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Qualitative behavior of solutions of difference equations with several oscillating coefficients

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Abstract Sufficient conditions which guarantee the convergence of the nonoscillatory solutions or oscillation of all solutions of a difference equation with several deviating arguments and oscillating coefficients are presented. Corresponding difference equations of both retarded and advanced type are studied. Examples illustrating the results are also given.

Mathematics Subject Classification 39A10 · 39A21

الملخص

يتم عرض شروط كافية تضمن تقارب الحلول غير المتذبذبة أو تذبذب جميع حلول معادلة فرْقية في عدة متغيرات منحرفة ومعاملات متذبذبة. تمت دراسة المعادلات الفرقية المناظرة من النوعين المتخلف والمتقدم. تم أيضاً إعطاء أمثلة ترسخ النتائج.

1 Introduction

In this paper, we study the convergence and oscillation of all solutions of the retarded difference equation of the form

$$\Delta x(n) + \sum_{i=1}^{m} p_i(n) x(\tau_i(n)) = 0, \quad n \in \mathbb{N}_0, \tag{E$_\mathrm{R}$}$$

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and the (dual) advanced difference equation of the form

$$\nabla x(n) - \sum_{i=1}^{m} p_i(n)x(\sigma_i(n)) = 0, \quad n \in \mathbb{N}, \tag{$\mathrm{E_A}$}$$

where $m \in \mathbb{N}$, p_i, $1 \le i \le m$, are oscillating sequences of real numbers, $\{\tau_i(n)\}_{n\in\mathbb{N}_0}$, $1 \le i \le m$, are sequences of integers such that

$$\tau_i(n) \le n-1, \quad n \in \mathbb{N}_0 \quad \text{and} \quad \lim_{n\to\infty} \tau_i(n) = \infty, \quad 1 \le i \le m, \tag{1.1}$$

$\{\sigma_i(n)\}_{n\in\mathbb{N}}$, $1 \le i \le m$, are sequences of integers such that

$$\sigma_i(n) \ge n+1, \quad n \in \mathbb{N}, \quad 1 \le i \le m, \tag{1.2}$$

Δ denotes the forward difference operator $\Delta x(n) = x(n+1) - x(n)$ and ∇ denotes the backward difference operator $\nabla x(n) = x(n) - x(n-1)$.

In the last few decades, the asymptotic and oscillatory behavior of all solutions of difference equations has been extensively studied when the coefficients $p_i(n)$ are nonnegative. However, for the general case when $p_i(n)$ are allowed to oscillate, it is difficult to study the oscillation of ($\mathrm{E_R}$) and ($\mathrm{E_A}$), since the differences $\Delta x(n)$ and $\nabla x(n)$ of any nonoscillatory solution of ($\mathrm{E_R}$) or ($\mathrm{E_A}$) are always oscillatory. Therefore, the results on convergence and oscillation of difference and differential equations with oscillating coefficients are relatively scarce. Thus, a small number of papers are dealing with this case. See, for example, [2–4,6–8,10–19] and the references cited therein. For the general theory of difference equations, the reader is referred to the monographs [1,5,9].

By a *solution* of the retarded difference equation ($\mathrm{E_R}$), we mean a sequence of real numbers $\{x(n)\}_{n\ge -w}$ which satisfies ($\mathrm{E_R}$) for all $n \in \mathbb{N}_0$. Here,

$$w := -\min_{\substack{n\in\mathbb{N}_0\\ 1\le i\le m}} \tau_i(n) \in \mathbb{N}_0.$$

It is clear that, for each choice of real numbers c_{-w}, c_{-w+1}, …, c_{-1}, c_0, there exists a unique solution $\{x(n)\}_{n\ge -w}$ of ($\mathrm{E_R}$) which satisfies the initial conditions $x(-w) = c_{-w}$, $x(-w+1) = c_{-w+1}, \ldots, x(-1) = c_{-1}$, $x(0) = c_0$. By a *solution* of the advanced difference equation ($\mathrm{E_A}$), we mean a sequence of real numbers $\{x(n)\}_{n\in\mathbb{N}_0}$ which satisfies ($\mathrm{E_A}$) for all $n \in \mathbb{N}$.

A solution $\{x(n)\}_{n\ge -w}$ (or $\{x(n)\}_{n\in\mathbb{N}_0}$) of the difference equation ($\mathrm{E_R}$) (or ($\mathrm{E_A}$)) is called *oscillatory* if the terms $x(n)$ of the sequence are neither eventually positive nor eventually negative. Otherwise, the solution is said to be *nonoscillatory*.

Strong interest in ($\mathrm{E_R}$) with several variable retarded arguments is motivated by the fact that it represents a discrete analogue of the differential equation with several variable retarded arguments (see [4] and the references cited therein)

$$x'(t) + \sum_{i=1}^{m} p_i(t)x(\tau_i(t)) = 0, \quad t \ge 0,$$

where for every $i \in \{1, \ldots, m\}$, p_i is an oscillating continuous real-valued function in the interval $[0, \infty)$, and τ_i is a continuous real-valued function on $[0, \infty)$ such that

$$\tau_i(t) \le t, \quad t \ge 0 \quad \text{and} \quad \lim_{t\to\infty} \tau_i(t) = \infty,$$

while ($\mathrm{E_A}$) represents a discrete analogue of the advanced differential equation (see [4] and the references cited therein)

$$x'(t) - \sum_{i=1}^{m} p_i(t)x(\sigma_i(t)) = 0, \quad t \ge 1,$$

where, for every $i \in \{1, \ldots, m\}$, p_i is an oscillating continuous real-valued function in the interval $[1, \infty)$, and σ_i is a continuous real-valued function on $[1, \infty)$ such that

$$\sigma_i(t) \ge t, \quad t \ge 1.$$

For $m = 1$, (E_R) and (E_A) take the forms

$$\Delta x(n) + p(n)x(\tau(n)) = 0, \quad n \in \mathbb{N}_0$$

and

$$\nabla x(n) - p(n)x(\sigma(n)) = 0, \quad n \in \mathbb{N},$$

respectively, and they represent the discrete analogues of the differential equations (see [4] and the references cited therein)

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq 0 \tag{1.3}$$

and

$$x'(t) - p(t)x(\sigma(t)) = 0, \quad t \geq 1, \tag{1.4}$$

respectively, where $\tau(t) \leq t$, $\sigma(t) \geq t$, and the coefficient p is a continuous function which is allowed to oscillate. In the case of $m = 1$ and $\tau_1(n) = n - k$, $\sigma_1(n) = n + k$, (E_R) and (E_A) take the forms

$$\Delta x(n) + p(n)x(n-k) = 0, \quad n \in \mathbb{N}_0 \tag{1.5}$$

and

$$\nabla x(n) - p(n)x(n+k) = 0, \quad n \in \mathbb{N},$$

respectively. These equations represent the discrete analogues of the differential equations (see [8,13] and the references cited therein)

$$x'(t) + p(t)x(t-\tau) = 0, \quad t \geq 0 \tag{1.6}$$

and

$$x'(t) - p(t)x(t+\sigma) = 0, \quad t \geq 1, \tag{1.7}$$

respectively, where τ and σ are positive constants and the coefficient p is a continuous function which is allowed to oscillate.

In 1982, Ladas et al. [8] studied the differential equations (1.6) and (1.7) with constant arguments and established the following theorems.

Theorem 1.1 (See [8, Theorem 2.1]) *Assume that* $p(t) > 0$ *on a sequence of disjoint intervals* $\bigcup_{n\in\mathbb{N}} (\xi(n), t(n))$ *with* $t(n) - \xi(n) = 2\tau$. *If*

$$\limsup_{n\to\infty} \int_{t(n)-\tau}^{t(n)} p(s)\mathrm{d}s > 1,$$

then all solutions of (1.6) *oscillate.*

Theorem 1.2 (See [8, Theorem 2.1]) *Assume that* $p(t) > 0$ *on a sequence of disjoint intervals* $\bigcup_{n\in\mathbb{N}} (\xi(n), t(n))$ *with* $t(n) - \xi(n) = 2\sigma$. *If*

$$\limsup_{n\to\infty} \int_{\xi(n)}^{\xi(n)+\sigma} p(s)\mathrm{d}s > 1,$$

then all solutions of (1.7) *oscillate.*

In 1984, Fukagai and Kusano [4] extended the above results to the differential equations (1.3) and (1.4) as follows.

Theorem 1.3 (See [4, Theorem 4 (i)]) *Assume that* $\tau(t) \le t$ *for* $t \ge 0$. *If there exists a sequence of numbers* $\{t(n)\}_{n\in\mathbb{N}}$ *such that* $\lim_{n\to\infty} t(n) = \infty$, *the intervals* $\bigcup_{n\in\mathbb{N}} [\tau(\tau(t(n))), t(n)]$ *are disjoint,*

$$p(t) \ge 0 \quad \text{for all } t \in \bigcup_{n\in\mathbb{N}} [\tau(\tau(t(n))), t(n)]$$

and

$$\int_{\tau(t(n))}^{t(n)} p(s)\mathrm{d}s \ge 1,$$

then all solutions of (1.3) *oscillate.*

Theorem 1.4 (See [4, Theorem 4 (ii)]) *Assume that* $\sigma(t) \ge t$ *for* $t \ge 1$. *If there exists a sequence of numbers* $\{t(n)\}_{n\in\mathbb{N}}$ *such that* $\lim_{n\to\infty} t(n) = \infty$, *the intervals* $\bigcup_{n\in\mathbb{N}} [t(n), \sigma(\sigma(t(n)))]$ *are disjoint,*

$$p(t) \ge 0 \quad \text{for all } t \in \bigcup_{n\in\mathbb{N}} [t(n), \sigma(\sigma(t(n)))]$$

and

$$\int_{t(n)}^{\sigma(t(n))} p(s)\mathrm{d}s \ge 1,$$

then all solutions of (1.4) *oscillate.*

In 1992, Qian et al. [11] studied the differential equation (1.5) with constant retarded argument and established the following theorem.

Theorem 1.5 (See [11, Theorem 1]) *Assume that there exist two sequences* $\{r(m)\}$ *and* $\{s(m)\}$ *of positive integers such that* $s(m) - r(m) \ge 2k$ *for* $m \in \mathbb{N}$. *If*

$$p(n) \ge 0 \quad \text{for all } n \in \bigcup_{m\in\mathbb{N}} \{r(m), r(m)+1, \ldots, s(m)\}$$

and

$$\limsup_{m\to\infty} \sum_{n=s(m)-k}^{s(m)} p^+(n) > 1,$$

where $p^+(n) = \max\{p(n), 0\}$, *then all solutions of* (1.5) *oscillate.*

In this paper, we study the difference equations ($\mathrm{E_R}$) and ($\mathrm{E_A}$) with several variable (retarded or advanced) arguments and oscillating coefficients and establish sufficient conditions for the convergence and oscillation of all solutions of these equations. We also provide examples to illustrate the results derived in this paper. Throughout, we use the notation

$$\alpha^+ = \max\{\alpha, 0\} \quad \text{and} \quad \alpha^- = \max\{-\alpha, 0\},$$

where $\alpha \in \mathbb{R}$, and note that clearly

$$\alpha^+, \alpha^- \ge 0 \quad \text{and} \quad \alpha = \alpha^+ - \alpha^-. \tag{1.8}$$

2 Retarded equations

Convergence of all nonoscillatory solutions of (E_R) is described by the following result.

Theorem 2.1 *If there exists $n_0 \in \mathbb{N}$ such that*

$$\sum_{i=1}^{m} \sum_{n=n_0}^{\infty} p_i^-(n) < \infty, \tag{2.1}$$

then every nonoscillatory solution of (E_R) *tends to a finite limit, and this limit is zero provided*

$$\sum_{i=1}^{m} \sum_{n=n_0}^{\infty} p_i^+(n) = \infty. \tag{2.2}$$

Proof Assume that the solution $\{x(n)\}_{n \geq -w}$ of (E_R) is nonoscillatory. Then it is either eventually positive or eventually negative. As $\{-x(n)\}_{n \geq -w}$ is also a solution of (E_R), we may restrict ourselves to the case where $x(n) > 0$ for all large n. Let $n_1 \geq -w$ be an integer such that $x(n) > 0$ for all $n \geq n_1$. Then there exists $n_2 \geq n_1$ such that

$$x(\tau_i(n)) > 0, \quad n \geq n_2, \quad 1 \leq i \leq m.$$

Since (2.1) holds, there exists $n_3 \geq n_2$ such that

$$\alpha = \sum_{i=1}^{m} \sum_{n=n_3}^{\infty} p_i^-(n) < 1. \tag{2.3}$$

First we show that $\{x(n)\}$ is bounded. Assume, for the sake of contradiction, that $\{x(n)\}$ is unbounded. Then there exists a subsequence $\{x(\varphi(n))\}$ of $\{x(n)\}$ such that

$$\lim_{n\to\infty} x(\varphi(n)) = \infty \quad \text{and} \quad x(\varphi(n)) = \max\left\{x(k) : \ k \leq \varphi(n)\right\}. \tag{2.4}$$

In view of (1.8), (E_R) shows that

$$x(n+1) - x(n) = -\sum_{i=1}^{m} p_i(n) x(\tau_i(n)) = \sum_{i=1}^{m} \left[p_i^-(n) - p_i^+(n)\right] x(\tau_i(n)), \tag{2.5}$$

and thus

$$x(n+1) - x(n) \leq \sum_{i=1}^{m} p_i^-(n) x(\tau_i(n)) \tag{2.6}$$

for all $n \geq n_3$. Summing up (2.6) from n_3 to $\varphi(n) - 1$ and taking into account (2.3) and (2.4), for sufficiently large n, we obtain

$$\begin{aligned} x(\varphi(n)) - x(n_3) &\leq \sum_{i=1}^{m} \sum_{j=n_3}^{\varphi(n)-1} p_i^-(j) x(\tau_i(j)) \\ &\leq \left(\sum_{i=1}^{m} \sum_{j=n_3}^{\varphi(n)-1} p_i^-(j) \right) x(\varphi(n)) \\ &\leq \alpha x(\varphi(n)), \end{aligned}$$

i.e.,

$$x(\varphi(n)) \leq \frac{x(n_3)}{1-\alpha},$$

which means that $\{x(\varphi(n))\}$ is bounded. This contradicts (2.4). Therefore $\{x(n)\}$ is bounded. Hence there exists $L > 0$ such that $0 < x(n) \leq L$ for all $n \in \mathbb{N}_0$.

Now we show that $\lim_{n\to\infty} x(n)$ exists. Indeed, summing up (2.5) from n_3 to $n \geq n_3$, we obtain

$$\begin{aligned}\sum_{i=1}^{m}\sum_{j=n_3}^{n} p_i^+(j)x(\tau_i(j)) &= \sum_{i=1}^{m}\sum_{j=n_3}^{n} p_i^-(j)x(\tau_i(j)) + \sum_{j=n_3}^{n}(x(j)-x(j+1))\\ &= \sum_{i=1}^{m}\sum_{j=n_3}^{n} p_i^-(j)x(\tau_i(j)) + x(n_3) - x(n+1)\\ &\leq L\sum_{i=1}^{m}\sum_{j=n_3}^{\infty} p_i^-(j) + x(n_3) - x(n+1)\\ &< \alpha L + x(n_3)\end{aligned}$$

so that

$$\sum_{i=1}^{m}\sum_{j=n_3}^{\infty} p_i^+(j)x(\tau_i(j)) < \infty.$$

Therefore, by (2.5), we see that

$$\sum_{j=n_3}^{\infty}(x(j+1)-x(j)) = \sum_{i=1}^{m}\sum_{j=n_3}^{\infty} p_i^-(j)x(\tau_i(j)) - \sum_{i=1}^{m}\sum_{j=n_3}^{\infty} p_i^+(j)x(\tau_i(j))$$

exists, that is,

$$\lim_{n\to\infty} x(n) = x(n_3) + \sum_{i=1}^{m}\sum_{j=n_3}^{\infty} p_i^-(j)x(\tau_i(j)) - \sum_{i=1}^{m}\sum_{j=n_3}^{\infty} p_i^+(j)x(\tau_i(j))$$

exists and is finite.

If, additionally, (2.2) holds, then $\lim_{n\to\infty} x(n) = 0$. Indeed, assume, for the sake of contradiction, that $\lim_{n\to\infty} x(n) > 0$. Hence

$$\inf_{n\in\mathbb{N}_0} x(n) = d > 0.$$

Summing up (2.5) from n_3 to $n \geq n_3$, we obtain

$$\begin{aligned}x(n+1) - x(n_3) &= \sum_{i=1}^{m}\sum_{j=n_3}^{n} p_i^-(j)x(\tau_i(j)) - \sum_{i=1}^{m}\sum_{j=n_3}^{n} p_i^+(j)x(\tau_i(j))\\ &\leq \alpha L - d\sum_{i=1}^{m}\sum_{j=n_3}^{n} p_i^+(j),\end{aligned}$$

which, in view of (2.2), means that $\lim_{n\to\infty} x(n) = -\infty$, a contradiction. Therefore $\lim_{n\to\infty} x(n) = 0$. □

Remark 2.2 (2.2) is equivalent to the existence of $i \in \{1, \ldots, m\}$ such that

$$\sum_{n=n_0}^{\infty} p_i^+(n) = \infty.$$

The following result is an immediate corollary of Theorem 2.1.

Corollary 2.3 *If* (2.1) *holds, then every unbounded solution of* (E_R) *oscillates.*

Now we present a new sufficient condition for the oscillation of all solutions of (E_R), under the assumption that τ_i are increasing for all $i \in \{1, \ldots, m\}$.

Theorem 2.4 *Assume* (1.1) *and that* τ_i *are increasing for all* $i \in \{1, \dots, m\}$. *Suppose also that for each* $i \in \{1, \dots, m\}$, *there exists* $\{n_i(j)\}_{j\in\mathbb{N}} \subset \mathbb{N}$ *such that* $\lim_{j\to\infty} n_i(j) = \infty$ *and*

$$p_k(n) \geq 0 \quad \textit{for all } n \in \bigcap_{i=1}^{m} \left\{ \bigcup_{j\in\mathbb{N}} [\tau(\tau(n_i(j))), n_i(j)] \cap \mathbb{N} \right\} \neq \emptyset, \quad 1 \leq k \leq m, \tag{2.7}$$

where

$$\tau(n) = \max_{1\leq i\leq m} \tau_i(n), \quad n \in \mathbb{N}_0. \tag{2.8}$$

If, moreover

$$\limsup_{j\to\infty} \sum_{i=1}^{m} \sum_{q=\tau(n(j))}^{n(j)} p_i(q) > 1, \tag{2.9}$$

where $n(j) = \min\{n_i(j) : 1 \leq i \leq m\}$, *then all solutions of* (E_R) *oscillate.*

Proof Assume, for the sake of contradiction, that $\{x(n)\}_{n\geq -w}$ is an eventually positive solution of (E_R). Then, in view of (2.7) and (2.9), it is clear that there exists $j_0 \in \mathbb{N}$ such that

$$p_k(n) \geq 0 \quad \textit{for all } n \in \bigcap_{i=1}^{m} [\tau(\tau(n_i(j_0))), n_i(j_0)] \cap \mathbb{N}, \quad 1 \leq k \leq m, \tag{2.10}$$

$$x(\tau_k(n)) > 0 \quad \textit{for all } n \in \bigcap_{i=1}^{m} [\tau(\tau(n_i(j_0))), n_i(j_0)] \cap \mathbb{N}, \quad 1 \leq k \leq m \tag{2.11}$$

and

$$\sum_{i=1}^{m} \sum_{q=\tau(n(j_0))}^{n(j_0)} p_i(q) > 1. \tag{2.12}$$

In view of (2.10) and (2.11), (E_R) gives

$$x(n+1) - x(n) = -\sum_{i=1}^{m} p_i(n)x(\tau_i(n)) \leq 0$$

for every $n \in \bigcap_{i=1}^{m} [\tau(\tau(n_i(j_0))), n_i(j_0)] \cap \mathbb{N}$. This guarantees that the sequence x is decreasing on $\bigcap_{i=1}^{m} [\tau(\tau(n_i(j_0))), n_i(j_0)] \cap \mathbb{N}$.

Now, summing up (E_R) from $\tau(n(j_0))$ to $n(j_0)$ and taking into account (2.12) and that τ is increasing and x is decreasing, we obtain

$$\begin{aligned}
0 &= x(n(j_0)+1) - x(\tau(n(j_0))) + \sum_{i=1}^{m} \sum_{q=\tau(n(j_0))}^{n(j_0)} p_i(q)x(\tau_i(q)) \\
&\geq x(n(j_0)+1) - x(\tau(n(j_0))) + \sum_{i=1}^{m} \sum_{q=\tau(n(j_0))}^{n(j_0)} p_i(q)x(\tau(q)) \\
&\geq x(n(j_0)+1) - x(\tau(n(j_0))) + \left[\sum_{i=1}^{m} \sum_{q=\tau(n(j_0))}^{n(j_0)} p_i(q) \right] x(\tau(n(j_0))) \\
&= x(n(j_0)+1) + \left[\sum_{i=1}^{m} \sum_{q=\tau(n(j_0))}^{n(j_0)} p_i(q) - 1 \right] x(\tau(n(j_0))) \\
&> x(n(j_0)+1),
\end{aligned}$$

which is a contradiction. □

A slight modification in the proof of Theorem 2.4 leads to the following result about retarded difference inequalities.

Theorem 2.5 *Assume that all conditions of Theorem 2.4 hold. Then there are*

(i) *no eventually positive solutions of the difference inequality*

$$\Delta x(n) + \sum_{i=1}^{m} p_i(n) x(\tau_i(n)) \leq 0, \quad n \in \mathbb{N}_0;$$

(ii) *no eventually negative solutions of the difference inequality*

$$\Delta x(n) + \sum_{i=1}^{m} p_i(n) x(\tau_i(n)) \geq 0, \quad n \in \mathbb{N}_0.$$

3 Advanced equations

Convergence of all nonoscillatory solutions of $(\mathrm{E_A})$ is described by the following result.

Theorem 3.1 *If there exists $n_0 \in \mathbb{N}$ such that*

$$\sum_{i=1}^{m} \sum_{n=n_0}^{\infty} p_i^{+}(n) < \infty, \tag{3.1}$$

then every nonoscillatory solution of $(\mathrm{E_A})$ tends to a finite limit, and this limit is zero provided

$$\sum_{i=1}^{m} \sum_{n=n_0}^{\infty} p_i^{-}(n) = \infty. \tag{3.2}$$

Proof The proof is an easy modification of the proof of Theorem 2.1 and hence is omitted. □

Remark 3.2 (3.2) is equivalent to the existence of $i \in \{1, \ldots, m\}$ such that

$$\sum_{n=n_0}^{\infty} p_i^{-}(n) = \infty.$$

The following result is an immediate corollary of Theorem 3.1.

Corollary 3.3 *If (3.1) holds, then every unbounded solution of $(\mathrm{E_A})$ oscillates.*

Now we present a new sufficient condition for the oscillation of all solutions of $(\mathrm{E_A})$, under the assumption that σ_i is increasing for all $i \in \{1, \ldots, m\}$.

Theorem 3.4 *Assume (1.2) and that σ_i is increasing for all $i \in \{1, \ldots, m\}$. Suppose also that for each $i \in \{1, \ldots, m\}$, there exists $\{n_i(j)\}_{j \in \mathbb{N}} \subset \mathbb{N}$ such that $\lim_{j \to \infty} n_i(j) = \infty$ and*

$$p_k(n) \geq 0 \quad \textit{for all } n \in \bigcap_{i=1}^{m} \left\{ \bigcup_{j \in \mathbb{N}} [n_i(j), \sigma(\sigma(n_i(j)))] \cap \mathbb{N} \right\} \neq \emptyset, \quad 1 \leq k \leq m, \tag{3.3}$$

where

$$\sigma(n) = \min_{1 \leq i \leq m} \sigma_i(n), \quad n \in \mathbb{N}. \tag{3.4}$$

If, moreover

$$\limsup_{j \to \infty} \sum_{i=1}^{m} \sum_{q=n(j)}^{\sigma(n(j))} p_i(q) > 1, \tag{3.5}$$

where $n(j) = \max\{n_i(j) : 1 \leq i \leq m\}$, then all solutions of $(\mathrm{E_A})$ oscillate.

Proof Assume, for the sake of contradiction, that x is an eventually positive solution of (E_A). Then, in view of (3.3) and (3.5), it is clear that there exists $j_0 \in \mathbb{N}$ such that

$$p_k(n) \geq 0 \quad \text{for all } n \in \bigcap_{i=1}^{m} [n_i(j_0), \sigma(\sigma(n_i(j_0)))] \cap \mathbb{N}, \quad 1 \leq k \leq m, \tag{3.6}$$

$$x(\sigma_k(n)) > 0 \quad \text{for all } n \in \bigcap_{i=1}^{m} [n_i(j_0), \sigma(\sigma(n_i(j_0)))] \cap \mathbb{N}, \quad 1 \leq k \leq m \tag{3.7}$$

and

$$\sum_{i=1}^{m} \sum_{q=n(j_0)}^{\sigma(n(j_0))} p_i(q) > 1. \tag{3.8}$$

In view of (3.6) and (3.7), (E_A) gives

$$x(n) - x(n-1) = \sum_{i=1}^{m} p_i(n) x(\sigma_i(n)) \geq 0$$

for every $n \in \bigcap_{i=1}^{m} [n_i(j_0), \sigma(\sigma(n_i(j_0)))] \cap \mathbb{N}$. This guarantees that the sequence x is increasing on $\bigcap_{i=1}^{m} [n_i(j_0), \sigma(\sigma(n_i(j_0)))] \cap \mathbb{N}$.

Now, summing up (E_A) from $n(j_0)$ to $\sigma(n(j_0))$ and taking into account (3.8) and that both σ and x are increasing, we obtain

$$\begin{aligned}
0 &= x(n(j_0) - 1) - x(\sigma(n(j_0))) + \sum_{i=1}^{m} \sum_{q=n(j_0)}^{\sigma(n(j_0))} p_i(q) x(\sigma_i(q)) \\
&\geq x(n(j_0) - 1) - x(\sigma(n(j_0))) + \sum_{i=1}^{m} \sum_{q=n(j_0)}^{\sigma(n(j_0))} p_i(q) x(\sigma(q)) \\
&\geq x(n(j_0) - 1) - x(\sigma(n(j_0))) + \left[\sum_{i=1}^{m} \sum_{q=n(j_0)}^{\sigma(n(j_0))} p_i(q) \right] x(\sigma(n(j_0))) \\
&= x(n(j_0) - 1) + \left[\sum_{i=1}^{m} \sum_{q=n(j_0)}^{\sigma(n(j_0))} p_i(q) - 1 \right] x(\sigma(n(j_0))) \\
&> x(n(j_0) - 1),
\end{aligned}$$

which is a contradiction. □

A slight modification in the proof of Theorem 3.4 leads to the following result about advanced difference inequalities.

Theorem 3.5 *Assume that all conditions of Theorem 3.4 hold. Then there are*

(*i*) *no eventually positive solutions of the difference inequality*

$$\nabla x(n) - \sum_{i=1}^{m} p_i(n) x(\sigma_i(n)) \geq 0, \quad n \in \mathbb{N};$$

(*ii*) *no eventually negative solutions of the difference inequality*

$$\nabla x(n) - \sum_{i=1}^{m} p_i(n) x(\sigma_i(n)) \leq 0, \quad n \in \mathbb{N}.$$

4 Examples

We give four examples illustrating the four main results of this paper.

Example 4.1 Consider the retarded difference equation

$$\Delta x(n) + p_1(n)x(n-1) + p_2(n)x(n-2) = 0, \quad n \geq 3, \tag{4.1}$$

where

$$p_1(n) = \left(-\frac{1}{2}\right)^n \quad \text{and} \quad p_2(n) = -\frac{\Delta\alpha(n) + p_1(n)\alpha(n-1)}{\alpha(n-2)}, \quad n \geq 3$$

and

$$\alpha(1) = 1 \quad \text{and} \quad \alpha(n+1) = \alpha(n) + \left(-\frac{1}{3}\right)^n, \quad n \in \mathbb{N}.$$

Observe that

$$\frac{1}{2} \leq \alpha(n) = \frac{3}{4} + \frac{1}{4}\left(-\frac{1}{3}\right)^{n-1} \leq 1 \quad \text{for all } n \in \mathbb{N},$$

$$p_1(n) = (-1)^n \cdot \frac{1}{2^n}, \quad n \geq 3$$

is negative for odd n and positive for even n, while

$$p_2(n) = \frac{\frac{4}{27} + \frac{1}{3\cdot 2^n}\cdot\left(3^{n-1} + (-1)^n\right)}{1-(-3)^{n-2}}, \quad n \geq 3$$

is positive for odd n and negative for even n. Hence

$$\begin{aligned}\sum_{i=1}^{2}\sum_{n=3}^{\infty} p_i^-(n) &= \sum_{k=1}^{\infty}(-p_1(2k+1)) + \sum_{k=2}^{\infty}(-p_2(2k)) \\ &= \sum_{k=1}^{\infty}\frac{1}{2^{2k+1}} + \sum_{k=2}^{\infty}\frac{\frac{1}{3^{2k}} + \frac{1}{2^{2k}}\cdot\alpha(2k-1)}{\alpha(2k-2)} \\ &\leq \sum_{k=1}^{\infty}\frac{1}{2^{2k+1}} + \sum_{k=2}^{\infty}\frac{\frac{1}{3^{2k}} + \frac{1}{2^{2k}}}{\frac{1}{2}} = \frac{13}{36} < \infty,\end{aligned}$$

which means that (2.1) of Theorem 2.1 is satisfied. Hence, every nonoscillatory solution of (4.1) tends to a (finite) limit. In fact, α is one such solution since it satisfies (4.1) for all $n \geq 3$ and $\lim_{n\to\infty}\alpha(n) = 3/4$.

Example 4.2 Consider the retarded difference equation

$$\Delta x(n) + p_1(n)x(n-2) + p_2(n)x(n-3) = 0, \quad n \in \mathbb{N}_0, \tag{4.2}$$

where

$$p_1(n) = \cos\frac{n\pi}{4} \quad \text{and} \quad p_2(n) = \sin\frac{n\pi}{4}, \quad n \in \mathbb{N}_0.$$

In view of (2.8), it is obvious that $\tau(n) = n-2$. Observe that for

$$n_1(j) = 8j + 10, \quad j \in \mathbb{N},$$

we have $p_1(n) = \cos\frac{n\pi}{4} \geq 0$ for every $n \in A$, where

$$A = \bigcup_{j\in\mathbb{N}}[\tau(\tau(n_1(j))), n_1(j)] \cap \mathbb{N} = \bigcup_{j\in\mathbb{N}}[8j+6, 8j+10] \cap \mathbb{N}.$$

Also, for

$$n_2(j) = 8j + 12, \quad j \in \mathbb{N},$$

we have $p_2(n) = \sin \frac{n\pi}{4} \geq 0$ for every $n \in B$, where

$$B = \bigcup_{j \in \mathbb{N}} [\tau(\tau(n_2(j))), n_2(j)] \cap \mathbb{N} = \bigcup_{j \in \mathbb{N}} [8j + 8, 8j + 12] \cap \mathbb{N}.$$

Therefore

$$p_1(n) \geq 0 \quad \text{and} \quad p_2(n) \geq 0 \quad \text{for all } n \in A \cap B = \bigcup_{j \in \mathbb{N}} [8j + 8, 8j + 10] \cap \mathbb{N}.$$

Observe that

$$n(j) = \min \{n_i(j) : \ 1 \leq i \leq 2\} = 8j + 10, \quad j \in \mathbb{N}.$$

Now,

$$\begin{aligned}
\limsup_{j \to \infty} \sum_{i=1}^{2} \sum_{q=\tau(n(j))}^{n(j)} p_i(q) &= \limsup_{j \to \infty} \left[\sum_{q=8j+8}^{8j+10} p_1(q) + \sum_{q=8j+8}^{8j+10} p_2(q) \right] \\
&= \cos(2\pi) + \cos \frac{9\pi}{4} + \cos \frac{5\pi}{2} + \sin(2\pi) + \sin \frac{9\pi}{4} + \sin \frac{5\pi}{2} \\
&= 2 + \sqrt{2} > 1,
\end{aligned}$$

that is, (2.9) of Theorem 2.4 is satisfied, and therefore all solutions of (4.2) oscillate.

Example 4.3 Consider the advanced difference equation

$$\nabla x(n) - p_1(n)x(n^2 + 1) - p_2(n)x(2n) = 0, \quad n \in \mathbb{N}, \tag{4.3}$$

where

$$p_1(n) = \left(-\frac{1}{4}\right)^n \quad \text{and} \quad p_2(n) = \frac{\nabla \alpha(n) - p_1(n)\alpha(n^2 + 1)}{\alpha(2n)}, \quad n \in \mathbb{N}$$

and

$$\alpha(1) = 2 \quad \text{and} \quad \alpha(n) = \alpha(n - 1) - \left(-\frac{1}{2}\right)^n, \quad n \geq 2.$$

Observe that

$$\begin{gathered}
\frac{3}{2} \leq \alpha(n) = \frac{11}{6} - \frac{1}{3}\left(-\frac{1}{2}\right)^n \leq 2 \quad \text{for all } n \in \mathbb{N}, \\
p_1(n) = (-1)^n \cdot \frac{1}{4^n}, \quad n \in \mathbb{N}
\end{gathered}$$

is negative for odd n and positive for even n, while

$$p_2(n) = (-1)^{n+1} \frac{6 \cdot 2^n + 11 + \left(-\frac{1}{2}\right)^{n^2}}{11 \cdot 4^n - 2}, \quad n \in \mathbb{N}$$

is positive for odd n and negative for even n. Hence

$$\begin{aligned}\sum_{i=1}^{2}\sum_{n=1}^{\infty} p_i^{+}(n) &= \sum_{k=1}^{\infty} p_1(2k) + \sum_{k=0}^{\infty} p_2(2k+1)\\ &= \sum_{k=1}^{\infty}\frac{1}{4^{2k}} + \sum_{k=0}^{\infty}\frac{\frac{1}{4^{2k+1}}\cdot\alpha(4k^2+4k+2)+\frac{1}{2^{2k+1}}}{\alpha(4k+2)}\\ &\le \sum_{k=1}^{\infty}\frac{1}{4^{2k}} + \sum_{k=0}^{\infty}\frac{\frac{1}{4^{2k+1}}\cdot 2+\frac{1}{2^{2k+1}}}{\frac{3}{2}} = \frac{13}{15} < \infty,\end{aligned}$$

which means that (3.1) of Theorem 3.1 is satisfied. Hence, every nonoscillatory solution of (4.3) tends to a (finite) limit. In fact, α is one such solution since it satisfies (4.3) for all $n \in \mathbb{N}$ and $\lim_{n\to\infty}\alpha(n) = 11/6$.

Example 4.4 Consider the advanced difference equation

$$\nabla x(n) - p_1(n)x(n+1) - p_2(n)x(n+3) = 0, \quad n \in \mathbb{N}, \tag{4.4}$$

where

$$p_1(n) = \cos\frac{n\pi}{4} \quad \text{and} \quad p_2(n) = \sin\frac{n\pi}{4}, \quad n \in \mathbb{N}.$$

In view of (3.4), it is obvious that $\sigma(n) = n+1$. Observe that for

$$n_1(j) = 8j+8, \quad j \in \mathbb{N},$$

we have $p_1(n) = \cos\frac{n\pi}{4} \ge 0$ for every $n \in A$, where

$$A = \bigcup_{j\in\mathbb{N}} [n_1(j), \sigma(\sigma(n_1(j)))] \cap \mathbb{N} = \bigcup_{j\in\mathbb{N}} [8j+8, 8j+10] \cap \mathbb{N}.$$

Also, for

$$n_2(j) = 8j+9, \quad j \in \mathbb{N},$$

we have $p_2(n) = \sin\frac{n\pi}{4} \ge 0$ for every $n \in B$, where

$$B = \bigcup_{j\in\mathbb{N}} [n_2(j), \sigma(\sigma(n_2(j)))] \cap \mathbb{N} = \bigcup_{j\in\mathbb{N}} [8j+9, 8j+11] \cap \mathbb{N}.$$

Therefore

$$p_1(n) \ge 0 \quad \text{and} \quad p_2(n) \ge 0 \quad \text{for all } n \in A \cap B = \bigcup_{j\in\mathbb{N}} [8j+9, 8j+10] \cap \mathbb{N}.$$

Observe that

$$n(j) = \max\{n_i(j) : 1 \le i \le 2\} = 8j+9, \quad j \in \mathbb{N}.$$

Now,

$$\begin{aligned}\limsup_{j\to\infty}\sum_{i=1}^{2}\sum_{q=n(j)}^{\sigma(n(j))} p_i(q) &= \limsup_{j\to\infty}\left[\sum_{q=8j+9}^{8j+10} p_1(q) + \sum_{q=8j+9}^{8j+10} p_2(q)\right]\\ &= \cos\frac{9\pi}{4} + \cos\frac{5\pi}{2} + \sin\frac{9\pi}{4} + \sin\frac{5\pi}{2} = 1+\sqrt{2} > 1,\end{aligned}$$

that is, (3.5) of Theorem 3.4 is satisfied, and therefore all solutions of (4.4) oscillate.

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