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On widths of periodic functions in L_2

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Abstract Exact values are obtained of the n-widths of 2π-periodic functions of the form

$$f(x) = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{K}(x-t)\varphi(t)\mathrm{d}t = (\mathcal{K} * \varphi)(x)$$

in space $L_2[0, 2\pi]$ and satisfy condition

$$\left(\int_0^h \omega_m^p(\varphi; t)\sin^\gamma nt\mathrm{d}t\right)^{1/p} \le 1,\ \ 0 < h \le \pi/n, \gamma > 0, 0 < p \le 2,$$

where $\omega_m(\varphi; t)-m$th order modulus of continuity of function $\varphi(x) \in L_2[0, 2\pi]$. Some further generalizations are included.

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الملخص

يتم الحصول على القيم المضبوطة لـِ n-عرض الدوال الدروية-2π والتي تأخذ الشكل

$$f(x) = \frac{1}{2\pi}\int_0^{2\pi} \kappa(x-t)\varphi(t)dt = (\kappa * \varphi)(x)$$

في الفضاء $L_2[0, 2\pi]$ وتحقق الشرط

$$\left(\int_0^h \omega_m^p(\varphi; t) sin^\gamma nt dt\right)^{\frac{1}{p}} \le 1,\ \ 0 < h \le \frac{\pi}{n},\ \ \gamma > 0,\ \ 0 < p \le 2,$$

حيث $\omega_m(\varphi; t)$ يمثل رتبة الاستمرار-m للدالة $\varphi(x) \in L_2[0, 2\pi]$. نعرض أيضا بعض التعميمات الإضافية.

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1 Introduction

Let $L_2 \equiv L_2[0, 2\pi]$ denote a space of Lebesgue measurable 2π-periodic real functions $f(x)$ with finite norm

$$||f|| = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2 \mathrm{d}x \right\}^{1/2} < \infty.$$

We will study certain issues regarding best trigonometric polynomial approximation of $f(x) \in L_2$ which can be represented as convolution

$$f(x) \stackrel{df}{=} (\mathcal{K} * \varphi)(x) = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{K}(x-t)\varphi(t)\mathrm{d}t, \tag{1}$$

where $\mathcal{K}(t) \in L_2$, $\varphi(t) \in L_2$ with following Fourier series

$$\mathcal{K}(t) \sim \sum_{l=-\infty}^{+\infty} a_l \mathrm{e}^{ilt}, a_l = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{K}(t)\mathrm{e}^{-ilt}\mathrm{d}t, \quad l = 0, \pm 1, \pm 2, \pm 3, \ldots \tag{2}$$

$$\varphi(t) \sim \sum_{l=-\infty}^{+\infty} b_l \mathrm{e}^{ilt}, b_l = \frac{1}{2\pi} \int_0^{2\pi} \varphi(t)\mathrm{e}^{-ilt}\mathrm{d}t, \quad l = 0, \pm 1, \pm 2, \pm 3, \ldots \tag{3}$$

For $\varphi(t) \in L_2$ let us denote as $\Delta_m(\varphi; h)$ the L_2-difference norm of mth order with step h

$$\Delta_m(\varphi; h) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{k=0}^{m} (-1)^k \binom{m}{k} \varphi(x + (m-k)h) \right|^2 \mathrm{d}x \right\}^{1/2},$$

and denote *modulus of continuity* of m-order of function $\varphi(t) \in L_2$ as

$$\omega_m(\varphi; \delta) = \sup\{\Delta_m(\varphi; h) : |h| \le \delta\}.$$

A set of all trigonometric polynoms of order not higher than n we denote as

$$\mathcal{T}_n = \left\{ T_n(t) : T_n(t) = \sum_{|k| \le n} c_k \mathrm{e}^{ikt} \right\}.$$

Expression

$$E_n(f) = E(f, \mathcal{T}_{n-1}) = inf\left\{ \|f - T_{n-1}\| : T_{n-1}(t) \in \mathcal{T}_{n-1} \right\}$$

will denote the best approximation of function $f(x) \in L_2$ by subspace $\mathcal{T}_{n-1}$. From (2), (3), it immediately follows that

$$f(x) \sim \sum_{k=-\infty}^{+\infty} a_k b_k \mathrm{e}^{ikx}. \tag{4}$$

It is well known that the best approximation of function $f(x)$ by subspace $\mathcal{T}_{n-1}$ is expressed by a partial sum

$$S_{n-1}(f; x) = \sum_{k=-n+1}^{n-1} a_k b_k \mathrm{e}^{ikx}$$

of the Fourier series (4). Then

$$E_n(f) = \|f - S_{n-1}(f)\| = \left\{ \sum_{|k|\geq n} |a_k b_k|^2 \right\}^{1/2}. \tag{5}$$

In approximation theory in L_2, problems of finding the exact constants in Jackson-type inequalities

$$E_n(f) \leq \chi n^{-r} \omega_m \left(f^{(r)}, \frac{t}{n} \right),\ f(x) \in L_2^r,\ t > 0,$$

were studied, for instance, in [1–3,5,7–13,15–20], where various approximative characteristics which lead to improving bounds from above for estimates of constant χ, are considered.

Here, we study approximative properties of convolution (1) and consider the following extremal characteristic

$$\chi_{m,n,p,\gamma}(h) \overset{df}{=} \sup_{\substack{\varphi \in L_2 \\ \varphi \neq const}} \frac{2^m |a_n|^{-1} E_n(f)}{\left(\int_0^h \omega_m^p(\varphi; t) \sin^\gamma nt \mathrm{d}t \right)^{1/p}}, \tag{6}$$

where $m, n \in \mathbb{N}, \gamma \geq 0, 0 < p \leq 2, 0 < h \leq \pi/n$, a_n-Fourier coefficient of function $\mathcal{K}(t)$, defined by (2).

Theorem 1.1 *For arbitrary function $\mathcal{K}(t) \in L_2$, whose Fourier coefficients satisfy $|a_0| \neq 0$, $|a_k| k^{1/p} \geq |a_{k+1}|(k+1)^{1/p}$, $k \geq 1$, $0 < p \leq 2$ for any $m, n \in \mathbb{N}$ and arbitrary $\gamma \geq 0$, $0 < h \leq \pi/n$ it holds that*

$$\chi_{m,n,p,\gamma}(h) = \left\{ \int_0^h \left(\sin \frac{nt}{2} \right)^{mp} \sin^\gamma nt \mathrm{d}t \right\}^{-1/p}. \tag{7}$$

There exists a function $f_0(x) \in L_2$ which can be represented as convolution (1) for which the upper bound in (6) is attained and the equality (7) then holds.

Proof Obviously, for function $\varphi(t) \in L_2$ with Fourier representation (3) the following inequality holds

$$\omega_m(\varphi; t) \geq \Delta_m(\varphi; t) \geq \left\{ \sum_{|k|\geq n} |b_k|^2 \left(2 \sin \frac{kt}{2} \right)^{2m} \right\}^{1/2}.$$

Let us use the Minkowski inequality (e.g., [4], p. 32)

$$\left(\int_0^h \left(\sum_{|k|\geq n} |f_k(t)|^2 \right)^{p/2} \mathrm{d}t \right)^{1/p} \geq \left(\sum_{|k|\geq n} \left(\int_0^h |f_k(t)|^p \mathrm{d}t \right)^{2/p} \right)^{1/2},\ 0 < p \leq 2.$$

We obtain:

$$\begin{aligned} \left(\int_0^h \omega_m^p(\varphi; t) \sin^\gamma nt \mathrm{d}t \right)^{1/p} &\geq \left(\int_0^h \left\{ \sum_{|k|\geq n} |b_k|^2 \left(2 \sin \frac{kt}{2} \right)^{2m} \cdot (\sin nt)^{2\gamma/p} \right\}^{p/2} \mathrm{d}t \right)^{1/p} \\ &= \left(2^m \int_0^h \left\{ \sum_{|k|\geq n} |b_k|^2 \cdot (1 - \cos kt)^m \cdot (\sin nt)^{2\gamma/p} \right\}^{p/2} \mathrm{d}t \right)^{1/p} \\ &\geq \left(2^m \sum_{|k|\geq n} \left(|b_k|^p \cdot \int_0^h (1 - \cos kt)^{mp/2} \cdot \sin^\gamma nt \mathrm{d}t \right)^{2/p} \right)^{1/2}. \end{aligned} \tag{8}$$

In the work [9] particularly it's proved that the function natural argument

$$\varphi(k)=k\int\limits_0^h(1-\cos kt)^{mp/2}\cdot\sin^\gamma nt\mathrm{d}t$$

does not decrease under the pointed meanings of parameters p, h, γ in the sphere $Q=\{k:|k|\geq n\}$ derivative $\varphi^{'}(k)>0$, therefore

$$\min\{\varphi(k):|k|\geq n\}=\varphi(n)=n\int\limits_0^h(1-\cos nt)^{mp/2}\sin^\gamma nt\mathrm{d}t$$

where inequality follows

$$\int\limits_0^h(1-\cos kt)^{mp/2}\cdot\sin^\gamma nt\mathrm{d}t\geq\frac{n}{k}\int\limits_0^h(1-\cos nt)^{mp/2}\cdot\sin^\gamma nt\mathrm{d}t. \tag{9}$$

In accordance with theorem related to Fourier coefficients $\{a_k\}$ of series (2) follows that $|a_n|\cdot n^{1/p}\geq|a_k|\cdot k^{1/p},\ k\geq n,\ 0<p\leq 2$ and so we have $n\cdot|a_n|^p\geq k\cdot|a_k|^p$ or $\frac{n}{k}\geq\left|\frac{a_k}{a_n}\right|^p$ taking in account inequality (9) we get

$$\int\limits_0^h(1-\cos kt)^{mp/2}\cdot\sin^\gamma nt\mathrm{d}t\geq\left|\frac{a_k}{a_n}\right|^p\int\limits_0^h(1-\cos nt)^{mp/2}\cdot\sin^\gamma nt\mathrm{d}t$$

Using the last inequality let's continue (8)

$$\begin{aligned}&\geq\left(2^m\sum_{|k|\geq n}\left(|b_k|^p\cdot\left|\frac{a_k}{a_n}\right|^p\cdot\int\limits_0^h(1-\cos nt)^{mp/2}\cdot\sin^\gamma nt\mathrm{d}t\right)^{2/p}\right)^{1/2}\\&=\frac{2^m}{|a_n|}\cdot\left(\int\limits_0^h\left(\sin\frac{nt}{2}\right)^{mp}\sin^\gamma nt\mathrm{d}t\right)^{1/p}\cdot\left\{\sum_{|k|\geq n}|a_kb_k|^2\right\}^{1/2}\\&=\frac{2^m}{|a_n|}\cdot\left(\int\limits_0^h\left(\sin\frac{nt}{2}\right)^{mp}\sin^\gamma nt\mathrm{d}t\right)^{1/p}\cdot E_n(f),\end{aligned}$$

which implies

$$\frac{2^m|a_n|^{-1}E_n(f)}{\left(\int_0^h\omega_m^p(\varphi;t)\sin^\gamma nt\mathrm{d}t\right)^{1/p}}\leq\left\{\int\limits_0^h\left(\sin\frac{nt}{2}\right)^{mp}\sin^\gamma nt\mathrm{d}t\right)^{-1/p}, \tag{10}$$

or, equivalently

$$\chi_{m,n,p,\gamma}(h)\leq\left\{\int\limits_0^h\left(\sin\frac{nt}{2}\right)^{mp}\sin^\gamma nt\mathrm{d}t\right)^{-1/p}. \tag{11}$$

The upper bound for $\chi_{m,n,p,\gamma}(h)$ is obtained.

In order to obtain the lower bound, it suffices to consider in L_2 a function (convolution)

$$f_0(x) = (\mathcal{K} * \varphi_0)(x) = a_n \mathrm{e}^{inx}, \varphi_0(t) = \mathrm{e}^{int},$$

and easily verified relations

$$E_n(f_0) = |a_n|,$$
$$\omega_m(\varphi_0; t) = 2^m \left(\sin \frac{nt}{2}\right)^m, \ 0 < t \le \pi/n.$$

Using the definition (6) of $\chi_{m,n,p,\gamma}(h)$ we write

$$\chi_{m,n,p,\gamma}(h) \ge \frac{2^m \cdot |a_n|^{-1} \cdot E_n(f_0)}{\left(\int_0^h \omega_m^p(\varphi_0, t) \sin^\gamma nt \mathrm{d}t\right)^{1/p}} = \left\{ \int_0^h \left(\sin \frac{nt}{2}\right)^{mp} \sin^\gamma nt \mathrm{d}t \right)^{-1/p}, \tag{12}$$

Combining upper (11) and lower (12) bounds gives us the desired equality (7). Theorem 1.1 is proven. □

2 Main theorems

We recall the necessary concepts and definitions which will be used later.

Let S be the unit ball in L_2, $\mathfrak{M}$ a convex centrally symmetric set in L_2, $\Lambda_n \subset L_2$ an n-dimensional space, $\Lambda^n \subset L_2$ a subspace of codimension n, $\mathcal{L}: L_2 \to \Lambda_n$ a continuous linear operator, and $\mathcal{L}^\perp : L_2 \to \Lambda_n$ a continuous orthogonal projection operator. The quantities

$$b_n(\mathfrak{M}, L_2) = \sup\{\sup\{\varepsilon > 0;\ \varepsilon S \cap \Lambda_{n+1} \subset \mathfrak{M}\} : \Lambda_{n+1} \subset L_2\},$$
$$d_n(\mathfrak{M}, L_2) = \inf\{\sup\{\inf\{\|f - g\| :\ g \in \Lambda_n\} : f \in \mathfrak{M}\} : \Lambda_n \subset L_2\},$$
$$\delta_n(\mathfrak{M}, L_2) = \inf\{\inf\{\sup\{\|f - \mathcal{L}f\| :\ f \in \mathfrak{M}\} :\ \mathcal{L}L_2 \subset \Lambda_n\} : \Lambda_n \subset L_2\},$$
$$d^n(\mathfrak{M}, L_2) = \inf\left\{\sup\left\{\|f\| :\ f \in \mathfrak{M} \cap \Lambda^n\right\} : \Lambda^n \subset L_2\right\},$$
$$\Pi_n(\mathfrak{M}, L_2) = \inf\left\{\inf\left\{\sup\left\{\|f - \mathcal{L}^\perp f\| :\ f \in \mathfrak{M}\right\} :\ \mathcal{L}^\perp L_2 \subset \Lambda_n\right\} : \Lambda_n \subset L_2\right\},$$

are called, correspondingly, Bernstein, Kolmogorov, linear, Gelfand, and projection n-widths of the set $\mathfrak{M}$ in the space L_2. Since L_2 is a Hilbert space, the n-widths listed above are related by (see, e.g., [6,14]):

$$b_n(\mathfrak{M}, L_2) \le d^n(\mathfrak{M}, L_2) \le d_n(\mathfrak{M}, L_2) = \delta_n(\mathfrak{M}, L_2) = \Pi_n(\mathfrak{M}, L_2). \tag{13}$$

For $m, n \in \mathbb{N}$, arbitrary $0 < p \le 2$, $\gamma \ge 0$ and $0 < h \le \pi/n$ in L_2 let us define a class of functions

$$\mathcal{F} \equiv \mathcal{F}(m, n, p, \gamma, h) \stackrel{df}{=} \left\{ f(x) = (\mathcal{K} * \varphi)(x) : \left(\int_0^h \omega_m^p(\varphi; t) \sin^\gamma nt \mathrm{d}t \right)^{1/p} \le 1 \right\}.$$

Theorem 2.1 *It holds that*

$$\lambda_{2n}(\mathcal{F}; L_2) = \lambda_{2n-1}(\mathcal{F}; L_2) = E_n(\mathcal{F}) = 2^{-m}|a_n|\chi_{m,n,p,\gamma}(h), 0 < p \le 2, \tag{14}$$

where

$$E_n(\mathcal{F}) = \sup\{E_n(f) :\ f \in \mathcal{F}\},$$

$\lambda_n(\cdot)$-any of the above-listed n-widths $b_n(\cdot)$, $d^n(\cdot)$, $d_n(\cdot)$, $\lambda_n(\cdot)$ or $\Pi_n(\cdot)$. In particular, if $h = \pi/n$, then

$$\lambda_{2n}(\mathcal{F}; L_2) = \lambda_{2n-1}(\mathcal{F}; L_2) = E_n(\mathcal{F}) = 2^{-m}|a_n|\chi_{m,n,p,\gamma}(\pi/n)$$
$$= 2^{-\left(m+\frac{\gamma}{p}\right)}|a_n|n^{1/p}\left\{\frac{\Gamma\left(\frac{mp+\gamma+1}{2}\right)\Gamma\left(\frac{\gamma+1}{2}\right)}{\Gamma\left(\frac{mp}{2}+\gamma+1\right)}\right\}^{-1/p},$$

where $\Gamma(u)$-is Euler's gamma function.

Proof From inequality (10) for an arbitrary function $f(x) \in \mathcal{F}$ we obtain:

$$E_n(\mathcal{F}) = \sup\{E_n(f) : f \in \mathcal{F}\}$$
$$\leq 2^{-m}|a_n|\left\{\int_0^h \left(\sin\frac{nt}{2}\right)^{mp} \sin^\gamma nt\,\mathrm{d}t\right)^{-1/p} = 2^{-m}|a_n|\chi_{m,n,p,\gamma}(h)$$

from which, considering (12), we derive an upper bound for all listed widths

$$\lambda_{2n}(\mathcal{F}; L_2) \leq \lambda_{2n-1}(\mathcal{F}; L_2) \leq E_n(\mathcal{F}) \leq 2^{-m}|a_n|\chi_{m,n,p,\gamma}(h). \tag{15}$$

In order to obtain a lower bound in subspace $\mathcal{T}_n$, let us consider a ball

$$\mathcal{B}_{2n+1} \stackrel{df}{=} \left\{T_n(x) \in \mathcal{T}_n : \|T_n\| \leq 2^{-m}|a_n|\chi_{m,n,p,\gamma}(h)\right\}$$

and show that it belongs to class $\mathcal{F}$.

Let $T_n(x) = \sum_{k=-n}^{n} c_k \mathrm{e}^{ikx} \in \mathcal{B}_{2n+1}$. Since according to conditions in Theorem 1.1 $a_k \neq 0, k = -n, \ldots, n$, function

$$\varphi(t) = \sum_{k=-n}^{n} (c_k/a_k)\mathrm{e}^{ikt}$$

satisfies the convolution

$$T_n(x) = \frac{1}{2}\int_0^{2\pi} \mathcal{K}(x-t)\varphi(t)\mathrm{d}t,$$

we should prove that

$$\left(\int_0^h \omega_m^p(\varphi; t)\sin^\gamma nt\,\mathrm{d}t\right)^{1/p} \leq 1.$$

For this we need an inequality from [6, p.104]

$$\omega_m(\varphi; t) \leq 2^m\left(\sin\frac{nt}{2}\right)^m\left(\sum_{k=-n}^{n}\left|\frac{c_k}{a_k}\right|^2\right)^{1/2} \leq 2^m\left(\sin\frac{nt}{2}\right)^m\frac{\|T_n\|}{|a_n|}. \tag{16}$$

From (16) we immediately obtain

$$\left(\int_0^h \omega_m^p(\varphi; t)\sin^\gamma nt\,\mathrm{d}t\right)^{1/p} \leq \frac{2^m\|T_n\|}{|a_n|}\cdot\left(\int_0^h \left(\sin\frac{nt}{2}\right)^m \sin^\gamma nt\,\mathrm{d}t\right)^{1/p} \leq 1.$$

It is therefore proven that $\mathcal{B}_{2n+1} \subset \mathcal{F}$. This inclusion, relation (13) and the definition of Bernstein width, imply the lower bound

$$\lambda_{2n}(\mathcal{F}; L_2) \geq \lambda_{2n-1}(\mathcal{F}; L_2) \geq E_n(\mathcal{F}) \geq 2^{-m}|a_n|\chi_{m,n,p,\gamma}(h). \tag{17}$$

From inequalities (15), (17) we obtain equality (14), which concludes the proof of Theorem 2.1. □

Let $W^{(r)}L_2$ $(r \in \mathbb{N}, W^{(0)}L_2) = L_2$ denote a class of functions $f(x) \in L_2$, with absolutely continuous derivatives up to order $(r-1)$, and derivative $f^{(r)}(x) \in L_2$. In [14, p. 36] it is proven that function $f(x) \in W^{(r)}L_2$ can be represented as

$$f(x) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\mathrm{d}t + \frac{1}{\pi}\int_0^{2\pi} \mathcal{D}_r(x-t)f^{(r)}(t)\mathrm{d}t,$$

where $\mathcal{D}_r(u) - 2\pi$-periodic function, defined by

$$\mathcal{D}_r(u) = \sum_{k=1}^{\infty} \frac{\cos(ku - \pi r/2)}{k^r}.$$

Let

$$\mathcal{F}^{(r)} \equiv \mathcal{F}^{(r)}(m,n,p,\gamma,h) = \left\{ f : f \in W^{(r)}L_2, \left(\int_0^h \omega_m^p(f^{(r)}; t)\sin^\gamma nt\mathrm{d}t \right)^{1/p} \leq 1 \right\}.$$

Since for $f(x) \in W^{(r)}L_2$, $|a_n| = n^{-r}$, condition $|a_j|j^{1/p} \geq |a_{j+1}|(j+1)^{1/p}$ implies that $p \geq 1/r$ and the following holds.

Corollary 2.2 *For any $m, n, r \in \mathbb{N},\ 1/r \leq p \leq 2,\ 0 \leq \gamma \leq rp - 1$ and $0 < h \leq \pi/n$ it holds that*

$$\lambda_{2n}(\mathcal{F}^{(r)}; L_2) = \lambda_{2n-1}(\mathcal{F}^{(r)}; L_2) = E_n(\mathcal{F}^{(r)}) = 2^{-m}n^{-r}\chi_{m,n,p,\gamma}(h).$$

In particular, for $h = \pi/n$ we have:

$$\begin{aligned}\lambda_{2n}(\mathcal{F}^{(r)}; L_2) &= \lambda_{2n-1}(\mathcal{F}^{(r)}; L_2) = E_n(\mathcal{F}^{(r)}) = 2^{-m}n^{-r}\chi_{m,n,p,\gamma}(\pi/n) \\ &= 2^{-\left(m+\frac{\gamma}{p}\right)} n^{-r+\frac{1}{p}} \left\{ \frac{\Gamma\left(\frac{mp+\gamma+1}{2}\right)\Gamma\left(\frac{\gamma+1}{2}\right)}{\Gamma\left(\frac{mp}{2}+\gamma+1\right)} \right\}^{-1/p}.\end{aligned}$$

Note, that for $\gamma = 0$ the result of Theorem 2.1 for Kolmogorov width was already obtained in [6, p.102]. Set

$$(nh - \pi)_+ = \{0,\ \text{if}\ nh \leq \pi;\ \ 1,\ \text{if}\ nh > \pi\}.$$

Let $\Phi(u)$ be an arbitrary continuous increasing function on $[0, \infty)$ satisfying the condition

$$\lim\{\Phi(u) : u \to 0\} = \Phi(0) = 0.$$

We shall designate by $\mathcal{F}(\Phi) := \mathcal{F}(m,n,p,\gamma,h;\Phi)$ the class of functions $f(x) = (\mathcal{K} * \varphi)(x)$, where $m, n \in \mathbb{N},\ 0 < p \leq 2$ and $\gamma > 0$, satisfying the condition

$$\left(\int_0^h \omega_m^p(\varphi; t)|\sin nt|^\gamma \mathrm{d}t \right)^{1/p} \leq \Phi(h)$$

for all $h \in (0, 2\pi]$. Theorem 2.1 was proved under the condition for $nh \leq \pi$.

Theorem 2.3 *Let* $m, n \in \mathbb{N},\ 0 < p \le 2,\ \gamma > 0$. *Let* $\Phi(h) \in C[0, 2\pi]$ *and assume that the following infimum* Q *is attained at some* $h_* \in [0, \pi/n]$

$$\inf_{0<h\le 2\pi} \frac{\Phi(h)}{\left(\int_0^{\min(h,\frac{\pi}{n})} \left(\sin\frac{nt}{2}\right)^{mp} \sin^\gamma nt\,\mathrm{d}t + \left(h-\frac{\pi}{n}\right)_*\right)^{1/p}} = Q. \tag{18}$$

Then we have

$$\lambda_{2n}(\mathcal{F}(\Phi); L_2) = \lambda_{2n-1}(\mathcal{F}(\Phi); L_2) = E_n(\mathcal{F}(\Phi)) = 2^{-m}|a_n|Q,$$

where $\lambda_k(\cdot)$ *are any of the k-widths of* $b_k(\cdot), d_k(\cdot), d^k(\cdot), \delta_k(\cdot), \Pi_k(\cdot)$.

Proof Following the reasoning in [6, pp. 105–107], from inequality (10) and from relation (13) for every $h \in [0, \pi/n]$ we obtain

$$\begin{aligned}\lambda_{2n}(\mathcal{F}(\Phi); L_2) &\le \lambda_{2n-1}(\mathcal{F}(\Phi); L_2)\\ &\le \Pi_{2n-1}(\mathcal{F}(\Phi); L_2) \le E_n(\mathcal{F}(\Phi))\\ &\le E_n(\mathcal{F}(\Phi)) \le 2^{-m}\,|a_n| \left(\int_0^h \left(\sin\frac{nt}{2}\right)^{mp} \sin^\gamma nt\,\mathrm{d}t\right)^{-1/p} \Phi(h),\end{aligned}$$

from which

$$\Pi_{2n-1}(\mathcal{F}(\Phi); L_2) \le 2^{-m}\,|a_n|\,Q. \tag{19}$$

To obtain the lower bound for the Bernstein n-width consider

$$\widetilde{\mathcal{B}}_{2n+1} = \left\{T_n : T_n \in \mathcal{T}_n,\ \|T_n\| \le 2^{-m}\,|a_n|\,Q\right\}.$$

We wish to prove that $\widetilde{\mathcal{B}}_{2n+1} \subset \mathcal{F}(\Phi)$. Using equality (16), we have

$$\begin{aligned}\left(\int_0^h \omega_m^p(T_n^{(r)}; t)|\sin nt|^\gamma \mathrm{d}t\right)^{1/p} &= \left(\int_0^{\pi/n} \omega_m^p(T_n^{(r)}; t)\sin^\gamma nt\,\mathrm{d}t + \int_{\pi/n}^h \omega_m^p(T_n^{(r)}; t)|\sin nt|^\gamma \mathrm{d}t\right)^{1/p}\\ &\le 2^m|a_n|^{-1}\|T_n\| \left(\int_0^{\pi/n} \left(\sin\frac{nt}{2}\right)^{mp} \sin^\gamma nt\,\mathrm{d}t + \int_{\pi/n}^h |\sin nt|^\gamma \mathrm{d}t\right)^{1/p}\\ &\le \left(\int_0^{\pi/n} \left(\sin\frac{nt}{2}\right)^{mp} \sin^\gamma nt\,\mathrm{d}t + \left(h-\frac{\pi}{n}\right)\right)^{1/p} Q \le \Phi(h).\end{aligned}$$

The last inequality implies $\widetilde{\mathcal{B}}_{2n+1} \subset \mathcal{F}(\Phi)$ and therefore

$$b_{2n-1}(\mathcal{F}(\Phi); L_2) \ge b_{2n-1}(\widetilde{\mathcal{B}}_{2n+1}; L_2) \ge 2^{-m}|a_n|\,Q. \tag{20}$$

Theorem 2.3 follows from (19) and (20). □

A natural question that arises is: for which values of α does the function $\Phi(h) = h^\alpha$ satisfy the condition of Theorem 2.3. It is obvious that for all $h \in [\pi/n, 2\pi]$, the result will follow if

$$\frac{\mathrm{d}}{\mathrm{d}h}\left\{h^\alpha \left(\int_0^{\pi/n} \left(\sin\frac{nt}{2}\right)^{mp} \sin^\gamma nt\,\mathrm{d}t + \left(h-\frac{\pi}{n}\right)\right)^{-\frac{1}{p}}\right\} \ge 0. \tag{21}$$

Doing the differentiation we obtain an inequality which is equivalent to (21),

$$\alpha p\left\{\int_0^{\pi/n}\left(\sin\frac{nt}{2}\right)^{mp}\sin^{\gamma}nt\mathrm{d}t+\left(h-\frac{\pi}{n}\right)\right\}-h\geq 0. \qquad (22)$$

The inequality (22) we write in the following form

$$\alpha p\left\{\int_0^{\pi/n}\left(\sin\frac{nt}{2}\right)^{mp}\sin^{\gamma}nt\mathrm{d}t-\frac{\pi}{n}\right\}\geq h(1-\alpha p).$$

But as

$$\int_0^{\pi/n}\left(\sin\frac{nt}{2}\right)^{mp}\sin^{\gamma}nt\mathrm{d}t-\frac{\pi}{n}\leq 0,$$

it is necessary that we must have $1-\alpha p\leq 0$, so that $\alpha\geq\frac{1}{p}$.
Evidently, for all $h\in[\pi/n,2\pi]$ we have:

$$\max\{h(1-\alpha p):\ h\in[\pi/n,2\pi]\}=\frac{\pi}{n}(1-\alpha p).$$

So from (22) we get

$$\alpha\geq\frac{1}{p}\left\{\frac{n}{\pi}\int_0^{\pi/n}\left(\sin\frac{nt}{2}\right)^{mp}\sin^{\gamma}nt\mathrm{d}t\right\}^{-1}=\frac{1}{p}\left\{\frac{\pi}{2^{\gamma}}\frac{\Gamma\left(\frac{mp}{2}+\gamma+1\right)}{\Gamma\left(\frac{mp+\gamma+1}{2}\right)\Gamma\left(\frac{\gamma+1}{2}\right)}\right\}, \qquad (23)$$

where $\Gamma(u)$ is Euler's gamma-function.

Thus, it is proved that for the function $\Phi(h)=h^{\alpha}$, $\alpha\geq 0$, condition (18) is guaranteed if α satisfies inequality (23), which does not depend upon n.

Finally, we note that the results of Theorem 2.3 contain in particular results of papers [2,5,7,17,18,20].

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