Arab J Math (2014) 3:189–198
DOI 10.1007/s40065-013-0089-7

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Fredholm pseudo-gradients for the action functional on a sub-manifold of dual Legendrian curves of a three dimensional contact manifold (M^3, α)

Received: 23 July 2013 / Accepted: 5 November 2013 / Published online: 15 February 2014

Abstract We prove in this paper that the intersection numbers between periodic orbits have an intrinsic meaning for the variational problem (J, C_β) {Bahri (Pseudo-Orbits of Contact Forms Pitman Research Notes in Mathematics Series No. 173, 1984), Bahri (C R Acad Sci Paris 299, Serie I 15:757–760, 1984), Bahri (Classical and Quantic periodic motions of multiply polarized spin-manifolds. Pitman Research Notes in Mathematics Series No. 378, 1998)}, corresponding to the periodic orbit problem on a sub-manifold of the loop space of a three dimensional compact contact manifold (M, α).

Mathematics Subject Classification 37J45 · 37J55 · 53D10 · 55N99 · 58E10

الملخص

نثبت في هذه الورقة أن لأعداد التقاطع بين المدارات الدورية معنى جوهرياً لمسألة التغاير (J, C_β) ([2] وَ [3] وَ [4]) المقابلة لمسألة المدار الدوري على متنوعة جزئية لفضاء الأنشوطة لمتنوعة متراصة ثلاثية الأبعاد (M, α).

1 Introduction

Given a compact finite dimensional manifold without boundary N^n and a C^2 function $f : N^n \longrightarrow R$, with non-degenerate critical points, the intersection number $i(x_m, x_{m-1})$ of a critical point x_m of index m with a critical point x_{m-1} of index $(m-1)$ is defined, for a Morse–Smale [29,30] pseudo-gradient Z for f, to be the algebraic number of flow-lines in the intersection $W_u(x_m) \cap W_s(x_{m-1})$ of the unstable manifold of x_m with the stable manifold of x_{m-1}.

If there are in $f^{-1}([f(x_{m-1}), f(x_m)])$ critical points of index m y_m such that $i(y_m, x_{m-1})$ is non zero or if there are, in the same set, critical points y_{m-1} such that $i(x_m, y_{m-1})$ is non zero, then this intersection number is not intrinsic; it depends on the choice of Z. However, if $f^{-1}([f(x_{m-1}), f(x_m)])$ contains no such critical points, e.g. if it contains no critical point but x_m and x_{m-1}, or if it contains only critical points of higher index $p \gneq m$ or lower index $s \lneq (m-1)$, then this intersection number becomes intrinsic, independent of Z.

As we move away from the compact finite dimensional framework and as we consider manifolds N of infinite dimension, two difficulties arise: the first difficulty relates to the definition of a transverse intersection $W_u(x_m) \cap W_s(x_{m-1})$. This difficulty has been solved historically ([14,22,24,31], not exhaustively) with the imposition of a Fredholm framework on both the manifold N and on the pseudo-gradient Z. The second difficulty relates to the possible existence of asymptotes and to the verification of the Palais–Smale condition.

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To a certain extent, these two difficulties are intertwined.

In many problems of Conformal Geometry, e.g. the Yamabe and related problems, the associated variational problems [1] are (locally) Fredholm, but they do not verify the Palais–Smale condition. Suitable techniques [9–11,27] have then been developed to overcome, at least partially, this difficulty.

In the area of Hamiltonian Systems, the Fredholm assumption and the so-called $(P.S)$ condition are "easy" (in that they are now classical) to verify for Lagrangian formulations (e.g. [12,17,28] including brake-orbits [33]). In the new formulation developed by P.H. Rabinowitz [25] in 1978, with the introduction of the action functional $\int_0^1 \Sigma p_i \dot{q}_i$ on the space $H^{\frac{1}{2}}(S^1, R^{2n})$, both conditions are verified for example through a Galerkin approximation by finite dimensional spaces.

This framework has also been used by C. Conley and E. Zehnder [13] for the solution of the Arnold conjecture on tori. However, as mathematicians moved away from the R^{2n}-framework and tried to solve the Arnold conjecture in full generality or tried to solve the Weinstein conjecture [32][1], they found themselves without an appropriate variational formulation for the periodic orbits problem for contact vector-fields.

It is not an easy task, even in the framework of cotangent bundles of finite dimensional manifolds, for Hamiltonians that are not convex in the momentum variables (no Lagrangian formulation), to define a Fredholm framework for this problem, see e.g. [21]. The space $H^{\frac{1}{2}}(S^1, M^{2n})$, which is the natural space (e.g. in a symplectic formulation) for the action functional, is not well-defined because $H^{\frac{1}{2}}(S^1, R^{2n})$ does not embed in L^∞.

Several methods have been devised to overcome this difficulty. For example, A. Floer [16], using the pseudo-holomorphic framework introduced by M. Gromov [19], was successful in extending the results of C. Conley and E. Zehnder [13] to the framework of compact symplectic manifolds. Also, in the contact framework, H. Hofer [20] was able, using this pseudo-holomorphic framework and the construction of a special disk for over-twisted contact structures, to prove the existence of one periodic orbit for the related contact vector-field (and did in this way solve positively, to a great extent, the three-dimensional version of the Weinstein conjecture).

However, despite this progress, the full understanding of the Morse relations between the periodic orbits of a given contact vector-field could not be achieved.

For example, with pseudo-holomorphic curves (assuming their existence, a non-trivial matter), one can try to understand these Morse relations through the moduli spaces of such curves [15]. However, along deformations of contact forms, these moduli spaces are not stable. There are "blow-ups", with discontinuities in the Fredholm index and failure of compactness. Beyond the issue of existence of moduli spaces of pseudo-holomorphic curves, one finds himself facing again the two fundamental difficulties described above.

Very early, we have defined, in collaboration with D. Bennequin [3], a variational framework for the periodic orbits problem for contact vector-fields on a three-dimensional closed and compact contact manifold (M^3, α). In this variational framework see e.g. [2,5,6], the action functional $J(x) = \int_0^1 \alpha(\dot{x})$ was studied on a sub-manifold $C_\beta = \{x \in H^1(S^1, M); \beta(\dot{x}) = d\alpha(v, \dot{x}) = 0; \alpha(\dot{x}) = a\}$ of the loop space of M^3. a, in the definition of C_β, is a positive constant that is not prescribed, it varies with the curve $x(t)$; v is a non-singular vector-field in $ker\alpha$, β verifies the condition (A) : β is a contact form with the same orientation than α, see Sect. 6 below for a considerable weakening of this condition.

Very early on also [2], we had noted that this variational problem failed both the Fredholm assumption and the Palais–Smale condition.

We have overcome, in various (different) ways the second difficulty in our work, see [5–8] in particular.

However, we could never overcome the violation of the Fredholm assumption, although we did reduce it in [8] to a violation of this assumption at the periodic orbits themselves. We were able in [8] to formulate a simple condition. Under this condition and for a special pseudo-gradient, see [8], the intersection operators ∂_{per} and ∂_∞ do not mix in between creations and cancellations of periodic orbits.

We prove in the present paper that there is a pseudo-gradient flow for (J, C_β), that can be continuously tracked along deformations of contact forms, for which the Fredholm assumption at the periodic orbits is verified (after [8], this is all what is needed) and that, for this pseudo-gradient, the intersection number between two periodic orbits of consecutive indexes is defined intrinsically (as described above, in the compact, finite dimensional framework).

[1] This conjecture was formulated by A. Weinstein after a clever understanding of the star-shaped condition on the energy surfaces of R^{2n} introduced by P.H. Rabinowitz in [25]. Alan Weinstein [33] had himself independently established a similar result for convex energy surfaces of R^{2n}.

Accordingly, with the use of this flow and the additional work in [5,6] and [8], the variational problem (J, C_β) becomes a "Fredholm framework" for the finding of periodic orbits to the contact vector-field ξ of α ($\beta = d\alpha(v, .)$, $v \in ker\alpha$, see [2,4]), "stable" under deformation.

This is already a significant progress in the effort to find an appropriate framework for the problem of periodic orbits. However, further progress is much needed to extend these techniques to higher dimension and to entirely remove conditions (A) and $(A)_t$.

We do not claim here to have the final framework for this kind of variational problems. The present paper rather asserts a direction of research and states positive (global) results of existence related to this direction.

This is a short paper and its main result is stated in Proposition 2.1. This Proposition is about the intersection number of two periodic orbits when there are no other periodic orbit or critical point at infinity in between their energy levels (for the action functional). This can be readily extended to allow for intermediate critical points with zero intersection numbers with the dominating or with the dominated periodic orbit, depending on their index, see above, at the beginning of this Introduction.

The proof of this Proposition 2.1 assumes the knowledge of the results of [5,6,8]. We proceed now with our precise claims and proofs:

2 Statement of the results. Beginning of the proof of Proposition 2.1

Let α_t be a deformation of contact forms on a contact closed manifold M^3 and let v_t be a family of continuously varying vector-fields in their kernel ($\alpha_t(v_t) = 0$). Let us assume that the condition

$$(A)_t : d\alpha_t(v_t, .) \text{ is a contact form with the same orientation than } \alpha_t$$

is verified all along the deformation. We will indicate at the end of this paper how to get rid of this condition.

As in [2,4,5], Γ^t_{2m}, which we also denote Γ_{2m}, is the space of curves made of $m\xi_t$-pieces of orbits alternating with $m \pm v_t$-pieces of orbits. ξ_t is the Reeb vector-field of α_t.

Let $a \lneq b$ be two values such that J^t has no critical point at infinity in $(J^t)^{-1}([a, b])$ but for the $(\delta^{(m)}+w)_\infty$ maybe (these $(\delta^{(m)} + w)_\infty$ are the critical points at infinity built with "Dirac masses", i.e. back or forth or forth and back runs along v, above some point of the periodic orbit w, see [5] and [8] for more details, [8], Appendix 1 in particular). Assume furthermore that the deformation α_t has been "adjusted", using the techniques of [5], p 85–93, e.g. Proposition 15, see also [6], p 473–474 for an earlier use for this proposition to "adjust" the v-rotation along a simple periodic orbit, so that:

(i) Every w^t_{2m+1} such that $a \lneq J^t(w^t_{2m+1}) \lneq b$ is a simple elliptic periodic orbit; whereas every w^t_{2p} such that $a \lneq J^t(w^t_{2p}) \lneq b$ is a simple hyperbolic periodic orbit. w^t_{2m+1} has Morse index $(2m + 1)$, w^t_{2p} has Morse index $2p$.

(ii) Given two periodic orbits w^t_{2k+1} and w^t_{2k} in $(J^t)^{-1}([a, b])$, of Morse index $(2k+1)$ and $2k$ respectively, we assume that either a cancellation (w^t_{2k+1}/w^t_{2k}) occurs at the time $t = t_0$; or that the level $J^t(w^t_{2k+1})$ crosses the level $J^t(w^t_{2k})$ at the time $t = t_0$. α_t is then "adjusted", if needed, so that the v-rotation along the simple elliptic periodic orbit $w^{t_0}_{2k+1}$ is $2k\pi + \theta, \theta \in (0, \pi)$.

(iii) On the other hand, if instead of a cancellation/crossing (w^t_{2k+1}/w^t_{2k}), a cancellation/crossing (w^t_{2k}/w^t_{2k-1}) occurs in $(J^t)^{-1}([a, b])$, at the time $t = t_0$, the v-rotation along the simple elliptic periodic orbit $w^{t_0}_{2k-1}$ is $(2k - 1)\pi + \theta, \theta \in (0, \pi)$. In addition, we assume that the v-rotation along w^t_{2k}, starting from any point along w^t_{2k-1}, is $2k\pi + o(\pi)$, just as in Section 4 of [8].

There is no loss of generality in assuming that (i)–(ii)–(iii) holds, see Proposition 15 of [5] and [6], p 473–474. In the case of cancellations, these conditions are verified for a deformation α_t in general position, without the need for any further adjustment.

After (i)–(ii)–(iii), we claim that the following holds in $(J^t)^{-1}([a, b])$:

Proposition 2.1 *There is a decreasing pseudo-gradient Z^t for (J^t, C_{β^t}) in $J^t_\infty)^{-1}([a, b]) \subset \bigcup_m \Gamma_{2m}$ that is "Fredholm" or "symplectic" ([8], Definition 1): no tangency between $W_s(w^t_r)$ and $W^t_s(\delta + w^t_{r-1})^\infty)$ occurs over the deformation for any two periodic orbits w^t_r, w^t_{r-1}, of respective indexes r, $r - 1$, such that $w^t_r, w^t_{r-1} \in (J^t_\infty)^{-1}([a, b])$. The family Z^t varies in a differentiable way with t and defines a flow on each Γ_{2m}, $m \in N$.*

Proof of Proposition 2.1 In all our arguments below, the “Dirac masses” built over the various curves in the deformation process (on the stable and unstable manifolds of the various critical points and critical points at infinity involved in the arguments) are suitably approximated with back and forth or forth and back runs along the vector-field v, these runs along v being separated by tiny ξ-pieces that eventually become larger as the two $\pm v$-jumps become small and are “pushed away” one from the other one. It is to this set of approximating curves, rather than to the “infinitely” contracted curves with “Dirac masses”, that the arguments for elliptic orbits w_{2k-1} and hyperbolic periodic orbits w_{2k} are applied below: the infinitely contracted “Dirac masses” could otherwise resolve themselves into two confounded zero $\pm v$-jumps through the “pushing away” and “widening process” of [6] and the arguments for Proposition 2.1 would then become less transparent.

We start now the proof of Proposition 2.1: since there are no critical point at infinity in $(J^t_\infty)^{-1}([a, b])$, besides the $(\delta^{(m)} + w^t_r)^\infty$, the unstable manifold—which we denote $W^t_u(w^t_m)$—of a simple periodic orbit of index m in $(J^t_\infty)^{-1}([a, b])$ is modelized, see Proposition 2.1, p 469 of [6], with m single $\pm v$-jumps that can be tracked over decreasing flow-lines.

Under (i)–(ii)–(iii), let w^t_{2k} and w^t_{2k-1} be given in $(J^t)^{-1}([a, b])$. Flow-lines out of w^t_{2k} are built in $(J^t)^{-1}([a, b])$ with curves that support $2k$ simple $\pm v$-jumps separated by ξ-pieces of orbits.

If one of these flow-lines enters an L^∞-neighborhood (in graph) that is small enough of $(\delta + w^t_{2k-1})^\infty$, then the curves on this portion of flow-line must have at least two non-zero $\pm v$-jumps. Using the arguments of Section 3 of [8], “Bypassing a simple elliptic periodic orbit”, such a flow-line will never end at w^t_{2k-1}. Tangencies $W^t_u(w^t_{2k}) - W^t_s((\delta + w^t_{2k-1})^\infty)$ are thereby forbidden with such a flow and the intersection number $i(w^t_{2k}, w^t_{2k-1})$ does not change with t.

In addition, all these flows can be deformed one into the other, those that do not introduce any companions to existing single $\pm v$-jumps in $(J^t)^{-1}([a, b])$ as well. For all of these, the flow-lines that enter an L^∞-neighborhood (in graph) that is small enough of $(\delta + w^t_{2k-1})^\infty$ do not abut at w^t_{2k-1} later. As we deform continuously pseudo-gradients over the time of the deformation, we find a flow in $(J^t)^{-1}([a, b])$ which we may assume to not introduce companions to existing simple $\pm v$-jumps in this “energy slice”.

If we continuously deform this flow, among pseudo-gradients that have the same property in $(J^t)^{-1}([a, b])$, the intersection number $i(w^t_{2k}, w^t_{2k-1})$ does not change. All pseudo-gradients with this property can be deformed one into the other. Among these, there is a “compact” pseudo-gradient which is almost explicit on $W_u(w^t_{2k})$ as the two energy levels, the one of w^t_{2k} and the one of w^t_{2k-1} become closer and closer. The intersection number can be computed on this compact one. The claim of the Proposition 2.1 follows in this case (see some further precisions below, when considering configurations such that the $\pm v$-jumps of the “Dirac mass” are not well-defined).

A similar phenomenon occurs for a pair w^t_{2k+1}/w^t_{2k} in $(J^t)^{-1}([a, b])$, but the proof is different.

Again, the curves on the flow-lines out of w^t_{2k+1} support $(2k+1)$ simple $\pm v$-jumps that can be continuously tracked. If a flow-line enters a small L^∞-neighborhood of a $(\delta + w^t_{2k})^\infty$, then two of its $\pm v$-jumps are large. We also observe that we can take them to be consecutive $\pm v$-jumps.

When this flow-line reaches a small L^∞-neighborhood of w^t_{2k}, the behavior of the related configurations can be understood as follows: the two consecutive $\pm v$-jumps are still non-zero, but small. Completing the “widening process”, see [6], Proposition 20, p 518, between these two $\pm v$-jumps, we can bring the v-rotation on the ξ-piece separating them to be $\pi - \frac{\pi}{2k+1} + o(e^{-k})$ as in Section 1 of [8]. Their orientations have not been reversed and they are still non-zero $\pm v$-jumps at the end of this process. They are now “locked in their positions”. The remaining $(2k-1) \pm v$-jumps have to be “rearranged” through the process of “pushing away” so that the v-rotation on any ξ-piece separating two consecutive $\pm v$-jumps is $i(w^t_{2k}, w^t_{2k-1})$. The v-rotation on the “external” nearly ξ-piece (it is “broken” with the remaining $(2k-1) \pm v$-jumps. All $\pm v$-jumps are assumed, without loss of generality to be $0(e^{-k^2})$) separating the two non-zero consecutive $\pm v$-jumps that were involved in the formation of the “Dirac mass” is $2k(\pi - \frac{\pi}{2k+1} + o(e^{-k}))$. It follows in particular that this second step in the re-arrangement process can be completed so that the two consecutive $\pm v$-jumps involved in the formation of the “Dirac mass” do not change location.

We therefore need to understand better now the first part of this process as the “Dirac masses” build up and are still large. We have developed such an understanding in [8], Appendix 1. However, in our present line of arguments, we seek an understanding of these phenomena when no companions are introduced to existing $\pm v$-jumps, although we will be also indicating the modifications needed in the case companions are introduced as in [6] and [7]. Consequently, we also need to adjust to this framework our understanding of the process of formation of “Dirac masses”.

3 Zoology of “Dirac masses”

A slightly different “zoology” for these “Dirac masses” holds in this new context. We now describe this zoology: “Dirac masses” contain a back and forth or forth and back run along v. Let us assume, without loss of generality, that we are in the latter case.

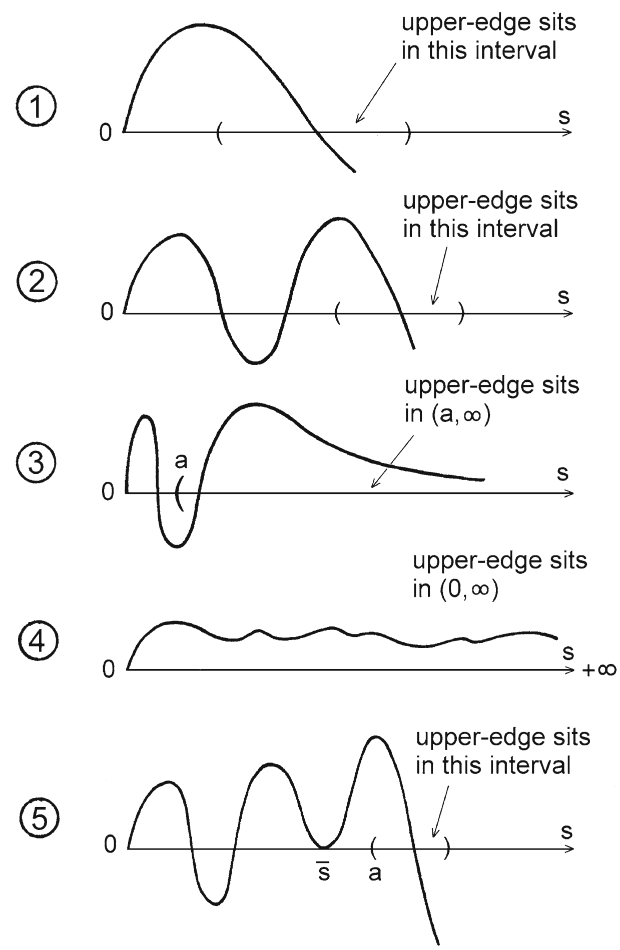
The function:

$$\theta(s) = 1 - \alpha_{x_0}(D\Phi_{-s}(\xi(x_s))),$$

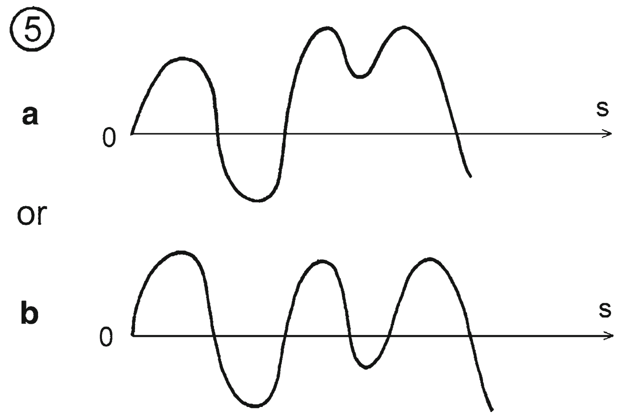
x_s is the v-orbit through x_0, is relevant to the formation of the “Dirac mass”, see [5], pp 28–29 and [8], Appendix 1 for more details.

A “Dirac mass” of the type indicated above can be built whenever $\theta(s)$ is negative for some positive s, see [8], Appendix 1 also.

However, depending on the behavior of $\theta(s)$ for $s \in (0, \infty)$, we may encounter different configurations involving different outcomes. $\theta(s)$, on sub-intervals of $[0, \infty)$, can behave in four basic ways, and also in a fifth way. These are best described with the drawings below:



I, II and III can be modified so that they will include more bumps, V also. However, more relevant to V is the behavior of $\theta(s)$ when the base point on x_{2k} varies. In general, V will break into:



However, in the case of circle-bundles along v, if $x_{\bar{s}}$ is x_0, then this behavior (the one described in V) survives the change.

Index at infinity of "Dirac masses":

In all the cases that we are considering, w_m with the addition of a "Dirac mass" may be viewed as a curve of Γ_4, with one ξ-piece reduced to zero, that is it can be viewed as a critical point at infinity of index i_∞ (at infinity, in Γ_4) equal to 0 or to 1. Along deformations, there is an additional parameter and the index at infinity can be equal to 2. Indeed the flow-lines of Γ_4 out of an x_{m+1} build a stratified space of top dimension 2. These flow-lines must dominate the critical point at infinity defined by this "Dirac mass" and this implies the conclusion.

More specifics about the "zoology", "energy levels":

The precise value that the function $\theta(s)$ takes at the edge of the "Dirac mass" is irrelevant. All these curves are at the same energy level for J_∞. This energy level depends only on the base curve w_{2k} or x upon which the "Dirac mass" is built.

However, the fact that $\theta(s)$ is positive or negative at the edges of the "Dirac mass" matters: if $\theta(s)$ is negative at the upper-edge, then, see Appendix 1 of [8], a small ξ-piece can be inserted at the "top" of this "Dirac mass" and J_∞ decreases substantially along this process.

Assuming now that $\theta(s)$ is non-negative at this upper-edge, we can make the "Dirac mass" longer or shorter. The first zeros for the function $\theta(s)$ that we encounter in either direction matter then: as soon as the edge of the "Dirac mass" enters an interval where θ is negative, the process of insertion described above can be completed and J_∞ decreases substantially.

This allows to understand better the behavior of the unstable manifold at infinity (recall that $i_\infty = 0$ or $i_\infty = 1$, see above and [8], if the "Dirac mass" is dominated by flow-lines of $W_u(w_{2k+1}) \cap \Gamma_4$).

If we are to discriminate between the "energy levels" defined by J_∞ for the curves of Γ_4 built with "Dirac masses" as above, then we can define a flow that follows the behavior of the function θ. Then, a "Dirac mass" such as I or III is higher than II and a "Dirac mass" such as $V(a)$ is higher than $V(b)$.

Once this flow on the "Dirac masses" is defined in Γ_4, the critical points at infinity of index $i_\infty = 0$ or $i_\infty = 1$ —they all turn out to be of index $i_\infty = 1$—are either isolated curves of the type $V(b)$ (their precise value depend on the full definition of the flow); or starting from "Dirac masses" of type I and following the analysis of Appendix 1 of [8], the flow-lines end at "Dirac masses" located at precise points x_0^i such that the upper-edge of the "Dirac mass" verifies:

$$(**)\alpha_{x_0^i}(D\Phi_{-s}([\xi, v](x_s^i))) = 0; \alpha_{x_0^i}(D\Phi_{-s}(\xi(x_s^i))) = 1$$

Both types of curves are of index $i_\infty = 1$. For II, the exit sets may be read on the drawing: The "Dirac mass" can be made longer and shorter so that the function θ is negative at its edge and, see Appendix 1 of [8], a small ξ-piece can be inserted at its top in a J_∞-decreasing process.

These curves do not therefore dominate w_{2k} at infinity. The arguments of Section 3 of [8] may therefore be applied to them. They fit in the framework of the critical points at infinity of [8].

If, instead of the arguments of [8], we allow the introduction of companions, as in [5] and [6], we can always "spare" the negative of the positive v-jump of the "Dirac mass" and choose not to introduce companions to one of them.

Introducing companions to the other one, the "Dirac masses" of type II do not appear since it is possible then to insert, there where θ is negative along the positive or the negative v-jump of the "Dirac mass", a small ξ-piece and J_∞ will thereby decrease substantially. The arguments for the expansion of J_∞ along this insertion can be found in [5] and [6].

Let us furthermore observe, along the same line of arguments that, whether the introduction of companions is allowed or not, critical points at infinity of type (III) do not appear: decreasing the size of the "Dirac mass", we reach an interval where $\theta(s)$ is negative. Inserting then a small ξ-piece, J_∞ decreases substantially.

For curves of type IV, we can make the "Dirac mass" shorter until it disappears: there is no exit set related to the behavior of $\theta(s)$ for $s \in (0, \infty)$.

Thus, through various arguments, depending on whether we are using the techniques of [7] or we are using the techniques of [8], critical points at infinity of type (III) either do not appear or can be treated as in Section 4 of [8].

4 Curves of type I and end of proof of Proposition 2.1

We are left with curves of type I.

Using the arguments of [5] and [8], they are viewed as critical points at infinity. Using companions, see [7], Section 8, they can be bypassed.

If we do not allow the use of companions, we still claim that no flow-line of $W_u(w_{2k+1}$, coming out of a neighborhood of a "Dirac mass" of type I, located at an x_0 along w_{2k} satisfying $(**)$, will abut at w_{2k}.

To see this, we resume our rearrangement/reordering argument above.

The positive v-jump of the "Dirac mass" is locked at x_0 along this process (it has been chosen as γ, see Section 3 of [8], in our argument). Indeed, this positive v-jump does not change location as the "Dirac mass" decreases in size along the unstable manifold. This holds also, as has been pointed out above, when the curves support several other small $\pm v$-jumps that are used to modelize the H_0^1-unstable manifold of this critical point at infinity.

We now re-scale the v-rotation along w_{2k} so that, all along the deformation, these points x_0 (there might be several such x_0s) are located in two consecutive intervals of "positive type": these are two consecutive intervals along which $J"_\infty(w_{2k})$ is positive along a curve defined with the insertion of a small $\pm v$-jump located at some point along x_{2k}, see Section 1 of [8]. Such "positive" intervals are separated by intervals where $J"_\infty(w_{2k})$ is negative along the same type of directions.

These two consecutive intervals of positivity for $J"_\infty(w_{2k})$ are evolved over the deformation of contact forms so that, at any time t along this deformation, all the x_0s (with $i_\infty = 1$) such that $(**)$ holds for some $s \in R - \{0\}$ are included in one of these intervals. This involves of course a continuous rescaling of the v-rotation along w_{2k} that can be completed as in [5], pp 85–93, Proposition 15 in particular.

In this way, we can include in intervals of positivity for $J"_\infty(w_{2k})$ all the base points of the "Dirac masses" such that their top level is not a local maximum in the set of "Dirac masses". Indeed, any "Dirac mass", as explained above, must be of index at least 1, this is embedded in the construction and a direction of negativity can be recognized as living along the "Dirac mass", as it changes size.

For these "Dirac masses", as we re-arrange the configuration when the $\pm v$-jumps become small, the positive v-jump of the "Dirac mass" can be kept locked at x_0. For configurations coming from "Dirac masses", the rearrangement can be completed by pushing all the $\pm v$-jumps away from the two $\pm v$-jumps of the "Dirac mass, then "pushing the negative $\pm v$-jump away from the positive one and adjusting then the rotation. Whatever happens does not change one fact: one $\pm v$-jump besides the positive v-jump of the "Dirac mass" remains non-zero, so that, after Section 3 of [8], if we choose the positive direction E^+ at this configuration to be modeled by a $\pm v$-jump located at x_0 (it can be done whenever x_0 is in a positivity interval for $J"_\infty(w_{2k})$), this configuration is not in the stable direction for w_{2k} and the configuration can be moved down, past w_{2k}.

We are left with the "Dirac masses" of higher index at infinity that have their base point located in an interval where $J"_\infty(w_{2k})$ is non-positive. If the base point is in an interval on negativity, then the index at infinity of the "Dirac mass" is at least 3: two directions of negativity are one along the "Dirac mass" as explained above, the other one because this "Dirac mass" top level is a local maximum among the top levels of the "Dirac masses". The third direction comes the possibility of changing the relative sizes of the positive and the negative large $\pm v$-jumps of the "Dirac mass", thereby creating a third negative direction since the base point is an interval of negativity for $J"_\infty(w_{2k})$.

Such curves of Γ_4 cannot be dominated along the deformation, the index at infinity is too large. Another $\pm v$-jump must be non-zero. The argument of Section 3 of [8], with the three edge rule can be applied and work without the need for Section 11 of [8] because γ_0 may be chosen as one of the two (initially) large $\pm v$-jumps of the "Dirac mass" since these two follow each other with opposite orientations: since these two $\pm v$-jumps are so close, rearrangement can be completed by "pushing away" all the other $\pm v$-jumps from this pair and never "pushing" one of them away from these $(2k-1)$ other $\pm v$-jumps. In addition, the final rearrangement, with these two special $\pm v$-jumps finding their final position can be completed with the use of the "widening process", see [6], between them, so that their respective orientations is never altered in this process. It can only be altered by the fact that we "pushed away" the other $(2k-1) \pm v$-jumps from them. Repetitions are then preserved and the rearrangements around either choice are the same. A choice for the positive direction at a configuration among these can be completed in a compatible way over the switch of choices for γ_0 amongst these two $\pm v$-jumps, see Section 3 and Section 11 of [8]. It follows that these configurations are also moved down, past w_{2k} (some further precisions are needed and provided below, when the two large $\pm v$-jumps of the "Dirac mass" are not well-defined).

At specific times, "Dirac masses" cancel themselves topologically; that is a "Dirac mass" whose top level is critical among top levels, but is not a local maximum cancels with one whose top level is a maximum. Since we are requiring that the first species have all their base point in intervals of type E^+, we have to allow for the second species to cross over, at certain times, from E^- to E^+, before cancellation.

At these specific times along the deformation, such a "Dirac mass" is located at a node, moving from E^- into E^+ or vice-versa. It has at least two non-zero decreasing directions at infinity in Γ_4, one as all "Dirac masses" do have; it is related to their length; the other one is related to its top level.

Flow-lines out of w_{2k+1} in Γ_4 are of dimension 2. Adding the deformation parameter, we find a set of dimension 3. Such a set cannot dominate a critical point at infinity-such as the above "Dirac mass"-of index at infinity larger than or equal to 2 but at specific times along the deformation. We need to warrant that these specific times are not the times at which these "Dirac masses" cross E_0. This amounts to check that, as these "Dirac masses" cross E_0, we can still perturb, far away, near w_{2k+1}, the deformation and our flow-lines in Γ_4 so that they do not dominate these "Dirac masses".

The argument is standard as the value of the contact form near w_{2k+1} is very much independent from the v-rotation on the hyperbolic orbit w_{2k}. The claim follows.

5 More complicated configurations and the verification of the Fredholm assumption

The arguments provided above rule out the violation of the Fredholm assumption for configurations where the large $\pm v$-jumps of the "Dirac mass" are well-defined. We prove that the arguments extend to the general case:

Assuming that we are considering here a Morse relation as above between a w_{2k} and an elliptic orbit w_{2k-1} and assume that the positive or the negative edge of the "Dirac mass" over this periodic orbit is not well-defined, i.e. two or more than two $*$s define it or there is a tiny or zero $\pm v$-jump in between the two large edges of the "Dirac mass". We will consider the case of one zero $\pm v$-jump in between these two large $\pm v$-jumps. The other cases force the occurrence of more repetitions, at least two as the ξ-piece in between these two large $\pm v$-jumps is tiny and does not support any H_0^1-index.

Under such an occurrence, there is a forced repetition in between the $2k \pm v$-jumps of the configuration. Two $\pm v$-jumps to the least are non-zero and their orientations force the existence of a repetition in between them.

The use of any $*$ among the $2k$ available ones as a γ will not change this fact along such configurations (repetition and two non zero $\pm v$-jumps). They can be moved down with any such γ and this deformation convex-combines with the decreasing deformation centered at the positive or at the negative (now well defined) $\pm v$-jump of the "Dirac mass" used over the remainder of the set of configurations.

For a hyperbolic orbit w_{2k}, the configurations out of w_{2k+1} having one zero $\pm v$-jump in between the two large $\pm v$-jumps of the "Dirac mass" correspond to a stratified set T of top dimension $2k$ (in Γ_{4k}).

Along a deformation of contact forms, this stratified set might undergo tangencies with the stable manifold of a "Dirac mass" D, assuming that the dimension of this stable manifold is $2k$.

Outside of the two large $\pm v$-jumps of the "Dirac mass", $(2k-2) \pm v$-jumps are available and they can be used, see Section 11 of [8], to build either the stable or the unstable H_0^1-manifold of the "Dirac mass". They provide at most $(2k-2)$ unstable directions.

The missing two dimensions in the co-index are related to the curve formed in Γ_4 by the two large $\pm v$-jumps of the "Dirac mass". This co-index must then be 2 for D and therefore the index must be two as well. It follows that the index of D is $2k$ and $W_u(D)$ is achieved in Γ_{4k} (the additional missing $\pm v$-jump is zero).

Considering this "Dirac mass" in Γ_{4k+2}, that is adding the zero or the nearly zero $\pm v$-jump in between the two large $\pm v$-jumps of the "Dirac mass", we easily see that it does not provide any additional index since this ξ-piece is very small, tiny in between two large $\pm v$-jumps.

Therefore the total index of this "Dirac mass" in Γ_{4k+2} is also $2k$. The hyperbolic orbit itself has an index equal to $2k$ and has, see Section 11, sub-section on Hyperbolic Periodic Orbits of [8], an unstable manifold of dimension $2k$ in Γ_{4k+2}, $2k$ in Γ_{4k} as well.

As tangency takes place between T and $W_s(D)$, $W_u(D)$ lives in Γ_{4k} and is of dimension $2k$. $W_s(w_{2k}) \cap \Gamma_{4k}$ is of dimension $2k$. Therefore, tangency between $W_u(D)$ and $W_s(w_{2k})$ occurs (in Γ_{4k}) at special times that can be made different from the times at which tangency occurs between T and $W_s(D)$. The conclusion follows.

If instead of a tangency as above, we have a domination, then the dominated chain is of dimension $(2k-1)$ at most, $(2k-2)$ transversally to the flow and the arguments used in Section 11 of [8] work over transitions, switches in γs etc., see [8] for more details.

Observe that a "Dirac mass" of index $(2k+1)$ does not have the "Dirac Mass" D in its boundary. Indeed, the additional $\pm v$-jump that is zero for D cannot be outside the two large $\pm v$-jumps of the dominating "Dirac mass" over the flow-lines of the domination: being small, it can be "pushed away" from them and it will never "enter" between them.

If this $\pm v$-jump is one of two sizable parts of a large $\pm v$-jump of the "Dirac mass" D, then the flow-lines that come to this configuration in Γ_6 can be seen to come from a level much higher than the level of the periodic orbit w_{2k} and therefore, they do not come from a "Dirac mass" associated to this periodic orbit since the energy level of such a "Dirac mass" is very close to the energy level of the corresponding periodic orbit.

It remains to study the case when this additional $\pm v$-jump is also in between the two large $\pm v$-jumps of the dominating "Dirac mass". Computing the H_0^1-index of such a "Dirac mass", we find at most $(2k-2)$, since there are at most $(2k-2) \pm v$-jumps outside of the two large $\pm v$-jumps of the "Dirac mass". To reach $(2k+1)$, we would need that the critical point of Γ_4 corresponding to the two large $\pm v$-jumps of this "Dirac mass" to be of index 3 in Γ_4. This is different from D and therefore the two tangencies $W_u(w_{2k+1})$ with the stable manifold of the dominating "Dirac mass" on one hand and the tangency of T with the stable manifold of D do not occur at the same time by general position arguments.

Therefore, we may assume that this additional $\pm v$-jump is not in between the two large $\pm v$-jumps of this dominating "Dirac mass". To reach D, this $\pm v$-jump would have to travel along a decreasing flow-line and that is ruled out by the previous arguments.

Let us observe, to conclude our argument that we can rule out as follows the case of double-tangency, T with $W_s(D)$ on one hand and $W_u(w_{2k+1})$ with the stable manifold of a "Dirac mass" of index $(2k+1)$ occurring together, over the same process. Indeed, then, the additional $\pm v$-jump is in between the two large ones and therefore it does not provide any additional index so that it is not possible to have a double-tangency.

6 Outline for the removal of condition $(A)_t$

The arguments used in [7] and [8], use $(A)_t$ in one basic fact: the unstable manifold for a simple periodic orbit w_m is achieved in the space Γ_{2m}.

This holds true under the weaker assumption that $(A)_t$ is verified in a neighborhood of w_m and this, in turn, holds true-after rescaling the v-rotation along w_m using the techniques of [5]-if there is a globally defined v in $ker\alpha$ such that its total rotation around w_m in a ξ-transported frame is positive.

If this does not hold and the total rotation for any globally defined, non-singular, v in $ker\alpha$ is negative, then one might have to modify the functional and use $-\int_0^1 \alpha(\dot{x})dt$ near w_m or use the same functional, but increase it instead of decreasing it. This would require further work.

However, under the assumption that $(A)_t$ is verified near w_m, the arguments of [7,8] can be carried out with one additional difficulty: the spaces C_{β_t} and the Γ_{2m}^t might have singularities. We have already understood the location of these singularities [4], p 19 for C_β, not yet for the Γ_{2m}s. However, we have not yet built a decreasing deformation for our variational problem through these singularities.

Would this be achieved, the homology would extend under a quite weakened version of $(A)_t$. This condition might be entirely removed after a modification of the functional, starting as indicated above, but this would require more work.

It is worth mentioning here that we can always assume that, at a given periodic orbit, the rotation of v is monotone, either positive or negative. Accordingly, one finds two Morse complexes; one for the positive rotation, as above and [8]. The other one is related to the functional $-J(x) = -\int_0^1 \alpha_x(\dot{x})$ and to the periodic orbits of the second type. The two Morse complexes are, by an argument of general position, independent of each other. Along this line, the present results and the results of [8] can be generalized.

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