Arab J Math (2013) 2:313–320
DOI 10.1007/s40065-013-0077-y

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Another generalization of the gcd-sum function

Received: 12 October 2012 / Accepted: 22 May 2013 / Published online: 5 June 2013

Abstract We investigate an arithmetic function representing a generalization of the gcd-sum function, considered by Kurokawa and Ochiai in 2009 in connection with the multivariable global Igusa zeta function for a finite cyclic group. We show that the asymptotic properties of this function are closely connected to the Piltz divisor function. A generalization of Menon's identity is also considered.

Mathematics Subject Classification 11A25 · 11N37 · 11M32

الملخص

نبحث دالة حسابية تمثل تعميماً لدالة مجموع القواسم المشتركة العظمى، والتي تم اعتبارها من قبل كوروكاوا وأوشياي في 2009م من خلال ارتباطها بدالة إيقوسا زيتا الإجمالية متعددة المتغيرات لزمرة دورية منتهية. نثبت أن خصائص تقاربية لهذه الدالة ترتبط بشكل وثيق بدالة قواسم بلتز. نعتبر أيضاً تعميماً لمتطابقة مِنون.

1 Introduction

Let $r \in \mathbb{N} := \{1, 2, \ldots\}$ and define the arithmetic function A_r by

$$A_r(n) := \frac{1}{n^r} \sum_{k_1,\ldots,k_r=1}^{n} \gcd(k_1 \ldots k_r, n) \quad (n \in \mathbb{N}).$$

The function A_r was considered by Kurokawa and Ochiai [6] in connection with certain zeta functions. More exactly, the multivariable global Igusa zeta function for a group A is defined by

$$Z^{\text{group}}(s_1, \ldots, s_r; A) := \sum_{m_1,\ldots,m_r=1}^{\infty} \frac{\# \operatorname{Hom}(A, \mathbb{Z}/m_1 \ldots m_r\mathbb{Z})}{m_1^{s_1} \ldots m_r^{s_r}}. \tag{1}$$

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Consider the case $A = \mathbb{Z}/n\mathbb{Z}$ ($n \in \mathbb{N}$). Since the number of group homomorphisms $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m_1 \dots m_r\mathbb{Z}$ is $\gcd(n, m_1 \dots m_r)$, the function (1) reduces to

$$Z^{\text{group}}(s_1, \dots, s_r; \mathbb{Z}/n\mathbb{Z}) := \sum_{m_1,\dots,m_r=1}^{\infty} \frac{\gcd(m_1 \dots m_r, n)}{m_1^{s_1} \dots m_r^{s_r}}. \tag{2}$$

Kurokawa and Ochiai [6] derived two representations for (2), one of them being

$$Z^{\text{group}}(s_1, \dots, s_r; \mathbb{Z}/n\mathbb{Z}) = \frac{1}{n^{s_1+\dots+s_r}} \sum_{k_1,\dots,k_r=1}^{\infty} \gcd(k_1 \dots k_r, n)\zeta(s_1, k_1/n) \dots \zeta(s_r, k_r/n), \tag{3}$$

where $\zeta(s, a) := \sum_{m=0}^{\infty} 1/(m + a)^s$ denotes the Hurwitz zeta function. It follows from (3) that (2) has a meromorphic continuation to $\mathbb{C}^r$.

Proposition 1.1 [6, Cor. 1] *For every* $n = \prod_{p|n} p^{\nu_p(n)} \in \mathbb{N}$,

$$A_r(n) = \prod_{p|n} \sum_{j=0}^{r} \left(\!\!\binom{\nu_p(n)}{j}\!\!\right) \left(1 - \frac{1}{p}\right)^j, \tag{4}$$

where

$$\left(\!\!\binom{n}{k}\!\!\right) := \binom{n+k-1}{k} = (-1)^k \binom{-n}{k}$$

denotes the number of k*-multisets of an* n*-set.*

Proposition 1.2 [6, Cor. 2] *For every* $n \in \mathbb{N}$,

$$\lim_{r\to\infty} A_r(n) = n. \tag{5}$$

Formula (4) was obtained in [6] as an application of the representations given for (2), while (5) is a direct consequence of (4). Note that (4) was reproved in [7,8] using the arguments of the elementary probability theory.

In the case $r = 1$,

$$A_1(n) := \frac{1}{n} \sum_{k=1}^{n} \gcd(k, n) = \sum_{d|n} \frac{\phi(d)}{d}, \tag{6}$$

where ϕ is Euler's totient function. Here $A_1(n)$ represents the arithmetic mean of $\gcd(1, n), \dots, \gcd(n, n)$ and (4) reduces to

$$A_1(n) = \prod_{p|n} \left(1 + \nu_p(n) \left(1 - \frac{1}{p}\right)\right).$$

See [2,4,14,16] for various properties, analogs and other generalizations of the function (6).

In the present paper we derive a simple recursion formula for the functions A_r, offer a direct number-theoretic proof for the formula (4) and show that the asymptotic properties of the function $A_r(n)$ are closely connected to the Piltz divisor function $\tau_{r+1}(n)$, defined as the number of ways of expressing n as a product of $r + 1$ factors.

As a modification of $A_r(n)$ we also consider and evaluate the function

$$B_r(n) := \sum_{\substack{k_1,\dots,k_r=1 \\ \gcd(k_1 \dots k_r, n)=1}}^{n} \gcd(k_1 \dots k_r - 1, n) \quad (n, r \in \mathbb{N}). \tag{7}$$

Note that in the case $r = 1$,

$$B_1(n) := \sum_{\substack{k=1 \\ \gcd(k,n)=1}}^{n} \gcd(k-1, n) = \phi(n)\tau(n) \quad (n \in \mathbb{N}), \tag{8}$$

where $\tau(n)$ stands for the number of divisors of n, according to a result of Menon [9]. See [12,15] for other Menon-type identities.

Our results are given in Sect. 2, while their proofs are included in Sect. 3.

2 Results

Let $A_0(n) := \mathbf{1}(n) = 1$ $(n \in \mathbb{N})$.

Proposition 2.1 *The following recursion formula holds:*

$$A_r(n) = \sum_{d|n} \frac{\phi(d) A_{r-1}(d)}{d} \quad (n, r \in \mathbb{N}). \tag{9}$$

Let $\overline{\phi}(n) = \phi(n)/n$.

Corollary 2.2 *In terms of the Dirichlet convolution, $A_r = \overline{\phi} A_{r-1} * \mathbf{1}$ $(r \in \mathbb{N})$. Therefore, $A_1 = \overline{\phi} * \mathbf{1}$, $A_2 = \overline{\phi}(\overline{\phi} * \mathbf{1}) * \mathbf{1}$, $A_3 = \overline{\phi}(\overline{\phi}(\overline{\phi} * \mathbf{1}) * \mathbf{1}) * \mathbf{1}$, in general*

$$A_r = \overline{\phi}(\overline{\phi}(\ldots(\overline{\phi} * \mathbf{1})\ldots) * \mathbf{1}) * \mathbf{1}$$

including r times $\overline{\phi}$ and r times $\mathbf{1}$.

Corollary 2.3 *The function A_r is multiplicative for any $r \in \mathbb{N}$.*

Observe that from formula (4),

$$A_r(n) \le \prod_{p|n} \sum_{j=0}^{r} \binom{\nu_p(n) + j - 1}{j} = \prod_{p|n} \binom{\nu_p(n) + r}{r} = \tau_{r+1}(n) \tag{10}$$

for any $n \in \mathbb{N}$, using parallel summation of the binomial coefficients.

Also, $A_r(p^k) = \binom{k+r}{r} + \mathcal{O}(1/p) = \tau_{r+1}(p^k) + \mathcal{O}(1/p)$, as $p \to \infty$ (p prime) for any fixed $k, r \in \mathbb{N}$. This suggests that the asymptotic behavior of $A_r(n)$ is similar to that of $\tau_{r+1}(n)$.

Proposition 2.4 *The Dirichlet series of the function A_r has the representation*

$$\sum_{n=1}^{\infty} \frac{A_r(n)}{n^s} = \zeta^{r+1}(s) F_r(s) \quad (\Re(s) > 1),$$

where the Dirichlet series $F_r(s) := \sum_{n=1}^{\infty} f_r(n)/n^s$ is absolutely convergent for $\Re(s) > 0$. Moreover, for any prime power p^k, $f_r(p^k) = 0$ if $k \ge r+1$ and $f_r(p^k) \ll 1/p$, as $p \to \infty$ if $1 \le k \le r$.

For the function τ_k $(k \ge 2)$ one has

$$\sum_{n \le x} \tau_k(n) = \operatorname*{Res}_{s=1} x^s \frac{\zeta^k(s)}{s} + \Delta_k(x), \tag{11}$$

where the main term is $x P_{k-1}(\log x)$ with a suitable polynomial $P_{k-1}(t)$ in t of degree $k-1$ having the leading coefficient $1/(k-1)!$. For the error term, $\Delta_k(x) = \mathcal{O}(x^{\alpha_k + \varepsilon})$, with $\alpha_k \le (k-1)/(k+1)$ $(k \ge 2)$, $\alpha_k \le (k-1)/(k+2)$ $(k \ge 4)$. See [13, Ch. XII] and [5] for further results on $\Delta_k(x)$.

Proposition 2.5 *Let $r \in \mathbb{N}$. Then*

$$\sum_{n \le x} A_r(n) = x Q_r(\log x) + R_r(x), \tag{12}$$

where $Q_r(t)$ is a polynomial in t of degree r having the leading coefficient

$$\frac{1}{r!} \prod_p \left(1 + \sum_{k=1}^{r} \frac{f_r(p^k)}{p^k} \right),$$

and $R_r(x) = \mathcal{O}(x^{\alpha_{r+1}+\varepsilon})$ (valid for every $\varepsilon > 0$).
Also, $R_r(x) = O(x^{r/(r+2)+\varepsilon})$ and $R_r(x) = \Omega(b_r(x))$, where

$$b_r(x) = (x \log x)^{\frac{r}{2r+2}} (\log_2 x)^{\frac{r+2}{2r+2}((r+1)^{(2r+2)/(r+2)}-1)} (\log_3 x)^{-\frac{3r+2}{4r+4}},$$

$\log_j$ *denoting the j-fold iterated logarithm.*

Proposition 2.6 *For every $r \in \mathbb{N}$,*

$$\limsup_{n \to \infty} \frac{\log A_r(n) \log\log n}{\log n} = \log(r+1). \tag{13}$$

In the case $r = 1$, formulae (12), without the omega result, and (13) were obtained by Chidambaraswamy and Sitaramachandrarao [3, Th. 3.1, 4.1]. In fact, both results were proved in [3] for a slightly more general function, namely for $\psi_k(n) = \sum_{d|n} \phi_k(d)/d^k$, where $k \in \mathbb{N}$ and $\phi_k(n) = n^k \prod_{p|n}(1 - 1/p^k)$ is the Jordan function of order k. Here $A_1(n) = \psi_1(n)/n$.

For the function $B_r(n)$ defined by (7) we have

Proposition 2.7 *For every $n, r \in \mathbb{N}$,*

$$B_r(n) = \phi^r(n)\tau(n).$$

3 Proofs

Proof of Proposition 2.1

$$A_r(n) = \frac{1}{n^r} \sum_{k_1,\dots,k_r=1}^{n} \sum_{d|\gcd(k_1\dots k_r,n)} \phi(d) = \frac{1}{n^r} \sum_{d|n} \phi(d) \sum_{\substack{k_1,\dots,k_r=1 \\ k_1\dots k_r \equiv 0 \ (\mathrm{mod}\ d)}}^{n} 1,$$

where for fixed $k_1, \dots, k_{r-1}$ the congruence $k_1 \dots k_{r-1} k_r \equiv 0 \pmod d$ has $\gcd(k_1 \dots k_{r-1}, d)$ solutions $k_r \pmod d$ and has $(n/d) \gcd(k_1 \dots k_{r-1}, d)$ solutions $k_r \pmod d$. Therefore,

$$A_r(n) = \frac{1}{n^{r-1}} \sum_{d|n} \frac{\phi(d)}{d} \sum_{k_1,\dots,k_{r-1}=1}^{n} \gcd(k_1 \dots k_{r-1}, d), \tag{14}$$

and writing $k_j = dq_j + s_j$ with $1 \le s_j \le d$, $0 \le q_j \le n/d - 1$ $(1 \le j \le r-1)$ we see that the inner sum is

$$\sum_{\substack{1 \le s_1,\dots,s_{r-1} \le d \\ 0 \le q_1,\dots,q_{r-1} \le n/d-1}} \gcd(s_1 \dots s_{r-1}, d) = \left(\frac{n}{d}\right)^{r-1} d^{r-1} A_{r-1}(d),$$

and inserting this into (14) we obtain (9). □

Proof of Proposition 1.1 The function $n \mapsto A_r(n)$ is multiplicative by Corollary 2.3. Therefore, to obtain (4) it is sufficient to consider the case $n = p^k$ ($k \in \mathbb{N}$), a prime power. Let $x_r(k) := A_r(p^k)$ ($r \geq 0$) with a fixed prime p. From the recursion formula (9) we have

$$A_r(p^k) = 1 + \sum_{j=1}^{k} \left(1 - \frac{1}{p}\right) A_{r-1}(p^j),$$

that is, by denoting $t := 1 - 1/p$,

$$x_r(k) = 1 + t \sum_{j=1}^{k} x_{r-1}(j) \quad (r, k \in \mathbb{N}), \tag{15}$$

where $x_0(k) := A_0(p^k) = 1$ ($k \in \mathbb{N}$). Here $x_1(k) = 1 + t \sum_{j=1}^{k} x_0(j) = 1 + kt$, $x_2(k) = 1 + t \sum_{j=1}^{k} x_1(j) = 1 + t \sum_{j=1}^{k}(1 + jt) = 1 + kt + \frac{k(k+1)}{2}t^2$, $x_3(k) = 1 + t \sum_{j=1}^{k} x_2(j) = 1 + t \sum_{j=1}^{k} \left(1 + jt + \frac{j(j+1)}{2}t^2\right) = 1 + kt + \frac{k(k+1)}{2}t^2 + \frac{k(k+1)(k+2)}{6}t^3$.

We show by induction on r that $x_r(k)$ is a polynomial in t of degree r with integer coefficients which do not depend on r, more exactly,

$$x_r(k) = 1 + \sum_{i=1}^{r} \left(\!\!\binom{k}{i}\!\!\right) t^i. \tag{16}$$

Assume that (16) is valid for r. Then by (15) we obtain for $r+1$,

$$\begin{aligned} x_{r+1}(k) &= 1 + t \sum_{j=1}^{k} x_r(j) = 1 + t \sum_{j=1}^{k} \left(1 + \sum_{i=1}^{r} \left(\!\!\binom{j}{i}\!\!\right) t^i\right) \\ &= 1 + kt + \sum_{i=1}^{r} t^{i+1} \sum_{j=1}^{k} \binom{j+i-1}{i} = 1 + \sum_{i=0}^{r} \binom{k+i}{i+1} t^{i+1} \\ &= 1 + \sum_{i=1}^{r+1} \binom{k+i-1}{i} t^i = 1 + \sum_{i=1}^{r+1} \left(\!\!\binom{k}{i}\!\!\right) t^i, \end{aligned}$$

applying the upper summation formula. This completes the proof of (4). □

Proof of Proposition 2.4 We use the conventions $\binom{a}{0} = 1$ ($a \in \mathbb{Z}$), $\binom{a}{b} = 0$ ($a, b \in \mathbb{N}$, $a < b$). In terms of the Dirichlet convolution, $A_r = \tau_{r+1} * f_r$, $f_r = A_r * \mu^{(r+1)}$ with $\mu^{(r+1)} = \mu * \cdots * \mu$ ($r+1$ times), where $\mu^{(r+1)}(p^k) = (-1)^k \binom{r+1}{k}$ for any prime power p^k ($k \in \mathbb{N}$).

Hence for any $k \in \mathbb{N}$,

$$\begin{aligned} f_r(p^k) &= \sum_{\ell=0}^{k} \mu^{(r+1)}(p^\ell) A_r(p^{k-\ell}) = \sum_{\ell=0}^{k} (-1)^\ell \binom{r+1}{\ell} \sum_{j=0}^{r} \binom{j+k-\ell-1}{j} \left(1 - \frac{1}{p}\right)^j \\ &= \sum_{j=0}^{r} \left(1 - \frac{1}{p}\right)^j \sum_{\ell=0}^{k} (-1)^\ell \binom{r+1}{\ell} \binom{j+k-\ell-1}{j}, \end{aligned} \tag{17}$$

which is a polynomial in $1/p$ of degree r.

Here for any $k \geq r+1$,

$$f_r(p^k) = \sum_{j=0}^{r} \left(1 - \frac{1}{p}\right)^j \sum_{\ell=0}^{r+1} (-1)^\ell \binom{r+1}{\ell} \binom{j+k-\ell-1}{j} = 0,$$

since $\binom{j+k-\ell-1}{j}$ is a polynomial in ℓ of degree j and the inner sum is zero for any $0 \le j \le r$ using the identity

$$\sum_{\ell=0}^{n}(-1)^{\ell}\ell^{j}\binom{n}{\ell} = 0 \quad (0 \le j \le n-1).$$

Now for $1 \le k \le r$ we obtain from (17) that the constant term of the polynomial in $1/p$ giving $f_r(p^k)$ is

$$\begin{aligned} c &:= \sum_{j=0}^{r}\sum_{\ell=0}^{k}(-1)^{\ell}\binom{r+1}{\ell}\binom{j+k-\ell-1}{j} \\ &= \sum_{\ell=0}^{k}(-1)^{\ell}\binom{r+1}{\ell}\sum_{j=0}^{r}\binom{j+k-\ell-1}{j} \\ &= \sum_{\ell=0}^{k}(-1)^{\ell}\binom{r+1}{\ell}\binom{r+k-\ell}{r}, \end{aligned}$$

using parallel summation again.

Using now that $\binom{r+k-\ell}{r} = (-1)^{k-\ell}\binom{-r-1}{k-\ell}$ we obtain

$$c = (-1)^{k}\sum_{\ell=0}^{k}\binom{r+1}{\ell}\binom{-(r+1)}{k-\ell} = 0,$$

by Vandermonde's identity.

Therefore, $f_r(p^k) \ll 1/p$, as $p \to \infty$ for any $k \in \{1,\ldots,r\}$. This shows that the Dirichlet series $F_r(s)$ is absolutely convergent for $\Re(s) > 0$. □

Proof of Proposition 2.5 Using Proposition 2.4 and (11) for $k = r+1$,

$$\begin{aligned} \sum_{n\le x} A_r(n) &= \sum_{d\le x} f_r(d)\sum_{e\le x/d}\tau_{r+1}(e) \\ &= \sum_{d\le x} f_r(d)\left(\frac{x}{d}P_r(\log(x/d)) + \Delta_{r+1}(x/d)\right), \end{aligned}$$

and (12) follows by usual estimates.

To obtain the omega result let g_r denote the inverse under Dirichlet convolution of the function f_r. Then g_r is multiplicative, $\tau_{r+1} = g_r * A_r$, so that

$$\sum_{n\le x}\tau_{r+1}(n) = \sum_{d\le x} g_r(d)\sum_{e\le x/d} A_r(e),$$

and the Dirichlet series $\sum_{n=1}^{\infty} g_r(n)/n^s$ is absolutely convergent for $\Re(s) > 0$. Now apply the Ω-result concerning the function τ_k, due to Soundararajan [10], for $k = r+1$. In the case $r = 1$,

$$\sum_{n\le x}\tau(n) = \sum_{d\le x}\frac{1}{d}\sum_{e\le x/d} A_1(e) = x\log x + (2\gamma-1)x + \sum_{d\le x}\frac{1}{d}R_1(x/d) + O(\log x). \tag{18}$$

Assume that $R_1(x) = \Omega(b_1(x))$ does not hold. Then for every $c > 0$ there exists $x_c > 0$ such that $|R_1(x)| \le c\, b_1(x)$ for any $x \ge x_c$. Now inserting this into (18) contradicts that $\Delta(x) = \Omega(b(x))$. The same proof works out also for $r \ge 2$. □

Proof of Proposition 2.6 Similar to the proof of [3, Th. 4.1]. By (10), $A_r(n) \le \tau_{r+1}(n)$ $(n \in \mathbb{N})$. Therefore, using that (13) holds for $\tau_{r+1}(n)$ instead of $A_r(n)$ [11, Eq. 3.4] we obtain that the given lim sup is $\le \log(r+1)$.

Furthermore, for squarefree n,

$$\begin{aligned} A_r(n) = \prod_{p\mid n}\sum_{j=0}^{r}(1-1/p)^j &= \prod_{p\mid n} p(1-(1-1/p)^{r+1}) \\ &= \prod_{p\mid n}\left(r+1-\frac{r(r+1)}{2}\cdot\frac{1}{p}+O(1/p^2)\right) \\ &= (r+1)^{\omega(n)}\prod_{p\mid n}\left(1-\frac{r}{2}\cdot\frac{1}{p}+O(1/p^2)\right), \end{aligned}$$

as $p \to \infty$ (for every fixed r).

Let $n_x = \prod_{x/\log x < p \le x} p$. Then

$$\begin{aligned} &\frac{\log A_r(n_x)\log\log n_x}{\log n_x} \\ &\quad = \log(r+1)\frac{\omega(n_x)\log\log n_x}{\log n_x} + \frac{\log\log n_x}{\log n_x}\log\prod_{p\mid n_x}\left(1-\frac{r}{2}\cdot\frac{1}{p}+O(1/p^2)\right). \end{aligned}$$

By using familiar estimates, $\log n_x \sim x$, $\log\log n_x \sim \log x$ and $\omega(n_x) \sim x/\log x$. Hence $\omega(n_x)\log\log n_x/\log n_x \to 1$, as $x \to \infty$.

Also, $\prod_{p\le x}\left(1-\frac{r}{2}\cdot\frac{1}{p}+O(1/p^2)\right) \sim C_r/(\log x)^{r/2}$ with a suitable constant C_r. Therefore, $\prod_{p\mid n_x}\left(1-\frac{r}{2}\cdot\frac{1}{p}+O(1/p^2)\right) \to 1$ as $x \to \infty$, and the result follows. □

Proof of Proposition 2.7 We use the following lemma, which follows easily by the inclusion–exclusion principle (cf. [1, Th. 5.32]).

Lemma 3.1 *Let* $n, d, x \in \mathbb{N}$ *be such that* $d \mid n$, $1 \le x \le d$, $\gcd(x, d) = 1$. *Then*

$$\#\{k \in \mathbb{N} : 1 \le k \le n, k \equiv x (\text{mod } d), \gcd(k, n) = 1\} = \phi(n)/\phi(d).$$

We also need the following identity, which reduces to (8) in the case $a = 1$.

Lemma 3.2 *Let* $\gcd(a, n) = 1$. *Then*

$$\sum_{\substack{k=1 \\ \gcd(k,n)=1}}^{n} \gcd(ak-1, n) = \phi(n)\tau(n) \quad (n \in \mathbb{N}).$$

For the proof of Lemma 3.2 write

$$\sum_{\substack{k=1 \\ \gcd(k,n)=1}}^{n} \gcd(ak-1, n) = \sum_{\substack{k=1 \\ \gcd(k,n)=1}}^{n} \sum_{d\mid\gcd(ak-1,n)} \phi(d) = \sum_{d\mid n}\phi(d) \sum_{\substack{1\le k\le n \\ \gcd(k,n)=1 \\ ak\equiv 1 (\text{mod } d)}} 1,$$

and observe that for every $d \mid n$ the congruence $ak \equiv 1$ (mod d) has a unique solution (mod d), since $\gcd(a, n) = 1$. Therefore the inner sum is $\phi(n)/\phi(d)$ by Lemma 3.1. See also [15, Cor. 14].

Now for the proof of Proposition 2.7,

$$B_r(n) = \sum_{\substack{k_1,\ldots,k_{r-1}=1 \\ \gcd(k_1\ldots k_{r-1},n)=1}}^{n} \sum_{\substack{k_r=1 \\ \gcd(k_r,n)=1}}^{n} \gcd((k_1\ldots k_{r-1})k_r-1, n),$$

and applying Lemma 3.2 for $a = k_1 \dots k_{r-1}$ we obtain that the inner sum is $\phi(n)\tau(n)$. Hence,

$$B_r(n) = \sum_{\substack{k_1,\dots,k_{r-1}=1 \\ \gcd(k_1\dots k_{r-1},n)=1}}^{n} \phi(n)\tau(n) = \phi^r(n)\tau(n).$$

□

Acknowledgments The author gratefully acknowledges support from the Austrian Science Fund (FWF) under the projects No. P20847-N18 and M1376-N18. The author thanks Professor Werner Georg Nowak for very helpful discussions on the subject.

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