

Ali Jabari Moghadam · Asghar Baradaran Rahimi

A singular perturbation solution of viscous incompressible fluid flow between two eccentric rotating cylinders

Received: 24 December 2012 / Accepted: 5 June 2013 / Published online: 20 June 2013
© The Author(s) 2013. This article is published with open access at Springerlink.com

Abstract The flow inside two concentric cylinders is one dimensional and an exact solution for quantities is easily found. However, when the cylinders axes are displaced by a small distance, two dimensional effects become obvious. In this research, the equations governing an incompressible viscous flow between two rotating cylinders are considered in polar coordinates that can be simplified by introducing vorticity and stream functions. By taking the curl of the vector form of the momentum equation, the pressure term is omitted. Because of the boundary conditions being in terms of perturbation parameter, a modified bi-polar coordinate system is introduced. This transforms the two eccentric cylinders into two concentric ones. By expanding the quantities in terms of the perturbation parameter up to second-order accuracy and by substitution into the vorticity-stream function, sets of differential equations are obtained to be solved for these functions. At the end, the closed-form velocity components are determined.

Mathematics Subject Classification 35Q30 · 76U05

المخلص

في البعد الواحد، يتم الحصول بسهولة على التدفق داخل أسطوانتين متحدتي المركز والحل الدقيق للكميات المتعلقة. ولكن، عند تحريك محوري الأسطوانتين بمسافات صغيرة تتضح تأثيرات البعد الثاني. في هذا البحث، نعتبر، في الإحداثيات القطبية، المعادلات الحاكمة لتدفق لزج غير قابل للانضغاط بين أسطوانتين تدوران والتي يمكن تبسيطها عن طريق إدخال الدوامية ودالة التدفق. وبأخذ دوران الشكل المتجهي لمعادلة الزخم، تم حذف حد الضغط. وبسبب الشروط الحدية المعطاة بدلالة وسائط الاضطراب، تم إدخال نظام إحداثيات ثنائي القطبية معدل. يمكن هذا من تحويل الأسطوانتين مختلفتي المركز إلى أسطوانتين متحدتي المركز. ومن خلال نشر الكميات بدلالة وسائط الاضطراب بدقة من الدرجة الثانية والتعويض في دالة الدوامية-التدفق، تم الحصول على مجموعات من المعادلات التفاضلية التي يجب حلها لتلك الدوال. أخيراً، تم تحديد شكل مغلق لمركبات السرعة.

1 Introduction

In the case of steady-state incompressible fluid flow, there is an exact solution for concentric rotating cylinders. Flow, in this case, is one-dimensional, and its solution is not dependent on Reynolds number. But flow between rotating eccentric cylinders is two-dimensional, and because of non-linearity of the Navier–Stokes equations, an exact solution is not achieved. This kind of flow is observed in journal bearings, in which the shaft axis is slightly displaced [7].

A. J. Moghadam (✉)
Department of Mechanical Engineering, Shahrood University of Technology, Post Box 316, Shahrood, Iran
E-mail: alijabari.m@aol.com

A. B. Rahimi
Faculty of Engineering, Ferdowsi University of Mashhad, P.O. Box 91775-1111, Mashhad, Iran
E-mail: rahimiab.m@yahoo.com



Many researchers tried to investigate the classical problem of lubrication theory [8,9]. They obtained an exact solution of the problem of two-dimensional creeping flow between two eccentrically arranged cylinders. These works describe the idealized flow in the gap of journal bearings which is filled by lubricating fluid. Chernyavsky [1] studied the problem of flow between circular cylinders with a free internal cylinder, in which, the dowel is under the action of gravity, the force from the fluid filling the space between the dowel and the bearing, and the action of the moment of forces with respect to the center of the dowel. Wannier [17] proved that the Reynolds equation may be obtained by expanding quantities in Stokes equation in terms of powers of very small thickness of film. Sommerfeld [15] solved the approximate equation of Reynolds in 1904, and Kamal [10] presented a brief of his solution. Wannier also represented an exact solution of Stokes equations for flow between eccentric cylinders which is in case of small Reynolds numbers. Wannier's solution does not include inertia effects. Kamal considered inertia effects, and expanded flow quantities in terms of inverse powers of kinematic viscosity. Kulinski and Ostrach [11] obtained velocity components for small eccentricity and moderate Reynolds numbers using perturbation method. Diprima and Stuart [4] worked on this and also got pressure distribution and torque. All these works are valid for small Reynolds numbers. Frene and Godet [5] studied the criteria for flow transition in a journal bearing. Wood [19] considered solutions for all ranges of Reynolds number, and expanded flow quantities in terms of eccentricity in polar coordinates. He claimed that if one of the cylinders is stationary, this expansion seems not to be valid for large Reynolds numbers, because inside the boundary layer attached to the stationary cylinders, flow quantities would not be analytic functions of eccentricity. In other words, the expansion is not valid inside the inner region near stationary cylinder, and secular terms appear in the regular expansion. Wood's solution is in Bessel functions and imaginary orders and due to its complexity it is not usually used to study flow stability. Meanwhile, it only includes first-order approximations. Diprima and Stuart proved that the Taylor number in which flow changes to vortex, is proportional to square eccentricity; hence, the second-order approximations are essential. Petrov [13] investigated the motion of an incompressible viscous fluid in a thin layer between two circular cylinders. He solved the equations of hydrodynamic lubrication for the unsteady plane parallel flow. Dai et al. [3] investigated the effect of eccentricity on Couette–Taylor transition for flow between infinite rotating cylinders. Their analysis method is Fourier expansion of the conservation equations in the axial direction, followed by projection onto a polynomial subspace.

In this research, a perturbation technique is used to solve the non-linear Navier–Stokes equations in a two-dimensional region. Second-order approximations together with first-order ones are represented in terms of simple functions. To do this, the boundary layer theory is applied, in which Reynolds number will be considered a large parameter [14]. Up to now, expanding the flow quantities in terms of inverse Reynolds number, or a function of it, has not been considered. If so, we deal with a singular perturbation problem.

2 Problem formulation

The governing equations in cylindrical coordinates for steady-state conditions and constant density and viscosity are given by:

$$\frac{1}{r^*} \frac{\partial}{\partial r^*} (r^* u^*) + \frac{1}{r^*} \frac{\partial}{\partial \theta} (v^*) = 0 \quad (1)$$

$$u^* \frac{\partial u^*}{\partial r^*} + \frac{v^*}{r^*} \frac{\partial u^*}{\partial \theta} - \frac{v^{*2}}{r^*} = \frac{-1}{\rho} \frac{\partial p^*}{\partial r^*} + \nu \left[\frac{\partial}{\partial r^*} \left(\frac{1}{r^*} \frac{\partial}{\partial r^*} (r^* u^*) \right) + \frac{1}{r^{*2}} \frac{\partial^2 u^*}{\partial \theta^2} - \frac{2}{r^{*2}} \frac{\partial v^*}{\partial \theta} \right] \quad (2)$$

$$u^* \frac{\partial v^*}{\partial r^*} + \frac{v^*}{r^*} \frac{\partial v^*}{\partial \theta} + \frac{u^* v^*}{r^*} = \frac{-1}{\rho r^*} \frac{\partial p^*}{\partial \theta} + \nu \left[\frac{\partial}{\partial r^*} \left(\frac{1}{r^*} \frac{\partial}{\partial r^*} (r^* v^*) \right) + \frac{1}{r^{*2}} \frac{\partial^2 v^*}{\partial \theta^2} + \frac{2}{r^{*2}} \frac{\partial u^*}{\partial \theta} \right] \quad (3)$$

The boundary conditions, according to the geometry given in Fig. 1, are:

$$\begin{aligned} u^*(a, \theta) &= 0 \\ v^*(a, \theta) &= a\Omega_1 \\ u^*(r_0^*, \theta) &= -e\Omega_2 \sin \theta \\ v^*(r_0^*, \theta) &= \Omega_2 \frac{b^2 + r_0^{*2} - e^2}{2r_0^*} \end{aligned} \quad (4)$$



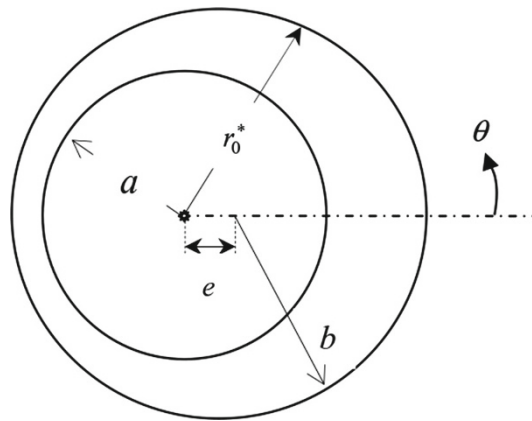


Fig. 1 Geometry of eccentric cylinders

where, e is the eccentricity, a and b are the inner and outer cylinders’ radii, respectively, and r_0^* is the distance of outer cylinder to the origin.

$$r_0^* = e \cos \theta + \sqrt{b^2 - e^2 \sin^2 \theta} \tag{5}$$

These cylinders are rotating at the same direction with constant angular velocities Ω_1 and Ω_2 and the origin of the coordinates system coincides with the center of the inner cylinder. If the following dimensionless variables are introduced:

$$r = \frac{r^*}{a} \tag{6}$$

$$u = \frac{u^*}{a\Omega_1} \tag{7}$$

$$v = \frac{v^*}{a\Omega_1} \tag{8}$$

$$t = t^* \Omega_1 \tag{9}$$

$$p = \frac{p^*}{\rho a^2 \Omega_1^2} \tag{10}$$

then, Eqs. (1)–(3) will be changed to the following dimensionless form:

$$\frac{1}{r} \frac{\partial}{\partial r} (ru) + \frac{1}{r} \frac{\partial}{\partial \theta} (v) = 0 \tag{11}$$

$$u \frac{\partial u}{\partial r} + \frac{v}{r} \frac{\partial u}{\partial \theta} - \frac{v^2}{r} = -\frac{\partial p}{\partial r} + \frac{1}{\text{Re}} \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (ru) \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial v}{\partial \theta} \right] \tag{12}$$

$$u \frac{\partial v}{\partial r} + \frac{v}{r} \frac{\partial v}{\partial \theta} + \frac{uv}{r} = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \frac{1}{\text{Re}} \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (rv) \right) + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial u}{\partial \theta} \right] \tag{13}$$

where, the Reynolds number is defined as:

$$\text{Re} = \frac{\Omega_1 a^2}{\nu} \tag{14}$$

The dimensionless form of the boundary conditions (4) is:

$$\begin{aligned} u(1, \theta) &= 0 \\ v(1, \theta) &= 1 \\ u(r_0, \theta) &= -\frac{\Omega_2}{\Omega_1} \varepsilon \sin \theta \\ v(r_0, \theta) &= \frac{\Omega_2}{\Omega_1} \frac{b^2 + r_0^{*2} - e^2}{2ar_0^*} \end{aligned} \tag{15}$$

where,

$$r_0 = \varepsilon \cos \theta + \sqrt{\frac{b^2}{a^2} - \varepsilon^2 \sin^2 \theta} \quad (16)$$

and

$$\varepsilon = \frac{e}{a} \quad (17)$$

Now, if the velocity components are defined by stream function [18]:

$$u = \frac{1}{r} \frac{\partial \psi}{\partial \theta}, \quad v = \frac{-\partial \psi}{\partial r} \quad (18)$$

the continuity equation (11) will automatically be satisfied. Then by taking derivative of Eqs. (12) and (13) with respect to θ and r , respectively, and combining them properly, the pressure term will be omitted and the following equation will be obtained in terms of vorticity function [6]:

$$u \frac{\partial \omega}{\partial r} + \frac{v}{r} \frac{\partial \omega}{\partial \theta} = \frac{1}{\text{Re}} \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \omega}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \omega}{\partial \theta^2} \right] \quad (19)$$

where,

$$\omega = \frac{-1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \psi}{\partial r} \right) - \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} \quad (20)$$

Boundary conditions (15) are now written in terms of ψ as follows:

$$\begin{aligned} \frac{\partial \psi}{\partial r}(1, \theta) &= -1 \\ \frac{\partial \psi}{\partial \theta}(1, \theta) &= 0 \\ \frac{\partial \psi}{\partial r}(r_0, \theta) &= -\frac{\Omega_2}{\Omega_1} \frac{b^2/a^2 + r_0^2 - \varepsilon^2}{2r_0} \\ \frac{\partial \psi}{\partial \theta}(r_0, \theta) &= -\frac{\Omega_2}{\Omega_1} \varepsilon r_0 \sin \theta \end{aligned} \quad (21)$$

Equations (19) and (20) with boundary conditions (21) are the mathematical model of motion between two eccentric rotating cylinders.

3 Equations of motion in modified bi-polar coordinates system

As the coordinate describing outer cylinder surface is a function of ε , finding approximate solution by expanding the quantities in terms of ε is not possible. Hence, the modified bi-polar coordinates system is introduced to overcome this problem [19]. Using conformal mapping, we have:

$$z = \frac{\zeta + \gamma}{1 + \gamma \zeta} \quad (22)$$

where, $z = r e^{i\theta}$ and $\zeta = \Re e^{i\varphi}$ and

$$\gamma = -2\varepsilon \left[\left(\frac{b}{a} \right)^2 - 1 - \varepsilon^2 + \sqrt{(b^2/a^2 - 1 - \varepsilon^2)^2 - 4\varepsilon^2} \right]^{-1} \quad (23)$$



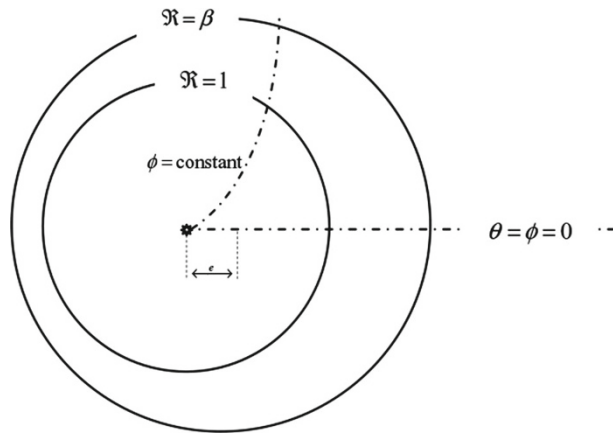


Fig. 2 The model geometric description in the modified bi-polar coordinates

By this mapping, eccentric cylinders in plane $z(r, \theta)$ are changed into concentric cylinders in plane $\zeta(\mathfrak{R}, \varphi)$. Consequently, the inner and outer cylinders will be changed to the coordinate lines $\mathfrak{R} = 1$ and $\mathfrak{R} = \beta$, respectively, in which [2]:

$$\beta = \frac{b/a + \epsilon - \gamma}{1 - (b/a)\gamma - \epsilon\gamma} \tag{24}$$

Figure 2 illustrates the superposition of modified bi-polar coordinates system on the cylindrical one. These new coordinates' axes are exactly the same as the two-dimensional bi-polar ones; however, variables \mathfrak{R} and φ have been modified in such a way that when the cylinders are concentric, they will be changed to r and θ . The stream function is related to the velocity components in this new coordinates system as follows [16]:

$$u_{\mathfrak{R}} = \frac{\sqrt{J}}{\mathfrak{R}} \frac{\partial \psi}{\partial \varphi} \tag{25}$$

$$u_{\varphi} = -\sqrt{J} \frac{\partial \psi}{\partial \mathfrak{R}}$$

$$J = \frac{(1 + 2\gamma \mathfrak{R} \cos \varphi + \gamma^2 \mathfrak{R}^2)^2}{(1 - \gamma^2)^2} \tag{26}$$

The equations of stream and vorticity functions are:

$$\frac{1}{\mathfrak{R}} \left(\frac{\partial \psi}{\partial \varphi} \frac{\partial \omega}{\partial \mathfrak{R}} - \frac{\partial \psi}{\partial \mathfrak{R}} \frac{\partial \omega}{\partial \varphi} \right) = \frac{1}{\text{Re}} \nabla^2 \omega \tag{27}$$

$$\omega = -J \nabla^2 \psi \tag{28}$$

where,

$$\nabla^2 \equiv \frac{\partial^2}{\partial \mathfrak{R}^2} + \frac{1}{\mathfrak{R}} \frac{\partial}{\partial \mathfrak{R}} + \frac{1}{\mathfrak{R}^2} \frac{\partial^2}{\partial \varphi^2} \tag{29}$$

Then, boundary conditions (21) become:

$$\begin{aligned} \frac{\partial \psi}{\partial \varphi}(1, \varphi) = 0, \quad \frac{\partial \psi}{\partial \mathfrak{R}}(1, \varphi) &= \frac{-1}{\sqrt{J(1, \varphi)}} \\ \frac{\partial \psi}{\partial \varphi}(\beta, \varphi) = 0, \quad \frac{\partial \psi}{\partial \mathfrak{R}}(\beta, \varphi) &= \frac{-q_2/q_1}{\sqrt{J(\beta, \varphi)}} \end{aligned} \tag{30}$$

For these boundary conditions, we have $q_1 = a\Omega_1$ and $q_2 = b\Omega_2$. Equations (27) and (28) with boundary conditions (30) are the governing equations of flow between two rotating cylinders in modified bi-polar coordinates system.

4 Linearization of the motion equations

As γ is a measure of eccentricity ε , it can be used to linearize the motion equations by asymptotic expansions of the stream and vorticity functions [12]:

$$\psi = \psi_0 + \gamma \psi_1 + \gamma^2 \psi_2 + \dots \quad (31)$$

$$\omega = \omega_0 + \gamma \omega_1 + \gamma^2 \omega_2 + \dots \quad (32)$$

Also, J may be expanded as follows:

$$J = J_0 + \gamma J_1 + \gamma^2 J_2 + \dots \quad (33)$$

where,

$$J_0 = 1, \quad J_1 = 4\Re \cos \varphi, \quad J_2 = 2(1 + 2\Re^2 + \Re^2 \cos 2\varphi) \quad (34)$$

Substituting Eqs. (31)–(33) into (27) and (28), and equating powers of γ to zero, gives:

$$\gamma^0: \quad \frac{1}{\Re} \frac{\partial \psi_0}{\partial \varphi} \frac{\partial \omega_0}{\partial \Re} - \frac{1}{\Re} \frac{\partial \psi_0}{\partial \Re} \frac{\partial \omega_0}{\partial \varphi} = \frac{1}{\text{Re}} \nabla^2 \omega_0$$

$$\omega_0 = -J_0 \nabla^2 \psi_0 \quad (35)$$

$$\gamma^1: \quad \frac{1}{\Re} \frac{\partial \psi_0}{\partial \varphi} \frac{\partial \omega_1}{\partial \Re} + \frac{1}{\Re} \frac{\partial \psi_1}{\partial \varphi} \frac{\partial \omega_0}{\partial \Re} - \frac{1}{\Re} \frac{\partial \psi_0}{\partial \Re} \frac{\partial \omega_1}{\partial \varphi} - \frac{1}{\Re} \frac{\partial \psi_1}{\partial \Re} \frac{\partial \omega_0}{\partial \varphi} = \frac{1}{\text{Re}} \nabla^2 \omega_1$$

$$\omega_1 = -J_0 \nabla^2 \psi_1 - J_1 \nabla^2 \psi_0 \quad (36)$$

$$\gamma^2: \quad \frac{1}{\Re} \frac{\partial \psi_0}{\partial \varphi} \frac{\partial \omega_2}{\partial \Re} + \frac{1}{\Re} \frac{\partial \psi_1}{\partial \varphi} \frac{\partial \omega_1}{\partial \Re} + \frac{1}{\Re} \frac{\partial \psi_2}{\partial \varphi} \frac{\partial \omega_0}{\partial \Re} - \frac{1}{\Re} \frac{\partial \psi_0}{\partial \Re} \frac{\partial \omega_2}{\partial \varphi}$$

$$- \frac{1}{\Re} \frac{\partial \psi_1}{\partial \Re} \frac{\partial \omega_1}{\partial \varphi} - \frac{1}{\Re} \frac{\partial \psi_2}{\partial \Re} \frac{\partial \omega_0}{\partial \varphi} = \frac{1}{\text{Re}} \nabla^2 \omega_2$$

$$\omega_2 = -J_0 \nabla^2 \psi_2 - J_1 \nabla^2 \psi_1 - J_2 \nabla^2 \psi_0 \quad (37)$$

When the parameter γ equals zero, the cylinders are concentric, and its flow is one-dimensional. Hence, flow quantities of zero order, that is, ψ_0 , and ω_0 , are independent of φ . In this case, vorticity is constant everywhere in the flow field. By considering these facts, Eqs. (35)–(37) are simplified as:

$$\omega_0 = \text{constant} \quad (38)$$

$$\omega_0 = -\frac{1}{\Re} \frac{\partial}{\partial \Re} \left(\Re \frac{\partial \psi_0}{\partial \Re} \right) = \text{constant} \quad (39)$$

$$-\frac{1}{\Re} \frac{\partial \psi_0}{\partial \Re} \frac{\partial \omega_1}{\partial \varphi} = \frac{1}{\text{Re}} \nabla^2 \omega_1 \quad (40)$$

$$\omega_1 = -J_0 \nabla^2 \psi_1 - J_1 \nabla^2 \psi_0 \quad (41)$$

$$\frac{1}{\Re} \frac{\partial \psi_1}{\partial \varphi} \frac{\partial \omega_1}{\partial \Re} - \frac{1}{\Re} \frac{\partial \psi_0}{\partial \Re} \frac{\partial \omega_2}{\partial \varphi} - \frac{1}{\Re} \frac{\partial \psi_1}{\partial \Re} \frac{\partial \omega_1}{\partial \varphi} = \frac{1}{\text{Re}} \nabla^2 \omega_2 \quad (42)$$

$$\omega_2 = -J_0 \nabla^2 \psi_2 - J_1 \nabla^2 \psi_1 - J_2 \nabla^2 \psi_0 \quad (43)$$

Expanding boundary conditions (30) in terms of powers of γ gives:

$$\psi_0(1, \varphi) = \text{constant}, \quad \frac{\partial \psi_0}{\partial \Re}(1, \varphi) = -1 \quad (44)$$

$$\psi_0(\beta, \varphi) = \text{constant}, \quad \frac{\partial \psi_0}{\partial \Re}(\beta, \varphi) = -\frac{q_2}{q_1}$$

$$\frac{\partial \psi_1}{\partial \varphi}(1, \varphi) = 0, \quad \frac{\partial \psi_1}{\partial \Re}(1, \varphi) = 2 \cos \varphi \quad (45)$$

$$\frac{\partial \psi_1}{\partial \varphi}(\beta, \varphi) = 0, \quad \frac{\partial \psi_1}{\partial \Re}(\beta, \varphi) = 2\beta \frac{q_2}{q_1} \cos \varphi$$

$$\frac{\partial \psi_2}{\partial \varphi}(1, \varphi) = 0, \quad \frac{\partial \psi_2}{\partial \Re}(1, \varphi) = -2 \cos 2\varphi \quad (46)$$

$$\frac{\partial \psi_2}{\partial \varphi}(\beta, \varphi) = 0, \quad \frac{\partial \psi_2}{\partial \Re}(\beta, \varphi) = \frac{q_2}{q_1} (1 - \beta^2) - 2\beta^2 \frac{q_2}{q_1} \cos 2\varphi$$



5 Solution of zeroth-order equations

Solution of Eq. (38) with boundary conditions (44) is:

$$\psi_0 = \frac{-1}{2}A\Re^2 - B \ln \Re, \quad \omega_0 = 2A \tag{47}$$

where,

$$A = \frac{\beta(q_2/q_1) - 1}{\beta^2 - 1}, \quad B = \frac{\beta^2 - \beta(q_2/q_1)}{\beta^2 - 1} \tag{48}$$

6 Solution of first-order equations

Substituting ψ_0 into Eqs. (40) and (41) gives:

$$\left(A + \frac{B}{\Re^2} \right) \frac{\partial \omega_1}{\partial \varphi} = \frac{1}{\text{Re}} \nabla^2 \omega_1 \tag{49}$$

$$\omega_1 = -\nabla^2(\psi_1 - A\Re^3 \cos \varphi) \tag{50}$$

As can be seen in the above equations and boundary conditions (45), the flow quantities must be functions of φ , hence, ψ_1 and ω_1 must take the following form:

$$\psi_1 = A\Re^3 \cos \varphi + 2f_c(\Re) \cos \varphi + 2f_s(\Re) \sin \varphi \tag{51}$$

$$\omega_1 = 2g_c(\Re) \cos \varphi + 2g_s(\Re) \sin \varphi \tag{52}$$

Substituting these functions into Eqs. (49) and (50), and rearranging them, gives:

$$\begin{aligned} & \frac{1}{\text{Re}} \left(f_s^{iv} + \frac{2}{\Re} f_s''' - \frac{3}{\Re^2} f_s'' + \frac{3}{\Re^3} f_s' - \frac{3}{\Re^4} f_s \right) \\ & + \left(A + \frac{B}{\Re^2} \right) \left(f_c'' + \frac{1}{\Re} f_c' - \frac{1}{\Re^2} f_c \right) = 0 \end{aligned} \tag{53}$$

$$\begin{aligned} & \frac{1}{\text{Re}} \left(f_c^{iv} + \frac{2}{\Re} f_c''' - \frac{3}{\Re^2} f_c'' + \frac{3}{\Re^3} f_c' - \frac{3}{\Re^4} f_c \right) \\ & + \left(A + \frac{B}{\Re^2} \right) \left(f_s'' + \frac{1}{\Re} f_s' - \frac{1}{\Re^2} f_s \right) = 0 \end{aligned} \tag{54}$$

Boundary conditions (45) are then changed for f_c and f_s as:

$$\begin{aligned} f_c(1) &= \frac{-1}{2}A, & f_c'(1) &= 1 - \frac{3}{2}A \\ f_c(\beta) &= \frac{-1}{2}A\beta^3, & f_c'(\beta) &= \beta \frac{q_2}{q_1} - \frac{3}{2}A\beta^2 \\ f_s(1) &= 0, & f_s'(1) &= 0 \\ f_s(\beta) &= 0, & f_s'(\beta) &= 0 \end{aligned} \tag{55}$$

Equations (53) and (54) are now solved for large Reynolds numbers. So we deal with a boundary layer problem, or singular perturbation in which a small parameter, named perturbation parameter, is multiplied by the greatest derivative of the equation. In this case, first, using an ordinary series expansion, an outer solution is obtained that is only valid outside the boundary layer. Then, by a proper variable change, an inner solution will be obtained that is valid inside the boundary layer. Finally, these inner and outer solutions will be matched to get a valid composite solution throughout the flow field.

6.1 Outer solution

If the following expansions:

$$f_c^0 = f_{c0} + \lambda f_{c1} + \dots \quad (56)$$

$$f_s^0 = f_{s0} + \lambda f_{s1} + \dots \quad (57)$$

are replaced into Eqs. (53) and (54), then by equating the powers of $\lambda = \text{Re}^{-1/2}$ to zero, we get the following equations:

$$\begin{aligned} f''_{c0} + \frac{f'_{c0}}{\Re} - \frac{f_{c0}}{\Re^2} = 0, \quad f''_{s0} + \frac{f'_{s0}}{\Re} - \frac{f_{s0}}{\Re^2} = 0 \\ f''_{c1} + \frac{f'_{c1}}{\Re} - \frac{f_{c1}}{\Re^2} = 0, \quad f''_{s1} + \frac{f'_{s1}}{\Re} - \frac{f_{s1}}{\Re^2} = 0 \end{aligned} \quad (58)$$

Their solutions are:

$$\begin{aligned} f_{c0} = c_1 \Re^{-1} + c_2 \Re, \quad f_{s0} = d_1 \Re^{-1} + d_2 \Re \\ f_{c1} = c_3 \Re^{-1} + c_4 \Re, \quad f_{s1} = d_3 \Re^{-1} + d_4 \Re \end{aligned} \quad (59)$$

where, the constants c and d will be determined by matching outer solution with inner one. So the outer expansions for f_c and f_s are as follows:

$$f_c^0 = c_1 \Re^{-1} + c_2 \Re + \lambda(c_3 \Re^{-1} + c_4 \Re) + \dots \quad (60)$$

$$f_s^0 = d_1 \Re^{-1} + d_2 \Re + \lambda(d_3 \Re^{-1} + d_4 \Re) + \dots \quad (61)$$

Constants c and d can be determined by matching these with inner solutions.

6.2 Inner solution near $\Re = 1$

To obtain a solution near $\Re = 1$, the following variable change is introduced:

$$\xi = \frac{\Re - 1}{\lambda} \quad (62)$$

Note that, here the thickness of boundary layer is of order of λ . To find inner solution, we assume the following expansions:

$$f_c^i = f_{c0} + \lambda f_{c1} + \dots \quad (63)$$

$$f_s^i = f_{s0} + \lambda f_{s1} + \dots \quad (64)$$

Substituting these expressions into Eqs. (53) and (54), and equating the powers of λ to zero, gives:

$$\begin{aligned} \frac{d^4 f_{s0}}{d\xi^4} + \frac{d^2 f_{c0}}{d\xi^2} = 0 \\ \frac{d^4 f_{c0}}{d\xi^4} - \frac{d^2 f_{s0}}{d\xi^2} = 0 \end{aligned} \quad (65)$$

$$\begin{aligned} \frac{d^4 f_{s1}}{d\xi^4} + \frac{d^2 f_{c1}}{d\xi^2} = -4\xi \frac{d^4 f_{s0}}{d\xi^4} - 2 \frac{d^3 f_{s0}}{d\xi^3} - 2(2A + B)\xi \frac{d^2 f_{c0}}{d\xi^2} - \frac{d f_{c0}}{d\xi} \\ \frac{d^4 f_{c1}}{d\xi^4} + \frac{d^2 f_{s1}}{d\xi^2} = -4\xi \frac{d^4 f_{c0}}{d\xi^4} - 2 \frac{d^3 f_{c0}}{d\xi^3} - 2(2A + B)\xi \frac{d^2 f_{s0}}{d\xi^2} - \frac{d f_{s0}}{d\xi} \end{aligned} \quad (66)$$



The boundary conditions (55), relating to $\mathfrak{R} = 1$ or $\xi = 0$, become:

$$\begin{aligned} f_{c0}(0) &= \frac{-1}{2}A, & f'_{c0}(0) &= 0 \\ f_{s0}(0) &= 0, & f'_{s0}(0) &= 0 \\ f_{c1}(0) &= 0, & f'_{c1}(0) &= 1 - \frac{3}{2}A \\ f_{s1}(0) &= 0, & f'_{s1}(0) &= 0 \end{aligned} \tag{67}$$

Solving Eqs. (65) and (66) with boundary conditions (67) leads to the following results for the inner solution:

$$\begin{aligned} f_c^i &= \frac{-1}{2}(A + 2c_0) + \frac{\sqrt{2}}{2}(c_0 - d_0)\xi \\ &+ c_0 e^{(-\frac{\sqrt{2}}{2}\xi)} \cos\left(\frac{\sqrt{2}}{2}\xi\right) + d_0 e^{(-\frac{\sqrt{2}}{2}\xi)} \sin\left(\frac{\sqrt{2}}{2}\xi\right) \\ &+ \lambda \left\{ \begin{aligned} & -c_1 + \left[1 - \frac{3}{2}A - \frac{1}{2}(5B - 1)c_0 + \frac{\sqrt{2}}{2}(c_1 - d_1) \right] \xi + \frac{\sqrt{2}}{4}(d_0 - c_0)\xi^2 \\ & + \left[\frac{\sqrt{2}}{4}B(c_0 - d_0)\xi^2 + \frac{1}{2}(5B - 1)c_0\xi + c_1 \right] e^{(-\frac{\sqrt{2}}{2}\xi)} \cos\left(\frac{\sqrt{2}}{2}\xi\right) \\ & + \left[\frac{\sqrt{2}}{4}B(c_0 + d_0)\xi^2 + \frac{1}{2}(5B - 1)d_0\xi + d_1 \right] e^{(-\frac{\sqrt{2}}{2}\xi)} \sin\left(\frac{\sqrt{2}}{2}\xi\right) \end{aligned} \right\} + \dots \end{aligned} \tag{68}$$

$$\begin{aligned} f_s^i &= d_0 - \frac{\sqrt{2}}{2}(c_0 + d_0)\xi - d_0 e^{(-\frac{\sqrt{2}}{2}\xi)} \cos\left(\frac{\sqrt{2}}{2}\xi\right) + c_0 e^{(-\frac{\sqrt{2}}{2}\xi)} \sin\left(\frac{\sqrt{2}}{2}\xi\right) \\ &+ \lambda \left\{ \begin{aligned} & d_1 + \left[\frac{1}{2}(5B - 1)d_0 - \frac{\sqrt{2}}{2}(c_1 + d_1) \right] \xi + \frac{\sqrt{2}}{4}(c_0 + d_0)\xi^2 \\ & + \left[-\frac{\sqrt{2}}{4}B(c_0 + d_0)\xi^2 - \frac{1}{2}(5B - 1)d_0\xi - d_1 \right] e^{(-\frac{\sqrt{2}}{2}\xi)} \cos\left(\frac{\sqrt{2}}{2}\xi\right) \\ & + \left[\frac{\sqrt{2}}{4}B(c_0 - d_0)\xi^2 + \frac{1}{2}(5B - 1)c_0\xi + c_1 \right] e^{(-\frac{\sqrt{2}}{2}\xi)} \sin\left(\frac{\sqrt{2}}{2}\xi\right) \end{aligned} \right\} + \dots \end{aligned} \tag{69}$$

Constants c_0, d_0 and c_1, d_1 can be determined by matching with outer solution.

6.3 Inner solution near $\mathfrak{R} = \beta$

To obtain a solution near $\mathfrak{R} = \beta$, the following variable change is introduced:

$$\eta = \frac{\beta - \mathfrak{R}}{\lambda} \tag{70}$$

If the following expansions:

$$\begin{aligned} f_c^I &= f_{c0}^I + \lambda f_{c1}^I + \dots \\ f_s^I &= f_{s0}^I + \lambda f_{s1}^I + \dots \end{aligned} \tag{71}$$

are substituted into Eqs. (53) and (54), and equating the powers of λ to zero, the following equations will be obtained:

$$\begin{aligned} \beta^2 \frac{d^4 f_{s0}^I}{d\eta^4} + \delta^2 \frac{d^2 f_{c0}^I}{d\eta^2} &= 0 \\ \beta^2 \frac{d^4 f_{c0}^I}{d\eta^4} - \delta^2 \frac{d^2 f_{s0}^I}{d\eta^2} &= 0 \end{aligned} \tag{72}$$

$$\begin{aligned}
\beta^2 \frac{d^4 f_{s1}^I}{d\eta^4} + \delta^2 \beta^2 \frac{d^2 f_{c1}^I}{d\eta^2} &= 4\beta^3 \eta \frac{d^4 f_{s0}^I}{d\eta^4} + 2\beta^3 \frac{d^3 f_{s0}^I}{d\eta^3} \\
&+ 2\beta(2A\beta^2 + B)\eta \frac{d^2 f_{c0}^I}{d\eta^2} + \delta^2 \beta \frac{d f_{c0}^I}{d\eta} \\
\beta^2 \frac{d^4 f_{c1}^I}{d\eta^4} - \delta^2 \beta^2 \frac{d^2 f_{s1}^I}{d\eta^2} &= 4\beta^3 \eta \frac{d^4 f_{c0}^I}{d\eta^4} + 2\beta^3 \frac{d^3 f_{c0}^I}{d\eta^3} \\
&- 2\beta(2A\beta^2 + B)\eta \frac{d^2 f_{s0}^I}{d\eta^2} - \delta^2 \beta \frac{d f_{s0}^I}{d\eta}
\end{aligned} \tag{73}$$

in which,

$$\delta = \sqrt{A\beta^2 + B} = \sqrt{\beta(q_2/q_1)} \tag{74}$$

As can be seen, for $q_2 = 0$, δ equals zero. In this case, solutions of each of the Eqs. (72) or (73) will not be decreased exponentially. Hence, these solutions cannot be matched with outer solution. Because of this, when one of the cylinders is stationary, expanding stream function will not be analytic. So this study will be limited to $q_2 > 0$. The boundary conditions (55), relating to $\mathfrak{R} = \beta$ or $\eta = 0$, become:

$$\begin{aligned}
f_{c0}^I(0) &= \frac{-1}{2} A\beta^3, \quad f_{c0}^I{}'(0) = 0 \\
f_{s0}^I(0) &= 0, \quad f_{s0}^I{}'(0) = 0 \\
f_{c1}^I(0) &= 0, \quad f_{c1}^I{}'(0) = \frac{3}{2} A\beta^2 - \beta \frac{q_2}{q_1} \\
f_{s1}^I(0) &= 0, \quad f_{s1}^I{}'(0) = 0
\end{aligned} \tag{75}$$

Solutions of Eqs. (72) and (73) with boundary conditions (75) are:

$$\begin{aligned}
f_c^I &= \frac{-1}{2} (A\beta^3 + 2C_0) + \frac{\sqrt{2}}{2} \frac{\delta}{\beta} (C_0 - D_0)\eta \\
&+ C_0 e^{\left(-\frac{\sqrt{2}}{2} \frac{\delta}{\beta} \eta\right)} \cos\left(\frac{\sqrt{2}}{2} \frac{\delta}{\beta} \eta\right) + D_0 e^{\left(-\frac{\sqrt{2}}{2} \frac{\delta}{\beta} \eta\right)} \sin\left(\frac{\sqrt{2}}{2} \frac{\delta}{\beta} \eta\right) \\
&+ \lambda \left\{ \begin{aligned} & \left[-c_1 + \frac{\sqrt{2}}{4} \frac{\delta}{\beta^2} (C_0 - D_0)\eta^2 \right. \\ & \left. + \left[\frac{3}{2} A\beta^2 - \beta \frac{q_2}{q_1} + \frac{\sqrt{2}}{2} \frac{\delta}{\beta} (C_1 - D_1) - \frac{C_0}{2\beta\delta^2} (\delta^2 - 5B) \right] \eta \right. \\ & \left. + \left[\frac{\sqrt{2}}{4} \frac{B}{\delta\beta^2} (D_0 - C_0)\eta^2 + \frac{C_0}{2\beta\delta^2} (\delta^2 - 5B)\eta + C_1 \right] e^{\left(-\frac{\sqrt{2}}{2} \frac{\delta}{\beta} \eta\right)} \cos\left(\frac{\sqrt{2}}{2} \frac{\delta}{\beta} \eta\right) \right. \\ & \left. + \left[-\frac{\sqrt{2}}{4} \frac{B}{\delta\beta^2} (D_0 + C_0)\eta^2 + \frac{D_0}{2\beta\delta^2} (\delta^2 - 5B)\eta + D_1 \right] e^{\left(-\frac{\sqrt{2}}{2} \frac{\delta}{\beta} \eta\right)} \sin\left(\frac{\sqrt{2}}{2} \frac{\delta}{\beta} \eta\right) \right] \end{aligned} \right\} + \dots \tag{76} \\
f_s^I &= D_0 - \frac{\sqrt{2}}{2} \frac{\delta}{\beta} (C_0 + D_0)\eta \\
&+ D_0 e^{\left(-\frac{\sqrt{2}}{2} \frac{\delta}{\beta} \eta\right)} \cos\left(\frac{\sqrt{2}}{2} \frac{\delta}{\beta} \eta\right) + C_0 e^{\left(-\frac{\sqrt{2}}{2} \frac{\delta}{\beta} \eta\right)} \sin\left(\frac{\sqrt{2}}{2} \frac{\delta}{\beta} \eta\right) \\
&+ \lambda \left\{ \begin{aligned} & \left[D_1 - \frac{\sqrt{2}}{4} \frac{\delta}{\beta^2} (C_0 + D_0)\eta^2 \right. \\ & \left. + \left[-\frac{\sqrt{2}}{2} \frac{\delta}{\beta} (C_1 + D_1) + \frac{D_0}{2\beta\delta^2} (\delta^2 - 5B) \right] \eta \right. \\ & \left. + \left[\frac{\sqrt{2}}{4} \frac{B}{\delta\beta^2} (D_0 + C_0)\eta^2 - \frac{D_0}{2\beta\delta^2} (\delta^2 - 5B)\eta - D_1 \right] e^{\left(-\frac{\sqrt{2}}{2} \frac{\delta}{\beta} \eta\right)} \cos\left(\frac{\sqrt{2}}{2} \frac{\delta}{\beta} \eta\right) \right. \\ & \left. + \left[\frac{\sqrt{2}}{4} \frac{B}{\delta\beta^2} (D_0 - C_0)\eta^2 + \frac{C_0}{2\beta\delta^2} (\delta^2 - 5B)\eta + C_1 \right] e^{\left(-\frac{\sqrt{2}}{2} \frac{\delta}{\beta} \eta\right)} \sin\left(\frac{\sqrt{2}}{2} \frac{\delta}{\beta} \eta\right) \right] \end{aligned} \right\} + \dots \tag{77}
\end{aligned}$$

Constants C_0 , D_0 and C_1 , D_1 can be determined by matching with outer solution.



6.4 Composite solution

The above three solutions are now combined to get the composite solution which is given as follows:

$$f_c^c = f_c^o + f_c^i + f_c^I - (f_c^o)^i - (f_c^o)^I \tag{78}$$

$$f_s^c = f_s^o + f_s^i + f_s^I - (f_s^o)^i - (f_s^o)^I \tag{79}$$

in which, $(f^o)^i$ is inner expansion of outer solution, and the same for $(f^o)^I$. According to the matching principle of these solutions, inner expansion of outer solution must be equal to outer expansion of inner solution. So we have:

$$\begin{aligned} (f_c^o)^i &= (f_c^i)^o, & (f_s^o)^i &= (f_s^i)^o \\ (f_c^o)^I &= (f_c^I)^o, & (f_s^o)^I &= (f_s^I)^o \end{aligned} \tag{80}$$

The unknown constants resulted in the outer and inner solutions can be determined by the above equalities. The composite solutions are:

$$f_c^c = \frac{1}{2}A[\beta^2\Re^{-1} - (1 + \beta^2)\Re] + \Re^{-1/2} \left\{ \begin{aligned} &\left[\frac{\sqrt{2}}{2\delta} \frac{\beta^2}{\beta^2-1} \left(\delta + \beta \frac{q_2}{q_1} \right) + \frac{\sqrt{2}}{2\delta} A\beta^2(\delta - 1) \right] \Re^{-1} \\ &+ \left[\frac{\sqrt{2}}{2\delta} \frac{\beta^2}{1-\beta^2} \left(\frac{\delta}{\beta^2} + \beta \frac{q_2}{q_1} \right) + \frac{\sqrt{2}}{2\delta} A(\beta^2 - \delta) \right] \Re \\ &+ \frac{\sqrt{2}}{2} [A(1 - \beta^2) - 1] e^{(-\frac{\sqrt{2}}{2}\xi)} \cos\left(\frac{\sqrt{2}}{2}\xi\right) \\ &+ \frac{\sqrt{2}}{2} [1 - A(1 - \beta^2)] e^{(-\frac{\sqrt{2}}{2}\xi)} \sin\left(\frac{\sqrt{2}}{2}\xi\right) \\ &+ \frac{\sqrt{2}}{2} \frac{\beta}{\delta} [A(1 - \beta^2) + \beta \frac{q_2}{q_1}] e^{(-\frac{\sqrt{2}}{2}\frac{\delta}{\beta}\eta)} \cos\left(\frac{\sqrt{2}}{2}\frac{\delta}{\beta}\eta\right) \\ &- \frac{\sqrt{2}}{2} \frac{\beta}{\delta} [A(1 - \beta^2) + \beta \frac{q_2}{q_1}] e^{(-\frac{\sqrt{2}}{2}\frac{\delta}{\beta}\eta)} \sin\left(\frac{\sqrt{2}}{2}\frac{\delta}{\beta}\eta\right) \end{aligned} \right\} \tag{81}$$

$$f_s^c = \Re^{-1/2} \left\{ \begin{aligned} &\left[\frac{\sqrt{2}}{2\delta} \frac{\beta^2}{\beta^2-1} \left(\delta + \beta \frac{q_2}{q_1} \right) + \frac{\sqrt{2}}{2\delta} A\beta^2(\delta - 1) \right] \Re^{-1} \\ &+ \left[\frac{\sqrt{2}}{2\delta} \frac{\beta^2}{1-\beta^2} \left(\frac{\delta}{\beta^2} + \beta \frac{q_2}{q_1} \right) + \frac{\sqrt{2}}{2\delta} A(\beta^2 - \delta) \right] \Re \\ &+ \frac{\sqrt{2}}{2} [A(1 - \beta^2) - 1] e^{(-\frac{\sqrt{2}}{2}\xi)} \cos\left(\frac{\sqrt{2}}{2}\xi\right) \\ &+ \frac{\sqrt{2}}{2} [A(1 - \beta^2) - 1] e^{(-\frac{\sqrt{2}}{2}\xi)} \sin\left(\frac{\sqrt{2}}{2}\xi\right) \\ &+ \frac{\sqrt{2}}{2} \frac{\beta}{\delta} [A(1 - \beta^2) + \beta \frac{q_2}{q_1}] e^{(-\frac{\sqrt{2}}{2}\frac{\delta}{\beta}\eta)} \cos\left(\frac{\sqrt{2}}{2}\frac{\delta}{\beta}\eta\right) \\ &+ \frac{\sqrt{2}}{2} \frac{\beta}{\delta} [A(1 - \beta^2) + \beta \frac{q_2}{q_1}] e^{(-\frac{\sqrt{2}}{2}\frac{\delta}{\beta}\eta)} \sin\left(\frac{\sqrt{2}}{2}\frac{\delta}{\beta}\eta\right) \end{aligned} \right\} \tag{82}$$

Finally, approximate solution of ψ_1 is:

$$\psi_1 = (A\Re^3 + 2f_c^c) \cos \varphi + 2f_s^c \sin \varphi \tag{83}$$

7 Solution of second-order equations

The second-order equations of problem are:

$$\left(A + \frac{B}{\Re^2} \right) \frac{\partial \omega_2}{\partial \varphi} - \frac{1}{\Re} \nabla^2 \omega_2 = H(\Re) + P(\Re) \cos(2\varphi) + Q(\Re) \sin(2\varphi) \tag{84}$$

$$\omega_2 + \nabla^2 \psi_2 = -8A\Re^2 + 4A + 4\Re g_c + (4\Re g_c - 12A\Re^2) \cos(2\varphi) + 4\Re g_s \sin(2\varphi) \tag{85}$$

in which,

$$\begin{aligned} H(\Re) &= \frac{1}{\Re} (-2f'_{s}g_c + 3A\Re^2g_s + 2f'_{c}g_s - 2f_s g'_c + A\Re^3g'_s + 2f_c g'_s) \\ P(\Re) &= \frac{1}{\Re} (2f'_{s}g_c + 3A\Re^2g_s + 2f'_{c}g_s - 2f_s g'_c + A\Re^3g'_s - 2f_c g'_s) \\ Q(\Re) &= \frac{1}{\Re} (-2f'_{c}g_c - 3A\Re^2g_c + 2f'_{s}g_s + 2f_c g'_c + A\Re^3g'_c - 2f_s g'_s) \end{aligned} \quad (86)$$

subject to the following boundary conditions:

$$\begin{aligned} \frac{\partial \psi_2}{\partial \varphi}(1, \varphi) &= 0, \quad \frac{\partial \psi_2}{\partial \Re}(1, \varphi) = -2 \cos 2\varphi \\ \frac{\partial \psi_2}{\partial \varphi}(\beta, \varphi) &= 0, \quad \frac{\partial \psi_2}{\partial \Re}(\beta, \varphi) = \frac{q_2}{q_1}(1 - \beta^2) - 2\frac{q_2}{q_1}\beta^2 \cos 2\varphi \end{aligned} \quad (87)$$

It is clear that ψ_2 and ω_2 vary as follows:

$$\psi_2 = Z(\Re) + X(\Re) \cos 2\varphi + Y(\Re) \sin 2\varphi \quad (88)$$

$$\omega_2 = S(\Re) + F(\Re) \cos 2\varphi + G(\Re) \sin 2\varphi \quad (89)$$

Substituting these relations into Eqs. (84) and (85), we have:

$$S + Z'' + \frac{Z'}{\Re} = -8A\Re^2 + 4A + 4\Re g_c \quad (90)$$

$$2\left(A + \frac{B}{\Re^2}\right)G - \frac{1}{\text{Re}}\left(F'' + \frac{F'}{\Re} - \frac{4}{\Re^2}F\right) = P \quad (91)$$

$$-2\left(A + \frac{B}{\Re^2}\right)F - \frac{1}{\text{Re}}\left(G'' + \frac{G'}{\Re} - \frac{4}{\Re^2}G\right) = Q \quad (92)$$

$$F + X'' + \frac{X'}{\Re} - \frac{4}{\Re^2}X = 4\Re g_c - 12A\Re^2 \quad (93)$$

$$G + Y'' + \frac{Y'}{\Re} - \frac{4}{\Re^2}Y = 4\Re g_s \quad (94)$$

When Re is finite, the vorticity function will be uniform, and S in Eq. (90) must be constant. Hence solution of Eq. (90) subject to related boundary conditions is:

$$Z = \frac{-1}{2}A\Re^4 + \left(A - \frac{1}{2}\beta\frac{q_2}{q_1} + A\beta^2\right)\Re^2 + \left(\beta\frac{q_2}{q_1} - 2A\beta^2\right)\ln(\Re) \quad (95)$$

If Eqs. (93) and (94) are substituted in (91) and (92), the following equations will be obtained:

$$\begin{aligned} 2\left(A + \frac{B}{\Re^2}\right)\left(4\Re g_s - Y'' - \frac{Y'}{\Re} + \frac{4}{\Re^2}Y\right) \\ - \lambda^2\left(\frac{-12}{\Re}g_c + 12g'_c + 4\Re g''_c - X^{iv} - \frac{2}{\Re}X''' + \frac{9}{\Re^2}X'' - \frac{9}{\Re^3}X'\right) = P \end{aligned} \quad (96)$$

$$\begin{aligned} -2\left(A + \frac{B}{\Re^2}\right)\left(4\Re g_c - X'' - \frac{X'}{\Re} + \frac{4}{\Re^2}X - 12A\Re^2\right) \\ - \lambda^2\left(\frac{-12}{\Re}g_s + 12g'_s + 4\Re g''_s - Y^{iv} - \frac{2}{\Re}Y''' + \frac{9}{\Re^2}Y'' - \frac{9}{\Re^3}Y'\right) = Q \end{aligned} \quad (97)$$

In which, $\lambda^2 = \frac{1}{\text{Re}}$, and the prime denotes differentiation with respect to \Re . To solve the above equations, we only consider the zeroth-order approximation.



7.1 Outer expansion

If the following single term approximations for X and Y

$$X = X_0 + \dots \quad \text{and} \quad Y = Y_0 + \dots \tag{98}$$

are substituted into Eqs. (96) and (97), and collecting equal terms, the zeroth-order equations results:

$$\begin{aligned} X''_0 + \frac{1}{\mathfrak{N}} X'_0 - \frac{4}{\mathfrak{N}^2} X_0 &= -12A\mathfrak{N}^2 \\ Y''_0 + \frac{1}{\mathfrak{N}} Y'_0 - \frac{4}{\mathfrak{N}^2} Y_0 &= 0 \end{aligned} \tag{99}$$

General solutions of these equations are:

$$\begin{aligned} X_0 &= a_0\mathfrak{N}^{-2} + b_0\mathfrak{N}^2 - A\mathfrak{N}^4 \\ Y_0 &= c_0\mathfrak{N}^{-2} + d_0\mathfrak{N}^2 \end{aligned} \tag{100}$$

In which, constants a_0, b_0, c_0, d_0 may be determined by matching with inner solutions.

7.2 Inner expansion near $\mathfrak{N} = 1$

By introducing the variable change:

$$\xi = \frac{\mathfrak{N} - 1}{\lambda} \tag{101}$$

and then, substituting the following single term expansions:

$$X^i = X_0^i(\xi) + \dots, \quad Y^i = Y_0^i(\xi) + \dots \tag{102}$$

into Eqs. (96) and (97), and equating similar powers of λ to zero, we have:

$$\begin{aligned} \frac{d^4 X_0^i}{d\xi^4} - 2 \frac{d^2 Y_0^i}{d\xi^2} &= 0 \\ \frac{d^4 Y_0^i}{d\xi^4} + 2 \frac{d^2 X_0^i}{d\xi^2} &= 0 \end{aligned} \tag{103}$$

Solutions of these equations subject to the related boundary conditions are:

$$\begin{aligned} X_0^i &= -A_2 + (A_2 - A_3)\xi + A_2 e^{-\xi} \cos \xi + A_3 e^{-\xi} \sin \xi \\ Y_0^i &= A_3 - (A_2 + A_3)\xi - A_3 e^{-\xi} \cos \xi + A_2 e^{-\xi} \sin \xi \end{aligned} \tag{104}$$

in which, constants A_2 and A_3 may be determined by matching with outer solution.

7.3 Inner expansion near $\mathfrak{N} = \beta$

By introducing the variable change:

$$\eta = \frac{\beta - \mathfrak{N}}{\lambda} \tag{105}$$

and then, substituting the following single term expansions:

$$X^I = X_0^I(\eta) + \dots, \quad Y^I = Y_0^I(\eta) + \dots \tag{106}$$

into Eqs. (96) and (97), and equating similar powers of λ to zero, we have:

$$\begin{aligned}\beta^2 \frac{d^4 X_0^I}{d\eta^4} - 2\delta \frac{d^2 Y_0^I}{d\eta^2} &= 0 \\ \beta^2 \frac{d^4 Y_0^I}{d\eta^4} + 2\delta \frac{d^2 X_0^I}{d\eta^2} &= 0\end{aligned}\quad (107)$$

in which, $\delta = \sqrt{A\beta^2 + B}$. Solutions of these equations subject to the related boundary conditions are:

$$\begin{aligned}X_0^I &= -B_2 + \frac{\delta}{\beta}(B_2 - B_3)\eta + B_2 e^{\left(\frac{-\delta}{\beta}\eta\right)} \cos\left(\frac{\delta}{\beta}\eta\right) + B_3 e^{\left(\frac{-\delta}{\beta}\eta\right)} \sin\left(\frac{\delta}{\beta}\eta\right) \\ Y_0^I &= B_3 - \frac{\delta}{\beta}(B_2 + B_3)\eta - B_3 e^{\left(\frac{-\delta}{\beta}\eta\right)} \cos\left(\frac{\delta}{\beta}\eta\right) + B_2 e^{\left(\frac{-\delta}{\beta}\eta\right)} \sin\left(\frac{\delta}{\beta}\eta\right)\end{aligned}\quad (108)$$

in which, constants B_2 and B_3 may be determined by matching with outer solution.

7.4 Composite solution

The composite solutions for X and Y are:

$$\begin{aligned}X^c &= X^o + X^i + X^I - (X^o)^i - (X^o)^I \\ Y^c &= Y^o + Y^i + Y^I - (Y^o)^i - (Y^o)^I\end{aligned}\quad (109)$$

Using the following matching conditions:

$$\begin{aligned}(X^i)^o &= (X^o)^i, & (X^I)^o &= (X^o)^I \\ (Y^i)^o &= (Y^o)^i, & (Y^I)^o &= (Y^o)^I\end{aligned}\quad (110)$$

the unknown constants in Eqs. (100), (104) and (108) are obtained, and so the composite solutions become:

$$\begin{aligned}X^c &= -A\beta^4 \frac{1-\beta^2}{1-\beta^4} \mathfrak{R}^{-2} + A \frac{1-\beta^6}{1-\beta^4} \mathfrak{R}^2 - A\mathfrak{R}^4 \\ Y^c &= 0\end{aligned}\quad (111)$$

Therefore ψ_2 becomes:

$$\begin{aligned}\psi_2 &= \frac{-1}{2} A\mathfrak{R}^4 + \left(A - \frac{\beta q_2}{2 q_1} + A\beta^2\right) \mathfrak{R}^2 + \left(\beta \frac{q_2}{q_1} - 2A\beta^2\right) \ln(\mathfrak{R}) \\ &+ \left(-A\beta^4 \frac{1-\beta^2}{1-\beta^4} \mathfrak{R}^{-2} + A \frac{1-\beta^6}{1-\beta^4} \mathfrak{R}^2 - A\mathfrak{R}^4\right) \cos 2\varphi + \dots\end{aligned}\quad (112)$$

8 Velocity components

Stream function ψ can be determined by Eqs. (47), (83) and (112) as follows:

$$\psi = \psi_0 + \gamma \psi_1 + \gamma^2 \psi_2 + O(\gamma^3) \quad (113)$$

Now, by considering velocity components in terms of ψ in Eq. (18), and also having conformal mapping in Eq. (22), we can compute the velocity field using the chain rule as follows:

$$u = \frac{1}{r} \frac{\partial \psi}{\partial \theta} = \frac{1}{r} \frac{\partial \psi}{\partial \varphi} \cdot \frac{\partial \varphi}{\partial \theta}; \quad v = -\frac{\partial \psi}{\partial r} = -\frac{\partial \psi}{\partial \mathfrak{R}} \cdot \frac{\partial \mathfrak{R}}{\partial r} \quad (114)$$

In the case of concentric rotating cylinders, the only non-zero velocity component is v , and the simplified non-dimensional Navier–Stokes equations give the following dimensionless rotational velocity component:

$$v = \frac{b \Omega_2}{a \Omega_1} \frac{1/r - r}{a/b - b/a} + \frac{(a/b)r - (b/a)(1/r)}{a/b - b/a} \quad (115)$$

Our analysis results coincide exactly with this one-dimensional state as the eccentricity vanishes.



9 Results and conclusions

In this study, a closed-form mathematical solution of flow between two eccentric rotating cylinders is presented which is of great importance in calculating velocity profiles inside the flow field. The velocity distributions for $\varepsilon = 0.1, b/a = 1.2, q_2/q_1 = 0.5$ and three different Reynolds number at four various angles ($\theta = 0, \pi/2, \pi, 3\pi/2$) are represented in Fig. 3. As the gap size is decreased from $\theta = 0$ to $\theta = \pi$, it is observed that the velocity values are increased to compensate the same mass flow rate. Figure 4 also illustrates the velocity contours of the whole domain.

As mentioned in [18], small-gap stability theory, for $0 \leq \Omega_2/\Omega_1 \leq 1$, predicts that instability occurs at $Ta_{crit} \approx 1708$. According to our notations, the Taylor number is defined as $Ta = \frac{a(b-a)^3(\Omega_1^2 - \Omega_2^2)}{v^2}$; it may be re-written as $Ta = Re^2(\frac{b}{a} - 1)^3[1 - \frac{q_2^2}{(b/a)^2}]$ in which, $q_{21} = \frac{q_2}{q_1} = \frac{b}{a} \frac{\Omega_2}{\Omega_1}$ and the Reynolds number is defined by (14). Hence, the critical Reynolds number for the case of $b/a = 1.2$ and $q_2/q_1 = 0.5$ is $Re_{crit} \approx 508$. The results presented here are therefore in the range of Taylor stability criterion.

In this study, we deal with a singular perturbation problem arising from the special interesting behavior of the flow field near the solid boundaries. Another advantage of this solution method is to identify the physical behavior of flow, especially near the solid cylindrical walls where the boundary layers are formed. In fact, a multi-layer region is developed in the flow field at high Reynolds numbers; an outer layer as well as some inner

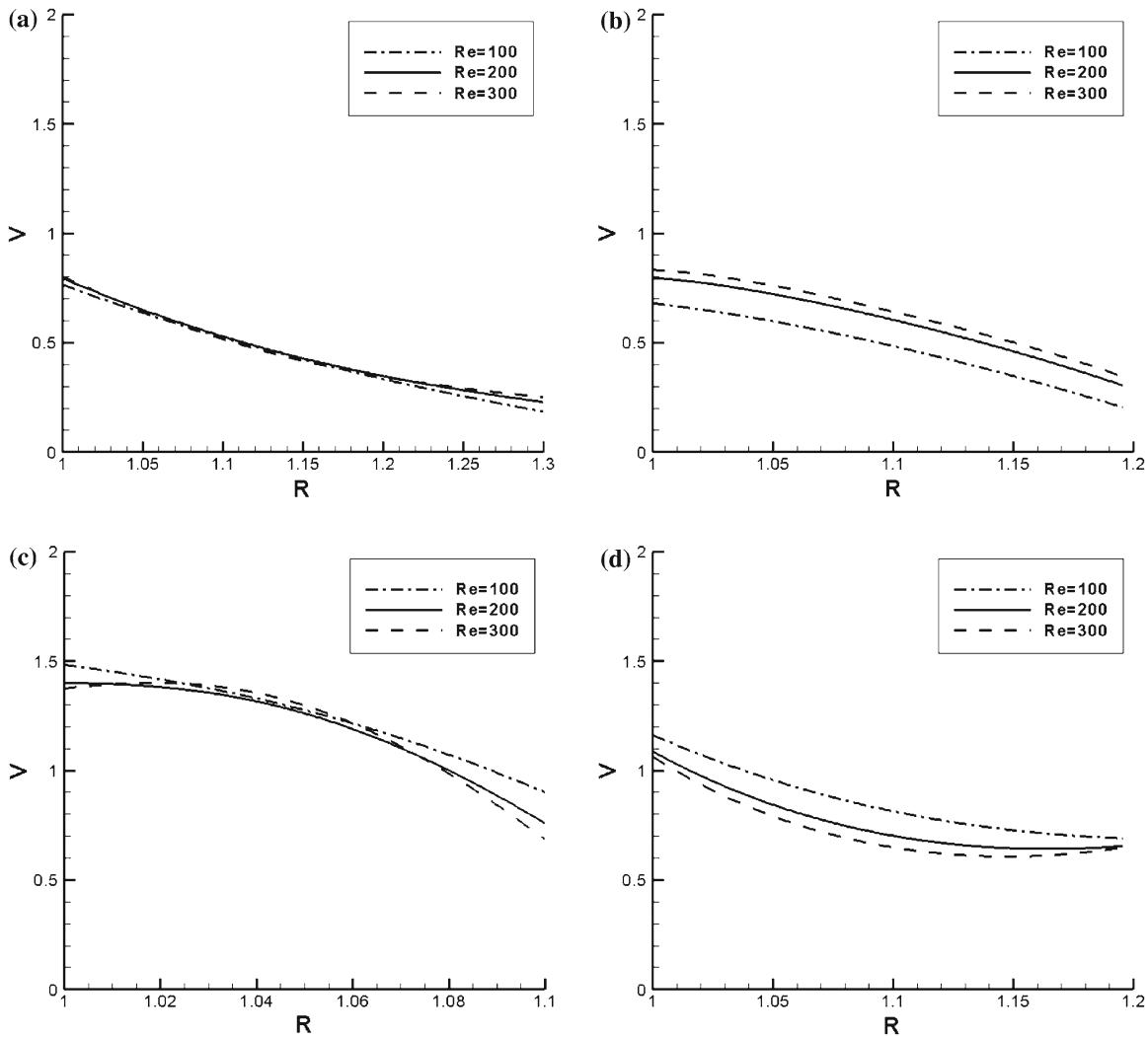


Fig. 3 Velocity profiles for $\varepsilon = 0.1, ba = 1.2, q_{21} = 0.5$ and **a** $\theta = 0$ **b** $\theta = 90$ **c** $\theta = 180$ **d** $\theta = 270$

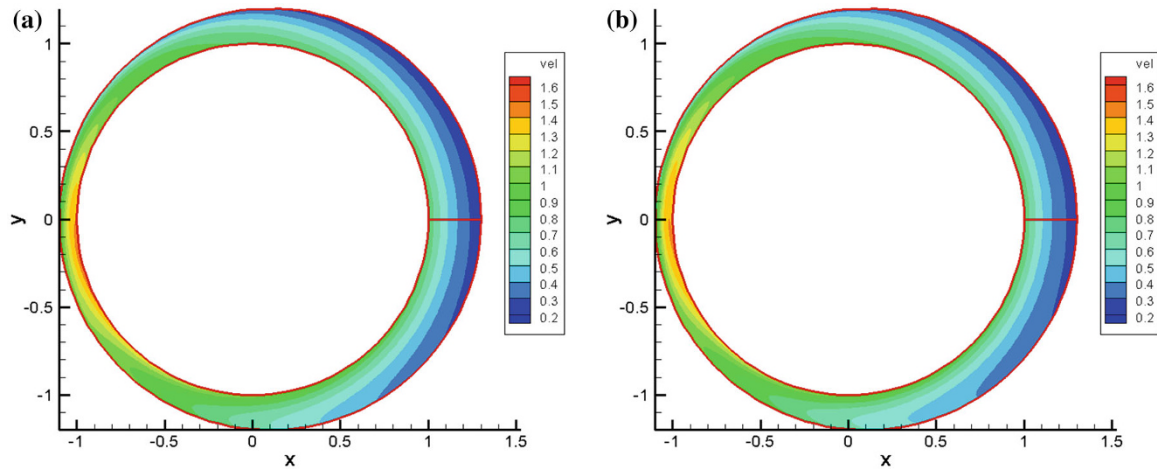


Fig. 4 Velocity distributions for $\varepsilon = 0.1$, $ba = 1.2$, $q_{21} = 0.5$ and **a** $Re = 200$ **b** $Re = 300$

layers may be distinguished inside the domain. This solution is valid for small eccentricity at high Reynolds numbers, when the two cylinders are rotating at the same direction. So this approximate analytic solution can be a good reference to compare some other solutions.

Open Access This article is distributed under the terms of the Creative Commons Attribution License which permits any use, distribution, and reproduction in any medium, provided the original author(s) and the source are credited.

References

- Chernyavsky, V.M.: Exact solution for creeping cylindrical flow in a free-dowel bearing. *Dokl. Phys.* **53**(1), 19–22 (2008)
- Churchill, R.V.; Brown, J.W.; Verhey, R.F.: *Complex Variables and Applications*. McGraw-Hill Book Company, New York (1976)
- Dai, R.X.; Dong, Q.; Szeri, A.Z.: Flow between eccentric rotating cylinders: bifurcation and stability. *Int. J. Eng. Sci.* **30**(10), 1323–1340 (1992)
- Diprima, R.C.; Stuart, J.T.: Flow between eccentric rotating cylinders. *Trans. ASME J. Lubr. Tech.* **94**(3), 266–274 (1972)
- Frene, J.; Godet, M.: Flow transition criteria in a journal bearing. *Trans. ASME J. Lubr. Tech.* **96**, 135–140 (1974)
- Graebel, W.P.: *Advanced Fluid Mechanics*. Elsevier, Amsterdam (2007)
- Hamrock, B.J.: *Fundamentals of Fluid Film Lubrication*. McGraw-Hill Book Company, New York (1994)
- Joukowski, N.E.: Motion of a viscous fluid contained between rotating eccentric cylindrical surfaces. In: *Proceedings of the Khar'kov Mathematical Society*, pp. 34–37 (1887)
- Joukowski, N.E.; Chaplygin, S.A.: Friction of a Lubricated layer between a shaft and its bearing. *Trudy old fiz nauk obshch. Lyub. Estest.* **13**, 24–36 (1904, in Russian)
- Kamal, M.M.: Separation in the flow between eccentric rotating cylinders. *Trans. ASME J. Basic Eng.* **88**(4), 717–724 (1966)
- Kulinski, E.S.; Ostrach, S.: Journal bearing velocity profiles for small eccentricity and moderate modified Reynolds numbers. *Trans. ASME J. Appl. Mech.* **34**(1), 16–22 (1967)
- Nayfeh, A.H.: *Introduction to Perturbation Techniques*. Wiley, New York (1981)
- Petrov, A.G.: The mixing of a viscous fluid in a layer between rotating eccentric cylinders. *J. Appl. Math. Mech.* **72**, 536–549 (2008)
- Schlichting, H.: *Boundary Layer Theory*. McGraw-Hill Book Company, New York (2004)
- Sommerfeld, A.: Zur hydrodynamischen theorie der schmiermittelreibung. *Zeitschrift fur mathematic und physic* **50**, 97 (1904)
- Spiegel, M.R.: *Mathematical Handbook*. Schaum's Outline Series. McGraw-Hill Book Company, New York (2009)
- Wannier, G.H.: A contribution to the hydrodynamics of lubrication. *Quat. Appl. Math.* **6**, 1–32 (1950)
- White, F.M.: *Viscous Fluid Flow*. McGraw-Hill Book Company, New York (1991)
- Wood, W.W.: The asymptotic expansions at large Reynolds numbers for steady motion between non-coaxial rotating cylinders. *J. Fluid Mech.* **3**, 159–175 (1957)

