$$

exists almost everywhere on $[a,b]$ ($[\alpha]$ is the integral part of α).

Lemma 2.3 ([13]) *Let* $\alpha, \beta > 0$ *and* $n = [\alpha] + 1$. *Then the following relations hold:*

$${}^c D_{a^+}^{\alpha} t^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} t^{\beta-\alpha-1}, \beta > n$$

and

$${}^c D_{a^+}^{\alpha} t^k = 0, \quad k = 0, 1, 2, \ldots, n-1.$$

Lemma 2.4 ([13]) *For* $\alpha > 0$, $g(t) \in C([a,b])$, *the homogenous fractional differential equation* ${}^c D_{a+}^{\alpha} g(t) = 0$, *has a solution*

$$g(t) = c_1 + c_2 t + c_3 t^2 + \cdots + c_n t^{n-1},$$

where, $c_i \in \mathbb{R}, i = 0, 1, 2, \ldots, n$, *and* $n = [\alpha] + 1$.

Denote by $L^1([0,1], \mathbb{R})$ the Banach space of Lebesgue integrable functions from $[0,1]$ into $\mathbb{R}$ with the norm $\|Y\|_{L^1} = \int_0^1 |Y(t)| \, \mathrm{d}t$. Let E be the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$ such that ${}^c D_{0+}^{\sigma} u \in C([0,1], \mathbb{R})$, $0 < \sigma < 1$, endowed with the norm

$$\|y\| = \max_{t \in [0,1]} |y(t)| + \max_{t \in [0,1]} \left| {}^c D_{0+}^{\sigma} y(t) \right|.$$

The following lemmas give some properties of Riemann–Liouville fractional integrals and Caputo fractional derivative.

Lemma 2.5 ([13]) *Let* $p, q \geq 0$, *and* $f \in L_1([a,b])$. *Then*

$$I_{0^+}^{P} I_{0^+}^{q} f(t) = I_{0^+}^{P+q} f(t) = I_{0^+}^{q} I_{0^+}^{P} f(t)$$

and

$${}^c D_{a^+}^{q} I_{0^+}^{q} f(t) = f(t), \forall t \in [a,b].$$

Lemma 2.6 ([13]) *Let* $\beta > \alpha > 0$, *and* $f \in L_1([a,b])$. *Then for all* $t \in [a,b]$ *we have*

$${}^c D_{a^+}^{\alpha} I_{0^+}^{\beta} f(t) = I_{0^+}^{\beta-\alpha} f(t).$$

The following lemma is fundamental in the proof of the existence theorem.

Lemma 2.7 ([4]) (Leray–Schauder nonlinear alternative) *Let* F *be a Banach space and* Ω *a bounded open subset of* F, $0 \in \Omega$ *and let* $T : \overline{\Omega} \longrightarrow F$ *be a completely continuous operator. Then, either there exist* $x \in \partial\Omega$, *and* $\lambda > 1$ *such that* $T(x) = \lambda x$, *or there exists a fixed point* $x^* \in \overline{\Omega}$.

We start by solving an auxiliary problem and we give the Green function.

Lemma 2.8 *Let* $2 < q < 3$, $0 < \sigma < 1$. *The unique solution of fractional problem (P0)*

$$\begin{cases} {}^c D_{0+}^{q} u(t) = y(t), \; 0 < t < 1 \\ u(0) = u''(0) = 0, u'(1) = {}^c D_{0+}^{\sigma} u(1), \end{cases} \tag{2.1}$$

is given by

$$u(t) = \int_0^1 G(t,s)\, y(s)\, \mathrm{d}s,$$

where

$$G(t,s) = \begin{cases} \frac{(t-s)^{q-1}}{\Gamma(q)} + \frac{\Gamma(2-\sigma)t(1-s)^{q-2}}{\Gamma(2-\sigma)-\Gamma(2)} \left(\frac{(1-s)^{1-\sigma}}{\Gamma(q-\sigma)} - \frac{1}{\Gamma(q-1)} \right), & 0 \le s \le t \le 1 \\ \frac{\Gamma(2-\sigma)t(1-s)^{q-2}}{\Gamma(2-\sigma)-\Gamma(2)} \left(\frac{(1-s)^{1-\sigma}}{\Gamma(q-\sigma)} - \frac{1}{\Gamma(q-1)} \right), & 0 \le t \le s \le 1. \end{cases} \tag{2.2}$$

Proof Assume that u is a solution of the fractional boundary value problem (*P0*). Then using Lemma 2.5, we have

$$u(t) = I_{0+}^{q} y(t) + C + Bt + At^2, \tag{2.3}$$

from the conditions $u(0) = u''(0) = 0$, we obtain $C = A = 0$. Therefore, differentiating (2.3) gives

$$u'(t) = I_{0+}^{q-1} y(t) + B,$$

otherwise, we have

$${}^{c}D_{0+}^{\sigma} u(t) = I_{0+}^{q-\sigma} y(t) + B\, {}^{c}D_{0+}^{\sigma} t.$$

The condition $u'(1) = {}^{c}D_{0+}^{\sigma} u(1)$ implies that

$$B = \frac{\Gamma(2-\sigma)}{\Gamma(2-\sigma) - \Gamma(2)} \int_0^1 \left(\frac{(1-s)^{q-\sigma-1}}{\Gamma(q-\sigma)} - \frac{(1-s)^{q-2}}{\Gamma(q-1)} \right) y(s)\, \mathrm{d}s,$$

so $u(t)$ can be written as

$$u(t) = I_{0+}^{q} y(t) + \frac{\Gamma(2-\sigma)\, t}{\Gamma(2-\sigma) - \Gamma(2)} \int_0^1 \left(\frac{(1-s)^{q-\sigma-1}}{\Gamma(q-\sigma)} - \frac{(1-s)^{q-2}}{\Gamma(q-1)} \right) y(s)\, \mathrm{d}s, \tag{2.4}$$

where G is defined by (2.2). The proof is complete. □

Define the integral operator $T : E \to E$ by

$$Tu(t) = \int_0^1 G(t,s)\, f\left(s, u(s), {}^{c}D_{0+}^{\sigma} u(s)\right) \mathrm{d}s, \forall t \in [0,1].$$

Lemma 2.9 *Let* $f \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. *Then* $u \in E$ *is a solution of the fractional boundary value problem (P1) if and only if* $Tu(t) = u(t), \forall t \in [0,1]$.

Proof Let u be a solution of (P1). Then using the same method as used in Lemma 2.8, we can prove that

$$u(t) = \int_0^1 G(t,s)\, f\left(s, u(s), {}^{c}D_{0+}^{\sigma} u(s)\right) \mathrm{d}s = Tu(t).$$

Conversely u satisfies

$$u(t) = I_{0+}^{q} f\left(t, u(t), {}^{c}D_{0+}^{\sigma} u(t)\right)$$
$$+ \frac{\Gamma(2-\sigma)t}{\Gamma(2-\sigma) - \Gamma(2)} \int_0^1 \left(\frac{(1-s)^{q-\sigma-1}}{\Gamma(q-\sigma)} - \frac{(1-s)^{q-2}}{\Gamma(q-1)} \right) f\left(s, u(s), {}^{c}D_{0+}^{\sigma} u(s)\right) \mathrm{d}s,$$

and denotes the right-hand side of the equation by $v(t)$. Then by Lemma 2.5, we obtain

$$\begin{aligned} {}^cD_{0+}^q v(t) &= {}^cD_{0+}^q I_{0+}^q f\left(t, u(t), {}^c D_{0+}^\sigma u(t)\right) \\ &\quad + \frac{\Gamma(2-\sigma) {}^cD_{0+}^q t}{\Gamma(2-\sigma) - \Gamma(2)} \int_0^1 \left(\frac{(1-s)^{q-\sigma-1}}{\Gamma(q-\sigma)} - \frac{(1-s)^{q-2}}{\Gamma(q-1)} \right) f\left(s, u(s), {}^c D_{0+}^\sigma u(s)\right) ds \\ &= f\left(t, u(t), {}^c D_{0+}^\sigma u(t)\right). \end{aligned}$$

Hence, $v(t)$ is a solution of the fractional differential Eq. (1.1). Also it is easy to verify that v satisfies conditions (1.2), then it is a solution for the problem (P1). This achieves the proof. □

3 Existence and uniqueness results

In this section, we prove the existence and uniqueness of solutions in the Banach space E.

Theorem 3.1 *Assume that there exist nonnegative functions* $g, h \in L^1([0,1], \mathbb{R}_+)$ *such that for all* $x, y \in \mathbb{R}$ *and* $t \in [0,1]$:

$$|f(t, x, \overline{x}) - f(t, y, \overline{y})| \le g(t)|x-y| + h(t)|\overline{x} - \overline{y}|, \tag{3.1}$$

$$A_1\left(\|g\|_{L^1} + \|h\|_{L^1}\right) < \frac{1}{2}, \; A_2\left(\|g\|_{L^1} + \|h\|_{L^1}\right) < \frac{\Gamma(2-\sigma)}{2}, \tag{3.2}$$

where

$$A_1 = \frac{1}{\Gamma(q)} + \frac{\Gamma(2-\sigma)}{\Gamma(2) - \Gamma(2-\sigma)} \left(\frac{1}{\Gamma(q-\sigma)} + \frac{1}{\Gamma(q-1)} \right),$$

and

$$A_2 = \frac{1}{\Gamma(q-1)} + \frac{\Gamma(2-\sigma)}{\Gamma(2) - \Gamma(2-\sigma)} \left(\frac{1}{\Gamma(q-\sigma)} + \frac{1}{\Gamma(q-1)} \right).$$

Then the FBVP (P1) has a unique solution u in E.

Proof We shall use Banach fixed-point theorem. For this, we need to verify that T is a contraction. Let $u, v \in E$. Applying (2.4) we get

$$Tu(t) - Tv(t) = \int_0^1 G(t,s)\left(f\left(s, u(s), {}^cD_{0+}^\sigma u(s)\right) - f\left(s, v(s), {}^cD_{0+}^\sigma v(s)\right)\right) ds,$$

taking (3.1) into account, we obtain

$$\begin{aligned} |Tu(t) - Tv(t)| &\le \int_0^1 g(s)|G(t,s)||u(s) - v(s)| ds \\ &\quad + \int_0^1 h(s)|G(t,s)| \left| {}^cD_{0+}^\sigma u(s) - {}^cD_{0+}^\sigma v(s) \right| ds \\ &\le \max_{0\le t\le 1} |u(t) - v(t)| \int_0^1 |G(t,s)| g(s) ds \\ &\quad + \max_{0 \le t \le 1} \left| {}^cD_{0+}^\sigma u(t) - {}^cD_{0+}^\sigma v(t) \right| \int_0^1 |G(t,s)| h(s) ds. \end{aligned}$$

Let us estimate the term $\int_0^1 |G(t,s)| g(s)\,ds$:

$$\int_0^1 |G(t,s)| g(s)\,ds = \int_0^t |G(t,s)| g(s)\,ds + \int_t^1 |G(t,s)| g(s)\,ds$$
$$\leq \int_0^t A_1 g(s)\,ds + \int_t^1 A_0 g(s)\,ds$$
$$\leq A_1 \|g\|_{L_1}$$

where $A_0 = \frac{\Gamma(2-\sigma)}{\Gamma(2)-\Gamma(2-\sigma)}\left(\frac{1}{\Gamma(q-\sigma)} + \frac{1}{\Gamma(q-1)}\right)$, hence from (3.2) we get

$$\max_{0\leq t\leq 1} |Tu - Tv| \leq \frac{1}{2}\|u-v\|. \tag{3.3}$$

On the other hand, we have

$${}^cD_{0+}^{\sigma} Tu - {}^cD_{0+}^{\sigma} Tv = \frac{1}{\Gamma(1-\sigma)} \int_0^t \frac{(Tu)'(s) - (Tv)'(s)}{(t-s)^{\sigma}} ds,$$

where

$$(Tu)'(t) = \int_0^1 \frac{\partial G(t,s)}{\partial t} f\left(s, u(s), {}^cD_{0+}^{\sigma} u(s)\right) ds$$

and

$$\frac{\partial G(t,s)}{\partial t} = \begin{cases} \frac{(t-s)^{q-2}}{\Gamma(q-1)} + \frac{\Gamma(2-\sigma)(1-s)^{q-2}}{\Gamma(2-\sigma)-\Gamma(2)}\left(\frac{(1-s)^{1-\sigma}}{\Gamma(q-\sigma)} - \frac{1}{\Gamma(q-1)}\right), 0 \leq s \leq t \leq 1 \\ \frac{\Gamma(2-\sigma)(1-s)^{q-2}}{\Gamma(2-\sigma)-\Gamma(2)}\left(\frac{(1-s)^{1-\sigma}}{\Gamma(q-\sigma)} - \frac{1}{\Gamma(q-1)}\right), 0 \leq t \leq s \leq 1. \end{cases}$$

From the above, we deduce

$${}^cD_{0+}^{\sigma} Tu - {}^cD_{0+}^{\sigma} Tv = \frac{1}{\Gamma(1-\sigma)} \int_0^t \int_0^1 (t-s)^{-\sigma} \frac{\partial G(s,r)}{\partial s} \left(f\left(r, u(r), {}^cD_{0+}^{\sigma} u(r)\right)\right.$$
$$\left. - f\left(r, v(r), {}^cD_{0+}^{\sigma} v(r)\right)\right) dr\,ds.$$

Applying (3.1) again it yields

$$\left|{}^cD_{0+}^{\sigma} Tu - {}^cD_{0+}^{\sigma} Tv\right| \leq \frac{\max_{0\leq t\leq 1}|u-v|}{\Gamma(1-\sigma)} \int_0^t \int_0^1 (t-s)^{-\sigma} \left|\frac{\partial G(s,r)}{\partial s}\right| g(r)\,dr\,ds$$
$$+ \frac{\max_{0\leq t\leq 1}\left|{}^cD_{0+}^{\sigma} u - {}^cD_{0+}^{\sigma} v\right|}{\Gamma(1-\sigma)} \int_0^t \int_0^1 (t-s)^{-\sigma} \left|\frac{\partial G(s,r)}{\partial s}\right| h(r)\,dr\,ds, \tag{3.4}$$

and using (3.2), we obtain

$$\max_{0\leq t\leq 1}\left|{}^cD_{0+}^{\sigma} Tu - {}^cD_{0+}^{\sigma} Tv\right| \leq \|u-v\| \frac{1}{\Gamma(2-\sigma)} A_2\left(\|g\|_{L^1} + \|h\|_{L^1}\right)$$
$$\leq \frac{1}{2}\|u-v\|. \tag{3.5}$$

Taking (3.3) and (3.5) into account, we acquire

$$\|Tu - Tv\| \leq \|u - v\|,$$

then, T is a contraction. As a consequence of Banach fixed-point theorem, we deduce that T has a fixed point which is the unique solution of the FBVP (P1). The proof is complete. □

Now, we give an existence result for the fractional boundary value problem (P1).

Theorem 3.2 *Assume that $f(t,0,0) \neq 0$ and there exist nonnegative functions $k, h, g \in L^1([0,1], \mathbb{R}_+)$, $\phi, \psi \in C(\mathbb{R}_+, (0,+\infty))$ nondecreasing on $\mathbb{R}_+$ and $r > 0$, such that*

$$|f(t,x,\overline{x})| \leq k(t)\psi(|x|) + h(t)\phi(|\overline{x}|) + g(t), \tag{3.6}$$

$$(\psi(r) + \phi(r) + 1)\left(C_1 + \frac{C_2}{\Gamma(2-\sigma)}\right) < r, \tag{3.7}$$

where

$$C_1 = \max\{C_k, C_h, C_g\}, \quad C_2 = \max\{A_k, A_h, A_g\},$$
$$\text{where } C_k = A_1\|k\|_{L^1}, \quad C_h = A_1\|h\|_{L^1}, C_g = A_1\|g\|_{L^1},$$
$$A_g = A_2\|g\|_{L^1}, \quad A_k = A_2\|k\|_{L^1}, A_h = A_2\|h\|_{L^1}.$$

Then the FBVP (P1) has at least one nontrivial solution $u^* \in E$.

Proof In view of the continuity of f and G, the operator T is continuous. Let $B_r = [u \in E, ||u|| \leq r\}$ be a bounded subset in E.

(i) For $u \in B_r$, using (3.6) and the fact that ϕ and ψ are nondecreasing, we obtain

$$\begin{aligned}|Tu(t)| &\leq \int_0^1 |G(t,s)|\left(k(s)\psi(|u(s)|) + h(s)\phi(\left|{}^cD_{0+}^{\sigma}u(s)\right|) + g(s)\right)\mathrm{d}s\\ &\leq \psi(r)\int_0^1 |G(t,s)|k(s)\,\mathrm{d}s + \phi(r)\int_0^1 |G(t,s)|h(s)\,\mathrm{d}s + \int_0^1 |G(t,s)|g(s)\,\mathrm{d}s\\ &\leq \psi(r)A_1\left(\int_0^1 k(s)\,\mathrm{d}s\right) + \phi(r)A_1\left(\int_0^1 h(s)\,\mathrm{d}s\right) + A_1\left(\int_0^1 g(s)\,\mathrm{d}s\right)\\ &\leq \psi(r)C_k + \phi(r)C_h + C_g.\end{aligned}$$

Thus, we have

$$|Tu(t)| \leq C_1(\psi(r) + \phi(r) + 1). \tag{3.8}$$

In addition,

$$\begin{aligned}\left|(Tu(t))'\right| \leq \psi(r)&\int_0^1 \left|\frac{\partial G(t,s)}{\partial t}\right| k(s)\mathrm{d}s\\ &+\phi(r)\int_0^1 \left|\frac{\partial G(t,s)}{\partial t}\right| h(s)\,\mathrm{d}s + \int_0^1 \left|\frac{\partial G(t,s)}{\partial t}\right| g(s)\,\mathrm{d}s.\end{aligned} \tag{3.9}$$

Hence, it follows that

$$\begin{aligned}\left|{}^cD_{0+}^{\sigma}Tu\right| &\leq \frac{1}{\Gamma(2-\sigma)}\left(\psi(r)A_2\int_0^1 k(s)\,\mathrm{d}s + \phi(r)A_2\int_0^1 h(s)\mathrm{d}s) + A_2\int_0^1 g(s)\mathrm{d}s\right)\\ &\leq \frac{C_2}{\Gamma(2-\sigma)}(\psi(r) + \phi(r) + 1),\end{aligned} \tag{3.10}$$

therefore,

$$\|Tu\| \leq \left(C_1 + \frac{C_2}{\Gamma(2-\sigma)}\right)(\psi(r) + \phi(r) + 1),$$

which implies that $T(B_r)$ is uniformly bounded.

(ii) For all $t_1, t_2 \in [0, 1]$, $t_1 < t_2$ and $u \in B_r$ we have

$$|Tu(t_1) - Tu(t_2)| = \left|\int_{t_1}^{t_2} (Tu)'(t)\mathrm{d}t\right| \leq \int_{t_1}^{t_2} \left|(Tu)'(t)\right| \mathrm{d}t,$$

since

$$\begin{aligned}\left|(Tu)'(t)\right| &= \left|\int_0^1 \frac{\partial G(t,s)}{\partial t} f\left(s, u(s), {}^cD_{0+}^{\sigma}u(s)\right)\mathrm{d}s\right| \\ &\leq (\psi(r) + \phi(r) + 1)\, C_2,\end{aligned} \tag{3.11}$$

we obtain

$$|Tu(t_1) - Tu(t_2)| \leq (\psi(r) + \phi(r) + 1)\, C_2 (t_2 - t_1). \tag{3.12}$$

The following estimate holds

$$\begin{aligned}\left|{}^cD_{0+}^{\sigma}Tu(t_1) - {}^cD_{0+}^{\sigma}Tu(t_2)\right| &= \left|\frac{1}{\Gamma(1-\sigma)}\left(\int_0^{t_1} \frac{(Tu)'(s)}{(t_1-s)^{\sigma}}\mathrm{d}s - \int_0^{t_2} \frac{(Tu)'(s)}{(t_2-s)^{\sigma}}\mathrm{d}s\right)\right| \\ &\leq \frac{1}{\Gamma(1-\sigma)}\left(\int_0^{t_1} \left((t_1-s)^{-\sigma} - (t_2-s)^{-\sigma}\right)\left|(Tu)'(s)\right|\mathrm{d}s\right. \\ &\quad \left. + \int_{t_1}^{t_2} (t_2-s)^{-\sigma}\left|(Tu)'(s)\right|\mathrm{d}s\right).\end{aligned}$$

From (3.11) we get

$$\begin{aligned}\left|{}^cD_{0+}^{\sigma}Tu(t_1) - {}^cD_{0+}^{\sigma}Tu(t_2)\right| &\leq \tfrac{(\psi(r)+\phi(r)+1)C_2}{\Gamma(1-\sigma)}\left(\int_0^{t_1} \left((t_1-s)^{-\sigma} - (t_2-s)^{-\sigma}\right)\mathrm{d}s + \int_{t_1}^{t_2} (t_2-s)^{-\sigma}\,\mathrm{d}s\right) \\ &\leq \frac{(\psi(r)+\phi(r)+1)\, C_2}{\Gamma(2-\sigma)}\left(\left(t_1^{1-\sigma} - t_2^{1-\sigma}\right) + 2(t_2-t_1)^{1-\sigma}\right).\end{aligned} \tag{3.13}$$

As $t_1 \to t_2$, the right-hand sides of the above inequalities (3.12) and (3.13) tend to 0, consequently $T(B_r)$ is equicontinuous.

By means of the Arzela–Ascoli Theorem, we conclude that T is completely continuous.

In what follows, we establish an existence result using the nonlinear alternative of Leray–Schauder.

Setting $\Omega = \{u \in E : \|u\| < r\}$ then for $u \in \partial\Omega$, such that $u = \lambda Tu$, $0 < \lambda < 1$.

Using (3.8) we get

$$|u(t)| = \lambda\, |Tu(t)| \leq |Tu(t)| \leq (\psi(r) + \phi(r) + 1)\, C_1. \tag{3.14}$$

In addition,

$$\left|{}^cD_{0+}^{\sigma}u(t)\right| = \lambda\left|{}^cD_{0+}^{\sigma}Tu(t)\right| \leq \left|{}^cD_{0+}^{\sigma}Tu(t)\right| \leq \frac{C_2}{\Gamma(2-\sigma)}(\psi(r) + \phi(r) + 1). \tag{3.15}$$

From (3.7), (3.14) and (3.15) we deduce that

$$\|u\| \leq (\psi(r) + \phi(r) + 1)\left(C_1 + \frac{C_2}{\Gamma(2-\sigma)}\right) < r,$$

this contradicts the fact that $u \in \partial\Omega$. By Lemma 2.7, we conclude that T has a fixed point $u^* \in \overline{\Omega}$ and then the FBVP (P1) has a nontrivial solution u^* in E. This achieves the proof □

We illustrate our work with two examples.

Example 3.3 Let us consider the fractional boundary value problem

$$\begin{cases} {}^cD_{0^+}^{\frac{14}{5}}u = \frac{t^4}{9}u + \left(\frac{1-t}{3}\right)^4 {}^cD_{0^+}^{\frac{4}{5}}u + \cos t, \; 0 < t < 1, \\ u(0) = u''(0) = 0, u'(1) = {}^cD_{0^+}^{\frac{4}{5}}u(1). \end{cases}$$

We have

$$f(t,x,y) = \frac{t^4}{9}x + y\left(\frac{1-t}{3}\right)^4 + \cos t, 2 < q = \frac{14}{5} < 3, \sigma = \frac{4}{5} < 1$$

and

$$|f(t,x,\overline{x}) - f(t,y,\overline{y})| \leq \frac{t^4}{9}|x-y| + \left(\frac{1-t}{3}\right)^4 |\overline{x}-\overline{y}|,$$

then

$$|f(t,x,\overline{x}) - f(t,y,\overline{y})| \leq g(t)|x-y| + h(t)|\overline{x}-\overline{y}|, \forall x, y \in \mathbb{R}, t \in [0,1],$$

where $g(t) = \frac{t^4}{9}$ and $h(t) = \left(\frac{1-t}{3}\right)^4$. Simple calculus gives:

$$\|g\|_{L^1} = 0,025, \|h\|_{L^1} = 0,0024, A_1 = 23,8716, A_2 = 24,3489$$
$$A_1\left(\|g\|_{L^1} + \|h\|_{L^1}\right) = 0,6540 < 1,$$
$$A_2(\|g\|_{L^1} + \|h\|_{L^1}) = 0,6671 < \Gamma(2-\sigma) = 0.9181.$$

Hence from Theorem 3.1, we conclude that the FBVP has a unique solution u^* in E.

Example 3.4 For the fractional boundary value problem

$$\begin{cases} {}^cD_{0^+}^{\frac{14}{5}}u = \frac{\exp(-t)}{90}\frac{u^2}{120} + \left(\frac{1-t}{3}\right)^4\left(\frac{{}^cD_{0^+}^{\frac{4}{5}}u}{10}\right)^3 + \frac{1-t}{100}, 0 < t < 1, \\ u(0) = u''(0) = 0, u'(1) = {}^cD_{0^+}^{\frac{4}{5}}u(1), \end{cases}$$

we have

$$f(t,x,y) = \frac{\exp(-t)}{90}\frac{x^2}{120} + \frac{y^3}{1{,}000}\left(\frac{1-t}{3}\right)^4 + \frac{1-t}{100}, 2 < q = \frac{14}{5} < 3, \sigma = \frac{4}{5} < 1.$$

Therefore,

$$|f(t,x,\overline{x})| \leq \frac{\exp(-t)}{90}\left(\frac{|x|^2}{120} + 1\right) + \left(\frac{|\overline{x}|^3}{1{,}000} + 1\right)\left(\frac{1-t}{3}\right)^4 + \frac{1-t}{100},$$

where $g(t) = \frac{\exp(-t)}{90}$, $h(t) = \left(\frac{1-t}{3}\right)^4$, $k(t) = \frac{1-t}{100}$, $\psi(x) = \frac{x^2}{120} + 1$, $\phi(\overline{x}) = \frac{(\overline{x})^3}{1{,}000} + 1$. Let us evaluate $\left[(\psi(r) + \phi(r) + 1)\left(C_1 + \frac{C_2}{\Gamma(2-\sigma)}\right) - r\right]$. Simple calculus gives:

$$\|g\|_{L^1} = 0,00702, \|h\|_{L^1} = 0,0024, \|k\|_{L^1} = 0.005, A_1 = 23,8716,$$
$$A_2 = 24,3489, C_1 = 0.1671, C_2 = 0.17044.$$

Then

$$\left[(\psi(r)+\phi(r)+1)\left(C_1+\frac{C_2}{\Gamma(2-\sigma)}\right)-r\right]=0.352\,73\left(\frac{r^2}{120}+\frac{r^3}{1,000}+3\right)-r<0,$$

for $r = 2$. Theorem 3.2 implies that the BVP has at least one nontrivial solution u^* in E.

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